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# Diffusion as a singular homogenization of the Frenkel-Kontorova model

N. Alibaud <sup>\*</sup>, A. Briani<sup>†</sup>, R. Monneau <sup>‡</sup>

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## Abstract

In this work, we consider a general fully overdamped Frenkel-Kontorova model. This model describes the dynamics of a infinite chain of particles, moving in a periodic landscape. Our aim is to describe the macroscopic behavior of this system. We study a singular limit corresponding to a high density of particles moving in a vanishing periodic landscape. We identify the limit equation which is a nonlinear diffusion equation. Our homogenization approach is done in the framework of viscosity solutions.

**AMS Classification:** 35B27, 40L25, 35B10.

**Keywords:** particle systems, periodic homogenization, Frenkel-Kontorova models, Hamilton-Jacobi equations, nonlinear diffusion.

## 1 Introduction

In this paper we consider the dynamics of particles on the real line, moving with interactions with their neighbors and with a periodic landscape. For each  $i \in \mathbb{Z}$ , the position of the  $i$ -th particle is called  $U_i(\tau) \in \mathbb{R}$ , where  $\tau$  is the time. These particles are assumed to solve a generalized fully overdamped Frenkel-Kontorova (FK) model. In this model, the velocity is proportional to the force acting on the particles. The classical physical model was introduced by Kontorova, Frenkel in [12] to describe the plasticity at a microscopic level. For a good overview on the FK model and on its applications, we refer to the recent book [4] of Braun and Kivshar and the article of Floria and Mazo [7]. In the present paper, our generalized fully overdamped FK model is the following infinite system of ODEs:

$$(1.1) \quad \begin{cases} \frac{dU_i}{d\tau}(\tau) = \varepsilon^{2(\alpha-1)} F \left( \left[ \frac{U_{i+j}(\tau) - U_i(\tau)}{\varepsilon^{2(\alpha-1)}} \right]_j^m, U_i(\tau) \right) & \text{for } \tau \in (0, \infty) \text{ and } i \in \mathbb{Z} \\ U_i(0) = \frac{1}{\varepsilon} u_0(i\varepsilon^\alpha) & \text{for } i \in \mathbb{Z}, \end{cases}$$

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where  $\varepsilon > 0$  is a small parameter and we use the notation

$$[V_j]_j^m = V = (V_{-m}, V_{-m+1}, \dots, V_{-1}, V_1, \dots, V_{m-1}, V_m) \in \mathbb{R}^{2m}$$

for some fixed  $m \in \mathbb{N}$ . This dynamics corresponds to general interactions with the  $m$  nearest neighbors on the left and the  $m$  nearest neighbors on the right.

Our goal is to study the *macroscopic behaviour of the chain of particles as  $\varepsilon$  goes to zero*. To this end we make the following assumptions the initial data  $u_0$ , on the interactions  $F$  and on the exponent  $\alpha$ .

**(H0) (Initial gradient bounded from below and above)**

The function  $u_0$  is in  $W^{3,\infty}(\mathbb{R})$  and satisfies

$$\delta_0 \leq (u_0)_x \leq \frac{1}{\delta_0} \quad \text{on } \mathbb{R},$$

for some constant  $\delta_0 > 0$ .

**(H1) (i) (Regularity)**

$F(V, V_0) \in C^1(\mathbb{R}^{2m} \times \mathbb{R})$  and is globally Lipschitz continuous in  $(V, V_0)$ .

**(ii) (Periodicity)**

For all  $(V, V_0) \in \mathbb{R}^{2m} \times \mathbb{R}$ ,  $F(V, V_0 + 1) = F(V, V_0)$ .

**(iii) (Monotonicity)**

For all  $1 \leq |i| \leq m$ ,  $F_i := \frac{\partial F}{\partial V_i} \geq 0$  on  $\mathbb{R}^{2m} \times \mathbb{R}$ .

**(iv) (Invariance by linear additions)**

For all  $(V, V_0, q) \in \mathbb{R}^{2m} \times \mathbb{R} \times \mathbb{R}$ ,  $F([V_j + jq]_j^m, V_0) = F([V_j]_j^m, V_0)$ .

**(v) (Non-degeneracy w.r.t. quadratic displacements)**

For all  $(V_0, q) \in \mathbb{R} \times \mathbb{R}$ ,  $\frac{d}{dq} F \left( \left[ \begin{array}{c} j^2 \\ 2q \end{array} \right]_j^m, V_0 \right) \geq \nu$ ,

for some positive constant  $\nu$ .

**(vi) (Zero mean value of  $f$ )**

For all  $h \in \mathbb{R}$ , define  $f(h)$  as the unique real such that

$$(1.2) \quad F \left( \left[ \begin{array}{c} -\frac{j^2}{2} f(h) \\ j \end{array} \right]_j^m, h \right) = 0;$$

the 1-periodic function  $f : \mathbb{R} \rightarrow \mathbb{R}$  thus defined is assumed to satisfy

$$\int_0^1 f(h) dh = 0.$$

**(H2) (The exponent  $\alpha$ )** We suppose  $\alpha > 2$ .

Let us make some comments on these assumptions.

In assumption **(H0)** the relevant information is the boundedness of the gradient. The regularity can be weakened to  $C^1$  by standard approximation arguments. For the sake of simplicity in the proof of our main result (Theorem 1.1 below) we decide to ask for  $u_0$  in  $W^{3,\infty}(\mathbb{R})$ .

Let us now discuss assumption **(H1)**. First, assumption **(ii)** takes into account the existence of the periodic landscape. Second, the monotonicity assumption **(iii)** will play a crucial role in our analysis: this will guarantee the existence of a comparison principle for our system of particles. Remark also that assumption **(vi)** is always satisfied for  $F$  replaced by  $a + F$  for a suitable constant  $a \in \mathbb{R}$ . The zero mean value property of the function  $f$  means that the periodic landscape is balanced and does not induce any drift in the motion of the particles. The most striking condition is assumption **(iv)** which is technically used to remove the contribution of linear displacements of the particles, and to focus on quadratic displacements. This assumption is for instance always satisfied if  $F(V, V_0)$  is a function depending only on the symmetric part  $V_{-j} + V_j$  for each  $j = 1, \dots, m$  and on  $V_0$ . Assumption **(iv)** is also satisfied if  $F(V, V_0)$  is linear in  $V$ , with suitable coefficients vanishing the linear contributions  $[jq]_j^m$ . The non-degeneracy assumption w.r.t. the quadratic contributions  $\left[\frac{j^2}{2}q\right]_j^m$  **(v)** is used to ensure the existence of the periodic potential  $f$  by the inverse function theorem. In particular, our analysis would remain true if  $\nu$  depends on  $(V_0, q)$  and if we are sure that Equation (1.2) admits at least one solution.

Finally, let us remark that the reason why we need to restrict ourself to the case  $\alpha > 2$  in assumption **(H2)** is the verification of our Ansatz. More precisely, only to obtain the estimate (3.16) in the proof of Lemma 3.2 in Section 3. (Note that all the other computations remain true for  $\alpha > 1$ ). For the sake of completeness the limit case  $\alpha = 2$  is discussed in Subsection 3.2. It is in particular seen that the homogenised equation obtained for  $\alpha = 2$  is different in general from the one obtained for  $\alpha > 2$ .

The classical (and simplest) example of the fully overdamped Frenkel Kontorova model is

$$\frac{dU_i}{d\tau} = U_{i+1} + U_{i-1} - 2U_i + \varepsilon^{2(\alpha-1)} \sin(2\pi U_i)$$

corresponding to interactions with the first nearest neighbors with  $m = 1$  and

$$(1.3) \quad F(V_{-1}, V_1, V_0) = V_{-1} + V_1 + f(V_0)$$

with  $f(V_0) = \sin(2\pi V_0)$ . Remark that for an initial data  $U_i^\varepsilon(0) = \frac{1}{\varepsilon} u_0(i\varepsilon^\alpha)$ , we see that the high density of particles behaves like  $\varepsilon^{1-\alpha} \frac{1}{(u_0)_x}$ . This shows that our scaling corresponds to the case where *the small amplitude  $\varepsilon^{2(\alpha-1)}$  of the periodic potential behaves exactly as the square of the inverse of the particle density*. This can be rephrased, saying that we are interested in a distinguished limit with a weak periodic landscape and high density particles.

To study the macroscopic behavior of the system in the limit as  $\varepsilon$  goes to zero, it is convenient to introduce a parabolic rescaling, defining the function

$$(1.4) \quad U^\varepsilon(t, x) := \varepsilon U_{\lfloor \frac{x}{\varepsilon^\alpha} \rfloor} \left( \frac{t}{\varepsilon^{2\alpha}} \right)$$

where  $\lfloor \cdot \rfloor$  denotes the floor integer part. Note that if  $U$  is a solution of the system (1.1) then  $U^\varepsilon$  has to solve the following equation:

$$(1.5) \quad \begin{cases} U_t^\varepsilon(t, x) = \frac{1}{\varepsilon} F \left( \left[ \frac{U^\varepsilon(t, x + j\varepsilon^\alpha) - U^\varepsilon(t, x)}{\varepsilon^{2\alpha-1}} \right]_j, \frac{U^\varepsilon(t, x)}{\varepsilon} \right) & \text{for } (t, x) \in (0, \infty) \times \mathbb{R} \\ U^\varepsilon(0, x) = u_0(\lfloor \frac{x}{\varepsilon^\alpha} \rfloor \varepsilon^\alpha) & \text{for } x \in \mathbb{R}, \end{cases}$$

which will be understood in the viscosity sense.

Our main result, which studies the singular limit as  $\varepsilon$  goes to zero, is the following.

**Theorem 1.1. (Diffusive limit by homogenization)**

Assume **(H0)**, **(H1)** and **(H2)**. There exist a unique  $U^\varepsilon$  viscosity solution of (1.5) and a function  $G : (0, \infty) \rightarrow \mathbb{R}$  such that

$$(1.6) \quad G \in C^0(0, \infty) \text{ is positive, } \lim_{p \rightarrow 0} G(p) = 0 \text{ and } \lim_{p \rightarrow \infty} G(p) = l > 0.$$

Moreover, the function  $U^\varepsilon$  converges to  $u^0$  locally uniformly on compact sets of  $[0, \infty) \times \mathbb{R}$  as  $\varepsilon \rightarrow 0$ , where  $u^0$  is the unique viscosity solution of

$$(1.7) \quad \begin{cases} u_t^0 = G(u_x^0) u_{xx}^0 & \text{for } (t, x) \in (0, \infty) \times \mathbb{R}, \\ u^0(0, x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Remark that if we introduce the inverse  $g$  of  $u^0$ , defined by  $u^0(t, g(t, y)) = y$ , then the rescaled particle density  $\rho(t, y) = g_y(t, y)$  solves formally the following nonlinear diffusion equation  $\rho_t = \left( \frac{1}{\rho^2} G \left( \frac{1}{\rho} \right) \rho_y \right)_y$ . This shows a very small diffusion coefficient for large densities of particles, and a huge diffusion coefficient for small rescaled densities.

Theorem 1.1 is an homogenization result and is obtained in the framework of viscosity solutions. Let us mention the pioneering work of Lions, Papanicolau and Varadan [13] where homogenization of Hamilton-Jacobi equations has been done for the first time.

In Theorem 1.1, the effective equation satisfied by the limit is parabolic and is naturally associated to the parabolic rescaling (1.4). This result has to be compared to the homogenization result given in Forcadel, Imbert, Monneau [8] for a hyperbolic scaling (in the case  $\alpha = 1$ ), where the limit equation is a first order Hamilton-Jacobi equation. We can also refer the interested reader to [9], where a study of an overdamped dynamics of particles with two-body interactions is done, with particular applications to dislocation dynamics.

The main idea of the proof of Theorem 1.1 is that the solution  $U^\varepsilon$  is locally - close to a point  $(t_0, x_0)$  - well approximated by the following Ansatz

$$(1.8) \quad U^\varepsilon(t, x) = \varepsilon h \left( \frac{\tilde{U}^\varepsilon(t, x)}{\varepsilon}, u_x^0(t, x) \right) \quad \text{with} \quad \tilde{U}^\varepsilon(t, x) := u^0(t, x) + \varepsilon^2 v \left( \frac{u^0(t, x)}{\varepsilon} \right),$$

where  $u^0$  is the limit solution,  $h$  is a hull function and  $v$  is a corrector. Our proof follows the lines of the ‘‘perturbed test function’’ introduced by Evans in [6], based on the ansatz (1.8).

The functions  $h, v$  are defined below:

**Proposition 1.2. (Hull function, corrector and diffusion coefficient)**

Assume **(H1)**.

**(a) The hull function  $h$**

There exists a unique function  $h : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$  such that  $h$  is  $C^2$  w.r.t its first variable and satisfies for all  $(z, p) \in \mathbb{R} \times (0, \infty)$ ,

$$(1.9) \quad \begin{cases} F \left( \left[ \frac{j^2}{2} p^2 h_{zz}(z, p) \right]_j^m, h(z, p) \right) = 0, \\ h(z+1, p) = h(z, p) + 1, \\ h_z(z, p) > 0, \\ h(0, p) = 0. \end{cases}$$

Moreover, we have  $h \in C^3(\mathbb{R} \times (0, \infty))$ .

**(b) The corrector  $v$**

For all  $(z, p) \in \mathbb{R} \times (0, \infty)$ , define

$$(1.10) \quad A(z, p) := \frac{1}{2} + p \frac{h_{pz}(z, p)}{h_z(z, p)} \quad \text{and} \quad K(z, p) := \sum_{1 \leq |i| \leq m} i^2 F_i \left( \left[ \frac{j^2}{2} p^2 h_{zz}(z, p) \right]_j^m, h(z, p) \right).$$

Then, for all  $p_0 > 0$  and  $M_0 \in \mathbb{R}$  there exists a unique  $\lambda \in \mathbb{R}$  such that there exists  $v \in C^2(\mathbb{R})$  satisfying

$$(1.11) \quad \begin{cases} \lambda = K(z, p_0)A(z, p_0)M_0 + K(z, p_0)p_0^2 \left( \frac{v_{zz}(z)}{2} + \frac{h_{zz}(z, p_0)}{h_z(z, p_0)} v_z(z) \right) & \text{for } z \in \mathbb{R}, \\ v(z+1) = v(z) & \text{for } z \in \mathbb{R}. \end{cases}$$

**(c) The diffusion coefficient  $G$**

Finally, we have  $\lambda = G(p_0)M_0$  where  $G$  is the function in (1.7) and is defined for  $p > 0$  by

$$(1.12) \quad G(p) := \frac{\int_0^1 A(z, p) h_z^2(z, p) dz}{\int_0^1 \frac{h_z^2(z, p)}{K(z, p)} dz}.$$

In the ansatz (1.8) around the point  $(t_0, x_0)$ , we use the definition of  $v$  given in Proposition 1.2 with  $p_0 = u_x^0(t_0, x_0)$  and  $M_0 = u_{xx}^0(t_0, x_0)$ . Remark also that the expression of the diffusion coefficient  $G$  in (1.12) is explicit, which can be underlined, in comparison to general homogenization results, where usually the effective Hamiltonian is not explicit.

Let us now list the main properties of  $G$ .

**Proposition 1.3. (Qualitative properties of  $G$ )**

Assume **(H1)**. Then, the function  $G$  defined by (1.12) fulfills (1.6). Moreover  $G$  satisfies:

i) the limit  $l$  can be computed explicitly as follows:

$$(1.13) \quad l := \frac{1}{2} \left\{ \int_0^1 \left[ \sum_{1 \leq |i| \leq m} i^2 F_i \left( - \left[ \frac{j^2}{2} f(z) \right]_j^m, z \right) \right]^{-1} dz \right\}^{-1}$$

( $f$  being the 1-periodic function in (1.2)).

ii) For each  $k \in \mathbb{N}$ , if  $F \in C^{k+1}(\mathbb{R}^{2m} \times \mathbb{R})$  then  $G \in C^k(0, \infty)$ .

In general, we do not know if  $G$  is a monotone function of  $p$ . Nevertheless, for instance in the special case of the classical FK model, we have the following additional result:

**Proposition 1.4. (Monotonicity of  $G$  for the classical FK)**

For the classical FK model given in (1.3) satisfying assumptions **(H1)** and **(H2)**, the function  $G$  defined in (1.12) is analytic on  $(0, +\infty)$  and non-decreasing.

After finishing this work, we were aware of the work of Jerrard [11], where he studies general homogenization problems. A special case of his results concerns the homogenization of the particular equation

$$(1.14) \quad u_t^\varepsilon(t, x) = u_{xx}^\varepsilon(t, x) + \frac{1}{\varepsilon} f \left( \frac{u^\varepsilon}{\varepsilon} \right) \quad \text{for } (t, x) \in (0, +\infty) \times \mathbb{R},$$

with  $f$  periodic with zero mean value. This model is the continuous analogue of the classical FK model with (1.3), and the limit equation is the first equation in (1.7), i.e.

$$u_t^0 = G(u_x^0) u_{xx}^0$$

It turns out that the hull function and the corrector found in [11] are the same as in Proposition 1.2 for the particular expression (1.3) of  $F$ . Therefore, as a byproduct, Proposition 1.4 shows also that the diffusion coefficient  $G(p)$  found in [11] is monotone in  $p$ .

Let us finally remark that we are studying the homogenization of equations with periodic terms in  $u/\varepsilon$ , for which only few results exist. In this direction, let us mention the work of Boccardo, Murat [3] about the homogenization of elliptic equations, and the two recent works of Barles [1] and Imbert, Monneau [10].

**Organization of the paper**

The paper is organized as follows. Section 2 recalls some basic facts concerning viscosity solutions. Sections 3-4 are devoted to proof of Theorem 1.1. More precisely, in Section 3 we discuss our Ansatz (1.8), in Subsection 3.2 we give a formal discussion on the range of exponent  $\alpha \in [3/2, 2]$  and in Section 4 we actually give the proof of Theorem 1.1. For the sake of clarity we postpone the proofs of the existence and the regularity of the hull function  $h$  and the corrector  $v$  as well as the properties of  $G$  (Proposition 1.2, Proposition 1.3 and Proposition 1.4) in the last Section 5.

## Notations

Let us introduce the notations that are used throughout.

**Integer parts.** The floor and the ceiling integer part of a real number  $a$  are denoted by  $\lfloor a \rfloor$  and  $\lceil a \rceil$ , respectively.

**Cylinders of  $(0, \infty) \times \mathbb{R}$ .** For  $(t_0, x_0) \in (0, \infty) \times \mathbb{R}$  and  $r, R > 0$ , define  $Q_{r,R}(t_0, x_0) := (t_0 - r, t_0 + r) \times (x_0 - R, x_0 + R)$ . For the sake of simplicity,  $Q_{r,R}(t_0, x_0)$  will be sometimes denoted  $Q_{r,R}$ .

**Semi-continuous envelopes.** Let  $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  be a locally bounded function. We let  $u^* : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  denote the smallest upper semi-continuous (u.s.c. for short) function above  $u$ . The largest lower semi-continuous (l.s.c. for short) function below  $u$  is denoted  $u_*$ .

**Relaxed semi-limits.** Let  $u^\varepsilon : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  be indexed by  $\varepsilon > 0$ . Assume that the family  $\{u^\varepsilon : \varepsilon > 0\}$  is uniformly locally bounded. The upper relaxed semi-limit of  $u^\varepsilon$  as  $\varepsilon \rightarrow 0$  is the function  $\limsup_{\varepsilon \rightarrow 0}^* u^\varepsilon : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\limsup_{\varepsilon \rightarrow 0}^* u^\varepsilon(t, x) = \limsup_{(\tau, y, \varepsilon) \rightarrow (t, x, 0)} u^\varepsilon(\tau, y).$$

The lower relaxed semi-limit is defined by:  $\liminf_{\varepsilon \rightarrow 0}^* u^\varepsilon := -\limsup_{\varepsilon \rightarrow 0}^*(-u^\varepsilon)$ .

## 2 On viscosity solutions

In this section, we list some basic facts concerning viscosity solutions that are needed throughout. We skip almost all of the proofs, since they are either well-known or straightforward adaptations of classical ones. For a survey on the classical viscosity solutions theory, we refer the reader to the book of Barles [2] and to the user's guide of Crandall, Ishii and Lions [5].

Moreover, let us stress the fact that for  $\varepsilon = 1$  our system is a special case of the one studied by Forcadel, Imbert and Monneau, therefore as more related reference one can see Section 2 in [8].

We need to introduce the concept of viscosity solution because in order to study the system of ODEs (1.1) we embed it into a single PDE. Indeed, by considering,  $\mathcal{U}(\tau, y) := U_{\lfloor y \rfloor}(\tau)$  we are lead to the following "finite-difference like" PDE:

$$(2.1) \quad \begin{cases} \mathcal{U}_\tau(\tau, y) = \varepsilon^{2(\alpha-1)} F\left(\frac{[\mathcal{U}(\tau, y+j) - \mathcal{U}(\tau, y)]_j^m}{\varepsilon^{2(\alpha-1)}}, \mathcal{U}(\tau, y)\right) & \text{for } (\tau, y) \in (0, \infty) \times \mathbb{R} \\ \mathcal{U}(0, y) = u_0^\varepsilon(y) & \text{for } y \in \mathbb{R}, \end{cases}$$

where  $u_0^\varepsilon(y) := \frac{1}{\varepsilon} u_0(\lfloor y \rfloor \varepsilon^\alpha)$ . Since the results we are going to give are valid for each  $\alpha$  and  $\varepsilon$  fixed, we will not stress the dependence of the function  $\mathcal{U}$  on them.

Since the initial datum is *sublinear* by **(H0)**, it is natural to restrict to the study of sublinear solutions.

**Definition 2.1. (Sublinear functions)** *A function  $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is sublinear iff for all  $T > 0$ ,  $\sup_{(\tau, y) \in [0, T] \times \mathbb{R}} \frac{|u(\tau, y)|}{1+|y|} < \infty$ .*

Let us recall the notion of viscosity solutions.



**Definition 2.2. (Viscosity solutions)**

Let  $\mathcal{U} : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  be a sublinear function. We say that:

1. An upper semi-continuous (resp. lower-semicontinuous) function  $\mathcal{U} : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is a viscosity subsolution (resp. supersolution) of (2.1) on an open set  $\Omega \subset (0, \infty) \times \mathbb{R}$  if for any  $(\tau_0, y_0) \in \Omega$  and any test function  $\phi \in C^\infty(\Omega)$  such that  $\mathcal{U} - \phi$  attains a strict local maximum (resp. local minimum) at the point  $(\tau_0, y_0)$ , we have

$$\phi_\tau(\tau_0, y_0) - \varepsilon^{2(\alpha-1)} F \left( \frac{[\mathcal{U}(\tau_0, y_0 + j) - \mathcal{U}(\tau_0, y_0)]_j^m}{\varepsilon^{2(\alpha-1)}}, \mathcal{U}(\tau_0, y_0) \right) \leq 0$$

$$\text{(resp. } \phi_\tau(\tau_0, y_0) - \varepsilon^{2(\alpha-1)} F \left( \frac{[\mathcal{U}(\tau_0, y_0 + j) - \mathcal{U}(\tau_0, y_0)]_j^m}{\varepsilon^{2(\alpha-1)}}, \mathcal{U}(\tau_0, y_0) \right) \geq 0.)$$

2. The function  $\mathcal{U}$  is said to be a viscosity subsolution (resp. supersolution) of (2.1) in  $[0, \infty) \times \mathbb{R}$ , if it is a viscosity subsolution (resp. supersolution) on  $\Omega = (0, \infty) \times \mathbb{R}$  and if moreover it satisfies

$$\mathcal{U}(0, y) \leq (u_0^\varepsilon)^*(y) \quad \forall y \in \mathbb{R}$$

$$\text{(resp. } \mathcal{U}(0, y) \geq (u_0^\varepsilon)_*(y) \quad \forall y \in \mathbb{R}).$$

3. A function  $\mathcal{U} : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is said to be a viscosity solution of (2.1) if  $\mathcal{U}^*$  is a subsolution and  $\mathcal{U}_*$  is a supersolution. ( $\mathcal{U}^*$  and  $\mathcal{U}_*$  being the u.s.c. and l.s.c. envelope of  $\mathcal{U}$ , respectively.)

Here is a comparison principle for the Cauchy problem.

**Theorem 2.3. (Comparison principle)**

Assume **(H0)**-**(H1)**. Let  $u$  and  $v$  be respectively a sub- and supersolution of (2.1). Then we have  $u \leq v$  on  $[0, \infty) \times \mathbb{R}$ .

In the sequel, we shall also need a comparison principle on bounded subdomains. Precisely, let us consider cylinders  $Q_{r,R} := Q_{r,R}(t_0, x_0)$  and  $Q_{r,R+m} := Q_{r,R+m}(t_0, x_0)$  such that  $Q_{r,R} \subset (0, \infty) \times \mathbb{R}$ . Then, we have the following result.

**Theorem 2.4. (Comparison principle on bounded sets)**

Assume **(H1)**. Let  $u$  and  $v$  be respectively a sub- and a supersolution of (2.1) in  $Q_{r,R}$  such that

$$(2.2) \quad u \leq v \quad \text{on} \quad \overline{Q_{r,R+m}} \setminus Q_{r,R}.$$

Then we have  $u \leq v$  on  $Q_{r,R}$ .

**Remark 2.5.** Let us recall that the PDE in (1.5) is obtained from the one in (2.1), after rescaling by (1.4). In particular, Proposition 2.4 remains true for (1.5) by replacing Condition (2.2) by “ $u \leq v$  on  $\overline{Q_{r,R+\varepsilon^\alpha m}} \setminus Q_{r,R}$ ”.

Let us now give an existence and uniqueness result.

**Theorem 2.6. (Existence and uniqueness)**

Under **(H0)**-**(H1)** there exists a unique continuous viscosity solution to (2.1).

**Proof.** To get the existence by the classical Perron's method we need to construct barriers. Since this construction will be done in the proof of convergence we will skip the details here. (See Step 1 in the proof of Theorem 1.1 in Section 4 below.) The uniqueness in the class of viscosity solution will be then a direct consequence of existence and comparison.  $\square$

To prove the homogeneization result (Theorem 1.1) we will consider the relaxed semilimits of the sequence of solutions  $u^\varepsilon$  and we will need to prove that those are solutions of the limit equation in the viscosity sense. For the sake of completeness let us recall the definition of viscosity solution in this case and the comparison result we will need in the proof. (The proof of the comparison principle for the limit equation being completely classical can be found in [5], Theorem 8.2.)

**Definition 2.7. (Viscosity solution: second order.)**

Let  $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  locally bounded and sublinear. We say that:

1. The function  $u$  is a viscosity subsolution (resp. supersolution) to (1.7) in  $[0, \infty) \times \mathbb{R}$  iff  $u$  is u.s.c. (resp. l.s.c.) and for any  $(t_0, x_0) \in (0, \infty) \times \mathbb{R}$  and any test function  $\phi \in C^\infty((0, \infty) \times \mathbb{R})$  such that  $u - \phi$  attains a strict local maximum (resp. local minimum) at the point  $(t_0, x_0)$ , we have

$$\phi_t(t_0, x_0) - G(\phi_x(t_0, x_0))\phi_{xx}(t_0, x_0) \leq 0$$

$$\text{(resp. } \phi_t(t_0, x_0) - G(\phi_x(t_0, x_0))\phi_{xx}(t_0, x_0) \geq 0\text{.)}$$

Moreover at time zero we have  $u(0, x) \leq (u_0)^*(x) \forall x \in \mathbb{R}$  (resp.  $u(0, x) \geq (u_0)_*(x) \forall x \in \mathbb{R}$ ).

2. A continuous function  $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ , is said to be a viscosity solution of (1.7) if it is both viscosity sub and supersolution.

**Theorem 2.8. (Comparison principle.)** Assume (H0) and (1.6). Let  $u$  and  $v$  be a viscosity subsolution and a viscosity supersolution of (1.7) in  $[0, \infty) \times \mathbb{R}$ , respectively. Then  $u(t, x) \leq v(t, x)$  for all  $(t, x) \in [0, \infty) \times \mathbb{R}$ .

### 3 The ansatz

In order to prove Theorem 1.1, we try to guess the behavior of  $U^\varepsilon$  around a fixed point  $(t_0, x_0)$ . Our ansatz has been described in (1.8). To verify this thesis, we first have to answer to the following questions:

**Question 1.** Does this ansatz almost satisfy the PDE in (1.5) around  $(t_0, x_0)$ ?

**Question 2.** Does this ansatz converge toward  $u^0$  as  $\varepsilon$  goes to zero, around  $(t_0, x_0)$ ?

This section is devoted to the verification of these points. Let us first enumerate some required properties on  $h$  and  $v$  that easily follows from the fact that  $h$  (resp.  $v$ ) is regular and linear plus periodic (resp. periodic).

**Lemma 3.1. (Hull function and corrector properties)**

Assume **(H1)**. For each  $\delta > 0$ , the hull function satisfies:

- (a)  $C_0(\delta) := \max_{\mathbb{R} \times [\delta, \frac{1}{\delta}]} |h(z, p) - z| < \infty$ .
- (b)  $C_1(\delta) := \|h_z\|_{C_b^2(\mathbb{R} \times [\delta, \frac{1}{\delta}])} + \|h_p\|_{C_b^2(\mathbb{R} \times [\delta, \frac{1}{\delta}])} < \infty$ .
- (c)  $m(\delta) := \min_{\mathbb{R} \times [\delta, \frac{1}{\delta}]} h_z > 0$ .

The functions  $A$  and  $K$  defined in (1.10) satisfy:

- (d)  $A$  and  $K$  are bounded and uniformly continuous on  $\mathbb{R} \times [\delta, \frac{1}{\delta}]$ .

Moreover, all corrector (associated to fixed  $p_0 > 0$  and  $M_0 \in \mathbb{R}$ ) satisfies

- (e)  $C_2 := \|v\|_{C_b^2(\mathbb{R})} < \infty$  and  $v_{zz}$  is uniformly continuous on  $\mathbb{R}$ .

### 3.1 Verification of the ansatz

Let us now state and prove two lemmata, that could be seen in some sense as answers to **Questions 1-2** above. In these lemmata, we replace  $u^0$  in (1.8) by general test functions  $\phi$  (see (3.3)). This test function will be used in Definition 2.7, when we shall prove the convergence of  $U^\varepsilon$  by the perturbed test function method in the next section.

For the sake of clarity, we introduce the following notation for the PDE operator in (1.5): for all  $\varepsilon > 0$ ,  $\phi \in C^\infty((0, \infty) \times \mathbb{R})$  and  $(t, x) \in (0, \infty) \times \mathbb{R}$ , define

$$(3.1) \quad \mathcal{L}_\varepsilon[\phi](t, x) := \phi_t(t, x) - \frac{1}{\varepsilon} F \left( \left[ \frac{\phi(t, x + j\varepsilon^\alpha) - \phi(t, x)}{\varepsilon^{2\alpha-1}} \right]_j, \frac{\phi(t, x)}{\varepsilon} \right).$$

With this notation in hand, the PDE in (1.5) can be rewritten as  $\mathcal{L}_\varepsilon[U^\varepsilon] = 0$ .

Here is the main result of this section that we will prove below.

**Lemma 3.2. (Local error estimate)**

**(Settings)** Assume **(H1)** and **(H2)**. Let  $\phi \in C^\infty((0, \infty) \times \mathbb{R})$  be such that

$$(3.2) \quad \phi_x(t_0, x_0) > 0 \quad \text{for some fixed } (t_0, x_0) \in (0, \infty) \times \mathbb{R}.$$

Let  $\lambda$  be defined by Proposition 1.2 item **(b)** with  $(p_0, M_0) := (\phi_x(t_0, x_0), \phi_{xx}(t_0, x_0))$ , and let  $v$  be an associated corrector. Let  $h$  be the hull function of Proposition 1.2 and for  $\varepsilon > 0$ , define

$$(3.3) \quad \phi^\varepsilon(t, x) := \varepsilon h \left( \frac{\tilde{\phi}^\varepsilon(t, x)}{\varepsilon}, \phi_x(t, x) \right) \quad \text{with} \quad \tilde{\phi}^\varepsilon(t, x) := \phi(t, x) + \varepsilon^2 v \left( \frac{\phi(t, x)}{\varepsilon} \right).$$

**(Result)** Then,  $\phi^\varepsilon(t, x)$  and  $\mathcal{L}_\varepsilon[\phi^\varepsilon](t, x)$  are well-defined for  $(t, x, \varepsilon)$  sufficiently close to  $(t_0, x_0, 0)$ . Moreover, we have

$$(3.4) \quad \mathcal{L}_\varepsilon[\phi^\varepsilon](t, x) = (\phi_t(t, x) - \lambda + \mathcal{E}(t, x, \varepsilon)) h_{\tilde{z}} \quad \text{with} \quad \lim_{(t, x, \varepsilon) \rightarrow (t_0, x_0, 0)} |\mathcal{E}(t, x, \varepsilon)| = 0,$$

where we set  $h_{\tilde{z}} = h_{\tilde{z}}(\tilde{z}, \phi_x(t, x))$  with  $\frac{\tilde{\phi}^\varepsilon(t, x)}{\varepsilon}$ .

**Lemma 3.3. (Convergence toward  $u^0$ )**

Assume **(H1)** and **(H2)** and let  $\phi^\varepsilon$  be defined by (3.3). Then, there exists  $r > 0$  such that for all  $\varepsilon > 0$ ,  $\phi^\varepsilon$  is well-defined on  $Q_{r,r} = Q_{r,r}(t_0, x_0)$  and converges toward  $\phi$ , as  $\varepsilon \rightarrow 0$ , uniformly on  $Q_{r,r}$ .

**Proof.** The thesis easily follows from Lemma 3.1 items **(a)** and **(e)**.  $\square$

**Remark 3.4.** Taking  $\phi = u^0$ , our ansatz in (1.8) writes  $U^\varepsilon \approx (u^0)^\varepsilon$  around  $(t_0, x_0)$ . These lemmata then mean formally that we indeed have  $\mathcal{L}_\varepsilon[(u^0)^\varepsilon] \approx 0$  and  $(u^0)^\varepsilon \approx u^0$  around  $(t_0, x_0)$ .

**Proof of Lemma 3.2.** Let us first remark that without loss of generality we can replace the assumption  $\phi \in C^\infty((0, \infty) \times \mathbb{R})$  by the following:

$$(3.5) \quad \begin{cases} \phi \text{ is Lipschitz-continuous on } (0, \infty) \times \mathbb{R} \text{ with } \delta \leq \phi_x \leq \frac{1}{\delta} \text{ for some } \delta > 0. \\ |\phi_x| + |\phi_{xx}| + |\phi_{xxx}| + |\phi_{xt}| + |\phi_t| \leq C_\phi \text{ on } (0, \infty) \times \mathbb{R} \text{ for some constant } C_\phi. \\ |\phi_t| + |\phi_{xt}| \leq C'_\phi \text{ on } (0, \infty) \times \mathbb{R} \text{ for some constant } C'_\phi. \end{cases}$$

Indeed, it is clear from (3.2) that  $\delta \leq \phi_x \leq \frac{1}{\delta}$  on  $Q_{r,2r} = Q_{r,2r}(t_0, x_0)$ , for some  $\delta > 0$  and  $r > 0$ . It follows that  $\phi^\varepsilon$  is well-defined on  $Q_{r,2r}$ , since the second argument of  $h$  in (3.3) is positive. By (3.1), it is also obvious that for all  $\varepsilon \leq \left(\frac{r}{m}\right)^\frac{1}{\alpha}$ ,  $\mathcal{L}_\varepsilon[\phi^\varepsilon]$  is well-defined on  $Q_{r,r}$  and only depends on the values of  $\phi$  on  $Q_{r,r+\varepsilon^\alpha m} \subseteq Q_{r,2r}$ . In particular, it is easy to modify  $\phi$  outside  $Q_{r,2r}$  in order to verify (3.5), without changing the value of  $\mathcal{L}_\varepsilon[\phi^\varepsilon](t, x)$  for  $(t, x, \varepsilon)$  close to  $(t_0, x_0, 0)$ .

Therefore we will prove (3.4) under the additional assumption (3.5). Our strategy is the following. First, we develop  $\mathcal{L}_\varepsilon[\phi^\varepsilon]$  w.r.t.  $\varepsilon$  by using Taylor's formula. This will give an expression of the form:

$$(3.6) \quad \mathcal{L}_\varepsilon[\phi^\varepsilon](t, x) = \frac{1}{\varepsilon} (\dots) + \varepsilon^0 (\dots) + o(1),$$

where  $o(1)$  denotes an error that vanishes as  $\varepsilon \rightarrow 0$ . Second, we use the cell equations (1.9) and (1.11), in order to vanish respectively the terms of order  $\frac{1}{\varepsilon}$  and  $\varepsilon^0$ .

The rest of the proof is organized in several steps. In the first step, we introduce some notations (slow and fast variables) that shall be used throughout. In Step 2, we give a first computation of  $\mathcal{L}_\varepsilon[\phi^\varepsilon]$  that has to be developed w.r.t.  $\varepsilon$ . In Steps 3-5, we detail the successive Taylor's expansions used to get (3.6). Finally, we conclude in Step 6 by using the cell equations.

*Step 1: slow and fast variables.* Let us introduce the new variables

$$\tilde{z} := \frac{\tilde{\phi}^\varepsilon(t, x)}{\varepsilon}, \quad z := \frac{\phi(t, x)}{\varepsilon} \quad \text{and} \quad p := \phi_x(t, x).$$

The functions  $\phi^\varepsilon$  and  $\tilde{\phi}^\varepsilon$  then write

$$(3.7) \quad \phi^\varepsilon(t, x) = \varepsilon h(\tilde{z}, p) \quad \text{and} \quad \tilde{\phi}^\varepsilon(t, x) = \phi(t, x) + \varepsilon^2 v(z).$$

In order to simplify the notations, we will not precise if not necessary the variables of  $\phi$ ,  $\tilde{\phi}^\varepsilon$ ,  $\phi^\varepsilon$ ,  $h$  and  $v$ . Without any more precision, the reader will have to understand that these functions and their derivatives are expressed at the variables as in (3.7).

*Step 2: first equation in  $\phi^\varepsilon$ .* We have

$$\phi_t^\varepsilon = h_z \tilde{\phi}_t^\varepsilon + \varepsilon h_p \phi_{xt} = h_z \phi_t + \varepsilon (h_z v_z \phi_t + h_p \phi_{xt}).$$

Consequently, we get the first following equation for  $\phi^\varepsilon$ :

$$(3.8) \quad \mathcal{L}_\varepsilon[\phi^\varepsilon] = h_z \phi_t + \varepsilon (h_z v_z \phi_t + h_p \phi_{xt}) - \frac{1}{\varepsilon} F \left( \frac{1}{\varepsilon^{2\alpha-2}} V, h \right),$$

where we set:

$$(3.9) \quad V := [V_j]_j^m = \left[ h \left( \frac{\tilde{\phi}^\varepsilon(t, x + j\varepsilon^\alpha)}{\varepsilon}, \phi_x(t, x + j\varepsilon^\alpha) \right) - h(\tilde{z}, p) \right]_j^m.$$

*Step 3: expansion of  $V$ .* In this step, we develop the first argument  $\frac{1}{\varepsilon^{2\alpha-2}} V$  of  $F$ . Let us begin by developing  $\phi_x$  and  $\tilde{\phi}^\varepsilon$  around  $(t, x)$ .

We claim that for all  $1 \leq |j| \leq m$ ,  $(t, x) \in (0, \infty) \times \mathbb{R}$  and  $\varepsilon > 0$  we have

$$(3.10) \quad \begin{cases} \tilde{\phi}^\varepsilon(t, x + j\varepsilon^\alpha) = \tilde{\phi}^\varepsilon + j\varepsilon^\alpha \tilde{\phi}_x^\varepsilon + \frac{j^2}{2} \varepsilon^{2\alpha} \tilde{\phi}_{xx}^\varepsilon + \tilde{R}(t, x, j, \varepsilon), \\ \text{with } \frac{|\tilde{R}(t, x, j, \varepsilon)|}{\varepsilon^{2\alpha}} \text{ as } \varepsilon \rightarrow 0 \text{ uniformly in } (t, x, j). \end{cases}$$

By the classical Taylor's formula, we have:

$$(3.11) \quad |\tilde{R}(t, x, j, \varepsilon)| \leq j^2 \varepsilon^{2\alpha} \frac{\omega_\varepsilon(|j|\varepsilon^\alpha)}{2},$$

where  $\omega_\varepsilon(\cdot)$  is the modulus of continuity in  $x$  of  $\tilde{\phi}_{xx}^\varepsilon = \phi_{xx} + v_{zz} \phi_x^2 + \varepsilon v_z \phi_{xx}$  (by Definition (3.3)). Let us estimate the modulus of continuity of these three terms. The most difficult is the middle term  $v_{zz} \phi_x^2$ . Let  $(x, x') \in \mathbb{R}$  be such that  $|x - x'| \leq \beta$ . We have

$$\begin{aligned} I := \left| v_{zz} \left( \frac{\phi(t, x)}{\varepsilon} \right) \phi_x^2(t, x) - v_{zz} \left( \frac{\phi(t, x')}{\varepsilon} \right) \phi_x^2(t, x') \right| \\ \leq C_2 C_\phi \beta + C_\phi^2 \left| v_{zz} \left( \frac{\phi(t, x)}{\varepsilon} \right) - v_{zz} \left( \frac{\phi(t, x')}{\varepsilon} \right) \right|, \end{aligned}$$

where  $C_2$  is the bound on  $v_{zz}$  in Lemma 3.1 (e) and  $C_\phi$  is the bound in (3.5). We have

$$\left| v_{zz} \left( \frac{\phi(t, x)}{\varepsilon} \right) - v_{zz} \left( \frac{\phi(t, x')}{\varepsilon} \right) \right| \leq \omega_v \left( \frac{\phi(t, x) - \phi(t, x')}{\varepsilon} \right) \leq \omega_v \left( \frac{C_\phi \beta}{\varepsilon} \right),$$

where  $\omega_v(\cdot)$  is the modulus of continuity of  $v_{zz}$  on  $\mathbb{R}$  from Lemma 3.1 (e). It follows that  $I \leq C_2 C_\phi \beta + C_\phi^2 \omega_v \left( \frac{C_\phi \beta}{\varepsilon} \right)$ , which proves that the modulus of continuity  $\omega(\cdot)$  of  $v_{zz} \phi_x^2$  satisfies

$$\omega(\beta) \leq C_2 C_\phi \beta + C_\phi^2 \omega_v \left( \frac{C_\phi \beta}{\varepsilon} \right) \quad \text{for all } \beta > 0.$$

Since (3.5) and Lemma 3.1 (e) imply that  $\phi_{xx} + \varepsilon v_z \phi_{xx}$  is Lipschitz-continuous in  $x$  with a Lipschitz-constant  $L_\phi$  independent on small  $\varepsilon$ , we get

$$\omega_\varepsilon(\beta) \leq L_\phi \beta + C_2 C_\phi \beta + C_\phi^2 \omega_v \left( \frac{C_\phi}{\varepsilon} \beta \right).$$

Thus (3.11) and  $\alpha > 1$  imply  $\tilde{R}(t, x, j, \varepsilon) = o(\varepsilon^{2\alpha})$  and the proof of (3.10) is complete.

Recall that  $\tilde{\phi}_x^\varepsilon = p + \varepsilon v_z p$  and  $\tilde{\phi}_{xx}^\varepsilon = \phi_{xx} + v_{zz} p^2 + \varepsilon v_z \phi_{xx}$ . Dividing (3.10) by  $\varepsilon$ , we get

$$\begin{aligned} \frac{\tilde{\phi}^\varepsilon(t, x + j\varepsilon^\alpha)}{\varepsilon} &= \frac{\tilde{\phi}^\varepsilon}{\varepsilon} + j\varepsilon^{\alpha-1} \tilde{\phi}_x^\varepsilon + \frac{j^2}{2} \varepsilon^{2\alpha-1} \tilde{\phi}_{xx}^\varepsilon + o(\varepsilon^{2\alpha-1}) \\ (3.12) \quad &= \tilde{z} + j\varepsilon^{\alpha-1} p + j\varepsilon^\alpha v_z p + \frac{j^2}{2} \varepsilon^{2\alpha-1} (\phi_{xx} + v_{zz} p^2) + o(\varepsilon^{2\alpha-1}) \\ (3.13) \quad &:= \tilde{z} + \mathcal{E}_1. \end{aligned}$$

Next, from (3.5) we deduce that

$$(3.14) \quad \phi_x(t, x + j\varepsilon^\alpha) = p + j\varepsilon^\alpha \phi_{xx} + o(\varepsilon^\alpha) := p + \mathcal{E}_2.$$

We can now develop the term  $h(\dots, \dots)$  in (3.9) around  $(\tilde{z}, p)$ . We get the following expansion

$$\begin{aligned} (3.15) \quad h \left( \frac{\tilde{\phi}^\varepsilon(t, x + j\varepsilon^\alpha)}{\varepsilon}, \phi_x(t, x + j\varepsilon^\alpha) \right) - h(\tilde{z}, p) &= h(\tilde{z} + \mathcal{E}_1, p + \mathcal{E}_2) - h(\tilde{z}, p) \\ &= h_{\tilde{z}} \mathcal{E}_1 + h_p \mathcal{E}_2 + \frac{h_{\tilde{z}\tilde{z}}}{2} \mathcal{E}_1^2 + \frac{h_{pp}}{2} \mathcal{E}_2^2 + h_{\tilde{z}p} \mathcal{E}_1 \mathcal{E}_2 + R, \end{aligned}$$

for some rest  $R = R(t, x, j, \varepsilon)$ . Since  $h$  is  $C^3$ , Taylor's Young formula implies that  $|R| \leq (|\mathcal{E}_1| + |\mathcal{E}_2|)^3 \frac{1}{3!} \|D^3 h\|_{\mathcal{D}}$ , where  $\mathcal{D}$  is the segment of extremities  $(\tilde{z}, p)$  and  $\left( \frac{\tilde{\phi}^\varepsilon(t, x + j\varepsilon^\alpha)}{\varepsilon}, \phi_x(t, x + j\varepsilon^\alpha) \right)$ . But, both these points belong to  $\mathbb{R} \times [\delta, \frac{1}{\delta}]$  by (3.5). Hence, Lemma 3.1 implies that

$$|R| \leq (|\mathcal{E}_1| + |\mathcal{E}_2|)^3 \frac{C_1(\delta)}{3!}.$$

Since the term of lowest order in  $|\mathcal{E}_1| + |\mathcal{E}_2|$  is  $j\varepsilon^{\alpha-1} p$ , the better estimate we can have for  $R$  is  $|R| \leq C\varepsilon^{3(\alpha-1)}$ . (Notice that the constant  $C$  does not depend on  $\varepsilon$  sufficiently small,  $(t, x) \in (0, \infty) \times \mathbb{R}$  and  $1 \leq |j| \leq m$ , since the other terms of  $|\mathcal{E}_1| + |\mathcal{E}_2|$  are controlled by (3.5) and Lemma 3.1). Since by **(H2)**  $\alpha$  is assumed greater than 2, we get *a fortiori*

$$(3.16) \quad R = o(\varepsilon^{2\alpha-1}).$$

*Let us stress the fact that this is the only point in which **(H2)** is used. All the other estimates in the paper are valid for  $\alpha > 1$ .*

Let us now develop all the terms in (3.15). In order to control all the negligible terms, we use (3.5) and Lemma 3.1. We get:

$$\begin{aligned}
h_{\bar{z}}\mathcal{E}_1 &= jh_{\bar{z}}(\varepsilon^{\alpha-1}p + \varepsilon^\alpha v_z p) + \varepsilon^{2\alpha-1} \frac{j^2}{2} h_{\bar{z}}(\phi_{xx} + v_{zz}p^2) + o(\varepsilon^{2\alpha-1}). \\
h_p\mathcal{E}_2 &= j\varepsilon^\alpha h_p \phi_{xx} + o(\varepsilon^\alpha). \\
\frac{h_{\bar{z}\bar{z}}}{2} \mathcal{E}_1^2 &= \varepsilon^{2\alpha-2} \frac{j^2}{2} h_{\bar{z}\bar{z}} p^2 + \varepsilon^{2\alpha-1} j^2 h_{\bar{z}\bar{z}} v_z p^2 + o(\varepsilon^{2\alpha-1}). \\
\frac{h_{pp}}{2} \mathcal{E}_2^2 &= o(\varepsilon^{2\alpha-1}). \\
h_{\bar{z}p} \mathcal{E}_1 \mathcal{E}_2 &= \varepsilon^{2\alpha-1} j^2 h_{\bar{z}p} p \phi_{xx} + o(\varepsilon^{2\alpha-1}).
\end{aligned}$$

Plugging this into (3.15), we get the following expansion:

$$\begin{aligned}
(3.17) \quad h \left( \frac{\tilde{\phi}^\varepsilon(t, x + j\varepsilon^\alpha)}{\varepsilon}, \phi_x(t, x + j\varepsilon^\alpha) \right) - h(\tilde{z}, p) &= \\
&= j \{ h_{\bar{z}}(\varepsilon^{\alpha-1}p + \varepsilon^\alpha v_z p) + \varepsilon^\alpha h_p \phi_{xx} \} + \varepsilon^{2\alpha-2} \left\{ \frac{j^2}{2} h_{\bar{z}\bar{z}} p^2 \right\} \\
&\quad + \varepsilon^{2\alpha-1} \left\{ \frac{j^2}{2} (h_{\bar{z}}(\phi_{xx} + v_{zz}p^2) + 2h_{\bar{z}\bar{z}} v_z p^2 + 2h_{\bar{z}p} p \phi_{xx}) \right\} + o(\varepsilon^{2\alpha-1}).
\end{aligned}$$

Dividing (3.17) by  $\varepsilon^{2\alpha-2}$ , we get the following expansion of  $V$  in (3.9):

$$\begin{aligned}
(3.18) \quad \frac{1}{\varepsilon^{2\alpha-2}} V_j &= \frac{1}{\varepsilon^{2\alpha-2}} \left\{ h \left( \frac{\tilde{\phi}^\varepsilon(t, x + j\varepsilon^\alpha)}{\varepsilon}, \phi_x(t, x + j\varepsilon^\alpha) \right) - h(\tilde{z}, p) \right\} \\
&= jq + \frac{j^2}{2} h_{\bar{z}\bar{z}} p^2 + \varepsilon j^2 h_{\bar{z}} \left\{ A \phi_{xx} + p^2 \left( \frac{v_{zz}}{2} + \frac{h_{\bar{z}\bar{z}}}{h_{\bar{z}}} v_z \right) \right\} + o(\varepsilon),
\end{aligned}$$

where  $A = A(\tilde{z}, p)$  is defined in (1.10) and  $q := \frac{1}{\varepsilon^{2\alpha-2}} \{ h_{\bar{z}}(\varepsilon^{\alpha-1}p + \varepsilon^\alpha v_z p) + \varepsilon^\alpha h_p \phi_{xx} \}$  is a linear displacement.

*Step 4: removing linear displacement.* Before doing the last expansion of  $F$ , we have to remove the linear displacements. This step is crucial, since  $q$  contains terms of order  $\varepsilon^{-\alpha+1}$  and  $\varepsilon^{-\alpha+2}$  that can not be controlled. We thus use **(H1)** (iv) to see from (3.18) that

$$\begin{aligned}
(3.19) \quad F \left( \frac{1}{\varepsilon^{2\alpha-2}} V, h \right) &= F \left( \left[ jq + \frac{j^2}{2} h_{\bar{z}\bar{z}} p^2 + \varepsilon j^2 h_{\bar{z}} \left\{ A \phi_{xx} + p^2 \left( \frac{v_{zz}}{2} + \frac{h_{\bar{z}\bar{z}}}{h_{\bar{z}}} v_z \right) \right\} + o(\varepsilon) \right]_j^m, h \right) \\
&= F \left( \left[ \frac{j^2}{2} h_{\bar{z}\bar{z}} p^2 \right]_j^m + \varepsilon \left[ j^2 h_{\bar{z}} \left\{ A \phi_{xx} + p^2 \left( \frac{v_{zz}}{2} + \frac{h_{\bar{z}\bar{z}}}{h_{\bar{z}}} v_z \right) \right\} \right]_j^m + o(\varepsilon), h \right).
\end{aligned}$$

*Step 5: expansion of  $F$ .* We develop  $F$  around the point  $\left( \left[ \frac{j^2}{2} h_{\bar{z}\bar{z}} p^2 \right]_j^m, h \right)$ .

We claim that there exists a constant  $R_\phi$  such that for all  $\varepsilon$  small enough and all  $(t, x) \in$

$(0, \infty) \times \mathbb{R}$ , the first argument of  $F$  in the third line of (3.19) is bounded by the constant  $R_\phi$ . To see this, we simply use one more time (3.5) and Lemma 3.1. With these observations in hands, it is clear from Taylor's formula that

$$F\left(\frac{1}{\varepsilon^{2\alpha-2}}V, h\right) = F\left(\left[\frac{j^2}{2}h_{\bar{z}\bar{z}}p^2\right]_j^m, h\right) + \varepsilon \left\{ \sum_{1 \leq |i| \leq m} i^2 F_i\left(\left[\frac{j^2}{2}h_{\bar{z}\bar{z}}p^2\right]_j^m, h\right) \right\} h_{\bar{z}} \left\{ A\phi_{xx} + p^2 \left( \frac{v_{zz}}{2} + \frac{h_{\bar{z}\bar{z}}}{h_{\bar{z}}} v_z \right) \right\} + o(\varepsilon),$$

where  $o(\varepsilon)$  depends only on  $R_\phi$ , and on the modulus of continuity and the  $L^\infty$ -norms of the first-order derivatives of  $F$  on  $\bar{B}_{R_\phi} \times \mathbb{R}$ . Notice that this modulus and this norm are finite, thanks to **(H1)** **(i)-(ii)**. Dividing this equation by  $\varepsilon$ , we get:

$$(3.20) \quad \frac{1}{\varepsilon} F\left(\frac{1}{\varepsilon^{2\alpha-2}}V, h\right) = \frac{1}{\varepsilon} F\left(\left[\frac{j^2}{2}h_{\bar{z}\bar{z}}p^2\right]_j^m, h\right) + \left\{ \sum_{1 \leq |i| \leq m} i^2 F_i\left(\left[\frac{j^2}{2}h_{\bar{z}\bar{z}}p^2\right]_j^m, h\right) \right\} h_{\bar{z}} \left\{ A\phi_{xx} + p^2 \left( \frac{v_{zz}}{2} + \frac{h_{\bar{z}\bar{z}}}{h_{\bar{z}}} v_z \right) \right\} + o(1) \\ = \frac{1}{\varepsilon} F\left(\left[\frac{j^2}{2}h_{\bar{z}\bar{z}}p^2\right]_j^m, h\right) + h_{\bar{z}} \left\{ K A\phi_{xx} + K p^2 \left( \frac{v_{zz}}{2} + \frac{h_{\bar{z}\bar{z}}}{h_{\bar{z}}} v_z \right) \right\} + o(1),$$

where  $K = K(\tilde{z}, p)$  is defined in (1.10).

*Step 6: conclusion.* By (3.20), Equation (3.8) becomes:

$$(3.21) \quad \mathcal{L}_\varepsilon[\phi^\varepsilon] = h_{\bar{z}}\phi_t + \varepsilon(h_{\bar{z}}v_z\phi_t + h_p\phi_{xt}) - \frac{1}{\varepsilon} F\left(\left[\frac{j^2}{2}h_{\bar{z}\bar{z}}p^2\right]_j^m, h\right) - h_{\bar{z}} \left\{ K A\phi_{xx} + K p^2 \left( \frac{v_{zz}}{2} + \frac{h_{\bar{z}\bar{z}}}{h_{\bar{z}}} v_z \right) \right\} + o(1).$$

Our aim is now to use the cell systems (1.9) and (1.11) to control the terms of order  $\frac{1}{\varepsilon}$  and  $\varepsilon^0$ . First, we see from (1.9) that

$$\frac{1}{\varepsilon} F\left(\left[\frac{j^2}{2}h_{\bar{z}\bar{z}}\right]_j^m, h\right) = \frac{1}{\varepsilon} F\left(\left[\frac{j^2}{2}h_{\bar{z}\bar{z}}(\tilde{z}, p)\right]_j^m, h(\tilde{z}, p)\right) = 0.$$

Moreover, by (3.5) and Lemma 3.1, we have

$$(3.22) \quad |h_{\bar{z}}v_z\phi_t + h_p\phi_{xt}| \leq C_1(\delta)C'_\phi(C_2 + 1).$$

Hence, the  $\varepsilon$ -term in  $x$  can be included in the  $o(1)$ . Therefore, we obtain:

$$(3.23) \quad \mathcal{L}_\varepsilon[\phi^\varepsilon](t, x) = h_{\bar{z}}(\tilde{z}, p) [\phi_t(t, x) - \mathcal{G}(z, \tilde{z}, p, M)] + o(1),$$

where we set

$$\mathcal{G}(z, \tilde{z}, p, M) := K(\tilde{z}, p) A(\tilde{z}, p) M + K(\tilde{z}, p) p^2 \left( \frac{v_{zz}(z)}{2} + \frac{h_{\bar{z}\bar{z}}(\tilde{z}, p)}{h_{\bar{z}}(\tilde{z}, p)} v_z(z) \right)$$



with  $M := \phi_{xx}(t, x)$ . By (1.11), we have  $\lambda = \mathcal{G}(z, z, p_0, M_0)$  and thus

$$(3.24) \quad \mathcal{L}_\varepsilon[\phi^\varepsilon](t, x) = h_{\tilde{z}}(\tilde{z}, p) (\phi_t(t, x) - \lambda + \mathcal{G}(z, z, p_0, M_0) - \mathcal{G}(z, \tilde{z}, p, M)) + o(1).$$

The thesis follows now from the regularity of  $\phi$ , the estimate (3.5) and the fact that by Lemma 3.1 (b)-(e),  $\mathcal{G}$  is uniformly continuous on  $\mathbb{R} \times (0, \infty) \times [\delta, \frac{1}{\delta}] \times [-C_\phi, C_\phi]$ .  $\square$

### 3.2 Formal discussion in the case $\alpha = 2$ .

We use the same notations than in the proof of Lemma 3.2. The main difference is the Taylor's development of the hull function  $h$  in (3.15). Indeed, as explained just below (3.15), the rest satisfies  $R = O(\varepsilon^{3\alpha-3})$  because of the term  $j\varepsilon^{\alpha-1}p$  in (3.12). In particular, if  $\alpha \leq 2$  this rest is not of order  $o(\varepsilon^{2\alpha-1})$  as in (3.16). Consequently, the rest of the proof of Lemma 3.2 fails in this case.

To avoid this difficulty, we have to replace (3.15) by a Taylor's expansion of  $h$  of order greater than 2. We claim that by doing an expansion of order 3 of the hull function, we get the following equation (see also (3.17)) for  $\varepsilon > 1$ :

$$(3.25) \quad h \left( \frac{\tilde{\phi}^\varepsilon(t, x + j\varepsilon^\alpha)}{\varepsilon}, \phi_x(t, x + j\varepsilon^\alpha) \right) - h(\tilde{z}, p) = \\ = j \left\{ h_{\tilde{z}} (\varepsilon^{\alpha-1}p + \varepsilon^\alpha v_z p) + \varepsilon^\alpha h_p \phi_{xx} \right\} + \varepsilon^{2\alpha-2} \left\{ \frac{j^2}{2} h_{\tilde{z}\tilde{z}} p^2 \right\} \\ + \varepsilon^{2\alpha-1} \left\{ \frac{j^2}{2} (h_{\tilde{z}} (\phi_{xx} + v_{zz} p^2) + 2h_{\tilde{z}\tilde{z}} v_z p^2 + 2h_{\tilde{z}p} p \phi_{xx}) \right\} \\ + \varepsilon^{3\alpha-3} \frac{1}{3!} j^3 p^3 h_{\tilde{z}\tilde{z}\tilde{z}} + o(\varepsilon^{2\alpha-1}).$$

Then we have:

$$(3.26) \quad \mathcal{L}_\varepsilon[\phi^\varepsilon] = h_{\tilde{z}} \left[ \phi_t - \left\{ K A \phi_{xx} + K p^2 \left( \frac{v_{zz}}{2} + \frac{h_{\tilde{z}\tilde{z}}}{h_{\tilde{z}}} v_z \right) \right\} \right] \\ - \varepsilon^{\alpha-2} \sum_{1 \leq |i| \leq m} \frac{i^3}{3!} F_i \left( \left[ \frac{j^2}{2} h_{\tilde{z}\tilde{z}} p^2 \right]_j^m, h \right) h_{\tilde{z}\tilde{z}\tilde{z}} p^3 + o(1).$$

This is the same than before with an additional term of order  $\varepsilon^{\alpha-2}$  produced by the displacements of order 3. Notice that this term still comes from the worst term  $j\varepsilon^{\alpha-1}p$  in (3.12).

**Case  $\alpha = 2$ .** In this case, the additional term is of order  $\varepsilon^{\alpha-2} = 1$  and has to be taken into account in the equation of the corrector  $v$ . Therefore the cell equation (1.11) on the corrector  $v$  becomes:

$$(3.27) \quad \lambda = \tilde{K}(z, p_0) \frac{h_{zzz}(z, p_0)}{h_z(z, p_0)} p_0^3 + K(z, p_0) A(z, p_0) M_0 + K(z, p_0) p_0^2 \left( \frac{v_{zz}(z)}{2} + \frac{h_{zz}(z, p_0)}{h_z(z, p_0)} v_z(z) \right),$$

where  $\tilde{K}$  is defined by:

$$(3.28) \quad \tilde{K}(z, p) := \sum_{1 \leq |i| \leq m} \frac{i^3}{3!} F_i \left( \left[ \frac{j^2}{2} h_{zz}(z, p) p^2 \right]_j^m, h(z, p) \right).$$

Consequently an homogenization result could be formally performed. Notice that, here again, we can completely explicit the effective Hamiltonian. Indeed, if we multiply (3.27) by  $\frac{h_z^2}{K}$  as before and integrate on one period  $z \in [0, 1]$  we get:

$$\lambda = \lambda(p_0, M_0) = b(p_0)p_0^3 + G(p_0)M_0,$$

where  $G$  is the same coefficient than in (1.12) and  $b$  is defined by:

$$(3.29) \quad b(p_0) := \frac{\int_0^1 \frac{\tilde{K}(z, p_0)}{K(z, p_0)} h_{zzz}(z, p_0) h_z(z, p_0) dz}{\int_0^1 \frac{h_z^2(z, p_0)}{K(z, p_0)} dz}.$$

The limit  $u^0$  of  $U^\varepsilon$  then would have to solve the following equation:

$$(3.30) \quad u_t^0 = b(u_x^0)(u_x^0)^3 + G(u_x^0)u_{xx}^0.$$

Remark that in the case  $F(V, V_0) = G(L(V), V_0)$  with  $L(V) = \sum_{1 \leq |i| \leq m} l_i V_i$ ,  $m = 2$ ,  $l_i > 0$ , and

$l_1 + l_2 = l_{-1} + l_{-2}$  we have:

$$b(p_0) = B \left( \frac{\int_0^1 h_{zz}^2(z, p_0) dz}{\int_0^1 \frac{h_z^2(z, p_0)}{K(z, p_0)} dz} \right) \quad \text{with} \quad B := -\frac{l_1 - l_{-1} + 8(l_2 - l_{-2})}{6(l_1 + l_{-1} + 4(l_2 - l_{-2}))} \neq 0.$$

This simple example shows that equation (3.29) is different from (1.7) in the case  $\alpha = 2$ .

## 4 Proof of the homogenization result.

**Proof of theorem 1.1.** Let us first note that Proposition 1.3 states the existence and the regularity of the function  $G$  required in (1.6). Moreover, this regularity allows us to apply the now classical results on viscosity solution to obtain the existence and uniqueness of a viscosity solution of (1.7). (See for instance [5].)

Let us first remark now that by the 1-periodicity of  $F$  in **(H1) (ii)**, the PDE in (1.5) is invariant w.r.t.  $\varepsilon \times$  *integer additions*: that is to say, if  $u$  is solution (or semi-solution) of (4.1) in  $(0, \infty) \times \mathbb{R}$ , then so do  $u + \varepsilon n$  for all  $n \in \mathbb{Z}$ . Therefore, we can deal with the discontinuous initial solution of (1.5) by proving the result for a  $u^\varepsilon$  satisfying the same equation with the regular initial datum  $u_0$ , more precisely:

$$(4.1) \quad \begin{cases} u_t^\varepsilon(t, x) = \frac{1}{\varepsilon} F \left( \frac{1}{\varepsilon^{2\alpha-1}} [u^\varepsilon(t, x + j\varepsilon^\alpha) - u^\varepsilon(t, x)]_j^m, \frac{u^\varepsilon(t, x)}{\varepsilon} \right) & \text{for } (t, x) \in (0, \infty) \times \mathbb{R} \\ u^\varepsilon(0, x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Indeed, once we have it, we can conclude by simply remarking that the following inequality

$$u^\varepsilon(t, x) - \varepsilon \left\lceil \frac{\varepsilon^\alpha}{\varepsilon \delta_0} \right\rceil \leq U^\varepsilon(t, x) \leq u^\varepsilon(t, x) \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}$$

is a direct consequence of Theorem 2.3 (and the invariance w.r.to  $\varepsilon \times$  integer additions), once we observe that by **(H0)** at time 0 we have  $u_0(x) - \varepsilon \left\lceil \frac{\varepsilon^\alpha}{\varepsilon \delta_0} \right\rceil \leq u_0\left(\left\lfloor \frac{x}{\varepsilon^\alpha} \right\rfloor \varepsilon^\alpha\right) \leq u_0(x)$  for each fixed  $x$ .

Let us now prove the result for a function  $u^\varepsilon$  solution of (4.1).

The proof will follow the classical method of building suitable perturbed test functions. Let us define the relaxed semilimit of the sequence  $u^\varepsilon$ :

$$\bar{u}(t, x) = \limsup_{\varepsilon \rightarrow 0} {}^*u^\varepsilon(t, x), \quad \underline{u}(t, x) = \liminf_{\varepsilon \rightarrow 0} {}_*u^\varepsilon(t, x).$$

Our aim is to prove that  $\bar{u}$  and  $\underline{u}$  are respectively viscosity sub and super solution of the limit Problem (1.7) in  $[0, \infty) \times \mathbb{R}$ . Indeed, if this is true by the comparison result for the limit equation (Theorem 2.8), we have  $\bar{u}(t, x) \leq \underline{u}(t, x)$  for all  $(t, x)$  in  $(0, \infty) \times \mathbb{R}$ . By construction we have the reverse inequality, thus, as  $\varepsilon$  tends to 0 we will have

$$u^\varepsilon(t, x) \rightarrow \bar{u}(t, x) = \underline{u}(t, x) := u^0(t, x) \text{ uniformly on compact set of } [0, \infty) \times \mathbb{R},$$

which will be the unique solution of (1.7) and this will give us the thesis.

We will proceed in three steps.

First, we will construct  $\varepsilon$ -uniform barriers on  $u^\varepsilon$ , with the double aim of ensuring the applicability of the Perron's method to get the existence of a viscosity solution of (4.1) and to be sure that the relaxed semi-limit are well defined. Secondly, we will prove a uniform bound on the gradients of  $\bar{u}$  and  $\underline{u}$  in order to be allowed to choose  $p = \phi_x = \bar{u}_x > 0$  (or  $p = \phi_x = \underline{u}_x > 0$ ) in the third step. Finally, we will prove that  $\bar{u}$  is a subsolution of (1.7). (The proof of  $\underline{u}$  being a supersolution being completely similar will be not detailed.)

### Step 1. Barriers on $u^\varepsilon$ .

The idea is to construct a sub- and a supersolution of (4.1) not depending on  $\varepsilon$ , or at least locally bounded uniformly in  $\varepsilon$ . To do this, we look for semi-solutions in the form of the ansatz (1.8). Precisely, for  $(t, x) \in [0, \infty) \times \mathbb{R}$  and  $\varepsilon > 0$ , define

$$u^{\pm, \varepsilon}(t, x) := \varepsilon h \left( \frac{u_0(x) \pm Ct}{\varepsilon}, (u_0)_x(x) \right) \pm \varepsilon [C_0(\delta_0)],$$

where  $C$  is a positive constant that will be appropriately chosen later. Let us recall that these functions are well-defined, since  $(u_0)_x \geq \delta_0 > 0$  on  $\mathbb{R}$ .

*Proof of  $u^{+, \varepsilon}$  is a supersolution of (1.5).* We begin by verifying the initial condition. By Lemma 3.1 item (a), we have for all  $x \in \mathbb{R}$ ,

$$u^{+, \varepsilon}(0, x) = \varepsilon h \left( \frac{u_0(x)}{\varepsilon}, (u_0)_x(x) \right) + \varepsilon [C_0(\delta_0)] \geq \varepsilon \left( \frac{u_0(x)}{\varepsilon} - C_0(\delta_0) \right) + \varepsilon [C_0(\delta_0)] \geq 0.$$

Let us now verify the equation. Since the equation is invariant w.r.t.  $\varepsilon \times$  *integer additions* we only need to prove that  $\phi^\varepsilon(t, x) := \varepsilon h \left( \frac{u_0(x) + Ct}{\varepsilon}, (u_0)_x(x) \right)$  is a supersolution of (1.5). Actually, all the computations have already been made during the proof of Lemma 3.2. Indeed, define  $\phi(t, x) := u_0(x) + Ct$ . We see that  $\phi^\varepsilon$  is of the form (3.3) with  $v \equiv 0$ . Moreover,  $\phi$  satisfies (3.5) with  $\delta := \delta_0$  and

$$C_\phi := \|(u_0)_x\|_\infty + \|(u_0)_{xx}\|_\infty + \|(u_0)_{xxx}\|_\infty.$$

We then argue exactly as in the proof of Lemma 3.2 to show (3.21), which in this case is:

$$\mathcal{L}_\varepsilon[\phi^\varepsilon] = h_z \phi_t + \varepsilon h_p \phi_{xt} - h_z K A \phi_{xx} + o(1).$$

Note that by (3.5) and Lemma 3.1**(b)** we have  $|h_p \phi_{xt}| \leq C_1(\delta)C'_\phi$  therefore

$$\mathcal{L}_\varepsilon[\phi^\varepsilon] = h_{\tilde{z}}(\tilde{z}, \phi_x(t, x)) \phi_t(t, x) - h_{\tilde{z}}(\tilde{z}, \phi_x(t, x)) K(\tilde{z}, \phi_x(t, x)) A(\tilde{z}, \phi_x(t, x)) \phi_{xx}(t, x) + o(1).$$

where  $\lim_{\varepsilon \rightarrow 0} o(1) = 0$  uniformly in  $(t, x) \in (0, \infty) \times \mathbb{R}$ . By (3.5) and Lemma 3.1, it is clear that there exists a constant  $M$  not depending on  $C$  such that for all  $\varepsilon$  small enough,  $\mathcal{L}_\varepsilon[\phi^\varepsilon] \geq m(\delta_0)C - M$  on  $(0, \infty) \times \mathbb{R}$ . We conclude that  $u^{+, \varepsilon}$  is a supersolution of (4.1) for  $C \geq \frac{M}{m(\delta_0)}$ .

The same way, we prove that  $u^{-, \varepsilon}$  is a subsolution of (4.1). By the comparison principle, we deduce that  $u^{-, \varepsilon} \leq u^\varepsilon \leq u^{+, \varepsilon}$ . Moreover, it is easy to show from Lemma 3.1 item **(a)** that

$$\left| \varepsilon h \left( \frac{u_0(x) \pm Ct}{\varepsilon}, (u_0)_x(x) \right) - (u_0(x) \pm Ct) \right| \leq \varepsilon C_0(\delta_0),$$

for all  $(t, x) \in (0, \infty) \times \mathbb{R}$  and  $\varepsilon > 0$ . It follows that  $|u^{\pm, \varepsilon}(t, x) - (u_0(x) \pm Ct)| \leq 2\varepsilon [C_0(\delta_0)]$ . Then, there exist  $\varepsilon_0 > 0$  and a positive constant  $C$  such that for all  $0 < \varepsilon \leq \varepsilon_0$  and all  $(t, x) \in (0, \infty) \times \mathbb{R}$ ,

$$(4.2) \quad |u^\varepsilon(t, x) - u_0(x)| \leq Ct + 2\varepsilon [C_0(\delta_0)],$$

where  $\delta_0$  is the bound in **(H0)** and  $C_0(\delta_0)$  the bound in Lemma 3.1 **(a)**, and this guarantees us the existence of the desired  $\varepsilon$ -uniform barriers on  $u^\varepsilon$ .

### Step 2. Uniform bounds on the gradients.

Using again the invariance of the PDE in (4.1) w.r.t.  $\varepsilon \times$  integer additions, we see that (for fixed  $a > 0$  and  $\varepsilon > 0$ ) the functions  $u^\varepsilon(t, x) + \varepsilon \left\lfloor \frac{\delta_0 a}{\varepsilon} \right\rfloor$  and  $u^\varepsilon(t, x) + \varepsilon \left\lceil \frac{a}{\delta_0 \varepsilon} \right\rceil$  are solutions of (4.1) in  $(0, \infty) \times \mathbb{R}$ . But, hypothesis **(H0)** implies for all  $x \in \mathbb{R}$ ,

$$u_0(x) + \varepsilon \left\lfloor \frac{\delta_0 a}{\varepsilon} \right\rfloor \leq u_0(x + a) \leq u_0(x) + \varepsilon \left\lceil \frac{a}{\delta_0 \varepsilon} \right\rceil$$

at the initial time. By the comparison principle, we deduce that for all  $(t, x) \in [0, \infty) \times \mathbb{R}$ ,

$$u^\varepsilon(t, x) + \varepsilon \left\lfloor \frac{\delta_0 a}{\varepsilon} \right\rfloor \leq u^\varepsilon(t, x) \leq u^\varepsilon(t, x) + \varepsilon \left\lceil \frac{a}{\delta_0 \varepsilon} \right\rceil.$$

Thus, letting  $\varepsilon \rightarrow 0$  we obtain the bound on the gradient for the limits, i.e.  $\frac{1}{\delta} \geq (\bar{u})_x \geq \delta$  and  $\frac{1}{\delta} \geq (\underline{u})_x \geq \delta$ .

### Step 3. Proof that $\bar{u}$ is a subsolution of (1.7).

First note that the initial condition is trivially satisfied because of Step 1. We will argue now by contradiction. So let  $\phi \in C^\infty((0, \infty) \times \mathbb{R})$  be a test function such that  $\bar{u} - \phi$  has a strict local maximum at  $(t_0, x_0)$ , i.e.

$$(4.3) \quad \bar{u}(t_0, x_0) - \phi(t_0, x_0) = 0, \quad \bar{u}(t, x) \leq \phi(t, x) \text{ on } Q_{r, 2r}(t_0, x_0) \text{ with } r > 0,$$

$$(4.4) \quad \bar{u}(t, x) \leq \phi(t, x) - 2\theta \text{ on } \overline{Q_{r, 2r}(t_0, x_0)} \setminus Q_{r, r}(t_0, x_0) \text{ with } \theta > 0,$$

we will suppose that there exists a  $\eta > 0$  such that

$$(4.5) \quad \phi_t(t_0, x_0) = \phi_{xx}(t_0, x_0)G(\phi_x(t_0, x_0)) + \eta.$$

In order to construct our perturbed test function we choose  $p = \phi_x(t_0, x_0)$  and  $M = \phi_{xx}(t_0, x_0)$  in Proposition 1.2 and thus obtain the existence of  $h$ ,  $v$  and a unique real number  $\lambda$  fulfilling (1.11). Fix  $\varepsilon > 0$ , we define in each point  $(t, x)$

$$(4.6) \quad \phi^\varepsilon(t, x) = \varepsilon h \left( \frac{\tilde{\phi}^\varepsilon(t, x)}{\varepsilon}, \phi_x(t, x) \right), \quad \tilde{\phi}^\varepsilon(t, x) = \phi(t, x) + \varepsilon^2 v \left( \frac{\phi(t, x)}{\varepsilon} \right).$$

Our aim is to prove now that for  $\varepsilon$  and  $r > 0$  small enough,  $\phi^\varepsilon$  is a supersolution of (4.1) in a open set  $Q_{r,r}(t_0, x_0)$  with  $r$  independent of  $\varepsilon$ .

Note that, also thanks to Step 2, we are fulfilling the assumptions of Lemma 3.2. Therefore, by (3.4), our thesis will be

$$0 \leq \mathcal{L}_\varepsilon(\phi^\varepsilon)(t, x) = h_{\tilde{z}}(\tilde{z}, p)[\phi_t(t, x) - \lambda + \mathcal{E}(t, x, \varepsilon)] \quad \forall (t, x) \in Q_{r,r}(t_0, x_0),$$

where now  $\tilde{z} = \frac{\tilde{\phi}^\varepsilon(t, x)}{\varepsilon}$ ,  $p = \phi_x(t_0, x_0)$  and  $z = \frac{\phi(t, x)}{\varepsilon}$ . Since, by (1.9),  $h_{\tilde{z}}(\cdot, \cdot) > 0$  we are left to prove

$$(4.7) \quad \phi_t(t, x) - \lambda + \mathcal{E}(t, x, \varepsilon) \geq 0 \quad \forall (t, x) \in Q_{r,r}(t_0, x_0) \text{ with } r \text{ independent of } \varepsilon.$$

We observe now that by the choice we made in Proposition 1.2, the real number  $\lambda$  verify  $\lambda = G(\phi_x(t_0, x_0))\phi_{xx}(t_0, x_0)$ . Therefore

$$\begin{aligned} \phi_t(t, x) - \lambda + \mathcal{E}(t, x, \varepsilon) &= \\ &= \phi_t(t, x) - \phi_t(t_0, x_0) + \phi_t(t_0, x_0) - \phi_{xx}(t_0, x_0)G(\phi_x(t_0, x_0)) + \mathcal{E}(t, x, \varepsilon) = \\ &= \phi_t(t, x) - \phi_t(t_0, x_0) + \eta + \mathcal{E}(t, x, \varepsilon), \end{aligned}$$

where we used also (4.5). Since  $\phi_t$  is continuous and  $\lim_{(t,x,\varepsilon) \rightarrow (t_0, x_0, 0)} \mathcal{E}(t, x, \varepsilon) = 0$  (4.7) follows. Therefore  $\phi^\varepsilon$  is a supersolution of (4.1) in  $Q_{r,r}(t_0, x_0)$  for  $\varepsilon$  and  $r$  small enough.

We remark now that by Lemma 3.1 **(a)**, **(b)** and the contradiction assumption (4.4) for  $\varepsilon$  small enough we have

$$(4.8) \quad \phi^\varepsilon(t, x) \geq u^\varepsilon(t, x) + \varepsilon \left\lfloor \frac{\theta}{\varepsilon} \right\rfloor \text{ on } \overline{Q_{r,2r}(t_0, x_0)} \setminus Q_{r,r}(t_0, x_0)$$

where we used also the definition of  $\bar{u}$ . Since  $u^\varepsilon$  is a solution of (4.1) in particular in the open set  $Q_{r,2r}(t_0, x_0)$  and thanks to by the invariance w.r.t. integer translation,  $u^\varepsilon(t, x) + \varepsilon \lfloor \frac{\theta}{\varepsilon} \rfloor$  is still a solution in  $Q_{r,2r}(t_0, x_0)$ . Our aim is to apply now the comparison result on bounded sets (Theorem 2.4). By Remark 2.5, this can be done for  $\varepsilon \leq \left(\frac{r}{m}\right)^{\frac{1}{\alpha}}$  ( $\Rightarrow 2r \geq r + \varepsilon^\alpha m$ ). Thus

$$\phi^\varepsilon(t, x) \geq u^\varepsilon(t, x) + \varepsilon \left\lfloor \frac{\theta}{\varepsilon} \right\rfloor \text{ in all } Q_{r,r}(t_0, x_0).$$

Letting  $\varepsilon$  going to 0 we are led to  $\phi(t, x) \geq \bar{u}(t, x) + \theta$  in  $Q_{r,r}(t_0, x_0)$  which evaluated at  $(t_0, x_0)$  is in contradiction with (4.3). Therefore (4.5) is false with  $\eta > 0$  and then  $\bar{u}$  is a subsolution of (1.7).  $\square$

## 5 Existence and main properties of $h, v, G$

Let us now prove Proposition 1.2 and Proposition 1.3 that have been admitted before. We also prove Proposition 1.4 below on the monotonicity of  $G$  for the classical FK model. Proposition 1.2 is proved in Subsection 5.2 just after doing some needed preliminaries in Subsection 5.1. Propositions 1.3-1.4 are proved in Subsection 5.3. The last subsection is devoted to the proofs of some technical facts.

### 5.1 Preliminaries on $f$

The following result establishes the well-definition of  $f$  defined by (1.2). This function is introduced in order to study Problem (1.9) that defines the hull function  $h$ . Indeed, as we shall see in the next subsection, this will allow to rewrite (1.9) in the “more readable” form (5.5) which can be solved by the classical separation variable method.

**Lemma 5.1. (Well-definition and main properties of  $f$ )**

Assume **(H1)** (i)-(v). Then, for each  $h \in \mathbb{R}$ , there exists a unique real  $f(h)$  such that

$$F \left( \left[ -\frac{j^2}{2} f(h) \right]_j^m, h \right) = 0.$$

Moreover, the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  thus defined is  $C^1$  and 1-periodic.

**Proof.** *Step 1: well-definition of  $f$ .* Let us define the function  $H = H(r, h)$  by

$$H(r, h) := F \left( \left[ -\frac{j^2}{2} r \right]_j^m, h \right) \quad \text{for } (r, h) \in \mathbb{R}^2.$$

For fixed  $h \in \mathbb{R}$ , consider the equation in  $r$ :

$$(5.1) \quad H(r, h) = F \left( \left[ -\frac{j^2}{2} r \right]_j^m, h \right) = 0.$$

By **(H1)** (i) and (v),  $H \in C^1(\mathbb{R}^2)$  and satisfies:

$$(5.2) \quad \frac{\partial H}{\partial r}(r, h) = - \sum_{1 \leq |i| \leq m} \frac{i^2}{2} F_i \left( \left[ \frac{j^2}{2} r \right]_j^m, h \right) \leq -\nu < 0 \quad \text{for all } (r, h) \in \mathbb{R}^2.$$

Equation (5.1) thus admits a unique solution  $r := f(h)$ .

*Step 2: 1-periodicity.* The function  $H$  is in fact 1-periodic w.r.t. the  $h$ -variable, thanks to the periodicity of  $F$  in **(H1)** (ii). It follows that for all  $h \in \mathbb{R}$ , we have

$$H(f(h+1), h+1) = 0 = H(f(h), h) = H(f(h), h+1)$$

and the uniqueness of the solution to (5.1) implies that  $f(h+1) = f(h)$ .

*Step 3: regularity.* By (5.2), the regularity result of the implicit function theorem implies that  $f$  defined by (5.1) has (at least) the same regularity than  $H$ . We conclude that  $f \in C^1(\mathbb{R})$  and complete the proof.  $\square$

Let us remark that, as a consequence of Lemma 5.1, the hull function solving (1.9), satisfies in particular:  $h_{zz} + \frac{1}{p^2}f(h) = 0$ . In order to solve this equation by the separation variable method, we have to introduce the function  $\mathbb{K} = \mathbb{K}(p)$  defined by the lemma below. This result also states the main properties of  $\mathbb{K}$  that will be needed for the qualitative study of  $G$  in Subsection 5.3.

**Lemma 5.2. (Definition of  $\mathbb{F}$  and  $\mathbb{K}$  and main properties of  $\mathbb{K}$ )**

Assume **(H1)** and let  $\mathbb{F}$  be the 1-periodic primitive of  $f$  with null mean. Then, for each  $p > 0$ , there exists a unique real  $\mathbb{K}(p) > \max \mathbb{F}$  such that

$$(5.3) \quad \int_0^1 \frac{dh}{\sqrt{2(\mathbb{K}(p) - \mathbb{F}(h))}} = \frac{1}{p}.$$

Moreover, the function  $\mathbb{K} : (0, \infty) \rightarrow (\max \mathbb{F}, \infty)$  thus defined is analytic and satisfies:

$$(5.4) \quad \mathbb{K}(p) \sim_{p \rightarrow \infty} \frac{p^2}{2}.$$

We skip the details of the proof of Lemma 5.2 which is an elementary result.

## 5.2 Proof of Proposition 1.2

We are now able to prove Proposition 1.2.

**Proof of item (a).** By Lemma 5.1, Problem (1.9) is equivalent to the following one: for all  $(z, p) \in \mathbb{R} \times (0, \infty)$ ,

$$(5.5) \quad \begin{cases} h_{zz}(z, p) + \frac{1}{p^2}f(h(z, p)) = 0, \\ h(z + 1, p) = h(z, p) + 1, \\ h_z(z, p) > 0, \\ h(0, p) = 0, \end{cases}$$

where  $f \in C^1(\mathbb{R})$  is 1-periodic with null mean. Let us solve this equation by the help of the separation variable method.

*Step 1: existence of the hull function.* For  $(h, p) \in \mathbb{R} \times (0, \infty)$ , define

$$(5.6) \quad \phi(h, p) := p \int_0^h \frac{d\tau}{\sqrt{2(\mathbb{K}(p) - \mathbb{F}(\tau))}},$$

where  $\mathbb{K}$  and  $\mathbb{F}$  are defined in Lemma 5.2. Since  $\mathbb{K} : (0, \infty) \rightarrow (\max \mathbb{F}, \infty)$  is analytic and  $\mathbb{F} \in C^2(\mathbb{R})$ , we have  $\phi \in C^3(\mathbb{R} \times (0, \infty))$  with

$$(5.7) \quad \frac{\partial \phi}{\partial h}(h, p) = \frac{p}{\sqrt{2(\mathbb{K}(p) - \mathbb{F}(h))}} \geq \frac{p}{\sqrt{2(\mathbb{K}(p) - \min \mathbb{F})}} > 0.$$

For fixed  $p > 0$ , the function  $\phi(\cdot, p) : \mathbb{R} \rightarrow \mathbb{R}$  is thus invertible,  $C^3$  and its inverse is also  $C^3$ , thanks to the regularity result of the inverse function theorem. Let us denote this inverse by

$$(5.8) \quad h(z, p) := (\phi(\cdot, p))^{-1}(z),$$

for each  $z \in \mathbb{R}$ , and let us prove that this function is solution to (5.5).

First, we have

$$(5.9) \quad h_z = \left( \frac{\partial \phi}{\partial h} \right)^{-1} = \frac{\sqrt{2(\mathbb{K}(p) - \mathbb{F}(h))}}{p};$$

in particular,  $\frac{h_z^2}{2} p^2 = \mathbb{K}(p) - \mathbb{F}(h)$  and derivating one time w.r.t.  $z$ , we get  $h_{zz} h_z p^2 + f(h) h_z = 0$ . Since (5.9) implies that  $h_z$  is positive, we can divide this equation by  $h_z$  to conclude that  $h$  satisfies the ODE of (5.5).

Moreover, the 1-periodicity of  $\mathbb{F}$  and (5.3) imply obviously that  $\phi(h+1, p) = \phi(h, p) + 1$  for all real  $h$ . Taking the inverse, the function  $h$  defined in (5.8) satisfies:

$$h(z+1, p) = (\phi(\cdot, p))^{-1}(z+1) = (\phi(\cdot, p))^{-1}(z) + 1 = h(z, p) + 1$$

for all  $z \in \mathbb{R}$ ,  $p > 0$ . Finally, we have already seen that  $h_z > 0$  and it is clear that  $h(0, p) = 0$  for all  $p > 0$ . Indeed,  $\phi(0, p) = 0$  and it follows that  $h(0, p) = (\phi(\cdot, p))^{-1}(0) = 0$ . We conclude that  $h$  is solution to (5.5) and *a fortiori* to (1.9).

*Step 2: uniqueness of the hull function.* Assume that  $\tilde{h}$  is another solution. Then multiplying the first equation (1.9) by  $\tilde{h}_z$  and integrating, we see that  $\frac{1}{2}(\tilde{h}_z)^2 + \frac{1}{p^2}\mathbb{F}(\tilde{h}) = \frac{C}{p^2}$  and then

$$\int_0^1 \frac{d\tilde{h}}{\sqrt{2(C - \mathbb{F}(\tilde{h}))}} = \int_0^1 \frac{dz}{p}$$

which implies that  $C = K(p)$  and then  $\tilde{h} = h$ .

*Step 3: regularity of the hull function.* For  $(h, p, z) \in \mathbb{R} \times (0, \infty) \times \mathbb{R}$ , define  $\psi(h, p, z) := \phi(h, p) - z$ . By (5.8),  $h(z, p)$  is the unique real that satisfies the equation

$$\psi(h(z, p), p, z) = 0.$$

Moreover,  $\psi \in C^3(\mathbb{R} \times (0, \infty) \times \mathbb{R})$  since  $\phi$  is  $C^3$ , and (5.7) implies that  $\frac{\partial \psi}{\partial h}(h, p, z) = \frac{\partial \phi}{\partial h}(h, p) > 0$ . By the regularity result of the implicit function theorem, we deduce that  $h \in C^3(\mathbb{R} \times (0, \infty))$ . The proof of the item **(a)** of Proposition 1.2 is now complete.  $\square$

**Proof of items (b) and (c).** Let  $p_0 > 0$  and  $M_0 \in \mathbb{R}$  be fixed. We begin by rewriting Equation (1.11) in the more “readable” form (5.10) below. To do this, observe that  $K$  defined by (1.10) is positive, thanks to **(H1)** **(v)**; since  $h_z^2$  is also positive, the ODE in (1.11) is equivalent to

$$\lambda \frac{h_z^2}{K} = M_0 A h_z^2 + \frac{h_z^2}{2} v_{zz} + h_{zz} h_z v_z.$$

Using now that  $\left( \frac{h_z^2}{2} v_{zz} + h_{zz} h_z v_z \right)_z = \left( \frac{h_z^2}{2} v_z \right)_z$ , we see that (1.11) is equivalent to

$$(5.10) \quad \begin{cases} \lambda \frac{h_z^2}{K} = M_0 A h_z^2 + \frac{p_0^2}{2} ((h_z)^2 v_z)_z, \\ v(z+1) = v(z). \end{cases}$$

We can now prove the existence and uniqueness of  $\lambda$ .



*Step 1: uniqueness of  $\lambda$ .* Assume that the equation above admits a solution  $v \in C^2(\mathbb{R})$ . Then, we can integrate (5.10) w.r.t.  $z \in [0, 1]$  and using the 1-periodicity of  $h_z(\cdot, p_0)$  and of  $v$ , we get:

$$(5.11) \quad \lambda \int_0^1 \frac{h_z^2(z, p_0)}{K(z, p_0)} dz = M_0 \int_0^1 A(z, p_0) h_z^2(z, p_0) dz.$$

This shows that there exists at most one  $\lambda \in \mathbb{R}$  such that (5.10) admits a  $C^2$  solution, and this  $\lambda$  is given by (5.11).

*Step 2: existence of  $\lambda$ .* Conversely, assume that  $\lambda$  satisfies (5.11) and let us prove that (1.11) has a solution. Define  $H = H(z)$  by

$$H := \frac{2}{p_0^2} \left( \lambda \frac{h_z^2}{K} - M_0 A h_z^2 \right).$$

By the regularity of  $h$  and Assumption **(H1)** (i),  $A$  and  $K$  are at least  $C^0$  w.r.t.  $z$  and so is  $H$ . Let  $\mathcal{H} = \mathcal{H}(z)$  be a primitive of  $H$  such that

$$(5.12) \quad \int_0^1 \frac{\mathcal{H}(z)}{h_z^2(z, p_0)} dz = 0.$$

Let  $v = v(z)$  be a primitive of  $\frac{\mathcal{H}}{h_z^2}$ , which is then 1-periodic. Since  $H$  is  $C^0$ ,  $v$  is  $C^2$ . Moreover, by construction, we see that  $v$  satisfies the ODE of (5.10).

*Step 3: conclusion.* To summarize, we have proved that (5.10) admits a solution iff (5.11) holds true; since (5.11) is equivalent to (1.12), we have completed the proof of both items **(b)** and **(c)** of Proposition 1.2.  $\square$

### 5.3 Qualitative properties of $G$

Let us now prove the properties of  $G$  in Propositions 1.3-1.4.

**Proof of Propositions 1.3-1.4.** The proof is based on the following decomposition for  $p > 0$ :

$$(5.13) \quad G(p) := \mathbb{G}(p)\mathbb{I}(p) \text{ with } \mathbb{G}(p) := \frac{\int_0^1 2A(z, p)h_z^2(z, p)dz}{\int_0^1 h_z^2(z, p)dz} \text{ and } \mathbb{I}(p) := \frac{1}{2} \frac{\int_0^1 h_z^2(z, p)dz}{\int_0^1 \frac{h_z^2(z, p)}{K(z, p)} dz}.$$

For the sake of clarity, the main properties of  $\mathbb{G}$  will be stated and proved in the next subsection (see Lemma 5.4).

*Step 1: positivity and regularity.* Recalling that  $K$  is positive by **(H1)** (v), we see that  $\mathbb{I}$  is positive. Therefore, Lemma 5.4 implies that  $G = \mathbb{G}\mathbb{I}$  is positive. Moreover, by the item **(a)** of Proposition 1.2 and (1.12), it is clear that  $G \in C^0(0, \infty)$ ; notice that  $G$  is only  $C^0$ , since  $F$  is assumed to be only  $C^1$  in **(H1)** (i). But, if  $F$  is  $C^{k+1}$ , then  $G$  is  $C^k$ . Indeed, the regularity result of the implicit function theorem applied in Step 3 (resp. Step 1) of the proof of Lemma 5.1 (resp. of the item **(a)** of Prop. 1.2), would imply that  $f$  is  $C^{k+1}$  (resp. that  $h$  is  $C^{k+3}$ ). Thus,  $A$  and  $K$  would be at least  $C^k$  and  $G$  also.

*Step 2: limit as  $p \rightarrow 0$ .* Because of **(H1)(i)** we remark that

$$(5.14) \quad M := \sup_{\mathbb{R} \times (0, \infty)} K < \infty.$$

It follows that

$$0 < \mathbb{I} \leq \frac{M \int_0^1 h_z^2(z, p) dz}{2 \int_0^1 h_z^2(z, p) dz} = \frac{M}{2}.$$

Since  $\lim_{p \rightarrow 0} \mathbb{G}(p) = 0$  by Lemma 5.4, we conclude that  $\lim_{p \rightarrow 0} G(p) = 0$ .

*Step 3: limit as  $p \rightarrow \infty$ .* Let us study the limits, as  $p \rightarrow \infty$ , of the different terms that define  $\mathbb{I}$ .

First, it is clear that  $h_{zz}(z, p) \rightarrow 0$  as  $p \rightarrow \infty$ , uniformly in  $z \in [0, 1]$ . Moreover, (5.9) and (5.4) easily imply that  $\lim_{p \rightarrow \infty} h_z(0, p) = 1$ . The primitives  $h_z$  of  $h_{zz}$  then have to satisfy:

$$(5.15) \quad h_z(z, p) \rightarrow 1, \text{ as } p \rightarrow \infty, \text{ uniformly in } z \in [0, 1].$$

Next, (5.15) and the initial condition  $h(0, p) = 0$  imply that  $h(z, p) \rightarrow z$ , as  $p \rightarrow \infty$ , uniformly in  $z \in [0, 1]$ . Consequently,  $p^2 h_{zz}(z, p) = -f(h(z, p)) \rightarrow -f(z)$  as  $p \rightarrow \infty$ , uniformly in  $z \in [0, 1]$ . By the regularity of  $F$  and (1.10), it is easy to deduce that

$$K(z, p) \rightarrow K(z, \infty) \text{ as } p \rightarrow \infty, \text{ uniformly in } z \in [0, 1],$$

where

$$(5.16) \quad K(z, \infty) := \sum_{1 \leq |i| \leq m} i^2 F_i \left( \left[ -j^2 \frac{f(z)}{2} \right]_j^m, z \right).$$

By **(H1)(i)** and **(v)**, the limit function  $K(\cdot, \infty)$  is still positive and continuous on  $\mathbb{R}$ . Then,  $\min_{[0, 1]} K(\cdot, \infty) =: \gamma > 0$ , which implies that  $\min_{z \in [0, 1]} K(z, p) \geq \gamma/2$  for  $p$  sufficiently large; in particular, we get:

$$(5.17) \quad \frac{1}{K(z, p)} \rightarrow \frac{1}{K(z, \infty)} \text{ as } p \rightarrow \infty, \text{ uniformly in } z \in [0, 1].$$

Passing finally to the limit under the integral signs that defines  $\mathbb{I}$  in (5.13), we deduce from (5.15)-(5.17) that

$$\lim_{p \rightarrow \infty} \mathbb{I}(p) = \frac{1}{2} \left\{ \int_0^1 \frac{1}{K(z, \infty)} dz \right\}^{-1} = l > 0,$$

where  $l$  is defined in (1.13). Since Lemma 5.4 states that  $\lim_{p \rightarrow \infty} \mathbb{G}(p) = 1$ , we have proved that  $\lim_{p \rightarrow \infty} G(p) = l$ .

*Step 4: analyticity and monotonicity for the classical FK model.* For the classical FK model (1.3), simple computations show that  $K(z, p)$  is constant equal to 2. Hence,  $\mathbb{I} = 1$  and  $G = \mathbb{G}$ . The proof of the analyticity and monotonicity of  $G$  is then an immediate corollary of Lemma 5.4.  $\square$

## 5.4 Technical results: properties of $\mathbb{G}$ .

Let us first prove the following technical lemma.

**Lemma 5.3.** *For  $\alpha \in \mathbb{R}$  and  $p > 0$ , define  $J^\alpha(p) := \int_0^1 (2(\mathbb{K}(p) - \mathbb{F}(h)))^\alpha dh$ , where  $\mathbb{K}(p)$  is defined by (5.3). Then,  $J^\alpha$  is positive, analytic w.r.t.  $p > 0$  and for  $i \in \mathbb{Z}$ ,*

$$(5.18) \quad (J^{\frac{i}{2}})' = i \mathbb{K}' J^{\frac{i-2}{2}},$$

$$(5.19) \quad J^{\frac{i}{2}} < \sqrt{J^{\frac{i+2}{2}} J^{\frac{i-2}{2}}} \quad \text{if } f \neq 0,$$

$$(5.20) \quad J^{-\frac{1}{2}} \leq \left(J^{-\frac{3}{2}}\right)^{\frac{1}{3}}.$$

**Proof.** By Lemma 5.1,  $J^\alpha$  is analytic. Of course  $J^\alpha$  is positive, since  $\mathbb{K} > \max \mathbb{F}$ . Moreover, Equality (5.18) is an immediate consequence of the theorem of derivation under the integral sign. To prove (5.19), we use Cauchy-Schwartz's inequality. We have

$$\begin{aligned} \int_0^1 (2(\mathbb{K}(p) - \mathbb{F}(h)))^{\frac{i}{2}} dh &= \int_0^1 (2(\mathbb{K}(p) - \mathbb{F}(h)))^{\frac{i+2}{4}} (2(\mathbb{K}(p) - \mathbb{F}(h)))^{\frac{i-2}{4}} dh \\ &\leq \sqrt{\int_0^1 (2(\mathbb{K}(p) - \mathbb{F}(h)))^{\frac{i+2}{2}} dh} \times \sqrt{\int_0^1 (2(\mathbb{K}(p) - \mathbb{F}(h)))^{\frac{i-2}{2}} dh}, \end{aligned}$$

which proves that  $J^{\frac{i}{2}} \leq \sqrt{J^{\frac{i+2}{2}} J^{\frac{i-2}{2}}}$ . But, these terms can not be equaled, because this would imply that

$$\forall h \in [0, 1], \quad (2(\mathbb{K}(p) - \mathbb{F}(h)))^{\frac{i+2}{4}} = C(2(\mathbb{K}(p) - \mathbb{F}(h)))^{\frac{i-2}{4}}$$

for some constant  $C$ ; since the primitive  $\mathbb{F}$  of  $f$  is non-constant, it is clear that such an equality does not hold true<sup>1</sup>. The same way, (5.20) follows by Hölder's inequality with exponents  $(p, q) = (3/2, 3)$  giving  $J^{-\frac{1}{2}} \leq (J^0)^{\frac{2}{3}} (J^{-\frac{3}{2}})^{\frac{1}{3}}$  and the proof is complete.  $\square$

**Lemma 5.4.** *Under (H1),  $\mathbb{G}$  is analytic, positive and increasing on  $(0, \infty)$ . Moreover, we have  $\lim_{p \rightarrow 0} \mathbb{G}(p) = 0$  and  $\lim_{p \rightarrow \infty} \mathbb{G}(p) = 1$ .*

**Proof.** *Step 1: new formula for  $\mathbb{G}$ .* By the definitions of  $\mathbb{G}$  and  $A$  in (5.13) and (1.10), simple computations show that

$$(5.21) \quad \mathbb{G} = \mathbb{G}(p) = 1 + p \frac{\int_0^1 2h_{zp}(z, p) h_z(z, p) dz}{\int_0^1 h_z^2(z, p) dz} = 1 + p \frac{\frac{d}{dp} \left( \int_0^1 h_z^2(z, p) dz \right)}{\int_0^1 h_z^2(z, p) dz} := 1 + p \frac{I'(p)}{I(p)},$$

where we set  $I := I(p) = \int_0^1 h_z^2(z, p) dz$ .

Note that Equation (5.9) implies that  $I(p) = \int_0^1 \frac{1}{p} \sqrt{2(\mathbb{K}(p) - \mathbb{F}(h(z, p)))} h_z(z, p) dz = \frac{J^{\frac{1}{2}}(p)}{p}$ , where the powers  $J^\alpha$  are defined and studied in Lemma 5.3.

Let us compute  $I'$ . By (5.18), it follows that  $I' = \frac{\mathbb{K}' J^{-\frac{1}{2}}}{p} - \frac{J^{\frac{1}{2}}}{p^2} = \frac{\mathbb{K}' J^{-\frac{1}{2}}}{p} - \frac{I}{p}$ . Equation (5.21) thus gives

$$\mathbb{G} = 1 + \mathbb{K}' \frac{J^{-\frac{1}{2}}}{I} - 1 = \frac{p \mathbb{K}' J^{-\frac{1}{2}}}{J^{\frac{1}{2}}}.$$

<sup>1</sup>Excepted in the trivial case  $f \equiv 0$ , which is not interesting in our settings.

But (5.3) implies that  $J^{-\frac{1}{2}} = \frac{1}{p}$ . Since  $(J^{-\frac{1}{2}})' = -\mathbb{K}' J^{-\frac{3}{2}}$ , we deduce that

$$(5.22) \quad \mathbb{K}' = \frac{1}{p^2 J^{-\frac{3}{2}}} = \frac{J^{-\frac{1}{2}}}{p J^{-\frac{3}{2}}};$$

hence, we get the following formula on  $\mathbb{G}$ :

$$(5.23) \quad \mathbb{G} = \frac{\left(J^{-\frac{1}{2}}\right)^2}{J^{\frac{1}{2}} J^{-\frac{3}{2}}}.$$

*Step 2: positivity and analyticity of  $\mathbb{G}$ .* By (5.23) and Lemma 5.3 below, it is readily seen that  $\mathbb{G}$  is positive and analytic w.r.t.  $p$ .

*Step 3: monotonicity of  $\mathbb{G}$ .* To compute  $\mathbb{G}'$ , we simply have to derivate (5.23). Using (5.18), we leave it to the reader to verify that this leads to the following formula:

$$(5.24) \quad \mathbb{G}' = \mathbb{K}' \left\{ 3 \frac{\left(J^{-\frac{1}{2}}\right)^2 J^{-\frac{5}{2}}}{J^{\frac{1}{2}} \left(J^{-\frac{3}{2}}\right)^2} - \frac{\left(J^{-\frac{1}{2}}\right)^3}{\left(J^{\frac{1}{2}}\right) 2J^{-\frac{3}{2}}} - 2 \frac{J^{-\frac{1}{2}}}{J^{\frac{1}{2}}} \right\}.$$

On denoting the three terms in brackets by  $I_1$ ,  $I_2$  and  $I_3$ , we get  $\mathbb{G}' = \mathbb{K}'(3I_1 - I_2 - 2I_3)$ . Let us prove that  $I_1 > I_2$  and  $I_1 > I_3$ , thus concluding the positivity of  $\mathbb{G}'$ , since  $\mathbb{K}'$  is positive by (5.22) and Lemma 5.3. To establish that  $I_1 > I_2$ , we have to prove that

$$\frac{\left(J^{-\frac{1}{2}}\right)^3}{\left(J^{\frac{1}{2}}\right)^2 J^{-\frac{3}{2}}} < \frac{\left(J^{-\frac{1}{2}}\right)^2 J^{-\frac{5}{2}}}{J^{\frac{1}{2}} \left(J^{-\frac{3}{2}}\right)^2},$$

which is equivalent to  $J^{-\frac{1}{2}} J^{-\frac{3}{2}} < J^{-\frac{5}{2}} J^{\frac{1}{2}}$ . But (5.19) implies that  $J^{-\frac{1}{2}} < \sqrt{J^{\frac{1}{2}} J^{-\frac{3}{2}}}$  and  $J^{-\frac{3}{2}} < \sqrt{J^{-\frac{1}{2}} J^{-\frac{5}{2}}}$ . Taking the product, we get the result.

The same way,  $I_1 > I_3$  is equivalent to  $\left(J^{-\frac{3}{2}}\right)^2 < J^{-\frac{1}{2}} J^{-\frac{5}{2}}$ , which is already given by (5.19).

*Step 4: limits as  $p \rightarrow 0, \infty$ .* By (5.4), we see that  $J^\alpha(p) \sim_{p \rightarrow \infty} p^{2\alpha}$ . By (5.23), we deduce that  $\lim_{p \rightarrow \infty} \mathbb{G}(p) = 1$ . To compute the limit as  $p \rightarrow 0$ , remark first that

$$J^{\frac{1}{2}} \geq \int_0^1 \sqrt{2(\max \mathbb{F} - \mathbb{F}(h))} dh =: C_0 > 0.$$

Next, recall that (5.3) implies  $J^{-\frac{1}{2}} = \frac{1}{p}$ . By (5.20), we get:  $\left(J^{-\frac{3}{2}}\right)^{\frac{1}{3}} \geq \frac{1}{p}$ . Finally, we deduce from (5.23) that  $\mathbb{G}(p) \leq \frac{p}{C_0}$  and thus  $\lim_{p \rightarrow 0} \mathbb{G}(p) = 0$ . The proof of Lemma 5.4 is complete.  $\square$

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