

Appendix of the paper: “Numerical solution of Boussinesq systems of the Bona-Smith family”

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Abstract

In this Appendix we provide proofs of Lemma 2(ii) and (iii), and of the estimate (16) of the paper “Numerical solution of Boussinesq systems of the Bona-Smith family”, using notation and results of that paper.

1. Proof of lemma 2(ii)

Let $v \in H^2$. Since $R_h v \in H^2$,

$$\|(R_h v)''\|^2 = \sum_{j=0}^J \int_{x_j}^{x_{j+1}} [(R_h v)'' - \psi'']^2 dx \leq Ch^{-2} \sum_{j=0}^J \int_{x_j}^{x_{j+1}} [(R_h v)' - \psi']^2 dx,$$

where ψ is the interpolant of v in the space of piecewise linear, continuous functions $S_h(0, 2)$, and where the second inequality follows from an inverse property on the polynomial space $\mathbb{P}_{r-2}([x_j, x_{j+1}])$ and the quasiuniformity of the mesh x_j . Therefore, by Lemma 1

$$\|(R_h v)''\| \leq Ch^{-1} \|(R_h v)' - \psi'\| \leq Ch^{-1} (\|(R_h v)' - v'\| + \|\psi' - v'\|) \leq C\|v\|_2,$$

from which there follows that $\|R_h v\|_2 \leq C\|v\|_2$ due to the H^1 -stability of R_h . The proof for R_h^0 is entirely analogous. \square

2. Proof of Lemma 2(iii)

Let $v \in W_\infty^1 \cap H_0^1$ and let $V \in S_h^0$ be such that

$$(V', \chi') = (v', \chi') \quad \forall \chi \in S_h^0. \tag{A1}$$

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Then, it is not hard to see that $V' = \tilde{P}_h v'$, where \tilde{P}_h is the L^2 -projection operator onto

$$S_h(\mu - 1, r - 1) = \frac{d}{dx} S_h^0(\mu, r) \oplus \{1\}.$$

Using now the definition of $a_D(\cdot, \cdot)$ and (A1) we conclude that

$$a_D(R_h^0 v - V, \chi) = (v - V, \chi) \quad \forall \chi \in S_h^0. \quad (\text{A2})$$

Consider now the problem

$$\begin{aligned} \Theta - b\Theta'' &= v - V \quad \text{in } \bar{I}, \\ \Theta(-L) &= \Theta(L) = 0. \end{aligned} \quad (\text{A3})$$

Since by (2)

$$a_D(R_h^0 \Theta, \chi) = a_D(\Theta, \chi) = (v - V, \chi) = a_D(R_h^0 v - V, \chi) \quad \forall \chi \in S_h^0,$$

we obtain that $R_h^0 \Theta = R_h^0 v - V$. Therefore,

$$\begin{aligned} \|(R_h^0 v)'\|_{L^\infty} &\leq \|(R_h^0 \Theta)'\|_{L^\infty} + \|V'\|_{L^\infty} = \|(R_h^0 \Theta)'\|_{L^\infty} + \|\tilde{P}_h v'\|_{L^\infty} \\ &\leq C(\|R_h^0 \Theta\|_2 + \|v'\|_{L^\infty}), \end{aligned}$$

using the stability of the L^2 -projection in L^∞ , [1]. By Lemma 2(ii), the elliptic regularity of the solution Θ of (A3) and the Poincaré inequality, we see that $\|(R_h^0 v)'\|_{L^\infty} \leq C\|v\|_{W_\infty^1}$, and the second inequality of lemma 2(iii) follows from the result (i) in the same Lemma.

To prove the analogous estimate for $v \in W_\infty^1$, we let now $V = \int_{-L}^x (\tilde{P}_h v')$. Then, $V \in S_h(\mu, r)$ with $V' = \tilde{P}_h v'$ and $V(-L) = 0$. As before, we have

$$a_N(R_h v - V, \chi) = (v - V, \chi) \quad \forall \chi \in S_h,$$

which gives, if Θ is a solution of the problem

$$\begin{aligned} \Theta - b\Theta'' &= v - V \quad \text{in } \bar{I}, \\ \Theta'(-L) &= \Theta'(L) = 0, \end{aligned}$$

the identity $R_h v = R_h \Theta + V$. We conclude that $\|R_h v\|_{W_\infty^1} \leq C\|v\|_{W_\infty^1}$, arguing as in the first part of the proof. \square

3. Proof of (16)

From (7) of Proposition 5 it follows that

$$\max_{0 \leq t \leq T} (\|\eta_h\|_2 + \|u_h\|_1) \leq C. \quad (\text{A4})$$

From (6), the definitions of f and g , and Lemma 4(i) we obtain, using (A4), for $0 \leq t \leq T$

$$\begin{aligned} \|\eta_{h,t}\|_2 &= \|f(\eta_h, u_h)\|_2 \leq \|\hat{f}(u_h)\|_2 + \|\hat{f}(\eta_h u_h)\|_2 \leq C(\|u_h\|_1 + \|\eta_h u_h\|_1) \\ &\leq C(\|u_h\|_1 + \|\eta_h\|_1 \|u_h\|_1) \leq C, \end{aligned} \quad (\text{A5})$$

and

$$\begin{aligned}
\|u_{ht}\|_1 &= \|g(\eta_h, u_h)\|_1 \leq |c| \|\hat{f}(\eta_{hx})\|_1 + \|\hat{f}(\eta_h)\|_1 + \frac{1}{2} \|\hat{f}(u_h^2)\|_1 \\
&\leq C (\|\eta_{hx}\| + \|\eta_h\| + \|u_h^2\|) \\
&\leq C (\|\eta_h\|_2 + \|u_h\|_1 \|u_h\|) \leq C.
\end{aligned} \tag{A6}$$

Differentiating in (6) with respect to t we see that

$$\begin{aligned}
\eta_{htt} &= \hat{f}(u_{ht}) + \hat{f}(\eta_{ht} u_h) + \hat{f}(\eta_h u_{ht}), \\
u_{htt} &= c \hat{f}(\eta_{hxt}) + \hat{f}(\eta_{ht}) + \hat{f}(u_h u_{ht}).
\end{aligned}$$

Therefore, using (A4)–(A6) we have as before, for $0 \leq t \leq T$

$$\begin{aligned}
\|\eta_{htt}\|_2 &\leq C (\|u_{ht}\|_1 + \|\eta_{ht} u_h\|_1 + \|\eta_h u_{ht}\|_1) \\
&\leq C (\|u_{ht}\|_1 + \|\eta_{ht}\|_1 \|u_h\|_1 + \|\eta_h\|_1 \|u_{ht}\|_1) \leq C,
\end{aligned}$$

and

$$\|u_{htt}\|_2 \leq C (\|\eta_{ht}\|_2 + \|\eta_{ht}\| + \|u_{ht}\|_1 \|u_h\|_1) \leq C.$$

Continuing inductively we see that (16) holds. \square

Additional reference

- [1] J. Douglas, T. Dupont, L. B. Wahlbin, Optimal L_∞ error estimates for Galerkin approximations to solutions of two-point boundary value problems, Math. Comp. 29 (1975) 475–483.