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Separating expansion from contraction and generalizing TOV condition in spherically symmetric models with pressure

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Abstract. We investigate spherically symmetric solutions with pressure and discuss the existence of a dividing shell separating expanding and collapsing regions. We perform a 3+1 splitting and obtain gauge invariant conditions relating not only the intrinsic spatial curvature of the shells to the ADM mass, but also a function of the pressure which we introduce that generalises the Tolman-Oppenheimer-Volkoff equilibrium condition. We consider the particular case of a Lemaître-Tolman dust models with a cosmological constant (a Λ-CDM model) as an example of our results.

Keywords: Cosmology – Birkhoff’s Theorem – trapped matter surfaces – Tolman-Oppenheimer-Volkoff – General Relativity and Gravitation

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INTRODUCTION

Cosmological formation of structure assumes the collapse of inhomogeneities, via gravitational instability, into "bound" structures, with the underlying idea that they depart from the cosmological expansion. This approach usually models overdensities with closed patches embedded in Friedman backgrounds, in particular in the spherical collapse included in the Press & Schechter scheme [1]. Birkhoff’s theorem is invoked to justify their independent evolutions [2], while it is proved for asymptotically flat spacetimes [3]. Such separation is reminiscent of the concept of trapped surfaces [4].

In this work we define general conditions for the existence of a shell separating contraction from expansion, and will illustrate our results with a simple example of perturbed Λ-CDM models. We adopt the Generalised Painlevé-Gullstrand (hereafter GPG) formalism used in Lasky & Lun [5], which involves a 3 + 1 splitting (ADM) and the consideration of gauge invariants kinematic quantities [6].

ADM APPROACH TO LTB MODELS IN GPG SYSTEM

We consider a spherically symmetric Generalised Lemaître-Tolman-Bondi (Generalised LTB, or GLTB) metric to include pressure. Performing an ADM 3+1 splitting in the GPG coordinates [5], the metric reads

\[ ds^2 = -\alpha(t,r)^2 dt^2 + \frac{1}{1 + E(t,r)} (\beta(t,r) dt + dr)^2 + r^2 d\Omega^2, \]

characterised where \( \alpha \) is a lapse function, \( \beta \) is a shift, and \( E \) is a curvature-energy characterising the curvature of the spatial surfaces orthogonal to the direction \( n_a = (-\alpha, 0, 0, 0) \) of the flow. For a perfect fluid, the projected Bianchi identities \( T_{bca} = 0 \)
yield the energy density conservation equation when we project along the flow $n^b$, and the Euler equation when we project orthogonally to the latter. Using the projection $h^b_a$, and denoting the density Lie derivative along the flow $\mathcal{L}_a \rho$, we have:

$$n^b T^a_{\mu \nu} = - \mathcal{L}_a \rho - (\rho + P) \Theta = 0, \quad h^b_a T^c_{\mu \nu} = 0 \Rightarrow P' = - (\rho + P) \frac{\alpha'}{\alpha},$$ (2)

where $P$ is the pressure, the radial derivatives are denoted by a prime, $'$, the time derivatives are represented by a dot, $\dot{}$, and the expansion is $\Theta$.

Introducing the ADM (also called Misner-Sharp) mass

$$M = r^2 (1 + E) (\ln \alpha)' - 4\pi P r^3 + \frac{1}{3} \Lambda r^3 + r^2 \mathcal{L}_n \left( \frac{\beta}{\alpha} \right),$$ (3)

where $\Lambda$ is the cosmological constant, we write the Einstein’s field equations (EFEs) as Lie derivatives along the flow

$$\mathcal{L}_n E = 2 \left( \frac{\beta}{\alpha} \right) \frac{1 + E}{\rho + P} \rho', \quad \Rightarrow \dot{E} = \beta \left( E' + 2 \frac{1 + E}{\rho + P} \rho' \right),$$ (4)

$$\mathcal{L}_n M = 4\pi P r^2 \left( \frac{\beta}{\alpha} \right), \quad \Rightarrow \dot{M} = \beta \left( M' + 4\pi P r^2 \right),$$ (5)

$$\left( \frac{\beta}{\alpha} \right)^2 = E + 2 \frac{M}{r} + \frac{1}{3} \Lambda r^2.$$ (6)

The system becomes then closed when an equation of state (EoS) $f(\rho, P) = 0$ is supplied. The $\Lambda$ term can be absorbed and many fluids mass equations written with each component using the $\frac{\beta}{\alpha}$ term for the overall sum of the masses.

SEPARATING COLLAPSE FROM EXPANSION

To characterise the separation of expansion from collapse, we prefer to use here GLTB coordinates in which the flow direction reduces to time by choosing $\beta = -\dot{r}$. We thus have

$$ds^2 = -\alpha(T, R)^2 \left( \frac{\partial_T}{\partial t} \right)^2 dT^2 + \frac{(\partial_R r)^2}{1 + E(T, R)} dR^2 + r^2 d\Omega^2,$$ (7)

$$\dot{M} = \beta 4\pi P r^2, \quad \dot{E} = 2\beta \frac{1 + E}{\rho + P} \rho', \quad \left( -\frac{\dot{r}}{\alpha} \right)^2 = E + 2 \frac{M}{r} + \frac{1}{3} \Lambda r^2.$$ (8)

There are two situations one should consider in parallel. On the one hand, we look for the vanishing of the gauge invariant expansion $\Theta$, defined as $n^{\mu}_{\mu a}$, since our goal is to...
separate an inner collapsing, spherical region from the outer expanding universe. On the other hand, the total ADM mass of the spherical region that departs from the expansion flow should be conserved. This is indeed suggested by the dust case where that happens for every shell.

We denote with $\star$ the dividing shell. The analysis of the kinematic quantities reveals that $\Theta$ is linked to the shear $\alpha$, in that at the $\star$ shell

$$\Theta = -3 \left( a + \frac{\beta}{\alpha r} \right) \iff \Theta_\star + 3a_\star = -3 \frac{\beta}{\alpha r_{\star}} \left( = 0 \text{ when } \mathcal{L}_n M(t, r_{\star}(t)) = 0 \right).$$

Moreover the generalized Friedman constraint

$$(3)^{\star}R + \frac{2}{3} \Theta^2 = 6a^2 + 16\pi \rho + 2\Lambda$$

(10)
tells us that the vanishing of $\Theta$ only happens in region of positive 3-curvature $(3)^{\star}R$. On the other hand, if we demand that the separating shell has a dust-like vanishing mass/energy $\rho$, we then need to choose initial conditions $\rho_\star$ as a parabola, and a Hubble flow for $v_{\star}^i$. Thus, the satisfaction of the Tolman-Oppenheimer-Volkoff equation of state \cite{7}, we are led to define a generalized TOV function

$$\text{TOV} = \left[ \frac{1 + E}{\rho + P'} + 4\pi Pr + \frac{M}{r^2} - \frac{1}{3} \Lambda r \right] \mathcal{L}_n \left( \frac{\beta}{\alpha} \right),$$

(12)

which reduces to the usual TOV equation when it vanishes.

The radial behaviour of the $\star$-shell is then similar to a turnaround shell. Indeed $r_{\star} = \frac{2M}{r_{\star}}$ leads to a null radial velocity $\dot{r}_{\star} = 0$ while its acceleration reveals the importance of the TOV parameter

$$\ddot{r}_{\star} = -\alpha^2 \left[ \text{TOV}_{\star} - r_{\star}^2 \text{TOV}^2_{\star} \right]; \quad \ddot{r}_{LTB,\star} = -\alpha^2 \text{TOV}_{\star}.$$  

(13)

The local staticity of the $\star$-shell is then shown to be equivalent to having a local TOV equation on this limit shell

$$\text{TOV}_{\star} = 0 \iff \frac{1}{\rho + P'} P' = \left[ \frac{4\pi Pr + \frac{M}{r^2}}{1 - \frac{2M}{r}} \right]_{\star}.$$  

(14)

**AN EXAMPLE: $\Lambda$CDM**

A simple illustration of our result is given by the case of dust with a $\Lambda$. There is then no pressure gradients and $M_{dust}$ and $E$ are conserved, which simplify the analysis (i.e. $\alpha = 1$), and allow us to perform a kinematic analysis (see Fig. \[1\]) per shell of Eq. (8-c). An effective potential is defined by $V(r) \equiv -\frac{2M}{r} - \frac{\Lambda}{3} r^2$, so that when $\ddot{r} = 0$ we have indeed $V(r) = E$:

$$r^2 = \frac{2M}{r} + \frac{1}{3} \Lambda r^2 + E, \quad \text{with } \ddot{r} = -\frac{M}{r^2} + \frac{\Lambda}{3} r.$$  

(15)

For each shell there is a virtual static state at $r_{\lim} = \sqrt{\frac{2M}{\Lambda}}, E_{\lim} = -\left(3M\right)^{\frac{2}{3}} \Lambda^{\frac{1}{3}}$, which only depends on $\Lambda$ and $M(R)$. We also recall that, in this model, $\text{TOV} = \frac{M}{r^2} - \frac{\Lambda}{3} r = -\ddot{r}$.

We then need to choose initial conditions $p_i$, which sets the $E_{\lim}$ profile, and $v_i$, which sets the $E_i$ profile. The intersection of $E_{\lim}$ with $E_i$ will be static at turnaround, hence defining our limit shell. We use two sets of cosmologically motivated initial conditions: one with $p_i$ as an NFW profile \cite{8} and $E_i$ as a parabola, and the second, with a cuspless power law as $p_i$ and a Hubble flow for $v_i$. 


CONCLUSIONS

Using non-singular, Generalized Painlevé-Gullstrand coordinate formulation of the ADM spherically symmetric, perfect fluid system [5] we have shown [9] that the existence of shells locally separating between inner collapsing and outer expanding regions, is governed by the condition that the combination of expansion scalar and shear $\theta + 3a$ should vanish on the shell. The ADM mass of the shell is then conserved. This condition requires that the separating shell must be located in an elliptic ($E < 0$) region. Moreover, for that shell to exist for certain over time, we have shown that the TOV equation must be locally satisfied. We argue that this local condition is global in a cosmological context (FLRW match at radial asymptote). Given appropriate initial conditions, this translates into global separations between an expanding outer region and an eventually collapsing inner region. We present simple but physically interesting illustrations of the results, a model of Lemaître-Tolman dust with $\Lambda$ representing spherical perturbations in a $\Lambda$CDM with two different initial sets of cosmologically interesting conditions consistent with known phenomenological constraints [4, and Refs. therein]: an NFW density profile with a simple curvature profile going from bound to unbound conditions, and a non cuspy power law fluctuation with initial Hubble flow. We show, for these models, the existence of a global separation. We argue that these shells are trapped matter surfaces [9] and that they constitute the validity locus to an analog to Birkhoff’s theorem.

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