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Utility functions and optimal investment in non-dominated models

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Abstract:

In this paper, we provide a framework in which we can set the problem of maximization of utility function, taking into account the model uncertainty and encompassing the case of the UVM model. The uncertainty is specified by a family of orthogonal martingale laws which is typically non-dominated. We establish a duality theory for robust utility maximization in this framework.

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1. Introduction

The purpose of this article is to study the utility maximization and to construct optimal investment strategies taking into account the model uncertainty in mathematical finance. The model uncertainty is specified by a very general set of laws which represents all the possible laws of the underlying assets. The usual approach of utility maximization is due to J. Von Neumann, O.Morgenstern ([24]). It provides conditions on an investor's preferences which guarantee that the utility of a contingent claim can be expressed as follows:

$$u(X) = E_Q[U(X)]$$

where Q is a given probability and U is a utility function, then, one has to maximize u over a set of admissible claims. D. Kramkov and W. Schachermayer ([16]) studied the problem of maximizing utility of final wealth in a general

semimartingale model by means of duality ([16]).

However, this paradigm of expected utility has clearly some deficiencies : it is not satisfactory in dealing with model uncertainty (see for example the Ellsberg paradox in [4] p.89-90). So, I. Gilboa and D. Schmeidler ([13]) introduced a robust version of the expected utility of the form

$$\inf_{Q \in \mathcal{P}} E_Q[U(X)]$$

where the infimum is taken over a whole class of possible probabilistic views of the given set of scenarios. Now, the aim of an investor is to maximize the robust utility function over the set of possible payoffs arising from admissible trading strategies.

There is a huge literature about this subject, let us mention [16], [19], [20]... In these papers, the authors assume that all the probabilities in \mathcal{P} are absolutely continuous with respect to a given reference measure P , but this condition on \mathcal{P} is violated in many examples.

The most famous example is the uncertain volatility model (UVM in short) introduced by Avellaneda and als. ([7]) and Lyons ([17]) which takes into account the difficulty of calibration of the volatility in the Black-Scholes model. In these papers, authors used stochastic control techniques to price European options. In a more recent paper, D. Talay and Z. Zheng ([22]), still using stochastic control methods, studied utility maximization in this context.

Then, in [11] a framework is introduced in a more general case, encompassing the case of the UVM model, which permits to evaluate the cheapest super-replication price of a European contingent claim under model uncertainty. L. Denis and C. Martini consider a set of probabilities \mathcal{P}_0 which is not necessarily dominated.

In this article, we are going to use the same kind of framework. So, we work with a set of probabilities \mathcal{P} which is not necessarily dominated. But, in this article, we also want to take into account uncertainty about the drift which is not important when one study prices (defined quasi-surely) but is important when one study utility maximization. So, we will take a set of probabilities \mathcal{P} including martingale laws belonging to the set \mathcal{P}_0 introduced in [11] and also probabilities which are equivalent to one of those laws.

Following ([11]), we are able to construct versions of functions and stochastic integrals defined *quasi-everywhere* i.e. defined P - almost surely under each probability $P \in \mathcal{P}$, even if \mathcal{P} is not dominated. So, for all $x > 0$, we shall consider $\chi(x)$ the set of admissible wealth processes with initial value x . Our aim is to maximize among $X \in \chi(x)$ the *robust utility function* and so we consider the value functions:

$$u(x) = \sup_{X \in \chi(x)} \inf_{P \in \mathcal{P}} E_P[U(X_T)],$$

given a utility function U .

In this article, we first establish a duality theory for robust utility maximization and then we show that there exists a probability \widehat{Q} which is *least favorable* in the sense that to solve the robust utility maximization, one just has to solve it

under \widehat{Q} .

The paper is organized as follows. First, in section 2, we set the framework using the theory of regular capacities and we formulate in a rigorous way the main results of the paper. In Section 3, we prove some useful properties and technical Lemmas. Sections 4 and 5 are devoted to the proves of the Theorems enunciated in Section 2.

2. Framework and main results

2.1. Framework

All along this article, we fix a terminal time $T > 0$.

Let $d \in \mathbb{N}^*$ be the dimension of the process we consider, i.e. the number of risky assets in our market.

We consider $\Omega = C_0([0; T], \mathbb{R}^d)$, the space of continuous functions defined on $[0, T]$ with values in \mathbb{R}^d , vanishing in 0, endowed with the uniform convergence norm, \mathcal{B} its Borel σ -field and $(\mathcal{F}_t)_{t \in [0, T]}$ the canonical filtration.

We denote by $(B_t)_{t \in [0, T]}$ the coordinates process, and for all $i \in \{1, \dots, d\}$, $(B_{i,t})_{t \in [0; T]}$ is the process of the i^{th} coordinate.

Finally, $C_b(\Omega)$ denotes the set of bounded and continuous functions defined on Ω equipped with the uniform norm topology.

Definition 1. Let P be a probability on (Ω, \mathcal{B}) , P is said to be an orthogonal martingale law if the coordinate process is a martingale with respect to (\mathcal{F}_t) under P , and if martingales $(B_i)_{1 \leq i \leq d}$ are orthogonal in the sense that for all $i \neq j$,

$$\forall t \in [0, T], \quad \langle B_i, B_j \rangle_t^P = 0 \quad P \text{ a.s.}$$

Where $\langle B_i, B_j \rangle^P$ denotes the quadratic variational process corresponding to B^i and B^j , under P . When $i = j$ we shall denote it by $\langle B_i \rangle^P$.

For all $i \in \{1, \dots, d\}$, we consider $\underline{\mu}_i$ and μ_i two finite deterministic measures on $[0, T]$. We also assume that for all i , μ_i is Hölder-continuous, i.e. there exist positive constants C_i, α_i such that:

$$\forall s, t \in [0, T], \quad s \leq t, \quad \mu_{i,t} - \mu_{i,s} \leq C_i |t - s|^{\alpha_i}$$

where for all $t \in [0, T]$, $\underline{\mu}_{i,t} = \underline{\mu}_i([0, t])$ and $\mu_{i,t} = \mu_i([0, t])$

From now on, we say that an orthogonal martingale law P verifies the hypothesis $H(\underline{\mu}, \mu)$ if:

$$\forall i \in \{1, \dots, d\}, \quad d\underline{\mu}_{i,t} \leq d \langle B_i \rangle_t^P \leq d\mu_{i,t}.$$

Let \mathcal{P}_0 be the set of orthogonal semi-martingale laws which satisfy $H(\underline{\mu}, \mu)$.

Remark : the case where $d\underline{\mu}_{i,t} = \underline{\sigma}_i^2 dt$ and $d\mu_{i,t} = \sigma_i^2 dt$ for some constants $\underline{\sigma}_i$ and σ_i , corresponds to the case where the volatility of the i^{th} risky asset belongs to $[\underline{\sigma}_i, \sigma_i]$ and the drift is equal to 0. Let us also mention that this encompasses the case where $\underline{\mu}_i = 0$.

The main property of \mathcal{P}_0 is that it is weakly compact. Indeed, for $Q \in \mathcal{P}_0$, we can consider that Q is the following linear mapping:

$$Q : \begin{array}{ccc} \mathcal{L}_1(c) & \longmapsto & \mathbb{R} \\ X & \longrightarrow & E_Q[X] \end{array}$$

Then, for all $X \in \mathcal{L}_1(c)$, $|E_Q[X]| \leq c(X)$. Therefore Q can be viewed as an element of the dual space of $(\mathcal{L}_1(c))$ denote $(\mathcal{L}_1(c))^*$ and so we can consider that \mathcal{P}_0 is a bounded subset of $(\mathcal{L}_1(c))^*$, endowed with the weak star topology.

Proposition 1. *Under the hypothesis $H(\underline{\mu}, \mu)$, \mathcal{P}_0 is weakly compact.*

Proof.

As a consequence of the Banach Alaoglu Theorem, \mathcal{P}_0 is weakly relatively compact for the weak star topology.

It remains to check that \mathcal{P}_0 is weakly closed. So, let us consider a sequence $(Q_n)_{n \in \mathbb{N}}$ in \mathcal{P}_0 converging to Q in $(\mathcal{L}_1(c))^*$ and let us prove that Q belongs to \mathcal{P}_0 .

Clearly, Q is a positive linear mapping so thanks to Proposition 11 in [12], Q is a measure which does not charge polar sets .

Moreover by passing to the limit, we can easily show that Q is a probability.

We now prove that Q is an orthogonal martingale law.

Let $0 \leq s < t \leq T$, $i \neq j \in \{1, \dots, d\}$, and $F : \mathbb{R}^{d \times n} \longrightarrow \mathbb{R}$ a continuous and bounded function. We consider

$$f = F(B_{t_1}, B_{t_2}, \dots, B_{t_n})$$

with, $0 \leq t_1 < t_2 < \dots < t_n \leq s$. Let us prove that $E_Q[(B_{i,t} - B_{i,s})f] = 0$ and $E_Q[(B_{i,t}B_{j,t} - B_{i,s}B_{j,s})f] = 0$.

As for each $n \in \mathbb{N}$, Q_n is an orthogonal martingale law

$$E_{Q_n}[(B_{i,t} - B_{i,s})f] = 0 \quad \text{and} \quad E_{Q_n}[(B_{i,t}B_{j,t} - B_{i,s}B_{j,s})f] = 0.$$

Using the same arguments as in Proposition 2.3 in [11] we can prove that $(B_{i,t} - B_{i,s})f$ and $(B_{i,t}B_{j,t} - B_{i,s}B_{j,s})f$ belong to $\mathcal{L}^1(c)$ and then by passing to the limit as $n \rightarrow \infty$, we get that:

$$E_{Q_n}[(B_{i,t} - B_{i,s})f] = 0 \quad \text{and} \quad E_{Q_n}[(B_{i,t}B_{j,t} - B_{i,s}B_{j,s})f] = 0$$

Hence Q is an orthogonal martingale law.

Now, thanks to Proposition 1, we can conclude that Q belongs to \mathcal{P}_0 . So the proof of the theorem is complete. \square

Now, we define a regular capacity using the set \mathcal{P}_0 by :

Definition 2. *$\forall p \in [1; +\infty[$, $\forall f \in C_b(\Omega)$, we set*

$$c_p(f) = \sup_{P \in \mathcal{P}_0} E_P[|f|^p]^{\frac{1}{p}}$$

Remark : In [11], the capacity considered is c_2 . Nevertheless, one can easily verify that all the proves and definitions hold in the case of an arbitrary $p \in [1; +\infty[$ (see [15], for more details).

So, these capacities are regular in the sense of Feyel-de la Pradelle ([12]) and we can consider their Lebesgue extensions (see [3] for more details):

- First, we extend c_p to lower semi-continuous positive function. For f lower semi-continuous and non negative, we set:

$$c_p(f) = \sup\{c_p(\phi); \phi \in C_b(\Omega), 0 \leq \phi \leq f\}$$

- Then, we extend c_p to arbitrary function. For $g : \Omega \rightarrow \bar{\mathbb{R}}$, we set:

$$c_p(g) = \inf\{c_p(f); f \text{ is lower-semicontinuous and } f \geq |g|\}$$

This permits to define a set capacity by setting for all $A \subset \Omega$

$$c_p(A) = c_p(\mathbf{1}_A).$$

We will use the standard capacity-related vocabulary:

- a set A is *polar* if $c_p(A) = 0$,
- a property holds *quasi-surely* (q.s.) or *quasi-everywhere* (q.e.) if it holds outside a polar set.

We do not precise with respect to which capacity a set is polar or a property holds because these notions do not depend on p . Indeed, we have the following

Theorem 1. *Let $A \in \mathcal{B}(\Omega)$, and $p \in [1, +\infty[$ then*

$$\forall P \in \mathcal{P}_0, c_p(A) = \sup_{P \in \mathcal{P}_0} (P(A))^{\frac{1}{p}}$$

As a consequence:

Corollary 1. *Let $A \in \mathcal{B}(\Omega)$, A is polar iff*

$$\forall P \in \mathcal{P}_0, P(A) = 0.$$

The proof of the theorem is given in appendix.

Remark : Thanks to this corollary, one can say that a borelian set is polar if and only if this set is negligible under each probability P belonging to \mathcal{P}_0 (or to \mathcal{P}).

Later in this article, we will only use the capacity c_1 and in order to simplify notation we denote it by c instead of c_1 . Now, using this theorem and the results of [11] (Proposition 2.12) applied to each coordinate, we can prove the following Proposition.

Proposition 2. *If Q is an orthogonal martingale law which doesn't charge polar sets, then Q belongs to \mathcal{P}_0 .*

We denote by \mathcal{L} the topological completion of $C_b(\Omega)$ with respect to the semi-norm c . \mathcal{L} is a set of functions defined quasi-everywhere (see [12]). Then, still following [11], we are able to construct a stochastic integral. Let \mathcal{H}_e be the space of elementary processes: h belongs to \mathcal{H}_e if

$$\forall s \in [0, T], h_s = \sum_{i=0}^N k_i \mathbf{1}_{]t_i; t_{i+1}]}(s),$$

where $(t_i)_i$ is a deterministic subdivision of $[0; T]$ and for all $i \in \{0, \dots, N\}$, k_{t_i} is \mathcal{F}_{t_i} -measurable, bounded and continuous.

For all $i \in \{1, \dots, d\}$, we denote by \mathcal{H}_i the completion of \mathcal{H}_e with respect to the semi-norm:

$$\|h\|_{\mathcal{H}_i} = c \left(\left(\int_0^T h_s^2 d\mu_{i,s} \right)^{\frac{1}{2}} \right) = \sup_{P \in \mathcal{P}_0} E_P \left[\left(\int_0^T h_s^2 d\mu_{i,s} \right)^{\frac{1}{2}} \right]$$

Let H_i be the quotient of \mathcal{H}_i with respect to the subspace of processes h such that $\|h\|_{\mathcal{H}_i} = 0$. Then, H_i is a Banach space with respect to the resulting norm. And, if $h = (h_i)_{1 \leq i \leq d}$ belongs to $\tilde{\mathcal{H}}_d = \mathcal{H}_1 \times \dots \times \mathcal{H}_d$, we put:

$$\|h\|_{\tilde{\mathcal{H}}_d} = \sum_{i=1}^d \|h_i\|_{\mathcal{H}_i}$$

Using the same arguments as in the proof of Theorem 2.8 in [11], we can prove the following theorem :

Theorem 2. *The linear mapping*

$$I_T : h = \left(\sum_{i=0}^N k_i^j \mathbf{1}_{]t_i; t_{i+1}]} \right)_{1 \leq j \leq d} \in (\mathcal{H}_e)^d \longmapsto \int_0^T h_s dB_s = \sum_{j=1}^d \left(\sum_{i=0}^N k_i^j (B_{j, t_{i+1}} - B_{j, t_i}) \right) \in \mathcal{L}_1(c)$$

admits the bound

$$c(I_T(h)) \leq K \|h\|_{\tilde{\mathcal{H}}_d} \quad (\star)$$

where K is the constant in the Burkholder-Davis-Gundy inequalities associated to $p=1$.

It can be extended uniquely to a continuous linear mapping from $\tilde{\mathcal{H}}_d$ to $\mathcal{L}^1(c)$, still denoted $I_T(h) = \int_0^T h_s dB_s$ and satisfying (\star) .

The main point is that all the functions or processes we consider (elements in \mathcal{L} , processes h , stochastic integrals...) are defined *quasi-everywhere* hence P almost-surely for any $P \in \mathcal{P}_0$. For the same reason, as in Lemma 2.10 of [11], by setting

$$\forall t \in [0, T], \langle B_i \rangle_t = B_{i,t}^2 - \int_0^t B_{i,s} dB_{i,s},$$

we can prove that for all i , the quadratic variation of B_i admits a quasi-everywhere defined version that we denote by $\langle B_i \rangle$.

2.2. A larger class of semimartingales laws

In order to take into account for example the uncertainty about the drift, we consider a larger set of laws denoted by \mathcal{P} which satisfies the following assumptions:

1. \mathcal{P} is convex and weakly compact.
2. For any $P \in \mathcal{P}$, there exists $P_0 \in \mathcal{P}_0$ such that P is equivalent to P_0 .
3. Conversely, for any $P_0 \in \mathcal{P}$, there exists $P \in \mathcal{P}$ such that P is equivalent to P_0 .

The main example we have in mind is the following, in which both volatility and drift are uncertain. Consider $d = 1$ for simplicity, $0 < \underline{\sigma} < \bar{\sigma}$ and $\underline{\alpha} < \bar{\alpha}$ some finite constants. We denote by $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, \underline{\alpha}, \bar{\alpha})$ the set of probability laws on (Ω, \mathcal{B}) of all the continuous semimartingales S which admit the decomposition:

$$S_t = \int_0^t \sigma_s dW_s + \int_0^t \alpha_s ds,$$

where W is a one-dimensional Brownian motion, σ is a predictable process and α an adapted one. These processes are defined on some filtered probability space and satisfy:

$$\underline{\sigma} \leq \sigma \leq \bar{\sigma} \text{ and } \underline{\alpha} \leq \alpha \leq \bar{\alpha} \text{ a.e.}$$

Here $\mathcal{P}_0 = \mathcal{P}(\underline{\sigma}, \bar{\sigma}, 0, 0)$.

Proposition 3. $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, \underline{\alpha}, \bar{\alpha})$ is convex and weakly compact.

Proof. Let us first prove that $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, \underline{\alpha}, \bar{\alpha})$ is convex. To this end, consider P_1, P_2 in $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, \underline{\alpha}, \bar{\alpha})$ and $\lambda \in [0, 1]$. Let $P = \lambda P_1 + (1 - \lambda)P_2$, as a consequence of a result due to Jacod (Théorème 7.42 in [5], see also Theorem 3 p.45 in [8]), we know that under P , B is a semimartingale. Since

$$\underline{\sigma}^2 dt \leq d\langle B, B \rangle_t^{P_1} \leq \bar{\sigma}^2 dt \text{ } P_1 - a.e.$$

and

$$\underline{\sigma}^2 dt \leq d\langle B, B \rangle_t^{P_2} \leq \bar{\sigma}^2 dt \text{ } P_2 - a.e.$$

it is clear that

$$\underline{\sigma}^2 dt \leq d\langle B, B \rangle_t^P \leq \bar{\sigma}^2 dt \text{ } P - a.e.$$

because the quadratic variation of a continuous semimartingale is defined almost surely.

So, under P , B admits the decomposition

$$B_t = M_t + A_t,$$

where M is a martingale whose law belongs to $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, 0, 0)$ and A is a finite variation process. Then for any non-negative, adapted process φ , we have:

$$\begin{aligned} E_P\left[\int_0^T \varphi(s)d(A_s - \underline{\alpha}s)\right] &= E_P\left[\int_0^T \varphi(s)d(A_s - \underline{\alpha}s)\right] \\ &= \lambda E_{P_1}\left[\int_0^T \varphi(s)d(B_s - \underline{\alpha}s)\right] + (1 - \lambda)E_{P_2}\left[\int_0^T \varphi(s)d(B_s - \underline{\alpha}s)\right] \\ &\geq 0 \end{aligned}$$

In the same way ,

$$E_P\left[\int_0^T \varphi(s)d(A_s - \bar{\alpha}s)\right] \leq 0,$$

considering a well-chosen countable family of processes φ , we conclude that

$$\underline{\alpha}dt \leq dA_t \leq \bar{\alpha}dt \quad P - a.e.$$

And this proves that P belongs to $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, \underline{\alpha}, \bar{\alpha})$.

Let us prove that $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, \underline{\alpha}, \bar{\alpha})$ is compact. First of all, thanks to the Girsanov's Theorem, it is clear that there exists a positive constant κ such that for any $P \in \mathcal{P}(\underline{\sigma}, \bar{\sigma}, \underline{\alpha}, \bar{\alpha})$, there exists $P_0 \in \mathcal{P}(\underline{\sigma}, \bar{\sigma}, 0, 0)$ such that if

$$Z = \frac{dP}{dP_0},$$

then $E_{P_0}[Z^2] \leq \kappa$.

This implies that if $K \subset \Omega$ is compact:

$$P(K) \leq \kappa^{1/2} P_0(K)^{1/2}.$$

Since $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, 0, 0)$ is compact, thanks to the Prokhorov's criterion, it is clear that $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, \underline{\alpha}, \bar{\alpha})$ is tight.

Consider now $(P_n)_{n \in \mathbb{N}}$ a sequence in $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, \underline{\alpha}, \bar{\alpha})$ which converges weakly to P . Then, (P_n) converges to P with respect to the topology introduced by Meyer and Zheng ([18]) so under P , B is still a semimartingale. Then by a slight modification of Theorems 5 and 6 in [21] or using similar arguments to those used in the proof of Proposition 1, one can prove that P belongs to $\mathcal{P}(\underline{\sigma}, \bar{\sigma}, \underline{\alpha}, \bar{\alpha})$. \square

2.3. Definitions and assumptions

As usual (see [16]), we consider a utility function $U :]0; +\infty[\rightarrow \mathbb{R}$ and we assume that it is strictly increasing, strictly concave, continuously differentiable and that it satisfies the Inada conditions :

$$U'(0) = \lim_{x \rightarrow 0} U'(x) = +\infty \quad \text{and} \quad U'(\infty) = \lim_{x \rightarrow \infty} U'(x) = 0.$$

We also assume that the asymptotic elasticity is strictly lower than 1, which means that:

$$AE(U) = \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1$$

Moreover, we assume that U is bounded, that is why we have assumed that it is defined in 0.

For example, one can consider $U(x) = K \cdot \arctan(x^\gamma/\gamma)$ where $\gamma \in (0, 1]$ and $K > 0$.

Hereafter, we will use the conjugate function of U :

$$\forall y > 0, V(y) = \sup_{x > 0} (U(x) - xy)$$

It is well known that the derivative of $U(x)$ is the inverse function of the derivative of $V(y)$ that, following [16], we denote by I :

$$I := -V' = (U')^{-1}$$

For all $Q \in \mathcal{P}$ and $x \geq 0$, we denote by $\chi_Q(x)$, the set of non-negative wealth processes X defined under Q and such that

$$X_t = x + \int_0^t h_s dB_s, \quad 0 \leq t \leq T$$

where h is predictable and B -admissible under Q .

As in [16], we set

$$u_Q(x) = \sup_{X \in \chi_Q(x)} E_Q[U(X_T)],$$

and the the dual value function of u_Q is defined by:

$$v_Q(y) = \inf_{Y \in \mathcal{Y}_Q(y)} E_Q[V(Y_T)]$$

where $\mathcal{Y}_Q(y) = \{Y \geq 0; Y_0 = y \text{ and } XY \text{ is a } Q\text{-supermartingale}, \forall X \in \chi_Q(1)\}$

Then, we define the value function of the robust problem by

$$u(x) = \inf_{Q \in \mathcal{P}} u_Q(x),$$

and the the dual function of the robust problem by

$$v(y) = \inf_{Q \in \mathcal{P}} v_Q(y).$$

As in [16], we will use an abstract version of utility function, in order to prove some results about optimization strategies. That is why we introduce the following set for all $Q \in \mathcal{P}$:

$$\mathcal{D}_Q(y) = \{h \in L_+^0(\Omega, \mathcal{F}_t, Q); 0 \leq h \leq Y_T \text{ with } Y_T \in \mathcal{Y}_Q(y)\}$$

Finally, for all $x \geq 0$, we denote by $\chi(x)$ the set of quasi-everywhere defined and non-negative wealth processes X such that

$$X_t = x + \int_0^t h_s dB_s, \quad 0 \leq t \leq T,$$

where $h \in \tilde{\mathcal{H}}_d$.

2.4. Main results

First in Section 4, we establish a duality Theorem:

Theorem 3. *Under the assumptions of Section 2, we have:*

1. *The value function u satisfies :*

$$\forall x \geq 0, u(x) = \sup_{X \in \chi(x)} \inf_{Q \in \mathcal{P}} E_Q [U(X_T)] = \inf_{Q \in \mathcal{P}} \sup_{X \in \chi(x)} E_Q [U(X_T)]$$

2. *The value functions u and v are conjugate i.e.*

$$u(x) = \inf_{y > 0} (v(y) + xy) \quad \text{and} \quad v(y) = \sup_{x > 0} (u(x) - xy)$$

The most important result of this paper is about the existence of optimization strategies.

Theorem 4. *Let $x_0 > 0$. Under the assumption of Section 2, there exists a probability measure $\hat{Q} \in \mathcal{P}$, an optimal strategy $\hat{X} \in \chi_{\hat{Q}}$ and a process $\hat{Y} \in \mathcal{Y}_{\hat{Q}}(\hat{y})$ with $\hat{y} = u'_{\hat{Q}}(x_0)$ such that*

1. $u(x_0) = u_{\hat{Q}}(x_0) = E_{\hat{Q}} [U(\hat{X})]$
2. $v(\hat{y}) = u(x_0) - \hat{y}x_0$
3. $v(\hat{y}) = v_{\hat{Q}}(\hat{y}) = E_{\hat{Q}} [\hat{Y}]$
4. $\hat{X}_T(x) = I(\hat{Y}_T(y))$ et $\hat{Y}_T(y) = U'(\hat{X}_T(x)) \quad \hat{Q} - p.s.$, and moreover $\hat{X}(x)\hat{Y}(y)$ is a uniformly integrable \hat{Q} -martingale.

We will prove this theorem in Section 5.

3. A density property

We put $\chi_e(x) = \left\{ x \exp \left(\int_0^T \alpha_s dB_s - \frac{1}{2} \sum_{i=1}^d \int_0^T \alpha_{i,s}^2 ds < B_i >_s \right); \alpha \in \mathcal{H}_e^d \right\}$.

Proposition 4. *Let $Q \in \mathcal{P}$.*

$$\chi_Q(x) \subset \overline{\chi_e(x)}^{L_0(Q)} \subset \overline{\chi(x)}^{L_0(Q)}$$

In order to prove this property, we first prove :

Lemma 1. *Let $X_T = x + \int_0^T h_s dB_s$ be an element in $\chi_Q(x)$. Assume that there exists a constant $\beta > 0$ such that $X_T \geq \beta Q$ - a.e. Then, there exists a sequence $(X_T^n)_{n \in \mathbb{N}}$ in $\chi_Q(x)$ with*

$$X_T^n = x + \int_0^T h_s^n dB_s,$$

$|h^n| \leq n$, for all $t \in [0, T]$, $x + \int_0^t h_s^n dB_s \geq \frac{\beta}{2}$, and such that:

$$\lim_{n \rightarrow \infty} X_T^n = X_T \quad \text{for the convergence in } Q \text{ - probability}$$

Proof.

Let $X_T = x + \int_0^T h_s dB_s \in \chi_Q(x)$ such that $X_T \geq \beta > 0 Q$ - a.e., then h is predictable and B-integrable, this means (see [2] p.122) that:

$$\lim_{n \rightarrow \infty} \int_0^T h_s \mathbf{1}_{|h_s| \leq n} dB_s = \int_0^T h_s dB_s,$$

for the norm D defined in [2] p122. Using the fact that our horizon is finite, we know that this implies

$$\lim_{n \rightarrow \infty} \sup_{|K| \leq 1} E_Q \left[\sup_{0 \leq t \leq T} \left| \int_0^t K_s (h_s \mathbf{1}_{|h_s| \leq n} - h_s) dB_s \right| \wedge 1 \right] = 0$$

and so:

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} \left| \int_0^t (h_s \mathbf{1}_{|h_s| \leq n} - h_s) dB_s \right| = 0 \quad \text{in } L^0(Q)$$

Let $0 < \epsilon < \frac{\beta}{2}$ and $\eta > 0$, then there exists $n_0 \in \mathbb{N}$ such that: $\forall n \geq n_0$,

$$Q \left(\sup_{0 \leq t \leq T} \left| \int_0^t (h_s \mathbf{1}_{|h_s| \leq n} - h_s) dB_s \right| > \epsilon \right) \leq \eta$$

Let $n > n_0$, we put:

$$\tau = \inf \left\{ t \geq 0, x + \int_0^t h_s \mathbf{1}_{|h_s| \leq n} dB_s < \frac{\beta}{2} \right\} \wedge T$$

and

$$h_s^n = h_s \mathbf{1}_{\{|h_s| \leq n, s < \tau\}}$$

Using the fact that $\epsilon < \frac{\beta}{2}$, we can remark that :

$$\left\{ \sup_{0 \leq t \leq T} \left| \int_0^t (h_s \mathbf{1}_{|h_s| \leq n} - h_s) dB_s \right| \leq \epsilon \right\} \subset \{\tau = T\}.$$

Besides, on $\{\tau = T\}$, $h_s^n = h_s \mathbf{1}_{\{|h_s| \leq n\}}$, as a consequence

$$\left\{ \left| \int_0^T (h_s - h_s^n) dB_s \right| > \epsilon \right\} \subset \left\{ \sup_{0 \leq t \leq T} \left| \int_0^t (h_s \mathbf{1}_{|h_s| \leq n} - h_s) dB_s \right| > \epsilon \right\}$$

and so we have proved:

$$\forall n > n_0, Q \left(\left| \int_0^T (h_s - h_s^n) dB_s \right| > \epsilon \right) \leq \eta.$$

The proof is now complete. \square

The following lemma is obvious:

Lemma 2. *For every bounded and predictable process h , there exists a sequence $(h^n)_{n \in \mathbb{N}}$ in \mathcal{H}_e^d such that:*

$$\lim_{n \rightarrow \infty} \int_0^T h_s^n dB_s = \int_0^T h_s dB_s \quad \text{in } L^0(Q)$$

and

$$\lim_{n \rightarrow \infty} \sum_{i=1}^d \int_0^T (h_{i,s}^n)^2 d \langle B_i \rangle_s = \sum_{i=1}^d \int_0^T (h_{i,s})^2 d \langle B_i \rangle_s \quad \text{in } L^0(Q)$$

Proof of Proposition 3.

We split the proof in several steps:

Step 1: We first prove that $\chi_e(x) \subset \chi(x)$.

Let $\alpha \in \mathcal{H}_e^d$, we put for all $t \in [0, T]$

$$X_t = x \exp \left(\int_0^t \alpha_s dB_s - \frac{1}{2} \sum_{i=1}^d \int_0^t \alpha_{i,s}^2 d \langle B_i \rangle_s \right)$$

and

$$Y_t = \int_0^t \alpha_s dB_s - \frac{1}{2} \sum_{i=1}^d \int_0^t \alpha_{i,s}^2 d \langle B_i \rangle_s.$$

As $\alpha \in \mathcal{H}_e^d$, it is clear that for all $n \in \mathbb{N}$, $\left(\int_0^t \alpha_s dB_s \right)^n$, $\left(\sum_{i=1}^d \int_0^t \alpha_{i,s}^2 d \langle B_i \rangle_s \right)^n$ belong to $\mathcal{L}_1(c)$ (see Proposition 2.3 in [11]). Then using Burkholder-Davis-Gundy and the boundedness of α , we see that there exists a constant K such that:

$$c(Y_t^n) \leq 2^{n-1} \left\{ c \left(\left(\int_0^t \alpha_s dB_s \right)^n \right) + c \left(\left(\sum_{i=1}^d \int_0^t \alpha_{i,s}^2 d \langle B_i \rangle_s \right)^n \right) \right\} \leq 2^{n-1} K^n$$

$\mathcal{L}_1(c)$ is a Banach space, so, thanks to the previous inequalities, $\sum_{n=0}^{\infty} \frac{Y_t^n}{n!} = \exp(Y_t)$ converges and belongs to $\mathcal{L}_1(c)$. In the same way, we can prove that there is some constant C such that $c(\exp(2Y_t)) < C$ and $c(X_t^2) < x^2 C$ which ensure that X belongs to \mathcal{H}_i for all $i \in \{1, \dots, d\}$. Indeed, for each $i \in \{1, \dots, d\}$, one can easily construct a sequence $(h_n)_{n \in \mathbb{N}}$ in \mathcal{H}_e such that:

$$\lim_{n \rightarrow \infty} \int_0^T c \left((X_s - h_{s,n})^2 \right) d\mu_{i,s} = 0,$$

and so

$$\|X - h_n\|_{\mathcal{H}_i}^2 \leq \sup_{P \in \mathcal{P}} E_P \left[\left(\int_0^T (X_s - h_{s,n})^2 d\mu_{i,s} \right)^{\frac{1}{2}} \right]^2 \leq \int_0^T \sup_{P \in \mathcal{P}} E_P \left[(X_s - h_{s,n})^2 \right] d\mu_{i,s}.$$

From this, it is clear that $\lim_{n \rightarrow \infty} h_n = X$ in \mathcal{H}_i and $X \in \mathcal{H}_i$.

Then, we apply the Itô formula to $\exp(Y_T)$ under each probability $P \in \mathcal{P}$, this yields:

$$X_T = x + \int_0^T x \alpha_s X_s dB_s$$

And, using the fact that $X \in \mathcal{H}_i$, it is clear that $x\alpha X \in \tilde{\mathcal{H}}_d$ and $X \in \chi(x)$.

Step 2: We prove that if $X \in \chi_Q(x)$ and $X \geq \beta$ Q-a.e. where β is a positive constant, then $X_T \in \overline{\chi_e(x)}^{L^0(Q)}$.

Let $X \in \chi_Q(x)$ such that $X \geq \beta$ Q - a.e., then, $X_T = x + \int_0^T h_s dB_s$ where h is B -integrable and predictable.

Let $(X^n)_{n \in \mathbb{N}}$ and $(h^n)_{n \in \mathbb{N}}$ defined in Lemma 1. Now, we apply the Itô formula to $\ln(X^n)$:

$$X_T^n = x \exp \left(\int_0^T \alpha_s^n dB_s - \frac{1}{2} \sum_{i=1}^d \int_0^T (\alpha_{i,s}^n)^2 d \langle B_i \rangle_s \right)$$

with $\alpha_s^n = \frac{h_s^n}{X_s^n}$ satisfying $|\alpha_s^n| \leq \frac{2n}{\beta}$. Thanks to the boundedness of α^n and the

Lemma 2, there exists a sequence $(\tilde{h}^{n,p})_{p \in \mathbb{N}} \in \mathcal{H}_e^d$ such that:

$$\lim_{p \rightarrow \infty} \int_0^T \tilde{h}_s^{n,p} dB_s = \int_0^T \alpha_s^n dB_s \quad \text{in } L^0(Q)$$

and

$$\lim_{p \rightarrow \infty} \sum_{i=1}^d \int_0^T (\tilde{h}_{i,s}^{n,p})^2 d \langle B_i \rangle_s = \sum_{i=1}^d \int_0^T (\alpha_{i,s}^n)^2 d \langle B_i \rangle_s \quad \text{in } L^0(Q)$$

We put:

$$\tilde{X}_T^{n,p} = x \exp \left(\int_0^T \tilde{h}_s^{n,p} dB_s - \frac{1}{2} \sum_{i=1}^d \int_0^T (\tilde{h}_{i,s}^{n,p})^2 d \langle B_i \rangle_s \right)$$

then

$$\lim_{p \rightarrow \infty} \tilde{X}_T^{n,p} = X_T^n \quad \text{in } L^0(Q).$$

So

$$\forall n \in \mathbb{N}, X_T^n \in \overline{\chi_e(x)}^{L^0(Q)}$$

Besides

$$\lim_{n \rightarrow \infty} X_T^n = X_T \text{ in } L_0(Q),$$

therefore,

$$X_T \in \overline{\chi_e(x)}^{L_0(Q)}$$

Step 3: Let's prove that $\chi_Q(x) \subset \overline{\chi_e(x)}^{L_0(Q)}$

Let $X \in \chi_Q(x)$, then, for all $n \in \mathbb{N}^*$, $X_n = X + \frac{1}{n} \in \overline{\chi_e(x)}^{L_0(Q)}$, thanks to step 2. It is now easy to conclude. \square

4. Duality theory

This Section is devoted to the proof of Theorem 2, for this we split it in 3 Lemmas.

First, we prove a mini-max Theorem for utility function, using a classical mini-max Theorem (see for example [1]).

Lemma 3. *Let $x > 0$.*

$$u(x) = \sup_{X \in \chi(x)} \inf_{Q \in \mathcal{P}} E_Q [U(X_T)] = \inf_{Q \in \mathcal{P}} \sup_{X \in \chi(x)} E_Q [U(X_T)].$$

Proof.

Using Proposition 3 and the fact that U is bounded and continuous, we get that:

$$u(x) = \inf_{Q \in \mathcal{P}} \sup_{X \in \chi_Q(x)} E_Q [U(X_T)] = \inf_{Q \in \mathcal{P}} \sup_{X \in \chi(x)} E_Q [U(X_T)]$$

We now apply the classical mini-max Theorem of [1] to the following function:

$$f : \begin{array}{ccc} \mathcal{P} \times \chi(x) & \longmapsto & \mathbb{R} \\ (Q, X) & \longrightarrow & E_Q [U(X_T)] \end{array}$$

where \mathcal{P} is endowed with the weak star topology.

First, we can remark that $\chi(x)$ is convex and, using Proposition 2, that \mathcal{P} is convex and compact for the weak star topology.

Then, thanks to the concavity of U , for all $Q \in \mathcal{P}$, $X \longrightarrow E_Q [U(X_T)]$ is concave. And, obviously, for all $X \in \chi(x)$, $Q \longrightarrow E_Q [U(X_T)]$ is convex and continuous on \mathcal{P} .

Therefore, we can apply the classical mini-max Theorem, and we get that:

$$u(x) = \inf_{Q \in \mathcal{P}} \sup_{X \in \chi(x)} E_Q [U(X_T)] = \sup_{X \in \chi(x)} \inf_{Q \in \mathcal{P}} E_Q [U(X_T)]$$

\square

Lemma 4. *Let $y > 0$.*

$$v(y) = \sup_{x > 0} (u(x) - xy)$$

Proof.

Using the Theorem 3.1. p.914 in [16], we deduce that:

$$\forall Q \in \mathcal{P}, v_Q(y) = \sup_{x>0} (u_Q(x) - xy)$$

Hence

$$v(y) = \inf_{Q \in \mathcal{P}} \sup_{x>0} (u_Q(x) - xy)$$

As for all $x > 0$, the mapping $Q \in \mathcal{P} \rightarrow u_Q(x)$ is lower semi-continuous and convex, we apply one more time the mini-max Theorem to the function:

$$\begin{array}{ccc} \mathcal{P} \times \mathbb{R}_+^* & \longmapsto & \mathbb{R} \\ (Q, x) & \longrightarrow & u_Q(x) - xy \end{array}$$

And we get that:

$$v(y) = \inf_{Q \in \mathcal{P}} \sup_{x>0} (u_Q(x) - xy) = \sup_{x>0} \inf_{Q \in \mathcal{P}} (u_Q(x) - xy) = \sup_{x>0} (u(x) - xy)$$

□

Lemma 5.

$$u(x) = \inf_{y>0} (v(y) + xy)$$

Proof.

We extend the function u on \mathbb{R} , in a function \bar{u} , putting $\bar{u}(x) = -\infty$ if $x < 0$. Then, $-\bar{u}$ is a convex and proper function, and we can apply Theorem 12.2. p.104 of [23]:

$$(-\bar{u})^{**} = -\bar{u}$$

Where for any convex function f , f^* denotes the conjugate function. We can easily prove that:

- if $y < 0$, then $(-\bar{u})^*(y) = \sup_{x \in \mathbb{R}} (\bar{u}(x) + xy) = \sup_{x \in \mathbb{R}} (u(x) + xy) = v(-y)$
- if $y > 0$, then $(-\bar{u})^*(y) = \sup_{x \in \mathbb{R}} (\bar{u}(x) + xy) = \sup_{x>0} (u(x) + xy) = +\infty$

So

$$\begin{aligned} -(-\bar{u})^{**}(x) &= \inf_{y \in \mathbb{R}} ((-\bar{u})^*(y) - xy) = \inf_{y < 0} ((-\bar{u})^*(y) - xy) \\ &= \inf_{y > 0} ((-\bar{u})^*(-y) + xy) = \inf_{y > 0} (v(y) + xy) \end{aligned}$$

Hence finally, we deduce that:

$$u(x) = \inf_{y > 0} (v(y) + xy)$$

□

5. Existence of optimization strategies

The aim of this section is to prove Theorem 3. To this end, we split the proof in 4 lemmas. All along this section, we keep the same assumptions as in Section 2.

Lemma 6. *Let $x_0 > 0$. There exist a probability $\widehat{Q} \in \mathcal{P}$ and a process $\widehat{X} \in \chi_{\widehat{Q}}(x_0)$ such that*

$$u(x_0) = u_{\widehat{Q}}(x_0) = E_{\widehat{Q}} \left[U \left(\widehat{X} \right) \right]$$

Moreover, u is strictly increasing.

Proof.

Applying the same mini-max theorem as in the proof of Lemma 3, we obtain that there exists a probability $\widehat{Q} \in \mathcal{P}$ which realizes the supremum, i.e. such that:

$$\begin{aligned} u(x_0) &= \sup_{X \in \chi(x)} \inf_{Q \in \mathcal{P}} E_Q [U(X_T)] = \inf_{Q \in \mathcal{P}} \sup_{X \in \chi(x)} E_Q [U(X_T)] \\ &= \sup_{X \in \chi(x)} E_{\widehat{Q}} [U(X_T)] = u_{\widehat{Q}}(x_0) \end{aligned}$$

Now, under \widehat{Q} , we can use (ii) of Theorem 2.2 in [16] (p.910) and we get that there exists a process $\widehat{X} \in \chi_{\widehat{Q}}(x)$ such that:

$$\sup_{X \in \chi_{\widehat{Q}}(x)} E_{\widehat{Q}} [U(X_T)] = E_{\widehat{Q}} \left[U \left(\widehat{X}_T \right) \right]$$

We now prove that u is strictly increasing. For this, consider $\epsilon > 0$, as U is strictly increasing, we deduce that $u_{\widehat{Q}}$ is also strictly increasing and:

$$\begin{aligned} u(x_0) - u(x_0 - \epsilon) &= u_{\widehat{Q}}(x_0) - u(x_0 - \epsilon) \\ &= u_{\widehat{Q}}(x_0) - \inf_{Q \in \mathcal{P}} u_Q(x_0 - \epsilon) \\ &\geq u_{\widehat{Q}}(x_0) - u_{\widehat{Q}}(x_0 - \epsilon) > 0 \end{aligned}$$

Therefore, u is strictly increasing and the proof is complete. □

Lemma 7. *Let $x_0 > 0$, we put $\widehat{y} = u'_{\widehat{Q}}(x_0)$ where \widehat{Q} is the probability of Lemma 6, then*

$$u'_+(x_0) \leq \widehat{y} \leq u'_-(x_0)$$

Proof.

First, by applying Theorem 3.1. p.914 of [16], we know that $u_{\widehat{Q}}$ is differentiable. By the concavity of U , $u_{\widehat{Q}}$ is also concave and thus the function $x \rightarrow \frac{u_{\widehat{Q}}(x) - u_{\widehat{Q}}(x_0)}{x - x_0}$ is non increasing. Therefore, we get that

- if $0 < x_1 < x_0$ then $\hat{y} = u'_{\hat{Q}}(x_0) \leq \frac{u_{\hat{Q}}(x_1) - u_{\hat{Q}}(x_0)}{x_1 - x_0}$
- if $x_0 < x_2$ then $\hat{y} = u'_{\hat{Q}}(x_0) \geq \frac{u_{\hat{Q}}(x_2) - u_{\hat{Q}}(x_0)}{x_2 - x_0}$

Moreover,

$$u_{\hat{Q}}(x_1) \geq u(x_1) = \inf_{Q \in \mathcal{P}} u_Q(x_1) \text{ et } u_{\hat{Q}}(x_2) \geq u(x_2)$$

So, for all x_1, x_2 such that $0 < x_1 < x_0 < x_2$,

$$\begin{aligned} u'_{\hat{Q}}(x_0) &\leq \frac{u_{\hat{Q}}(x_1) - u_{\hat{Q}}(x_0)}{x_1 - x_0} \leq \frac{u(x_1) - u_{\hat{Q}}(x_0)}{x_1 - x_0} \\ u'_{\hat{Q}}(x_0) &\geq \frac{u_{\hat{Q}}(x_2) - u_{\hat{Q}}(x_0)}{x_2 - x_0} \geq \frac{u(x_2) - u_{\hat{Q}}(x_0)}{x_2 - x_0} \end{aligned}$$

Now, by passing to the limit as $x_1 \rightarrow x_0$, and then as $x_2 \rightarrow x_0$, we obtain that:

$$u'_+(x_0) \leq \hat{y} \leq u'_-(x_0)$$

□

Lemma 8. *With notations of Lemmas 6 and 7, let $x_0 > 0$, then*

$$v_{\hat{Q}}(\hat{y}) = v(\hat{y}) = u(x_0) - \hat{y}x_0$$

Proof.

In the proof of Lemma 7, we show that

- if $0 < x_1 < x_0$ then $\hat{y} \leq \frac{u_{\hat{Q}}(x_1) - u_{\hat{Q}}(x_0)}{x_1 - x_0}$
- if $x_0 < x_2$ then $\hat{y} \geq \frac{u_{\hat{Q}}(x_2) - u_{\hat{Q}}(x_0)}{x_2 - x_0}$

So, for all $x > 0$,

$$u_{\hat{Q}}(x) - \hat{y}x \leq u_{\hat{Q}}(x_0) - \hat{y}x_0$$

Moreover,

$$v_{\hat{Q}}(\hat{y}) = \sup_{x > 0} (u_{\hat{Q}}(x) - \hat{y}x) \leq u_{\hat{Q}}(x_0) - \hat{y}x_0$$

Therefore,

$$v_{\hat{Q}}(\hat{y}) = u_{\hat{Q}}(x_0) - \hat{y}x_0$$

Now, we prove that u is concave. Let $s, t > 0$ and $\lambda \in [0, 1]$, then as $u_{\hat{Q}}$ is concave, we get

$$\begin{aligned} u(\lambda x_0 + (1 - \lambda)x_1) &= \inf_{Q \in \mathcal{P}} u_Q(\lambda x_0 + (1 - \lambda)x_1) \\ &\geq \inf_{Q \in \mathcal{P}} (\lambda u_Q(x_0) + (1 - \lambda)u_Q(x_1)) \\ &\geq \lambda \inf_{Q \in \mathcal{P}} u_Q(x_0) + (1 - \lambda) \inf_{Q \in \mathcal{P}} u_Q(x_1) \\ &\geq \lambda u(x_0) + (1 - \lambda)u(x_1) \end{aligned}$$

Hence u is concave. As a consequence the map $x \longrightarrow \frac{u(x)-u(x_0)}{x-x_0}$ is non-increasing and in the same way as in the proof of Lemma 7 for $u_{\widehat{Q}}$, we get that

- if $0 < x_1 < x_0$ then $u'_-(x_0) \leq \frac{u(x_1) - u(x_0)}{x_1 - x_0}$
- if $x_0 < x_2$ then $\widehat{y} = u'_+(x_0) \geq \frac{u(x_2) - u(x_0)}{x_2 - x_0}$

Thanks to Lemma 7, $u_-(x_0) \geq \widehat{y}$ so if $0 < x_1 < x_0$, $u(x_1) - \widehat{y}x_1 \leq u(x_0) - \widehat{y}x_0$. And, thanks to Lemma 7, $u_+(x_0) \leq \widehat{y}$ so if $x_0 < x_2$, $u(x_2) - \widehat{y}x_2 \leq u(x_0) - \widehat{y}x_0$. Finally,

$$\forall x > 0, \quad u(x) - \widehat{y}x \leq u(x_0) - \widehat{y}x_0.$$

Therefore,

$$v(\widehat{y}) = \sup_{x > 0} (u(x) - \widehat{y}x) \leq u(x_0) - \widehat{y}x_0$$

So, we get the following equality:

$$v(\widehat{y}) = u(x_0) - \widehat{y}x_0,$$

which is the desired result. □

Lemma 9. *With notations of Lemmas 6 and 7, let $x_0 > 0$, then there exists a process $\widehat{Y} \in \mathcal{Y}_{\widehat{Q}}(\widehat{y})$ such that*

$$v(\widehat{y}) = v_{\widehat{Q}}(\widehat{y}) = E_{\widehat{Q}} \left[V(\widehat{Y}) \right].$$

Moreover,

$$\widehat{X}_T(x) = I(\widehat{Y}_T(y)) \quad \text{and} \quad \widehat{Y}_T(y) = U'(\widehat{X}_T(x)) \quad \widehat{Q} - p.s.$$

and $\widehat{X}(x)\widehat{Y}(y)$ is a uniformly integrable \widehat{Q} -martingale.

Proof. In the proof of the previous lemma, we have proved that $v(\widehat{y}) = v_{\widehat{Q}}(\widehat{y})$. Now, we apply Theorem 2.1 p.910 of [16] under the probability \widehat{Q} :

$$\exists! \widehat{Y} \in \mathcal{Y}_{\widehat{Q}}(\widehat{y}) \quad \text{tel que} \quad v_{\widehat{Q}}(\widehat{y}) = E_{\widehat{Q}} \left[V(\widehat{Y}) \right]$$

Moreover, as $u_{\widehat{Q}}(x_0) = E_{\widehat{Q}} \left[U(\widehat{X}) \right]$ et $\widehat{y} = u'_{\widehat{Q}}(x_0)$, we apply Theorem 2.1 p.910 of [16] under the probability \widehat{Q} :

$$\widehat{X}_T(x) = I(\widehat{Y}_T(y)) \quad \text{et} \quad \widehat{Y}_T(y) = U'(\widehat{X}_T(x)) \quad \widehat{Q} - p.s.$$

and $\widehat{X}(x)\widehat{Y}(y)$ is a uniformly integrable \widehat{Q} -martingale. So, we obtain the result of the lemma and the proof of Theorem 3 is complete. □

6. appendix

The aim of this appendix is to prove Theorem 1.

Proof of Theorem 1.

Let $p \in [1, +\infty[$. Using the definition, it is obvious that

$$\forall f \in C_b(\Omega), c_p(f) = (c_1(|f|^p))^{\frac{1}{p}}.$$

Then, using the Lebesgue extension of c_p , it is clear that this equality is satisfied for any function f and taking for f the indicator function we get:

$$\forall A \in \mathcal{B}(\Omega), c_p(A) = (c_1(A))^{\frac{1}{p}}.$$

From this, we just need to consider the case $p = 1$.

As the map $f \in C_b(\Omega) \mapsto \sup_{P \in \mathcal{P}} E_P[f]$ is a regular non linear expectation, in the sense of [6], we can apply Theorem 42 of [6] and get that:

$$\forall A \in \mathcal{B}(\Omega), c_1(A) = \sup_{Q \in \mathcal{Q}} Q(A)$$

where \mathcal{Q} is the set of probability measures which do not charge polar sets and are dominated by c_1 on $C_b(\Omega)$.

Now, we put

$$\forall A \in \mathcal{B}(\Omega), \hat{c}(A) = \sup_{P \in \mathcal{P}} P(A)$$

Thanks to [6] and the compactness of \mathcal{P} , we know that \hat{c} is a regular capacity.

As $\mathcal{P} \subset \mathcal{Q}$,

$$\forall A \in \mathcal{B}(\Omega), c_1(A) \geq \hat{c}(A)$$

Conversely, if O is an open set in Ω , using the definition of the Lebesgue extension, we deduce that

$$c_1(O) = \sup \{c_1(\phi); \phi \in C_b(\Omega), 0 \leq \phi \leq \mathbf{1}_O\}$$

Let $\varepsilon > 0$, there exists $\phi \in C_b(\Omega)$, such that $0 \leq \phi \leq \mathbf{1}_O$ and $c_1(O) \leq c_1(\phi) + \varepsilon$. Moreover

$$c_1(\phi) = \sup_{P \in \mathcal{P}} E_P[\phi] \leq \sup_{P \in \mathcal{P}} P(O) = \hat{c}(O).$$

Therefore, $c_1(O) \leq \hat{c}(O) + \varepsilon$. As ε can be chosen arbitrary small, we have

$$c_1(O) \leq \hat{c}(O),$$

and so $c_1(O) = \hat{c}(O)$.

At this stage, we have proved that c_1 and \hat{c} are regular capacities which coincide on open sets, but thanks to Theorem 6 in [6], we have for all $A \in \mathcal{B}(\Omega)$

$$c_1(A) = \inf \{c_1(O), O \text{ open and } A \subset O\} = \inf \{\hat{c}(O), O \text{ open and } A \subset O\} = \hat{c}(A).$$

The proof is complete. □

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