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► **To cite this version:**

Cédric Bellis, Marc Bonnet. Crack identification by 3D time-domain elastic or acoustic topological sensitivity. *Comptes Rendus Mécanique*, 2009, 337, pp.124-130. 10.1016/j.crme.2009.03.015 . hal-00371796v2

**HAL Id: hal-00371796**

**<https://hal.science/hal-00371796v2>**

Submitted on 26 Apr 2009 (v2), last revised 19 May 2009 (v3)

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# Crack identification by 3D time-domain elastic or acoustic topological sensitivity

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Accepted after revision, March 30, 2009

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## Abstract

The topological sensitivity analysis, based on the asymptotic behaviour of a cost functional associated with the creation of a small trial flaw in a defect-free solid, provides a computationally-fast, non-iterative approach for identifying flaws embedded in solids. This concept is here considered for crack identification using time-dependent measurements on the external boundary. The topological derivative of a cost function under the nucleation of a crack of infinitesimal size is established, in the framework of time-domain elasticity or acoustics. The simplicity and efficiency of the proposed formulation is enhanced by the recourse to an adjoint solution. Numerical results obtained on a 3-D elastodynamic example using the conventional FEM demonstrate the usefulness of the topological derivative as a crack indicator function. *To cite this article: C. Bellis, M. Bonnet, C. R. Mecanique 337 (2009).*

## Résumé

**Identification de fissures par sensibilité topologique dynamique en élasticité ou acoustique tridimensionnelle.** L'analyse de sensibilité topologique, reposant sur le comportement asymptotique d'une fonction coût associée à la création d'un défaut virtuel infinitésimal dans un solide sain, fournit une méthode de calcul rapide et non itératif de construction d'une fonction indicatrice de défauts. Dans cette Note, consacrée à l'identification de fissures, le gradient topologique d'une fonctionnelle coût quelconque par rapport à l'apparition d'une fissure de taille infinitésimale est établi pour l'élastodynamique linéaire et l'acoustique. Les développements présentés reposent sur l'utilisation d'un état adjoint pour plus de simplicité et d'efficacité. Un exemple numérique en élastodynamique tridimensionnelle, basé sur une méthode d'éléments finis standard, valide l'intérêt de l'approche proposée. *Pour citer cet article : C. Bellis, M. Bonnet, C. R. Mecanique 337 (2009).*

*Key words:* Computational solid mechanics ; Topological sensitivity ; Elastodynamics ; Crack identification ; Adjoint solution

*Mots-clés :* Mécanique des solides numérique ; Sensibilité topologique ; Elastodynamique ; Identification de fissure ; Etat adjoint

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*Preprint submitted to Elsevier Science*

*April 26, 2009*

## Introduction

Consider an finite elastic body  $\Omega_\Gamma^* \subset \mathbb{R}^3$ , externally bounded by the piecewise-smooth closed surface  $S$ , characterized by its shear modulus  $\mu$ , Poisson's ratio  $\nu$  and mass density  $\rho$ , and containing an internal crack idealized by the smooth open surface  $\Gamma^*$  and with traction-free faces  $\Gamma^{*\pm}$ . Let  $\Omega$  denote the crack-free solid such that  $\Omega_\Gamma^* = \Omega \setminus \Gamma^*$ .

This Note is concerned with the identification of the crack  $\Gamma^*$  from available measured displacement  $\mathbf{u}_{\text{obs}}$  on a measurement surface  $S_{\text{obs}} \subset S$  resulting from the excitation of  $\Omega_\Gamma^*$  by known applied time-dependent tractions over the boundary  $S$ . The misfit between a trial cracked domain  $\Omega_\Gamma = \Omega \setminus \Gamma$  and the correct crack configuration  $\Omega_\Gamma^*$  is expressed by means of a cost functional  $\mathbb{J}$  of the form

$$\mathbb{J}(\Omega_\Gamma, T) = \int_0^T \int_{S_{\text{obs}}} \varphi[\mathbf{u}_\Gamma(\boldsymbol{\xi}, t), \boldsymbol{\xi}, t] dS_\xi dt \quad (1)$$

where the misfit function  $\varphi$  measures the distance between  $\mathbf{u}_{\text{obs}}$  and the displacement  $\mathbf{u}_\Gamma$  arising for a trial crack  $\Gamma$ . Straightforward adjustments of the formulation to be presented herein allow to consider types of boundary conditions and overdetermined data other than prescribed tractions and measured displacements, respectively.

Considering an infinitesimal trial crack  $\Gamma_\varepsilon = \mathbf{x} + \varepsilon \bar{\Gamma}$  centered at  $\mathbf{x} \in \Omega$  with characteristic size  $\varepsilon$ , whose shape is defined using a normalized open surface  $\bar{\Gamma}$ , and setting  $\Omega_\varepsilon = \Omega \setminus \Gamma_\varepsilon$ , the topological derivative  $\mathbb{T}$  of  $\mathbb{J}$  is defined through the expansion

$$\mathbb{J}(\Omega_\varepsilon, T) = \mathbb{J}(\Omega, T) + \eta(\varepsilon)\mathbb{T}(\mathbf{x}, T) + o(\eta(\varepsilon)) \quad (2)$$

where the function  $\eta(\varepsilon)$ , to be determined, vanishes in the limit  $\varepsilon \rightarrow 0$ .

The topological derivative field  $\mathbb{T}(\mathbf{x}, T)$  is considered here as a possible crack indicator function. The heuristic of this approach, following and generalizing upon previous investigations on void identification under transient dynamical conditions [7,9] or crack identification under 2-D static conditions [10], consists in seeking the (finite) crack  $\Gamma$  in regions of  $\Omega$  where  $\mathbb{T}(\mathbf{x}, T)$  reaches its most negative values, i.e. where  $\mathbb{J}(\Omega_\varepsilon, T)$  decreases most for sufficiently small crack size  $\varepsilon$ . The concept of topological sensitivity is a particular instance of the broader class of asymptotic methods, where unknown defects whose geometry involves a small parameter are sought by means of expansions with respect to that parameter. Relevant references include [3], where the identification of small elastic inclusions using dynamic data in the time domain is considered, and [4] where small crack-like perfectly-conducting defects are considered in the framework of the 2-D scalar Helmholtz equation.

## 1. Time domain formulation via adjoint field approach.

Under the prescribed dynamical loading over  $S$  an elastodynamic state  $\mathbf{u}_\varepsilon$  arises that can be conveniently decomposed into  $\mathbf{u}_\varepsilon = \mathbf{u} + \tilde{\mathbf{u}}_\varepsilon$  with  $\mathbf{u}$  being the response of the crack-free domain  $\Omega$  and the scattered field  $\tilde{\mathbf{u}}_\varepsilon$  being the solution of

$$\begin{aligned} \operatorname{div}[\mathbf{C} : \nabla \tilde{\mathbf{u}}_\varepsilon](\boldsymbol{\xi}, t) &= \rho \ddot{\tilde{\mathbf{u}}}_\varepsilon(\boldsymbol{\xi}, t) & (\boldsymbol{\xi} \in \Omega_\varepsilon, t \geq 0) \\ \mathbf{t}^\pm[\tilde{\mathbf{u}}_\varepsilon^\pm](\boldsymbol{\xi}, t) &= -\mathbf{t}^\pm[\mathbf{u}](\boldsymbol{\xi}, t) & (\boldsymbol{\xi} \in \Gamma_\varepsilon^\pm, t \geq 0) \\ \mathbf{t}[\tilde{\mathbf{u}}_\varepsilon](\boldsymbol{\xi}, t) &= 0 & (\boldsymbol{\xi} \in S, t \geq 0) \\ \tilde{\mathbf{u}}_\varepsilon(\boldsymbol{\xi}, 0) &= \dot{\tilde{\mathbf{u}}}_\varepsilon(\boldsymbol{\xi}, 0) = 0 & (\boldsymbol{\xi} \in \Omega_\varepsilon) \end{aligned} \quad (3)$$

where the traction vector  $\mathbf{t}[\mathbf{w}] = \mathbf{C} : \nabla \mathbf{w} \cdot \mathbf{n} = \boldsymbol{\sigma}[\mathbf{w}] \cdot \mathbf{n}$  is associated with a displacement  $\mathbf{w}$  and the outward normal  $\mathbf{n}$  through the fourth-order elastic Hooke's tensor  $\mathbf{C}$ . Moreover,  $\Gamma_\varepsilon^\pm$  denote the two faces of the crack, associated with the respective outward normals  $\mathbf{n}^\pm$  (such that  $\mathbf{n}^+ = -\mathbf{n}^-$ ) and supporting scattered displacements  $\tilde{\mathbf{u}}_\varepsilon^\pm$ , so that  $\mathbf{t}^\pm[\mathbf{w}^\pm] = \boldsymbol{\sigma}[\mathbf{w}^\pm] \cdot \mathbf{n}^\pm$  on  $\Gamma_\varepsilon^\pm$ .

Then, since the scattered field is expected to vanish for infinitesimal cracks, i.e.  $\lim_{\varepsilon \rightarrow 0} \|\tilde{\mathbf{u}}_\varepsilon\| = 0$ , the topological derivative defined in (2) can be expressed by means of the first order derivative of  $\varphi$  through

$$\mathbb{T}(\mathbf{x}, T) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\eta(\varepsilon)} \int_0^T \int_{S_{\text{obs}}} \frac{\partial \varphi}{\partial \mathbf{u}}[\mathbf{u}(\boldsymbol{\xi}, t), \boldsymbol{\xi}, t] \cdot \tilde{\mathbf{u}}_\varepsilon(\boldsymbol{\xi}, t) \, dS_\xi \, dt \quad (4)$$

Following the approach of [7], define the adjoint solution  $\hat{\mathbf{u}}$  in  $\Omega$  by the following adjoint problem

$$\begin{aligned} \operatorname{div}[\mathbf{C} : \nabla \hat{\mathbf{u}}](\boldsymbol{\xi}, t) &= \rho \ddot{\hat{\mathbf{u}}}(\boldsymbol{\xi}, t) & (\boldsymbol{\xi} \in \Omega, 0 \leq t \leq T) \\ \mathbf{t}[\hat{\mathbf{u}}](\boldsymbol{\xi}, t) &= \frac{\partial \varphi}{\partial \mathbf{u}}[\mathbf{u}(\boldsymbol{\xi}, T-t), \boldsymbol{\xi}, T-t] & (\boldsymbol{\xi} \in S_{\text{obs}}, 0 \leq t \leq T) \\ \mathbf{t}[\hat{\mathbf{u}}](\boldsymbol{\xi}, t) &= 0 & (\boldsymbol{\xi} \in S \setminus S_{\text{obs}}, 0 \leq t \leq T) \\ \hat{\mathbf{u}}(\boldsymbol{\xi}, 0) &= \dot{\hat{\mathbf{u}}}(\boldsymbol{\xi}, 0) = 0 & (\boldsymbol{\xi} \in \Omega) \end{aligned} \quad (5)$$

Using the well known dynamical reciprocity identity [1] between  $\tilde{\mathbf{u}}_\varepsilon$  and  $\hat{\mathbf{u}}$  and the boundary conditions in problems (3) and (5) one obtains

$$\int_0^T \int_{S_{\text{obs}}} \frac{\partial \varphi}{\partial \mathbf{u}}[\mathbf{u}(\boldsymbol{\xi}, t), \boldsymbol{\xi}, t] \cdot \tilde{\mathbf{u}}_\varepsilon(\boldsymbol{\xi}, t) \, dS_\xi \, dt = \int_0^T \int_{\Gamma_\varepsilon} \mathbf{t}[\hat{\mathbf{u}}](\boldsymbol{\xi}, T-t) \cdot \llbracket \tilde{\mathbf{u}}_\varepsilon \rrbracket(\boldsymbol{\xi}, t) \, dS_\xi \, dt \quad (6)$$

on noting  $\llbracket \tilde{\mathbf{u}}_\varepsilon \rrbracket = \tilde{\mathbf{u}}_\varepsilon^+ - \tilde{\mathbf{u}}_\varepsilon^-$  the crack opening displacement (COD) through the crack and using notations  $\mathbf{n}^- = \mathbf{n}$  and  $\mathbf{t}[\hat{\mathbf{u}}] = \boldsymbol{\sigma}[\hat{\mathbf{u}}] \cdot \mathbf{n}$  on  $\Gamma_\varepsilon$ .

## 2. Asymptotic behavior of cost function in the small-crack limit

Leading contributions as  $\varepsilon \rightarrow 0$ . The COD associated to problem (3) verifies the integral equation [6]

$$t_i^\pm[\mathbf{u}](\boldsymbol{\zeta}, t) = C_{ijkl} n_j^\pm(\boldsymbol{\zeta}) \int_0^t \left\{ \int_{\Gamma_\varepsilon}^* \Sigma_{ab}^k(\boldsymbol{\zeta}, \boldsymbol{\xi}, t-\tau) D_{lb} \llbracket \tilde{\mathbf{u}}_\varepsilon \rrbracket_a(\boldsymbol{\xi}, \tau) \, dS_\xi + \rho \int_{\Gamma_\varepsilon} \mathbf{U}_a^k(\boldsymbol{\zeta}, \boldsymbol{\xi}, t-\tau) \llbracket \tilde{\mathbf{u}}_\varepsilon \rrbracket_a(\boldsymbol{\xi}, \tau) \, dS_\xi \right\} d\tau \quad (\boldsymbol{\zeta} \in \Gamma_\varepsilon, t \geq 0) \quad (7)$$

where  $\mathbf{U}^k(\boldsymbol{\zeta}, \boldsymbol{\xi}, t)$  and  $\boldsymbol{\Sigma}^k(\boldsymbol{\zeta}, \boldsymbol{\xi}, t)$  are the elastodynamic Green's tensors, i.e. the displacement and stress at  $\boldsymbol{\xi}$  and time  $t$  due to a unit impulsive point force acting at  $\boldsymbol{\zeta}$  in the  $k$ -th direction and at time  $t = 0$  in  $\mathbb{R}^3$  and satisfying a traction-free boundary condition on  $S$ ,  $\int^*$  denotes a Cauchy principal value singular integral, and  $D_{lb}(\cdot) = n_l(\cdot)_{,b} - n_b(\cdot)_{,l}$  defines a curl-like tangential differential operator.

To determine the behavior of the solution  $\llbracket \tilde{\mathbf{u}}_\varepsilon \rrbracket$  on  $\Gamma_\varepsilon$  in the limit  $\varepsilon \rightarrow 0$ , scaled coordinates are introduced according to

$$\boldsymbol{\xi} \in \Gamma_\varepsilon \Rightarrow \bar{\boldsymbol{\xi}} = (\boldsymbol{\xi} - \mathbf{x})/\varepsilon \in \bar{\Gamma} \quad \text{and hence} \quad dS_\xi = \varepsilon^2 dS_{\bar{\xi}}. \quad (8)$$

One then seeks the COD in the asymptotic form

$$\llbracket \tilde{\mathbf{u}}_\varepsilon \rrbracket(\boldsymbol{\xi}, t) = \varepsilon^{d_e} \llbracket \mathbf{V} \rrbracket(\bar{\boldsymbol{\xi}}, t) + o(\varepsilon^{d_e}) \quad (\boldsymbol{\xi} \in \Gamma_\varepsilon, \bar{\boldsymbol{\xi}} \in \bar{\Gamma}) \quad (9)$$

where the order  $d_e \geq 0$  is to be determined. On substituting (9) into (7), using scaled coordinates (8), and invoking the following asymptotic properties of the elastodynamic Green's tensor:

$$\begin{aligned} \int_0^t \mathbf{U}^k(\zeta, \xi, t - \tau) \mathbf{f}(\xi, \tau) d\tau &= \frac{1}{\varepsilon} \mathbf{U}^k(\bar{\zeta}, \bar{\xi}) \cdot \mathbf{f}(\bar{\xi}, t) + o(1) \\ \int_0^t \Sigma^k(\zeta, \xi, t - \tau) \mathbf{g}(\xi, \tau) d\tau &= \frac{1}{\varepsilon^2} \Sigma^k(\bar{\zeta}, \bar{\xi}) \cdot \mathbf{g}(\bar{\xi}, t) + o(1) \end{aligned} \quad ((\zeta, \xi) \in \Gamma_\varepsilon, (\bar{\zeta}, \bar{\xi}) \in \bar{\Gamma}, t \geq 0) \quad (10)$$

where  $\mathbf{U}^k(\bar{\zeta}, \bar{\xi})$  and  $\Sigma^k(\bar{\zeta}, \bar{\xi})$  are the *elastostatic* fundamental displacement and stress (Kelvin solution), one obtains:

$$\begin{aligned} t_i^\pm [\mathbf{u}](\mathbf{x}, t) + o(1) &= c_{ijkl} n_j^\pm(\bar{\zeta}) \left\{ \varepsilon^{d_e-1} \int_{\bar{\Gamma}}^* \Sigma_{ab}^k(\bar{\zeta}, \bar{\xi}) \bar{D}_{lb} [\mathbf{V}]_a(\bar{\xi}, t) dS_{\bar{\xi}} \right. \\ &\quad \left. + \varepsilon^{d_e+1} \rho \int_{\bar{\Gamma}} U_a^k(\bar{\zeta}, \bar{\xi}) [\ddot{\mathbf{V}}]_a(\bar{\xi}, t) dS_{\bar{\xi}} \right\} + o(\varepsilon^{(d_e-1)}) \quad (\bar{\zeta} \in \bar{\Gamma}, t \geq 0) \end{aligned} \quad (11)$$

where the tangential differential  $\bar{D}_{lb}$  is analogous to  $D_{lb}$  but defined in terms of scaled coordinates. Then, (11) implies that  $d_e = 1$ . The COD therefore has the following asymptotic form

$$[\tilde{\mathbf{u}}_\varepsilon](\xi, t) = \varepsilon [\mathbf{V}](\bar{\xi}, t) + o(\varepsilon) \quad (\xi \in \Gamma_\varepsilon, \bar{\xi} \in \bar{\Gamma}) \quad (12)$$

Considering the leading contribution of (11) as  $\varepsilon \rightarrow 0$ , the field  $\mathbf{V}$  is thus found to be determined in terms of six canonical fields

$$\mathbf{V}(\bar{\xi}, t) = \sigma_{kl} [\mathbf{u}](\mathbf{x}, t) \mathcal{V}^{kl}(\bar{\xi}) \quad (\bar{\xi} \in \mathbb{R}^3 \setminus \bar{\Gamma}, t \geq 0, (k, l) \in \{1, 2, 3\}) \quad (13)$$

where the  $\mathcal{V}^{kl}$  solve exterior elastostatic problems related to the normalized crack  $\bar{\Gamma}$  embedded in an infinite elastic medium and described by

$$\begin{aligned} \nabla_{\bar{\xi}} \cdot [\mathcal{C} : \nabla_{\bar{\xi}} \mathcal{V}^{kl}](\bar{\xi}) &= 0 \quad (\bar{\xi} \in \mathbb{R}^3 \setminus \bar{\Gamma}) \\ [\mathcal{C} : \nabla_{\bar{\xi}} \mathcal{V}^{kl}](\bar{\xi}) \cdot \mathbf{n}^\pm &= -\frac{1}{2} (\mathbf{e}_k \otimes \mathbf{e}_l + \mathbf{e}_l \otimes \mathbf{e}_k) \cdot \mathbf{n} \quad (\bar{\xi} \in \bar{\Gamma}^\pm) \end{aligned} \quad (14)$$

Closed form of the topological derivative. Substituting (12) and (13) into (6) one obtains

$$\begin{aligned} \int_0^T \int_{\Gamma_\varepsilon} \mathbf{t}[\hat{\mathbf{u}}](\xi, T-t) \cdot [\tilde{\mathbf{u}}_\varepsilon](\xi, t) dS_\xi dt &= \varepsilon^3 \int_0^T \int_{\bar{\Gamma}} \mathbf{n} \cdot \boldsymbol{\sigma}[\hat{\mathbf{u}}](\bar{\xi}, T-t) \cdot [\mathbf{V}](\bar{\xi}, t) dS_{\bar{\xi}} dt + o(\varepsilon^3) \\ &= \varepsilon^3 \int_0^T \sigma_{ij}[\hat{\mathbf{u}}](\mathbf{x}, T-t) \sigma_{kl}[\mathbf{u}](\mathbf{x}, t) \int_{\bar{\Gamma}} n_j [\mathcal{V}_i^{kl}](\bar{\xi}) dS_{\bar{\xi}} dt + o(\varepsilon^3) \end{aligned} \quad (15)$$

Thus, with reference to (4), one obtains  $\eta(\varepsilon) = \varepsilon^3$  and

$$\mathbb{T}(\mathbf{x}, T) = \int_0^T \boldsymbol{\sigma}[\hat{\mathbf{u}}](\mathbf{x}, T-t) : \mathcal{A} : \boldsymbol{\sigma}[\mathbf{u}](\mathbf{x}, t) dt \quad (16)$$

where the fourth-order *polarization tensor*  $\mathcal{A}$  is defined by

$$\mathcal{A}_{ijkl} = \int_{\bar{\Gamma}} n_j [\mathcal{V}_i^{kl}](\bar{\xi}) dS_{\bar{\xi}} \quad (17)$$

Through the canonical solutions  $\mathcal{V}^{kl}$  defined by (14),  $\mathcal{A}$  depends on the assumed crack shape  $\bar{\Gamma}$  and the material properties synthesized in the elastic tensor  $\mathcal{C}$ .

Equations (10) and (15), which play a key role in the small-crack asymptotic analysis, rely on the implicit assumption that the probing excitation be sufficiently smooth as a function of time (which e.g. places restrictions on its Fourier content). Specifically, (10) and (15) rely on  $\mathbf{f}(\bar{\xi}, t)$ ,  $\mathbf{g}(\bar{\xi}, t)$  being continuous in time and on  $\boldsymbol{\sigma}[\hat{\mathbf{u}}](\bar{\xi}, T-t)$  being continuous over  $\Gamma_\varepsilon$ , respectively. Such issues are addressed in [2], where the order in  $\varepsilon$  of the leading perturbation by a small inclusion undergone by the fundamental solution of the transient scalar wave equation is shown to depend on the high-frequency content of the point source.

Canonical solution for circular planar crack. In the case of a circular planar crack (for which  $\bar{\Gamma}$  is the unit disk with unit normal  $\mathbf{n}$ ), the solutions of the elastic canonical problems (14) are known [8] as

$$\llbracket \boldsymbol{\nu}^{kl} \rrbracket(\bar{\boldsymbol{\xi}}) = \boldsymbol{\phi}^{kl}(\bar{\boldsymbol{\xi}}) + \frac{4(1-\nu)}{\pi\mu} \sigma_{nn} \sqrt{1-|\bar{\boldsymbol{\xi}}|^2} \mathbf{n} \quad (\bar{\boldsymbol{\xi}} \in \Gamma) \quad (18)$$

where  $\sigma_{nn} = \frac{1}{2} \mathbf{n} \cdot (\mathbf{e}_k \otimes \mathbf{e}_l + \mathbf{e}_l \otimes \mathbf{e}_k) \cdot \mathbf{n} = n_k n_l$ , and  $\boldsymbol{\phi}^{kl}$  denotes the in-plane contribution to  $\boldsymbol{\nu}^{kl}$ . Moreover, using integral equation formulations, it can be proved that  $\boldsymbol{\phi}^{kl}$  is skew-symmetric over  $\bar{\Gamma}$ , which implies

$$\int_{\bar{\Gamma}} n_j \phi_i^{kl}(\bar{\boldsymbol{\xi}}) dS_{\bar{\boldsymbol{\xi}}} = 0 \quad (19)$$

Hence, (17) reduces to

$$\mathcal{A} = \frac{8(1-\nu)}{3\mu} (\mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n}) \quad (20)$$

### 3. Numerical example

Consider the identification of a penny-shaped crack of radius 0.1 embedded in a unit cubic elastic body (characterized by  $\nu = 0.3$ ) at initial rest, with a unit normal given by  $\mathbf{n} = \sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_3$  (with coordinate directions defined as in Figs. 1, 2). A FE model with 28976 nodes had been used to produce synthetic data of boundary measurements  $\mathbf{u}_{\text{obs}}$  over the whole domain boundary. The bottom face ( $x_3 = 0$ ) is clamped, while a uniform and constant compressional loading is applied on the top face during time interval  $0 \leq t \leq T$ . The (synthetic) experiment duration  $T$  corresponds to the time for a longitudinal wave to travel from the top to the bottom face of the cube. The topological derivative is computed using (16) and (20). The cost function is of format (1) with  $S_{\text{obs}} = S$  and using the least-squares misfit function

$$\varphi[\mathbf{u}, \boldsymbol{\xi}, t] = \frac{1}{2} \|\mathbf{u} - \mathbf{u}_{\text{obs}}(\boldsymbol{\xi}, t)\|^2$$

Numerical results for (the negative part of)  $\mathbb{T}(\mathbf{x}, T)$  and the subset  $\Omega(\alpha) = \{\mathbf{x} \in \Omega \mid \mathbb{T}(\mathbf{x}, T) \leq \alpha \min_{\boldsymbol{\xi} \in \Omega} \mathbb{T}(\boldsymbol{\xi}, T)\}$  of  $\Omega$  over which  $\mathbb{T}(\mathbf{x}, T)$  is deemed sufficient low (with  $\alpha = 0.8$  here) are shown in Figures 1 (for a horizontal true crack, i.e.  $\theta = 0$ ) and 2 (for an inclined true crack, with  $\theta = \pi/4$ ). These results all indicate a

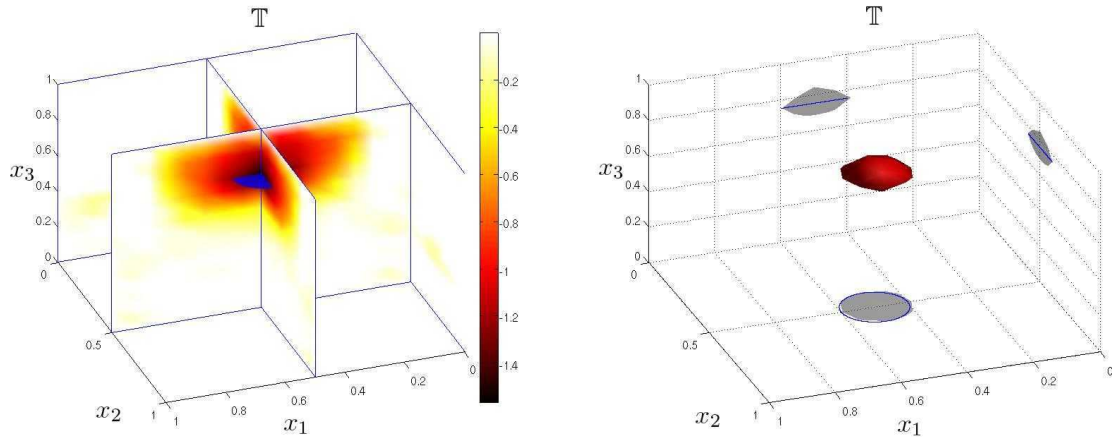


Figure 1. Identification of a horizontal circular crack ( $\theta = 0$ ): negative part of  $\mathbb{T}(\mathbf{x}, T)$  (left) and subset  $\Omega(0.8)$  (right).

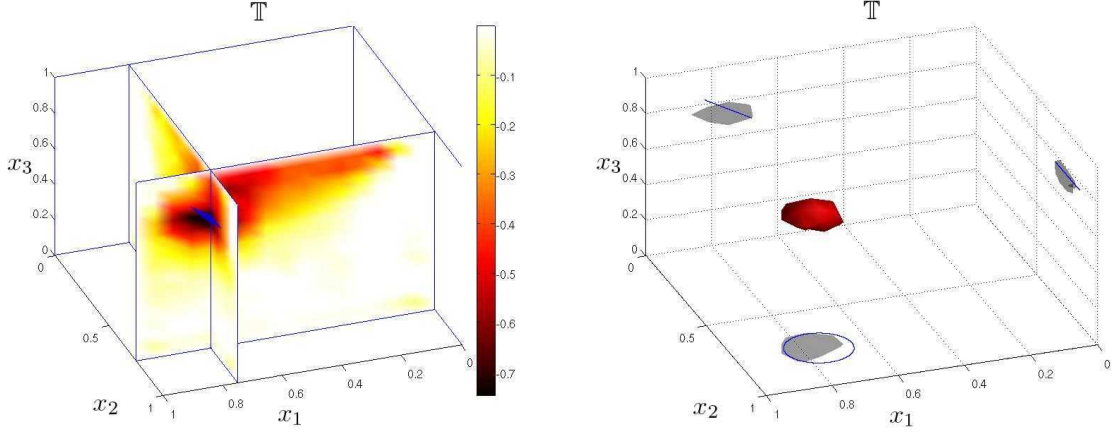


Figure 2. Identification of an inclined circular crack ( $\theta = \pi/4$ ): negative part of  $\mathbb{T}(\mathbf{x}, T)$  (left) and subset  $\Omega(0.8)$  (right).

qualitatively correct identification of the crack, and demonstrate the usefulness of the topological derivative  $\mathbb{T}(\mathbf{x}, T)$  of the misfit functional  $\mathbb{J}$  as a crack indicator function. Note that evaluating the field  $\mathbb{T}(\mathbf{x}, T)$  requires the numerical computation of just two states, namely the free and adjoint solutions, which are both defined on the same, crack-free, reference configuration. This procedure is therefore much faster than a full inversion, which would necessarily require an iterative solution procedure.

#### 4. Acoustic topological sensitivity

Framework. In the framework of time-domain acoustics, a “crack” is a rigid thin screen. Under prescribed time-dependent normal velocity on  $S$  the scattered acoustic pressure field  $\tilde{u}_\varepsilon$  arising in  $\Omega_\varepsilon$  solves the following set of equations

$$\begin{aligned}
\Delta \tilde{u}_\varepsilon(\boldsymbol{\xi}, t) &= \frac{1}{c^2} \ddot{\tilde{u}}_\varepsilon(\boldsymbol{\xi}, t) & (\boldsymbol{\xi} \in \Omega_\varepsilon, t \geq 0) \\
\nabla \tilde{u}_\varepsilon^\pm(\boldsymbol{\xi}, t) \cdot \mathbf{n}^\pm(\boldsymbol{\xi}) &= -\nabla u(\boldsymbol{\xi}, t) \cdot \mathbf{n}^\pm(\boldsymbol{\xi}) & (\boldsymbol{\xi} \in \Gamma_\varepsilon^\pm, t \geq 0) \\
\nabla \tilde{u}_\varepsilon(\boldsymbol{\xi}, t) \cdot \mathbf{n}(\boldsymbol{\xi}) &= 0 & (\boldsymbol{\xi} \in S, t \geq 0) \\
\tilde{u}_\varepsilon(\boldsymbol{\xi}, 0) = \dot{\tilde{u}}_\varepsilon(\boldsymbol{\xi}, 0) &= 0 & (\boldsymbol{\xi} \in \Omega_\varepsilon)
\end{aligned} \tag{21}$$

where  $c$  is the acoustic wave velocity and  $u$  is the free field arising in  $\Omega$  under the given excitation on  $S$  in the absence of “crack”. Defining the adjoint solution  $\hat{u}$  by

$$\begin{aligned}
\Delta \hat{u}(\boldsymbol{\xi}, t) &= \frac{1}{c^2} \ddot{\hat{u}}(\boldsymbol{\xi}, t) & (\boldsymbol{\xi} \in \Omega, 0 \leq t \leq T) \\
\nabla \hat{u}(\boldsymbol{\xi}, t) \cdot \mathbf{n}(\boldsymbol{\xi}) &= \frac{\partial \varphi}{\partial u}[u(\boldsymbol{\xi}, T-t), \boldsymbol{\xi}, T-t] & (\boldsymbol{\xi} \in S_{\text{obs}}, 0 \leq t \leq T) \\
\nabla \hat{u}(\boldsymbol{\xi}, t) \cdot \mathbf{n}(\boldsymbol{\xi}) &= 0 & (\boldsymbol{\xi} \in S \setminus S_{\text{obs}}, 0 \leq t \leq T) \\
\hat{u}(\boldsymbol{\xi}, 0) = \dot{\hat{u}}(\boldsymbol{\xi}, 0) &= 0 & (\boldsymbol{\xi} \in \Omega)
\end{aligned} \tag{22}$$

and using scalar dynamical reciprocity, in the same fashion as in equations (4) and (6), the topological derivative can be established from

$$\mathbb{T}(\mathbf{x}, T) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\eta(\varepsilon)} \int_0^T \int_{\Gamma_\varepsilon} \nabla \hat{u}(\boldsymbol{\xi}, T-t) \cdot \mathbf{n}(\boldsymbol{\xi}) \llbracket \tilde{u}_\varepsilon \rrbracket(\boldsymbol{\xi}, t) dS_\xi dt \quad (23)$$

where  $\llbracket \tilde{u}_\varepsilon \rrbracket = \tilde{u}_\varepsilon^+ - \tilde{u}_\varepsilon^-$  denotes the jump of acoustic pressure through the "crack".

Asymptotic analysis and closed form.

In the same fashion than in the elasticity case, a boundary integral formulation can be used to show that the leading behavior of  $\tilde{u}_\varepsilon$  in the limit  $\varepsilon \rightarrow 0$  is given by

$$\llbracket \tilde{u}_\varepsilon \rrbracket(\boldsymbol{\xi}, t) = \varepsilon \llbracket W \rrbracket(\bar{\boldsymbol{\xi}}, t) + o(\varepsilon) \quad (\boldsymbol{\xi} \in \Gamma_\varepsilon, \bar{\boldsymbol{\xi}} \in \bar{\Gamma}) \quad (24)$$

where  $W$  is given by

$$W(\bar{\boldsymbol{\xi}}, t) = u_{,k}(\mathbf{x}, t) \mathcal{W}^k(\bar{\boldsymbol{\xi}}) \quad (\bar{\boldsymbol{\xi}} \in \mathbb{R}^3 \setminus \bar{\Gamma}, t \geq 0, k \in \{1, 2, 3\}) \quad (25)$$

in terms of solutions  $\mathcal{W}^k$  of the following canonical exterior Laplace problems in  $\mathbb{R}^3 \setminus \bar{\Gamma}$ :

$$\begin{aligned} \Delta_{\bar{\boldsymbol{\xi}}} \mathcal{W}^k(\bar{\boldsymbol{\xi}}) &= 0 & (\bar{\boldsymbol{\xi}} \in \mathbb{R}^3 \setminus \bar{\Gamma}) \\ \nabla_{\bar{\boldsymbol{\xi}}} \mathcal{W}^k(\bar{\boldsymbol{\xi}}) \cdot \mathbf{n}^\pm &= -\mathbf{e}_k \cdot \mathbf{n} & (\bar{\boldsymbol{\xi}} \in \bar{\Gamma}^\pm) \end{aligned} \quad (26)$$

From (24) and (25) one has

$$\int_0^T \int_{\Gamma_\varepsilon} \nabla \hat{u}(\boldsymbol{\xi}, T-t) \cdot \mathbf{n}(\boldsymbol{\xi}) \llbracket \tilde{u}_\varepsilon \rrbracket(\boldsymbol{\xi}, t) dS_\xi dt = \varepsilon^3 \int_0^T \hat{u}_{,i}(\mathbf{x}, T-t) u_{,j}(\mathbf{x}, t) \int_{\Gamma} n_i \llbracket \mathcal{W}^j \rrbracket(\bar{\boldsymbol{\xi}}, t) dS_{\bar{\boldsymbol{\xi}}} dt + o(\varepsilon^3) \quad (27)$$

i.e.  $\eta(\varepsilon) = \varepsilon^3$  again, and the topological sensitivity can finally be expressed as

$$\mathbb{T}(\mathbf{x}, T) = \int_0^T \nabla \hat{u}(\mathbf{x}, T-t) : \mathcal{B} : \nabla u(\mathbf{x}, t) dt \quad (28)$$

with the polarization tensor  $\mathcal{B}$  given by

$$\mathcal{B}_{ij} = \int_{\bar{\Gamma}} n_i \llbracket \mathcal{W}^j \rrbracket(\bar{\boldsymbol{\xi}}, t) dS_{\bar{\boldsymbol{\xi}}} \quad (29)$$

In the case of a circular planar small screen, expression (29) takes the following explicit form, identical to that previously established in [5] under time-harmonic conditions:

$$\mathcal{B} = \frac{8}{3} (\mathbf{n} \otimes \mathbf{n}) \quad (30)$$

## Conclusion

In this Note, the time-domain topological derivative method had been formulated to crack identification in linear elasticity and acoustics. The corresponding topological derivatives are given in closed form in terms of the forward and adjoint solutions. They can then be easily implemented using standard numerical methods such as the FEM. Numerical examples show the usefulness of the topological derivative as a crack indicator. Qualitative identification results obtained using this fast, non-iterative approach may for example be used as good initial guesses in subsequent iterative identification algorithms.



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