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# On the multiplicative order of $a^n$ modulo $n$

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## Abstract

Let  $n$  be a positive integer and  $\alpha_n$  be the arithmetic function which assigns the multiplicative order of  $a^n$  modulo  $n$  to every integer  $a$  coprime to  $n$  and vanishes elsewhere. Similarly, let  $\beta_n$  assign the projective multiplicative order of  $a^n$  modulo  $n$  to every integer  $a$  coprime to  $n$  and vanishes elsewhere. In this paper, we present a study of these two arithmetic functions. In particular, we prove that for positive integers  $n_1$  and  $n_2$  with the same square-free part, there exists an exact relationship between the functions  $\alpha_{n_1}$  and  $\alpha_{n_2}$  and between the functions  $\beta_{n_1}$  and  $\beta_{n_2}$ . This allows us to reduce the determination of  $\alpha_n$  and  $\beta_n$  to the case where  $n$  is square-free. These arithmetic functions recently appeared in the context of an old problem of Molluzzo, and more precisely in the study of which arithmetic progressions yield a balanced Steinhaus triangle in  $\mathbb{Z}/n\mathbb{Z}$  for  $n$  odd.

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## 1 Introduction

We start by introducing some notation relating to the order of certain elements modulo  $n$ . For every positive integer  $n$  and every prime number  $p$ , we denote by  $v_p(n)$  the *p-adic valuation* of  $n$ , i.e., the greatest exponent  $e \geq 0$  for which  $p^e$  divides  $n$ . The prime factorization of  $n$  may then be written as

$$n = \prod_{p \in \mathbb{P}} p^{v_p(n)},$$

where  $\mathbb{P}$  denotes the set of all prime numbers. We denote by  $\text{rad}(n)$  the *radical* of  $n$ , i.e., the largest square-free divisor of  $n$ , namely

$$\text{rad}(n) = \prod_{\substack{p \in \mathbb{P} \\ p|n}} p.$$

For every positive integer  $n$  and every integer  $a$  coprime to  $n$ , we denote by  $\mathcal{O}_n(a)$  the *multiplicative order of  $a$  modulo  $n$* , i.e., the smallest positive integer  $e$  such that  $a^e \equiv 1 \pmod{n}$ , namely

$$\mathcal{O}_n(a) = \min \{e \in \mathbb{N}^* \mid a^e \equiv 1 \pmod{n}\}$$

and we denote by  $\mathcal{R}_n(a)$  the *multiplicative remainder of  $a$  modulo  $n$* , i.e., the multiple of  $n$  defined by

$$\mathcal{R}_n(a) = a^{\mathcal{O}_n(a)} - 1.$$

The multiplicative order of  $a$  modulo  $n$  also corresponds with the order of the element  $\pi_n(a)$ , where  $\pi_n : \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  is the canonical surjective morphism, in the multiplicative group  $(\mathbb{Z}/n\mathbb{Z})^*$ , the group of units of  $\mathbb{Z}/n\mathbb{Z}$ . Note that  $\mathcal{O}_n(a)$  divides  $\varphi(n)$ ,  $\varphi$  being the Euler's totient function.

For every positive integer  $n$ , we define and denote by  $\alpha_n$  the arithmetic function

$$\alpha_n : \mathbb{Z} \rightarrow \mathbb{N} \\ a \mapsto \begin{cases} \mathcal{O}_n(a^n), & \text{for all } \gcd(a, n) = 1; \\ 0, & \text{otherwise,} \end{cases}$$

where  $\gcd(a, n)$  denotes the greatest common divisor of  $a$  and  $n$ , with the convention that  $\gcd(0, n) = n$ . Observe that, for every  $a$  coprime to  $n$ , the integer  $\alpha_n(a)$  divides  $\varphi(n)/\gcd(\varphi(n), n)$ . This follows from the previous remark on  $\mathcal{O}_n(a)$  and the equality  $\alpha_n(a) = \mathcal{O}_n(a^n) = \mathcal{O}_n(a)/\gcd(\mathcal{O}_n(a), n)$ .

For every positive integer  $n$  and every integer  $a$  coprime to  $n$ , we denote by  $\mathcal{PO}_n(a)$  the *projective multiplicative order of  $a$  modulo  $n$* , i.e., the smallest positive integer  $e$  such that  $a^e \equiv \pm 1 \pmod{n}$ , namely

$$\mathcal{PO}_n(a) = \min \{e \in \mathbb{N}^* \mid a^e \equiv \pm 1 \pmod{n}\}.$$

The projective multiplicative order of  $a$  modulo  $n$  also corresponds with the order of the element  $\pi_n(a)$  in the multiplicative quotient group  $(\mathbb{Z}/n\mathbb{Z})^*/\{-1, 1\}$ .

For every positive integer  $n$ , we define and denote by  $\beta_n$  the arithmetic function

$$\beta_n : \mathbb{Z} \rightarrow \mathbb{N} \\ a \mapsto \begin{cases} \mathcal{PO}_n(a^n), & \text{for all } \gcd(a, n) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that we have the alternative  $\alpha_n = \beta_n$  or  $\alpha_n = 2\beta_n$ .

In this paper, we study in detail these two arithmetic functions. In particular, we prove that, for every positive integers  $n_1$  and  $n_2$  such that

$$\begin{cases} \text{rad}(n_1) \mid n_2 \text{ and } n_2 \mid n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1) \mid n_2 \text{ and } n_2 \mid n_1, & \text{if } v_2(n_1) \geq 2, \end{cases}$$

the integer  $\alpha_{n_1}(a)$  (respectively  $\beta_{n_1}(a)$ ) divides  $\alpha_{n_2}(a)$  (resp.  $\beta_{n_2}(a)$ ), for every integer  $a$ . More precisely, we determine the exact relationship between the functions  $\alpha_{n_1}$  and  $\alpha_{n_2}$  and between  $\beta_{n_1}$  and  $\beta_{n_2}$ . We prove that we have

$$\alpha_{n_1}(a) = \frac{\alpha_{n_2}(a)}{\gcd\left(\alpha_{n_2}(a), \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right)} \text{ for all } \gcd(a, n_1) = 1$$

in Theorem 2.5 of Section 2 and that we have

$$\beta_{n_1}(a) = \frac{\beta_{n_2}(a)}{\gcd\left(\beta_{n_2}(a), \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right)} \text{ for all } \gcd(a, n_1) = 1$$

in Theorem 3.3 of Section 3. Thus, for every integer  $a$  coprime to  $n$ , the determination of  $\alpha_n(a)$  is reduced to the computation of  $\alpha_{\text{rad}(n)}(a)$  and  $\mathcal{R}_{\text{rad}(n)}(a)$  if  $v_2(n) \leq 1$  and of  $\alpha_{2\text{rad}(n)}(a)$  and  $\mathcal{R}_{2\text{rad}(n)}(a)$  if  $v_2(n) \geq 2$ . These theorems on the functions  $\alpha_n$  and  $\beta_n$  are derived from Theorem 2.6 of Section 2, which states that

$$\mathcal{O}_{n_1}(a) = \mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))},$$

for all integers  $a$  coprime to  $n_1$  and  $n_2$ . This result generalizes the following theorem of Nathanson which, in the above notation, states that for every odd prime number  $p$  and for every positive integer  $k$ , we have the equality

$$\mathcal{O}_{p^k}(a) = \mathcal{O}_p(a) \cdot \frac{p^k}{\gcd(p^k, \mathcal{R}_p(a))}$$

for all integers  $a$  not divisible by  $p$ .

**Theorem 3.6 of [3].** *Let  $p$  be an odd prime, and let  $a \neq \pm 1$  be an integer not divisible by  $p$ . Let  $d$  be the order of  $a$  modulo  $p$ . Let  $k_0$  be the largest integer such that  $a^d \equiv 1 \pmod{p^{k_0}}$ . Then the order of  $a$  modulo  $p^k$  is  $d$  for  $k = 1, \dots, k_0$  and  $dp^{k-k_0}$  for  $k \geq k_0$ .*

For every finite sequence  $S = (a_1, \dots, a_m)$  of length  $m \geq 1$  in  $\mathbb{Z}/n\mathbb{Z}$ , we denote by  $\Delta S$  the *Steinhaus triangle* of  $S$ , that is the finite multiset of cardinality  $\binom{m+1}{2}$  in  $\mathbb{Z}/n\mathbb{Z}$  defined by

$$\Delta S = \left\{ \sum_{k=0}^i \binom{i}{k} a_{j+k} \mid 0 \leq i \leq m-1, 1 \leq j \leq m-i \right\}.$$

A finite sequence  $S$  in  $\mathbb{Z}/n\mathbb{Z}$  is said to be *balanced* if each element of  $\mathbb{Z}/n\mathbb{Z}$  occurs in its Steinhaus triangle  $\Delta S$  with the same multiplicity. For instance, the sequence  $(2, 2, 3, 3)$  of length 4 is balanced in  $\mathbb{Z}/5\mathbb{Z}$ . Indeed, as depicted in Figure 1, its Steinhaus triangle is composed by each element of  $\mathbb{Z}/5\mathbb{Z}$  occurring twice.

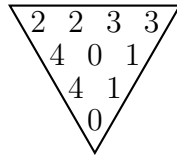


Figure 1: The Steinhaus triangle of a balanced sequence in  $\mathbb{Z}/5\mathbb{Z}$

Note that, for a sequence  $S$  of length  $m \geq 1$  in  $\mathbb{Z}/n\mathbb{Z}$ , a necessary condition on  $m$  for  $S$  to be balanced is that the integer  $n$  divides the binomial coefficient  $\binom{m+1}{2}$ . In 1976, John C. Molluzzo [2] posed the problem to determine whether this necessary condition on  $m$  is also sufficient to guarantee the existence of a balanced sequence. In [1], it was proved that, for each odd number  $n$ , *there exists a balanced sequence of length  $m$  for every  $m \equiv 0$  or  $-1 \pmod{\alpha_n(2) \cdot n}$  and for every  $m \equiv 0$  or  $-1 \pmod{\beta_n(2) \cdot n}$* . This was achieved by analyzing the Steinhaus triangles generated by arithmetic progressions. In particular, since  $\beta_{3^k}(2) = 1$  for all  $k \geq 1$ , the above result implies a complete and positive solution of Molluzzo's Problem in  $\mathbb{Z}/n\mathbb{Z}$  for all  $n = 3^k$ .

## 2 The arithmetic function $\alpha_n$

The table depicted in Figure 2 gives us the first values of  $\alpha_n(a)$  for every positive integer  $n$ ,  $1 \leq n \leq 20$ , and for every integer  $a$ ,  $-20 \leq a \leq 20$ .

$n \backslash a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
3	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
4	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
5	1	4	4	2	0	1	4	4	2	0	1	4	4	2	0	1	4	4	2	0
6	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
7	1	3	6	3	6	2	0	1	3	6	3	6	2	0	1	3	6	3	6	2
8	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
9	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
10	1	0	2	0	0	0	2	0	1	0	1	0	2	0	0	0	2	0	1	0
11	1	10	5	5	5	10	10	10	5	2	0	1	10	5	5	5	10	10	10	5
12	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
13	1	12	3	6	4	12	12	4	3	6	12	2	0	1	12	3	6	4	12	12
14	1	0	3	0	3	0	0	0	3	0	3	0	1	0	1	0	3	0	3	0
15	1	4	0	2	0	0	4	4	0	0	2	0	4	2	0	1	4	0	2	0
16	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
17	1	8	16	4	16	16	16	8	8	16	16	16	4	16	8	2	0	1	8	16
18	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
19	1	18	18	9	9	9	3	6	9	18	3	6	18	18	18	9	9	2	0	1
20	1	0	1	0	0	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0

Figure 2: The first values of  $\alpha_n(a)$

The positive integer  $\alpha_n(a)$  seems to be difficult to determine. Indeed, there is no general formula known to compute the multiplicative order of an integer modulo  $n$  but, however, we get the following helpful propositions.

**Lemma 2.1.** *Let  $n_1$  and  $n_2$  be two positive integers such that  $\text{rad}(n_1) = \text{rad}(n_2)$ . Then, an integer  $a$  is coprime to  $n_1$  if, and only if, it is also coprime to  $n_2$ .*

*Proof.* This follows from the definition of the greatest common divisor of two integers and from the definition of the radical of an integer.  $\square$

**Proposition 2.2.** *Let  $n_1$  and  $n_2$  be two positive integers such that  $\text{rad}(n_1) | n_2$  and  $n_2 | n_1$ . Then, for every integer  $a$ , the integer  $\alpha_{n_1}(a)$  divides  $\alpha_{n_2}(a)$ .*

*Proof.* If  $a$  is not coprime to  $n_1$  and  $n_2$ , then, by definition of the functions  $\alpha_{n_1}$  and  $\alpha_{n_2}$  and by Lemma 2.1, we have

$$\alpha_{n_1}(a) = \alpha_{n_2}(a) = 0.$$

Suppose that  $a$  is coprime to  $n_1$  and  $n_2$ . If  $v_p(n_1) = 1$  for all prime factors  $p$  of  $n_1$ , then  $n_2 = n_1$ . Otherwise, let  $p$  be a prime factor of  $n_1$  such that  $v_p(n_1) \geq 2$ . We shall show that  $\alpha_{n_1}(a)$  divides  $\alpha_{n_1/p}(a)$ . By definition of  $\alpha_{n_1/p}(a)$ , there exists an integer  $u$  such that

$$a^{\alpha_{n_1/p}(a) \cdot \frac{n_1}{p}} = 1 + u \cdot \frac{n_1}{p}.$$

Therefore, by the binomial theorem, we have

$$a^{\alpha_{n_1/p}(a) \cdot n_1} = \left( a^{\alpha_{n_1/p}(a) \cdot \frac{n_1}{p}} \right)^p = \left( 1 + u \cdot \frac{n_1}{p} \right)^p = 1 + u \cdot n_1 + \sum_{k=2}^p \binom{p}{k} \cdot u^k \cdot \left( \frac{n_1}{p} \right)^k.$$

Since  $v_p(n_1) \geq 2$ , it follows that  $(n_1/p)^k$  is divisible by  $n_1$  for every integer  $k \geq 2$  and so

$$a^{\alpha_{n_1/p}(a) \cdot n_1} \equiv 1 \pmod{n_1}.$$

Hence  $\alpha_{n_1}(a)$  divides  $\alpha_{n_1/p}(a)$ . This completes the proof.  $\square$

An exact relationship between  $\alpha_{n_1}(a)$  and  $\alpha_{n_2}(a)$ , for every integer  $a$  coprime to  $n_1$  and  $n_2$ , is determined at the end of this section. We first settle the easy prime power case.

**Proposition 2.3.** *Let  $p$  be a prime number and let  $a$  be an integer. Then we have*

$$\alpha_{p^k}(a) = \mathcal{O}_p(a)$$

for every positive integer  $k$ .

*Proof.* Let  $k$  be a positive integer. If  $a$  is not coprime to  $p$ , then we have  $\alpha_{p^k}(a) = \alpha_p(a) = 0$ . Suppose now that  $a$  is coprime to  $p$ . By Proposition 2.2, the integer  $\alpha_{p^k}(a)$  divides  $\alpha_p(a)$ . It remains to prove that  $\alpha_p(a)$  divides  $\alpha_{p^k}(a)$ . The congruence

$$a^{\alpha_{p^k}(a) \cdot p^k} \equiv 1 \pmod{p^k}$$

implies that

$$a^{\alpha_{p^k}(a) \cdot p^k} \equiv 1 \pmod{p},$$

and hence, by Fermat's Little Theorem, it follows that

$$a^{\alpha_{p^k}(a) \cdot p} \equiv a^{\alpha_{p^k}(a) \cdot p^k} \equiv 1 \pmod{p}.$$

Therefore  $\alpha_p(a)$  divides  $\alpha_{p^k}(a)$ . Finally, we have

$$\alpha_{p^k}(a) = \alpha_p(a) = \mathcal{O}_p(a^p) = \mathcal{O}_p(a).$$

This completes the proof.  $\square$

*Remark.* If  $p = 2$ , then, for every positive integer  $k$ , we obtain

$$\alpha_{2^k}(a) = \mathcal{O}_2(a) = \begin{cases} 0, & \text{for } a \text{ even;} \\ 1, & \text{for } a \text{ odd.} \end{cases}$$

**Proposition 2.4.** *Let  $n_1$  and  $n_2$  be two coprime numbers and let  $a$  be an integer. Then  $\alpha_{n_1 n_2}(a)$  divides  $\text{lcm}(\alpha_{n_1}(a), \alpha_{n_2}(a))$ , the least common multiple of  $\alpha_{n_1}(a)$  and  $\alpha_{n_2}(a)$ .*

*Proof.* If  $\text{gcd}(a, n_1 n_2) \neq 1$ , then  $\text{gcd}(a, n_1) \neq 1$  or  $\text{gcd}(a, n_2) \neq 1$  and so

$$\alpha_{n_1 n_2}(a) = \text{lcm}(\alpha_{n_1}(a), \alpha_{n_2}(a)) = 0.$$

Suppose now that  $\text{gcd}(a, n_1 n_2) = 1$  and hence that the integers  $a$ ,  $n_1$  and  $n_2$  are coprime pairwise. Let  $i \in \{1, 2\}$ . The congruences

$$a^{\alpha_{n_i}(a) \cdot n_i} \equiv 1 \pmod{n_i}$$

imply that

$$a^{n_1 n_2 \text{lcm}(\alpha_{n_1}(a), \alpha_{n_2}(a))} \equiv 1 \pmod{n_i}.$$

Therefore  $\alpha_{n_1 n_2}(a)$  divides  $\text{lcm}(\alpha_{n_1}(a), \alpha_{n_2}(a))$  by the Chinese remainder theorem.  $\square$

Let  $n_1$  and  $n_2$  be two positive integers such that

$$\begin{cases} \text{rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \geq 2. \end{cases}$$

By definition, we know that  $\alpha_{n_1}(a) = \alpha_{n_2}(a) = 0$  for every integer  $a$  not coprime to  $n_1$  and  $n_2$ . We end this section by determining the exact relationship between  $\alpha_{n_1}(a)$  and  $\alpha_{n_2}(a)$  for every integer  $a$  coprime to  $n_1$  and  $n_2$ .

**Theorem 2.5.** *Let  $n_1$  and  $n_2$  be two positive integers such that*

$$\begin{cases} \text{rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \geq 2. \end{cases}$$

*Then, for every integer  $a$  coprime to  $n_1$  and  $n_2$ , we have*

$$\alpha_{n_1}(a) = \frac{\alpha_{n_2}(a)}{\text{gcd}\left(\alpha_{n_2}(a), \frac{\text{gcd}(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right)}$$

This result is a corollary of the following theorem.

**Theorem 2.6.** *Let  $n_1$  and  $n_2$  be two positive integers such that*

$$\begin{cases} \text{rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \geq 2. \end{cases}$$

*Then, for every integer  $a$  coprime to  $n_1$  and  $n_2$ , we have*

$$\mathcal{O}_{n_1}(a) = \mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\text{gcd}(n_1, \mathcal{R}_{n_2}(a))}.$$

The proof of this theorem is based on the following lemma.

**Lemma 2.7.** *Let  $n$  be a positive integer and let  $a$  be an integer coprime to  $n$ . Let  $m$  be an integer such that  $\text{rad}(m) | \text{rad } n$ . Then, there exists an integer  $u_m$ , coprime to  $n$  if  $m$  is odd, or coprime to  $n/2$  if  $m$  is even, such that*

$$a^{\mathcal{O}_n(a) \cdot m} = 1 + u_m \cdot \mathcal{R}_n(a) \cdot m.$$

*Proof.* We distinguish different cases based upon the parity of  $m$ . First, we prove the odd case by induction on  $m$ . If  $m = 1$ , then, by definition of the integer  $\mathcal{R}_n(a)$ , we have

$$a^{\mathcal{O}_n(a)} = 1 + \mathcal{R}_n(a).$$

Therefore the assertion is true for  $m = 1$ .

Now, let  $p$  be a prime factor of  $m$  and suppose that the assertion is true for the odd number  $m/p$ , i.e., there exists an integer  $u_{m/p}$ , coprime to  $n$ , such that

$$a^{\mathcal{O}_n(a) \cdot \frac{m}{p}} = 1 + u_{m/p} \cdot \mathcal{R}_n(a) \cdot \frac{m}{p}.$$

Then, we obtain

$$\begin{aligned} a^{\mathcal{O}_n(a) \cdot m} &= \left( a^{\mathcal{O}_n(a) \cdot \frac{m}{p}} \right)^p = \left( 1 + u_{m/p} \cdot \mathcal{R}_n(a) \cdot \frac{m}{p} \right)^p \\ &= 1 + u_{m/p} \cdot \mathcal{R}_n(a) \cdot m + \sum_{k=2}^{p-1} \binom{p}{k} \left( u_{m/p} \cdot \mathcal{R}_n(a) \cdot \frac{m}{p} \right)^k + \left( u_{m/p} \cdot \mathcal{R}_n(a) \cdot \frac{m}{p} \right)^p \\ &= 1 + \left( u_{m/p} + \sum_{k=2}^{p-1} \frac{\binom{p}{k}}{p} \cdot (u_{m/p})^k \cdot \mathcal{R}_n(a)^{k-1} \cdot \left( \frac{m}{p} \right)^{k-1} + \right. \\ &\quad \left. + (u_{m/p})^p \cdot \frac{\mathcal{R}_n(a)^{p-1}}{p} \cdot \left( \frac{m}{p} \right)^{p-1} \right) \cdot \mathcal{R}_n(a) \cdot m \\ &= 1 + u_m \cdot \mathcal{R}_n(a) \cdot m. \end{aligned}$$

Since  $n$  divides  $\mathcal{R}_n(a)$  which divides

$$u_m - u_{m/p} = \sum_{k=2}^{p-1} \frac{\binom{p}{k}}{p} \cdot (u_{m/p})^k \cdot \mathcal{R}_n(a)^{k-1} \cdot \left( \frac{m}{p} \right)^{k-1} + (u_{m/p})^p \cdot \frac{\mathcal{R}_n(a)^{p-1}}{p} \cdot \left( \frac{m}{p} \right)^{p-1},$$

it follows that  $\gcd(u_m, n) = \gcd(u_{m/p}, n) = 1$ . This completes the proof for the odd case.

Suppose now that  $n$  and  $m$  are even. We proceed by induction on  $v_2(m)$ . If  $v_2(m) = 1$ , then  $m/2$  is odd and by the first part of this proof,

$$a^{\frac{m}{2} \cdot \mathcal{O}_n(a)} = 1 + u_{m/2} \cdot \frac{m}{2} \cdot \mathcal{R}_n(a)$$

where  $u_{m/2}$  is coprime to  $n$  and hence to  $n/2$ . Now assume that  $v_2(m) > 1$  and that

$$a^{\frac{m}{2} \cdot \mathcal{O}_n(a)} = 1 + u_{m/2} \cdot \frac{m}{2} \cdot \mathcal{R}_n(a)$$

with  $u_{m/2}$  coprime to  $n/2$ . Then, we obtain

$$\begin{aligned} a^{\mathcal{O}_n(a) \cdot m} &= \left( a^{\mathcal{O}_n(a) \cdot \frac{m}{2}} \right)^2 = \left( 1 + u_{m/2} \cdot \mathcal{R}_n(a) \cdot \frac{m}{2} \right)^2 \\ &= 1 + u_{m/2} \cdot \mathcal{R}_n(a) \cdot m + \left( u_{m/2} \cdot \mathcal{R}_n(a) \cdot \frac{m}{2} \right)^2 \\ &= 1 + \left( u_{m/2} + (u_{m/2})^2 \cdot \frac{\mathcal{R}_n(a)}{2} \cdot \frac{m}{2} \right) \cdot \mathcal{R}_n(a) m \\ &= 1 + u_m \cdot \mathcal{R}_n(a) \cdot m. \end{aligned}$$

Since  $n/2$  divides  $\mathcal{R}_n(a)/2$  which divides  $u_m - u_{m/2}$ , it follows that  $\gcd(u_m, n/2) = \gcd(u_{m/2}, n/2) = 1$ . This completes the proof.  $\square$



We are now ready to prove Theorem 2.6.

*Proof of Theorem 2.6.* The proof is by induction on the integer  $n_1/n_2$ . If  $n_1 = n_2$ , then we have

$$\frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))} = \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_1}(a))} = \frac{n_1}{n_1} = 1,$$

since  $\mathcal{R}_{n_1}(a)$  is divisible by  $n_1$ , and thus the statement is true. Let  $p$  be a prime factor of  $n_1$  and  $n_2$  such that  $n_2$  divides  $n_1/p$  and suppose that

$$\mathcal{O}_{n_1/p}(a) = \mathcal{O}_{n_2}(a) \cdot \frac{n_1/p}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))}.$$

First, the congruence

$$a^{\mathcal{O}_{n_1}(a)} \equiv 1 \pmod{n_1}$$

implies that

$$a^{\mathcal{O}_{n_1}(a)} \equiv 1 \pmod{\frac{n_1}{p}}$$

and so  $\mathcal{O}_{n_1/p}(a)$  divides  $\mathcal{O}_{n_1}(a)$ . We consider two cases.

**First Case:**  $v_p(n_1) \leq v_p(\mathcal{R}_{n_2}(a))$ .

Since  $n_2$  divides  $n_1/p$ , it follows that  $\mathcal{O}_{n_2}(a)$  divides  $\mathcal{O}_{n_1/p}(a)$ . Let  $r = \frac{\mathcal{O}_{n_1/p}(a)}{\mathcal{O}_{n_2}(a)}$ . Hence

$$\begin{aligned} \mathcal{R}_{n_1/p}(a) &= a^{\mathcal{O}_{n_1/p}(a)} - 1 = a^{\mathcal{O}_{n_2}(a) \cdot r} - 1 = (a^{\mathcal{O}_{n_2}(a)} - 1) \left( \sum_{k=0}^{r-1} a^{k\mathcal{O}_{n_2}(a)} \right) \\ &= \mathcal{R}_{n_2}(a) \left( \sum_{k=0}^{r-1} a^{k\mathcal{O}_{n_2}(a)} \right) \end{aligned}$$

and so  $\mathcal{R}_{n_1/p}(a)$  is divisible by  $\mathcal{R}_{n_2}(a)$ . This leads to

$$v_p(n_1) \leq v_p(\mathcal{R}_{n_2}(a)) \leq v_p(\mathcal{R}_{n_1/p}(a)).$$

Therefore  $\mathcal{R}_{n_1/p}(a)$  is divisible by  $n_1$  and hence we have

$$a^{\mathcal{O}_{n_1/p}(a)} = 1 + \mathcal{R}_{n_1/p}(a) \equiv 1 \pmod{n_1}.$$

This implies that  $\mathcal{O}_{n_1}(a) = \mathcal{O}_{n_1/p}(a)$ . Moreover, the hypothesis  $v_p(n_1) \leq v_p(\mathcal{R}_{n_2}(a))$  implies that  $\gcd(n_1/p, \mathcal{R}_{n_2}(a)) = \gcd(n_1, \mathcal{R}_{n_2}(a))/p$ . Finally, we obtain

$$\mathcal{O}_{n_1}(a) = \mathcal{O}_{n_1/p}(a) = \mathcal{O}_{n_2}(a) \cdot \frac{n_1/p}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))} = \mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))}.$$

**Second Case:**  $v_p(n_1) > v_p(\mathcal{R}_{n_2}(a))$ .

If  $v_2(n_1) \leq 1$ , then  $(n_1/p)/\gcd(n_1/p, \mathcal{R}_{n_2}(a))$  is odd. Otherwise, if  $v_2(n_1) \geq 2$ , then  $v_2(n_2) \geq 2$  and every integer coprime to  $n_2/2$  is also coprime to  $n_2$ . In both cases,

$v_2(n_1) \leq 1$  or  $v_2(n_1) \geq 2$ , we know, by Lemma 2.7, that there exists an integer  $u$ , coprime to  $n_2$ , such that

$$\begin{aligned} a^{\mathcal{O}_{n_1/p}(a)} &= a^{\mathcal{O}_{n_2}(a) \cdot \frac{n_1/p}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))}} = 1 + u \cdot \mathcal{R}_{n_2}(a) \cdot \frac{n_1/p}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))} \\ &= 1 + u \cdot \frac{\mathcal{R}_{n_2}(a)}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))} \cdot \frac{n_1}{p}. \end{aligned}$$

As  $v_p(\mathcal{R}_{n_2}(a)) \leq v_p(n_1/p)$ , it follows that  $\mathcal{R}_{n_2}(a)/\gcd(n_1/p, \mathcal{R}_{n_2}(a))$  is coprime to  $p$ , and hence  $\mathcal{O}_{n_1/p}(a)$  is a proper divisor of  $\mathcal{O}_{n_1}(a)$  since

$$a^{\mathcal{O}_{n_1/p}(a)} \not\equiv 1 \pmod{n_1}.$$

Moreover, by Lemma 2.7 again, there exists an integer  $u_p$  such that

$$a^{\mathcal{O}_{n_1/p}(a) \cdot p} = 1 + u_p \cdot \mathcal{R}_{n_1/p}(a) \cdot p \equiv 1 \pmod{n_1}.$$

This leads to

$$\mathcal{O}_{n_1}(a) = \mathcal{O}_{n_1/p}(a) \cdot p = \mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))} = \mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))}.$$

This completes the proof of Theorem 2.6.  $\square$

We may view Theorem 2.6 as a generalization of Theorem 3.6 of [3], where  $n_2 = p$  is an odd prime number and  $n_1 = p^k$  for some positive integer  $k$ . Note that the conclusion of Theorem 2.6 fails in general in the case where  $v_2(n_1) \geq 2$  and  $n_2 = \text{rad}(n_1)$ . For instance, for  $n_1 = 24 = 3 \cdot 2^3$ ,  $n_2 = 6 = 3 \cdot 2$  and  $a = 7$ , we obtain that  $\mathcal{O}_{n_1}(a) = 2$  while  $\mathcal{O}_{n_2}(a)n_1/\gcd(n_1, \mathcal{R}_{n_2}(a)) = 24/\gcd(24, 6) = 4$ .

We now turn to the proof of the main result of this paper.

*Proof of Theorem 2.5.* From Theorem 2.6, we obtain

$$\begin{aligned} \alpha_{n_1}(a) &= \mathcal{O}_{n_1}(a^{n_1}) = \frac{\mathcal{O}_{n_1}(a)}{\gcd(\mathcal{O}_{n_1}(a), n_1)} = \frac{\mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))}}{\gcd\left(\mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))}, n_1\right)} \\ &= \frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_1, \mathcal{R}_{n_2}(a))}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\alpha_{n_2}(a)}{\alpha_{n_1}(a)} &= \frac{\frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_2)}}{\frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_1, \mathcal{R}_{n_2}(a))}} = \frac{\gcd(\mathcal{O}_{n_2}(a), n_1, \mathcal{R}_{n_2}(a))}{\gcd(\mathcal{O}_{n_2}(a), n_2)} \\ &= \gcd\left(\frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_2)}, \frac{n_2}{\gcd(\mathcal{O}_{n_2}(a), n_2)} \cdot \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right). \end{aligned}$$

Finally, since we have

$$\gcd\left(\frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_2)}, \frac{n_2}{\gcd(\mathcal{O}_{n_2}(a), n_2)}\right) = \frac{\gcd(\mathcal{O}_{n_2}(a), n_2)}{\gcd(\mathcal{O}_{n_2}(a), n_2)} = 1,$$

it follows that

$$\frac{\alpha_{n_2}(a)}{\alpha_{n_1}(a)} = \gcd\left(\frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_2)}, \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right) = \gcd\left(\alpha_{n_2}(a), \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right).$$

□

Thus, the determination of  $\alpha_n$  is reduced to the case where  $n$  is square-free.

**Corollary 2.8.** *Let  $n$  be a positive integer such that  $v_2(n) \leq 1$ . Then, for every integer  $a$ , coprime to  $n$ , we have*

$$\alpha_n(a) = \frac{\alpha_{\text{rad}(n)}(a)}{\gcd\left(\alpha_{\text{rad}(n)}(a), \frac{\gcd(n, \mathcal{R}_{\text{rad}(n)}(a))}{\text{rad}(n)}\right)}.$$

**Corollary 2.9.** *Let  $n$  be a positive integer such that  $v_2(n) \geq 2$ . Then, for every integer  $a$ , coprime to  $n$ , we have*

$$\alpha_n(a) = \frac{\alpha_{2\text{rad}(n)}(a)}{\gcd\left(\alpha_{2\text{rad}(n)}(a), \frac{\gcd(n, \mathcal{R}_{2\text{rad}(n)}(a))}{2\text{rad}(n)}\right)}.$$

### 3 The arithmetic function $\beta_n$

First, we can observe that, by definition of the functions  $\alpha_n$  and  $\beta_n$ , we have

$$\alpha_n(a) = \beta_n(a) = 0$$

for every integer  $a$  not coprime to  $n$  and

$$\frac{\alpha_n(a)}{\beta_n(a)} \in \{1, 2\}$$

for every integer  $a$  coprime to  $n$ . There is no general formula known to compute  $\alpha_n(a)/\beta_n(a)$  but, however, we get the following proposition.

**Proposition 3.1.** *Let  $n_1$  and  $n_2$  be two positive integers such that  $\text{rad}(n_1) = \text{rad}(n_2)$ . Let  $a$  be an integer coprime to  $n_1$  and  $n_2$ . If  $v_2(n_1) \leq 1$ , then we have*

$$\frac{\alpha_{n_1}(a)}{\beta_{n_1}(a)} = \frac{\alpha_{n_2}(a)}{\beta_{n_2}(a)}.$$

If  $v_2(n_1) \geq 2$ , then we have

$$\alpha_{n_1}(a) = \beta_{n_1}(a).$$

*Proof.* Let  $n_1$  be a positive integer such that  $v_2(n_1) \leq 1$  and  $a$  be an integer coprime to  $n_1$ . Let  $p$  be an odd prime factor of  $n_1$  such that  $v_p(n_1) \geq 2$ . We will prove that

$$\frac{\alpha_{n_1}(a)}{\beta_{n_1}(a)} = \frac{\alpha_{n_1/p}(a)}{\beta_{n_1/p}(a)}.$$

If  $\alpha_{n_1}(a) = 2\beta_{n_1}(a)$ , then

$$a^{\beta_{n_1}(a) \cdot n_1} \equiv -1 \pmod{n_1}$$

and thus

$$a^{\beta_{n_1}(a) \cdot p \cdot \frac{n_1}{p}} \equiv -1 \pmod{\frac{n_1}{p}}.$$

This implies that  $\alpha_{n_1/p}(a) = 2\beta_{n_1/p}(a)$ . Conversely, if  $\alpha_{n_1/p}(a) = 2\beta_{n_1/p}(a)$ , then we have

$$a^{\beta_{n_1/p}(a) \cdot \frac{n_1}{p}} \equiv -1 \pmod{\frac{n_1}{p}}.$$

Since  $v_p(n_1) \geq 2$ , it follows that

$$a^{\beta_{n_1/p}(a) \cdot \frac{n_1}{p}} \equiv -1 \pmod{p}$$

and thus

$$a^{\beta_{n_1/p}(a) \cdot n_1} + 1 = 1 - \left(-a^{\beta_{n_1/p}(a) \cdot \frac{n_1}{p}}\right)^p = \left(1 + a^{\beta_{n_1/p}(a) \cdot \frac{n_1}{p}}\right) \sum_{k=0}^{p-1} \left(-a^{\beta_{n_1/p}(a) \cdot \frac{n_1}{p}}\right)^k \equiv 0 \pmod{n_1}.$$

This implies that  $\alpha_{n_1}(a) = 2\beta_{n_1}(a)$ . Continuing this process we have

$$\frac{\alpha_{n_1}(a)}{\beta_{n_1}(a)} = \frac{\alpha_{\text{rad}(n_1)}(a)}{\beta_{\text{rad}(n_1)}(a)}$$

and since  $\text{rad}(n_1) = \text{rad}(n_2)$ ,

$$\frac{\alpha_{n_1}(a)}{\beta_{n_1}(a)} = \frac{\alpha_{n_2}(a)}{\beta_{n_2}(a)}.$$

Now, let  $n_1$  be a positive integer such that  $v_2(n_1) \geq 2$ , and let  $a$  be a non-zero integer. Suppose that we have  $\alpha_{n_1}(a) = 2\beta_{n_1}(a)$ . Since

$$a^{\beta_{n_1}(a) \cdot n_1} \equiv -1 \pmod{n_1}$$

it follows that

$$\left(a^{\beta_{n_1}(a) \cdot \frac{n_1}{4}}\right)^4 \equiv -1 \pmod{4}$$

in contradiction with

$$\left(a^{\beta_{n_1}(a) \cdot \frac{n_1}{4}}\right)^4 \equiv 1 \pmod{4}.$$

Thus  $\alpha_{n_1}(a) = \beta_{n_1}(a)$ . □

If  $n$  is a prime power, then  $\beta_n = \beta_{\text{rad}(n)}$ , in analogy with Proposition 2.3 for  $\alpha_n$ .

**Proposition 3.2.** *Let  $p$  be a prime number and let  $a$  be an integer. Then we have*

$$\beta_{p^k}(a) = \beta_p(a)$$

for every positive integer  $k$ .

*Proof.* This result is trivial for every integer  $a$  not coprime to  $p$ . Suppose now that  $a$  is coprime to  $p$ . For  $p = 2$ , then, by Proposition 3.1, we have

$$\beta_{2^k}(a) = \alpha_{2^k}(a) = 1$$

for every positive integer  $k$ . For an odd prime number  $p \geq 3$ , Proposition 3.1 and Proposition 2.3 lead to

$$\beta_{p^k}(a) = \frac{\alpha_{p^k}(a)}{\alpha_p(a)} \cdot \beta_p(a) = \beta_p(a)$$

for every positive integer  $k$ . This completes the proof.  $\square$

Let  $n_1$  and  $n_2$  be two positive integers such that

$$\begin{cases} \text{rad}(n_1)|n_2 \text{ and } n_2|n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1)|n_2 \text{ and } n_2|n_1, & \text{if } v_2(n_1) \geq 2. \end{cases}$$

It immediately follows that  $\beta_{n_1}(a) = \beta_{n_2}(a) = 0$  for every integer  $a$  not coprime to  $n_1$  and  $n_2$ . Finally, we determine the relationship between  $\beta_{n_1}(a)$  and  $\beta_{n_2}(a)$  for every integer  $a$  coprime to  $n_1$  and  $n_2$ .

**Theorem 3.3.** *Let  $n_1$  and  $n_2$  be two positive integers such that*

$$\begin{cases} \text{rad}(n_1)|n_2 \text{ and } n_2|n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1)|n_2 \text{ and } n_2|n_1, & \text{if } v_2(n_1) \geq 2. \end{cases}$$

*Let  $a$  be an integer coprime to  $n_1$  and  $n_2$ . Then, we have*

$$\beta_{n_1}(a) = \frac{\beta_{n_2}(a)}{\text{gcd}\left(\beta_{n_2}(a), \frac{\text{gcd}(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right)}.$$

*Proof.* If  $v_2(n_1) \leq 1$ , then Theorem 2.5 and Proposition 3.1 lead to

$$\frac{\beta_{n_2}(a)}{\beta_{n_1}(a)} = \frac{\alpha_{n_2}(a)}{\alpha_{n_1}(a)} = \text{gcd}\left(\alpha_{n_2}(a), \frac{\text{gcd}(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right).$$

Since  $v_2(n_2) = v_2(n_1) \leq 1$ , it follows that  $\text{gcd}(n_1, \mathcal{R}_{n_2}(a))/n_2$  is odd and hence, we have

$$\frac{\beta_{n_2}(a)}{\beta_{n_1}(a)} = \text{gcd}\left(\alpha_{n_2}(a), \frac{\text{gcd}(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right) = \text{gcd}\left(\beta_{n_2}(a), \frac{\text{gcd}(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right).$$

If  $v_2(n_1) \geq 2$ , then  $\beta_{n_1}(a) = \alpha_{n_1}(a)$  and  $\beta_{n_2}(a) = \alpha_{n_2}(a)$  by Proposition 3.1 and the result follows from Theorem 2.5.  $\square$

Thus, as for  $\alpha_n$ , the determination of  $\beta_n$  is reduced to the case where  $n$  is square-free.

**Corollary 3.4.** *Let  $n$  be a positive integer such that  $v_2(n) \leq 1$ . Then, for every integer  $a$ , coprime to  $n$ , we have*

$$\beta_n(a) = \frac{\beta_{\text{rad}(n)}(a)}{\text{gcd}\left(\beta_{\text{rad}(n)}(a), \frac{\text{gcd}(n, \mathcal{R}_{\text{rad}(n)}(a))}{\text{rad}(n)}\right)}.$$

**Corollary 3.5.** *Let  $n$  be a positive integer such that  $v_2(n) \geq 2$ . Then, for every integer  $a$ , coprime to  $n$ , we have*

$$\beta_n(a) = \frac{\beta_{2 \text{ rad}(n)}(a)}{\text{gcd}\left(\beta_{2 \text{ rad}(n)}(a), \frac{\text{gcd}(n, \mathcal{R}_{2 \text{ rad}(n)}(a))}{2 \text{ rad}(n)}\right)}.$$

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