Inequivalent embeddings of the Koras-Russell cubic threefold

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INEQUIVALENT EMBEDDINGS OF THE KORAS-RUSSELL CUBIC THREEFOLD

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ABSTRACT. Let X be the hypersurface of the complex affine four-space $\supset^4$ defined by the equation $x^2y + z^2 + t^3 + x = 0$. It is well-known that X is an affine contractible smooth threefold, which is not algebraically isomorphic to affine three-space. The main result of this article is to show that there exists another hypersurface Y of $\supset^4$, which is isomorphic to X, but such that there exists no automorphism of the ambient four-space which restricts to an isomorphism between X and Y. In other words, the two hypersurfaces are inequivalent. To prove this result, we give a description of the automorphism group of X. We show that all algebraic automorphisms of X extend to automorphisms of $\mathbb{A}^4$. As a corollary, we find that every automorphism of X fixes the point $(0,0,0,0) \in X$.

1. INTRODUCTION

The Koras-Russell cubic threefold is the hypersurface X of the complex affine space $\mathbb{A}^4 = \mathrm{Spec}\,(\mathbb{C}\,[x,y,z,t])$ defined by the equation

$$P = x + x^2y + z^2 + t^3 = 0.$$

In this paper we will study certain properties of this threefold. The point of view comes from the elementary remark that the Koras-Russell threefold can be interpreted as a one-parameter family of Danielewski hypersurfaces. A *Danielewski hypersurface* is a subvariety of $\mathbb{A}^3 = \mathrm{Spec}\,(\mathbb{C}\,[x,y,z])$ defined by an equation of the form $x^n y = q(x,z)$, where n is a non-zero natural number and $q(x,z) \in \mathbb{C}[x,z]$ is a polynomial such that $q(0,z)$ is of degree at least two. Such hypersurfaces have been studied by the authors in [4], [13] and [14]. This interpretation allows us to deduce results similar to the ones for Danielewski hypersurfaces for this threefold.

An important question in affine algebraic geometry asks whether every embedding of complex affine k-space $\mathbb{A}^k$ in $\mathbb{A}^n$, where $k < n$, is rectifiable, i.e., is equivalent to an embedding as a linear subspace. The Abyhankar-Moh-Suzuki theorem shows that the answer is "yes" if $n = 2$ ([1], [17]), and, by a general result proved independantly by Kaliman [10] and Srinivas [16], if $n \geq 2k + 2$ the answer is also affirmative. However, all other cases remain open.

Here we are interested in the case of embeddings of hypersurfaces. It is easy to find affine varieties of dimension n admitting non-equivalent embeddings into $\mathbb{A}^{n+1}$. For example, the punctured line $\mathbb{A}^1 \setminus \{0\}$ has many non-equivalent embeddings in $\mathbb{A}^2$. For each $n \in \mathbb{N}$, let $P_n = x^n y - 1$. The subvariety defined by the zero set of P_n is isomorphic to $\mathbb{A}^1 \setminus \{0\}$, however, the induced embeddings for each n are inequivalent. This can be seen by the fact that the subschemes defined by $P_n - 1 = 0$ are all non-isomorphic. It is more difficult to find examples where all non-zero fibers of the defining polynomial are irreducible.

In example 6.3 of [9], Kaliman and Zaidenberg gave examples of acyclic surfaces which admit non-equivalent embeddings in three-space. In these cases, the obstruction for equivalence is essentially topological, the non-zero fibers of the polynomials which define the two hypersurfaces being non homeomorphic. Even if they are contractible, these surfaces are algebraically remote from the affine plane due to the fact that they have nonnegative logarithmic Kodaira dimension. In contrast, an example is given [6] for Danielewski hypersurfaces, where the non-zero fibers are algebraically non-isomorphic, but analytically isomorphic, and in [13] and [14], examples of Danielewski hypersurfaces are given where all fibers are algebraically isomorphic, but the embeddings are non-equivalent. Danielewski hypersurfaces are rational, whence close to the affine plane from an algebraic point of view, but they have non-trivial singular homology groups.

However, the techniques used in *loc. cit.* are purely algebraic and do not depend on the topological properties of these surfaces. As we shall see here, similar ideas can be used to treat the case of a variety diffeomorphic to $\mathbb{R}^6$. For other inequivalent embeddings of hypersurfaces, see, for example, [15].

In the present article, we use similar techniques as for the Danielewski hypersurfaces to study the Koras-Russell threefold X above. For a polynomial $f \in \mathbb{C}[x, y, z, t]$, we denote by $V(f)$ the subscheme of $\mathbb{A}^4$ defined by the zero set of f in $\mathbb{A}^4$, that is, $V(f) = \mathrm{Spec}(\mathbb{C}[x, y, z, t]/(f))$. The hypersurface $X = V(P)$ is smooth and contractible, and is therefore diffeomorphic to $\mathbb{A}^3$ [2]. However, it was shown by Makar-Limanov that it is not isomorphic to affine three-space [11]. We show that there is another hypersurface $Y = V(Q)$ which is isomorphic to X, but there is no algebraic automorphism of four-space which restricts to an isomorphism between X and Y. Thus we have at least two inequivalent embeddings of X. For this example, the two hypersurfaces are analytically equivalent by a holomorphic automorphism which preserves the fibers of P and Q, and, therefore, as for certain examples of Danielewski hypersurfaces, there is no topological obstruction to the existence of such an automorphism. In other words, the obstruction to extending automorphisms in this case is purely algebraic. Also, for all $c \in \mathbb{C}\backslash\{1\}$, the fibers $V(P+c)$ and $V(Q+c)$ are isomorphic. It is an open question if $V(P+1) \cong V(Q+1)$.

The methods to study the question of equivalent embeddings are similar to those used in the articles cited above for Danielewski hypersurfaces. However, they must be adapted in order to consider a higher dimensional variety. They are based on certain properties of the automorphism group of the varieties. The set of locally nilpotent derivations on a Danielewski hypersurface is explicitly known (see [4]; see also [12]), and the Makar-Limanov invariant is non trivial when $n \geq 2$. This restricts the possibilities for automorphisms of these surfaces. The Makar-Limanov invariant of the Koras-Russell threefold is also non trivial [11]. In section 3, we determine the complete automorphism group of X. For this case, new methods are needed, since the restrictions given by the Makar-Limanov invariant do not suffice to determine the automorphism group. As a corollary, we find the surprising result that any algebraic automorphism of X fixes the point $(0, 0, 0, 0) \in X$. (See corollary 4.5.) Also, all automorphisms of X extend to automorphisms of $\mathbb{A}^4$.

2. INEQUIVALENT EMBEDDINGS IN $\mathbb{A}^4$

We denote by P and Q the following polynomials of $\mathbb{C}^{[4]} = \mathbb{C}[x, y, z, t]$:

$$P = x^2 y + z^2 + x + t^3 \quad \text{and} \quad Q = x^2 y + (1 + x)\left(z^2 + x + t^3\right).$$

The Koras-Russell cubic threefold is the hypersurface $X \subset \mathbb{A}^4 = \mathrm{Spec}\left(\mathbb{C}\left[x, y, z, t\right]\right)$ defined by the equation $P = 0$ whereas Y denote the hypersurface defined by the equation $Q = 0$.

We start by giving some definitions to clarify the difference between *inequivalent embeddings* and *inequivalent hypersurfaces*.

Definition 2.1. A regular map $\phi : X \to Z$ between two algebraic varieties is a *closed embedding of X in Z* if:

(1) $\phi(X)$ is a closed subvariety of Z;
(2) $\phi : X \to \phi(X)$ is an isomorphism.

Definition 2.2. Two embeddings $\phi_1, \phi_2 : X \to Z$ are *equivalent* if there exists an automorphism Ψ of Z such that $\phi_2 = \Psi \circ \phi_1$.

Definition 2.3. Two subvarieties X_1 and X_2 of Z are *equivalent* if there exists an automorphism Ψ of Z such that $\Psi(X_1) = X_2$. If X_1 and X_2 are hypersurfaces, then we say they are *equivalent hypersurfaces*.

In this article, we will show that $V(P)$ and $V(Q)$ are isomorphic as abstract threefolds, however, they are inequivalent as hypersurfaces of $\mathbb{A}^4$, in the sense of definition 2.3. In other words, no isomorphism between $V(P)$ and $V(Q)$ extends to an automorphism of $\mathbb{A}^4$. We will do this in two steps. First, we will find an isomorphism ϕ between $V(P)$ and $V(Q)$ which does not extend to an automorphism of $\mathbb{A}^4$. This implies that the two embeddings $i_1 : V(P) \to \mathbb{A}^4$ and $i_2 \circ \phi : V(P) \to \mathbb{A}^4$ are inequivalent in the sense of definition 2.2. (Here, $V(P)$ is the scheme defined as $\mathrm{Spec}(\mathbb{C}[x,y,z,t]/(P))$, and i_1 is the embedding of $V(P)$ in $\mathbb{A}^4$ corresponding to the canonical homomorphism $i_1^* : \mathbb{C}[x,y,z,t] \to \mathbb{C}[x,y,z,t]/(P)$. The embedding i_2 is defined similarly, by replacing P by Q.)

The second step, discussed in section 3, will be to study the automorphism group of $V(P)$. We will show in section 4 that all automorphisms of this hypersurface extend to automorphisms of $\mathbb{A}^4$.

Finally, putting these two results together, we will show the stronger result that $V(P)$ and $V(Q)$ are inequivalent hypersurfaces.

In this article, we use the following key result concerning the Makar-Limanov invariant of an irreducible affine variety. Given an irreducible affine variety Z, denote by $\mathbb{C}[Z]$ the ring of regular functions on Z. A *locally nilpotent derivation* of $\mathbb{C}[Z]$ is a $\mathbb{C}$-derivation $\partial : \mathbb{C}[Z] \to \mathbb{C}[Z]$ such that for any $f \in \mathbb{C}[Z]$, there exists $n \in \mathbb{N}$ such that $\partial^n(f) = 0$. If ∂ is a locally nilpotent derivation of $\mathbb{C}[Z]$, then $\exp(\lambda\partial)$ defines an algebraic action of $(\mathbb{C}, +)$ on Z, and all $(\mathbb{C}, +)$ actions arise in this way. The kernel of a locally nilpotent derivation is the subalgebra of invariants in $\mathbb{C}[Z]$ of the corresponding action. The *Makar-Limanov invariant*, $\mathrm{ML}(\mathbb{C}[Z])$ is defined as the intersection of the kernels of all locally nilpotent derivations on $\mathbb{C}[Z]$. If Z is affine space, the Makar-Limanov invariant is simply $\mathbb{C}$. However, it was shown in [11] that $\mathrm{ML}(\mathbb{C}[X]) = \mathbb{C}[x] \neq \mathbb{C}$. This implies that the Koras-Russell threefold X is not isomorphic to the affine three-space. In theorem 9.1 and example 9.1 of [8], the result of Makar-Limanov is generalized, and it is shown in particular that for every $c \in \mathbb{C}$, and every $\lambda \in \mathbb{C}^*$,

$$\mathrm{ML}\left(\mathbb{C}[x,y,z,t]/\left(Q - c\right)\right) = \mathrm{ML}\left(\mathbb{C}[x,y,z,t]/\left(\lambda P - c\right)\right) = \mathbb{C}[x].$$

We will use this result throughout the article.

Theorem 2.4. *The following hold :*
 a) *The endomorphism of $\mathbb{A}^4$ defined by $(x,y,z,t) \mapsto (x, (1+x)y, z, t)$ induces an isomorphism $\phi : X \xrightarrow{\sim} Y$.*
 b) *ϕ cannot be extended to an algebraic automorphism of $\mathbb{A}^4$.*

Proof. Let Φ and Ψ be the endomorphisms of $\mathbb{A}^4$ defined by

$$\begin{cases} \Phi : & (x,y,z,t) \mapsto (x, (1+x)\,y, z, t) \\ \Psi : & (x,y,z,t) \mapsto (x, (1-x)\,y - x - z^2 - t^3, z, t) \end{cases}$$

Denote by Φ^* the endomorphism of $\mathbb{C}[x,y,z,t]$ corresponding to Φ (Φ^* fixes x, z and t, and $\Phi^*(y) = (1+x)y$), and by Ψ^* the corresponding endomorphism to Ψ. One checks that $\Phi^*(Q) = (1+x)P$ and $\Psi^*(P) = (1-x)Q$. Thus, Φ induces a well-defined morphism of algebraic varieties $\phi : X \to Y$, and Ψ induces a well-defined regular map $\psi : Y \to X$.

Since ϕ and ψ are morphisms of schemes over $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x,z,t])$, the identities $(\Phi^* \circ \Psi^*)(y) = y - P$ and $(\Psi^* \circ \Phi^*)(y) = y - Q$ guarantee that they are inverse isomorphisms.

The second assertion follows from the remark that $\phi((0,0,0,0)) = (0,0,0,0)$, and from the first part of proposition 2.5 below. $\qquad\square$

Proposition 2.5. *The following results hold.*
 (i) *Suppose that there exists Ξ an algebraic automorphism of $\mathbb{A}^4$ which restricts to an isomorphism between X and Y. Then Ξ does not fix the point $(0,0,0,0)$, i.e. $\Xi(0,0,0,0) \neq (0,0,0,0)$.*
 (ii) *If Ξ is an algebraic automorphism of $\mathbb{A}^4$ which restricts to an automorphism of X, then Ξ fixes the point $(0,0,0,0)$.*

(iii) *If Ξ is an algebraic automorphism of $\mathbb{A}^4$ which restricts to an automorphism of Y, then Ξ fixes the point $(0, -1, 0, 0)$.*

Proof. For part (i), suppose that Ξ is an automorphism of $\mathbb{A}^4$ which extends an isomorphism between X and Y.

Note that since $Q = x^2 y + (1 + x)(z^2 + x + t^3)$ is irreducible, it follows that $\Xi^*(Q) = \lambda P$ for a certain $\lambda \in \mathbb{C}^*$. Thus, Ξ maps $V(P - \lambda^{-1}c)$ isomorphically onto $V(Q - c)$ for every $c \in \mathbb{C}$.

Let us show first that $\Xi^*(x) = \lambda x$. As mentioned above, for any $c \in \mathbb{C}$, the Makar-Limanov invariant of $(\mathbb{C}[x, y, z, t]/(Q - c))$ and of $(\mathbb{C}[x, y, z, t]/(\lambda P - c))$ are $\mathbb{C}[x]$. This implies that for every $c \in \mathbb{C}$, Ξ^* restricts to an isomorphism

$$\mathbb{C}[x] = \mathrm{ML}\left(\mathbb{C}[x, y, z, t]/(Q - c)\right) \xrightarrow{\sim} \mathrm{ML}\left(\mathbb{C}[x, y, z, t]/(\lambda P - c)\right) = \mathbb{C}[x].$$

Combined with the fact that for any $c \in \mathbb{C}$, $V(x - a, Q - c)$ and $V(x - a, P - c)$ are isomorphic to $\mathbb{A}^2$ if and only if $a \neq 0$, we find that Ξ^* preserves the ideal (x).

Let $\Xi^*(x) = \mu x$ with $\mu \in \mathbb{C}^*$. Then $\Xi^*(Q - x) = \lambda P - \mu x$. One checks easily that the variety $Z = \mathrm{Spec}\left(\mathbb{C}[x, y, z, t]/(Q - x)\right)$ is singular along the line $L_z = \{x = z = t = 0\}$ in $\mathbb{A}^4$. On the other hand, it follows from the Jacobian criterion that the variety $\Xi^{-1}(Z) = V(\lambda P - \mu x)$ is singular if and only if $\lambda = \mu$. This implies that $\lambda = \mu$.

In other words, we have shown that $\Xi^*(Q - x) = \lambda(P - x)$. Thus, Ξ restricts to an isomorphism between $W_P = V(P - x)$ and $W_Q = V(Q - x)$.

For parts (ii) and (iii), a similar argument shows that any automorphism of $\mathbb{A}^4$ which extends an automorphism of X will preserve the subvariety W_P, and any automorphism of $\mathbb{A}^4$ which extends an automorphism of Y will preserve the subvariety W_Q.

Now we look more carefully at the two subvarieties W_P and W_Q. They are both singular along the line. $\{x = z = t = 0\}$. We look now at the tangent cone of each singular point of W_P and W_Q. Let $p_0 = (0, y_0, 0, 0, 0)$.

We deduce from the identity

$$P - x \;=\; y_0 x^2 + z^2 + x^2 (y - y_0) + t^3$$

that the tangent cone $TC_{p_0}(W_P)$ of W_P at p_0 is isomorphic to $\mathrm{Spec}\left(\mathbb{C}[x, y, z, t]/(z^2 + y_0 x^2)\right)$. In particular $TC_{p_0} W_P$ consists of two distinct hyperplanes for $y_0 \neq 0$ and of the double hyperplane $\{z = 0\}$ if $y_0 = 0$. On the other hand, we deduce from the identity

$$Q - x \;=\; (y_0 + 1)x^2 + z^2 + x^2 (y - y_0) + t^3 + xz^2 + xt^3$$

that the tangent cone of W_Q at a point $p_0 = (0, y_0, 0, 0)$ is isomorphic to $\mathrm{Spec}\left(\mathbb{C}[x, y, z, t]/(z^2 + (y_0 + 1)x^2)\right)$. Thus the tangent cone $TC_{p_0}(W_Q)$ consists of two distinct hyperplanes for $y_0 \neq -1$ and of the double hyperplane $\{z = 0\}$ if $y_0 = -1$.

For part (i), since $\Xi(W_P) = W_Q$, we have that $\Xi(0, 0, 0, 0) = (0, -1, 0, 0)$. For part (ii), any automorphism of W_P fixes the point $(0, 0, 0, 0)$. Finally, for part (iii), any automorphism of W_Q fixes the point $(0, -1, 0, 0)$. This completes the proof. □

Corollary 2.6. *The Koras-Russell cubic threefold admits at least two non-equivalent embeddings in $\mathbb{A}^4$.*

Proof. Consider the inclusions $i_1 : X \hookrightarrow \mathbb{A}^4$ and $i_2 : Y \hookrightarrow \mathbb{A}^4$. The embeddings i_1 and $i_2 \circ \phi$ are inequivalent embeddings by the theorem 2.4. □

3. THE AUTOMORPHISM GROUP OF X

We will now determine the structure of the automorphism group $\mathrm{Aut}(X)$. We start with some notation. If S is a ring and R is a subring, then $\mathrm{Aut}_R(S)$ denotes the group of ring automorphisms of S which fix R. Denote by $\mathbb{C}[X] = \mathbb{C}^{[4]}/(P)$ the ring of regular functions on X. The group $\mathrm{Aut}(X)$ is isomorphic to the group $\mathrm{Aut}(\mathbb{C}[X]) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[X])$.

Notation 3.1. Denote by $I = (x^2, z^2 + t^3 + x) \subset \mathbb{C}[x, z, t]$ the ideal generated by x^2 and $z^2 + t^3 + x$. Let $\mathcal{A}$ be the subgroup of $\mathrm{Aut}(\mathbb{C}[x, z, t])$ of automorphisms which preserve the ideals (x) and I. Let $\mathcal{A}_1$ be the subgroup of $\mathcal{A}$ of automorphisms $\varphi \in \mathcal{A}$ such that φ fixes x, and $\varphi \equiv \mathrm{id}$ mod (x). Finally, let $\mathcal{A}_2$ be the subgroup of $\mathrm{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x][z, t])$ which are equivalent to the identity modulo x^2:

$$\mathcal{A} = \{\varphi \in \mathrm{Aut}(\mathbb{C}[x, z, t]) \mid \varphi((x)) = (x), \varphi(I) = I\};$$
$$\mathcal{A}_1 = \{\varphi \in \mathrm{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x][z, t]) \mid \varphi(I) = I, \varphi \equiv \mathrm{id} \mod (x)\};$$
$$\mathcal{A}_2 = \{\varphi \in \mathrm{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x][z, t]) \mid \varphi \equiv \mathrm{id} \mod (x^2)\}.$$

It is clear that $\mathcal{A}_2$ is a normal subgroup of $\mathcal{A}_1$, and $\mathcal{A}_1$ is a normal subgroup of $\mathcal{A}$.

The ring $\mathbb{C}[X]$ can be viewed as the subalgebra of $\mathbb{C}[x, x^{-1}, z, t]$, generated by x, z, t and $(z^2 + t^3 + x)/x^2$. In particular, it contains $\mathbb{C}[x, z, t]$ as a subring.

The following proposition can be deduced from the results of Makar-Limanov concerning the set of locally nilpotent derivations on X. See [11] and [5].

Proposition 3.2. *The automorphism group* $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathbb{C}[X])$ *is isomorphic to the group* $\mathcal{A}$. *The isomorphism of* $\mathrm{Aut}(\mathbb{C}[X])$ *to* $\mathcal{A}$ *is induced by restriction of any automorphism of* $\mathbb{C}[X]$ *to the subalgebra* $\mathbb{C}[x, z, t]$.

Proof. In [11], it was shown that the Makar-Limanov invariant of $\mathbb{C}[X]$ is $\mathbb{C}[x]$. In fact, more was proven. It was shown that for any locally nilpotent derivation ∂ of $\mathbb{C}[X]$, we have that $\ker \partial^2 \subset \mathbb{C}[x, z, t]$. There exists a locally nilpotent derivation ∂_1 on $\mathbb{C}[X]$ such that $\partial_1(z) = 0$. Thus for any automorphism φ of $\mathbb{C}[X]$, $\varphi \circ \partial_1 \circ \varphi^{-1}$ has $\varphi(z)$ in its kernel. In particular, $\varphi(z) \in \mathbb{C}[x, z, t]$. Also $\varphi^{-1}(z)$ belongs to $\mathbb{C}[x, z, t]$, and therefore, $z = \varphi(\varphi^{-1}(z))$ is in the image $\varphi(\mathbb{C}[x, z, t])$. The same holds for t. Thus we have shown that any automorphism of $\mathbb{C}[X]$ restricts to an automorphism of the subalgebra $\mathbb{C}[x, z, t]$ of $\mathbb{C}[X]$. Also, any automorphism $\varphi \in \mathbb{C}[X]$ stabilizes the ideal (x). Indeed, since the Makar-Limanov invariant $\mathbb{C}[x] \subset \mathbb{C}[X]$ is stable, there exists $\lambda \in \mathbb{C}^*$ and $b \in \mathbb{C}$ such that $\varphi(x) = \lambda x + b$. Also, the zero set of $\lambda x + b$ in X is singular if and only if $b = 0$. Thus, since the zero set of x in X is singular, we have that $b = 0$, and thus the ideal (x) is preserved.

Now, to show that the ideal I is preserved, we show that $x^2 \mathbb{C}[X] \cap \mathbb{C}[x, z, t] = I$. It is clear that I is contained in the intersection. For the converse, note that any element in $\mathbb{C}[X]$ is represented in a unique way as a polynomial of the form $y f_0(y, z, t) + xy f_1(y, z, t) + g(x, z, t)$, where $f_0, f_1 \in \mathbb{C}[y, z, t]$ and $g \in \mathbb{C}[x, z, t]$. (In other words, all monomials containing $x^2 y$ are eliminated.) If we look now at the ideal generated by x^2, $x^2 y f_0 + x^3 y f_1 \in \mathbb{C}[x, z, t]$ if and only if f_0 and f_1 are independent of y, and in this case,

$$x^2 y f_0 + x^3 y f_1 = x^2 y(f_0 + x f_1) = (-z^2 - t^3 - x)(f_0 + x f_1) \in (z^2 + t^3 + x) \subset \mathbb{C}[x, z, t].$$

Finally, we show that any automorphism φ in $\mathcal{A}$ extends to a unique automorphism $\tilde{\varphi}$ of $\mathbb{C}[X]$. To prove uniqueness, note that any element of $\mathcal{A}$ induces a unique automorphism of $\mathbb{C}[x, x^{-1}, z, t]$, which contains $\mathbb{C}[X]$. To prove existence, we extend φ to an endomorphism Φ of $\mathbb{C}[x, y, z, t]$, such that $\Phi(P)$ is in the ideal generated by P. More precisely, we define $\Phi(h) = h$ if $h \in \mathbb{C}[x, z, t]$, and we construct $\Phi(y)$ as follows. By hypothesis, there exists $f, g \in \mathbb{C}[x, z, t]$ such that $\varphi(z^2 + t^3 + x) = (z^2 + t^3 + x)f + x^2 g$. Suppose also that $\varphi(x) = \lambda x$, with $\lambda \in \mathbb{C}^*$. Now we pose $\Phi(y) = (yf - g)/\lambda^2$. One checks easily that $\Phi(P) = fP$. Thus Φ induces a homomorphism $\tilde{\varphi}$ from $\mathbb{C}[X]$ to $\mathbb{C}[X]$. We will show it is an automorphism. Let ψ be the inverse of φ as an automorphism of $\mathbb{C}[x, z, t]$. Construct Ψ and $\tilde{\psi}$ as above. Then $\tilde{\psi} \circ \tilde{\varphi}$ and $\tilde{\varphi} \circ \tilde{\psi}$ are homomorphisms of $\mathbb{C}[X]$ which extend to the identity as homomorphisms of $\mathbb{C}[x, x^{-1}, z, t]$. Thus $\tilde{\psi}$ is the inverse of $\tilde{\varphi}$. $\square$

Remark 3.3. This result has the following geometric interpretation. The inclusion $\mathbb{C}[x, z, t] \subset \mathbb{C}[X]$ induces a dominant morphism $\sigma : X \to \mathbb{A}^3$. Any automorphism of X is the lifting by σ of a unique automorphism of $\mathbb{A}^3$. More precisely, if $\tilde{\varphi}$ is an automorphism of X, there is a unique automorphism φ of $\mathbb{A}^3$ such that $\varphi \circ \sigma = \sigma \circ \tilde{\varphi}$. Also, an automorphism φ of $\mathbb{A}^3$ has a lifting as an automorphism of X if and only if φ preserves the ideals (x) and I.

We will now discuss the structure of the group $\mathcal{A}$. Note that it contains a subgroup isomorphic to $\mathbb{C}^*$, (corresponding to a $\mathbb{C}^*$-action on X) given by the $\mathbb{C}^*$-action where x has weight 6, z has weight 3 and t has weight 2. .

Proposition 3.4. $\mathcal{A} = \mathcal{A}_1 \rtimes \mathbb{C}^*$.

Proof. It is clear that $\mathcal{A}_1 \rtimes \mathbb{C}^*$ is a subgroup of $\mathcal{A}$. We will now show that $\mathcal{A}_1$ and $\mathbb{C}^*$ generate $\mathcal{A}$. First note that if $\varphi \in \mathcal{A}$, then, since φ preserves the ideal (x), it induces an automorphism $\overline{\varphi}$ of $\mathbb{C}[x, z, t]/(x) \cong \mathbb{C}[z, t]$. Also, since I is preserved, the ideal $(z^2 + t^3)$ is preserved by $\overline{\varphi}$. By composing with an automorphism in $\mathbb{C}^*$, we can assume that $\overline{\varphi}(z^2 + t^3) = z^2 + t^3$. In particular, for all $c \in \mathbb{C}$, $\overline{\varphi}$ induces an automorphism of $V(z^2 + t^3 + c)$. If $c \neq 0$, this defines a smooth elliptic curve E with one point p removed. The group of automorphisms of this affine curve is the group of automorphisms of E which fix the point p. This group is of order 6, generated by the automorphism that fixes t and sends z to $-z$, and the automorphism that fixes z and sends t to $e^{i2\pi/3}t$ (see, for example, [7]). There are therefore only 6 automorphisms of $\mathbb{C}[z, t]$ which fixes the polynomial $z^2 + t^3$, and they are all in the image of $\mathbb{C}^*$. We can therefore suppose that $\overline{\varphi}(z) = z$ and $\overline{\varphi}(t) = t$. This means exactly that $\varphi \in \mathcal{A}_1$. $\square$

Now we are left with the problem of understanding the group $\mathcal{A}_1$. For this part, we will consider a more general situation. First note that the group $\mathcal{A}_1$ is exactly the group of automorphisms φ of $\mathbb{C}[x, z, t]$ which fix x, such that $\varphi \equiv \mathrm{id} \mod (x)$, and such that $\varphi(z^2 + t^3) \in (x^2, z^2 + t^3)$.

Notation 3.5. Let $r \in \mathbb{C}[z, t]$ be a polynomial. Denote by $\mathcal{A}_1(r)$ the group

$$\mathcal{A}_1(r) = \{\varphi \in \mathrm{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x][z, t]) \mid \varphi \equiv \mathrm{id} \mod (x), \varphi(r) \in (x^2, r)\}.$$

We have thus that $\mathcal{A}_1 = \mathcal{A}_1(z^2 + t^3)$, and that, for any r, $\mathcal{A}_2$ is a normal subgroup of $\mathcal{A}_1(r)$.

We use the following standard notation for partial derivatives. If $h \in \mathbb{C}[z, t]$, $h_z = \partial h/\partial z$, $h_t = \partial h/\partial t$, and if $h, f \in \mathbb{C}[z, t]$, then the Poisson bracket of h and f is given by $\{h, f\} = h_z f_t - h_t f_z$.

Proposition 3.6. *Let $r \in \mathbb{C}[z, t]$ be a polynomial with no multiple irreducible factor and such that the zero set $V(r) \in \mathbb{A}^2$ is connected. Then, for every automorphism there exists $\varphi \in \mathcal{A}_1(r)$, a polynomial $\alpha \in \mathbb{C}[z, t]$ such that $\varphi(z) \equiv z + x(r\alpha)_t \mod (x^2)$ and $\varphi(t) \equiv t - x(r\alpha)_z \mod (x^2)$.*

Moreover, $\theta : \mathcal{A}_1(r) \to \mathbb{C}[z, t]$, $\varphi \mapsto \alpha$ is a surjective group homomorphism whose kernel is $\mathcal{A}_2$. In particular, the quotient group $\mathcal{A}_1(r)/\mathcal{A}_2$ is isomorphic to the additive group $(\mathbb{C}[z, t], +)$.

Proof. For any $\varphi \in \mathcal{A}_1(r)$, we have that $\varphi(z) \equiv z + xf \mod (x^2)$ and $\varphi(t) \equiv t + xg \mod (x^2)$, where $f, g \in \mathbb{C}[z, t]$. By hypothesis, φ is an automorphism. Therefore, its Jacobian equals one. This implies in particular that $f_z + g_t = 0$. In other words, there exists $h \in \mathbb{C}[z, t]$ such that $h_t = f$ and $h_z = -g$.

Now consider $\varphi(r) \equiv r + x(\{r, h\}) \mod (x^2)$. Since $\varphi(r) \in (r, x^2)$, the Poisson bracket $\{r, h\}$ is in the ideal (r). This implies that there exists a constant $c \in \mathbb{C}$ such that $h - c \in (r)$. To see this, note that $dh \wedge dr$ is identically zero along the zero set $V(r)$ of r. Thus h is locally constant as a function on $V(r)$ in a neighborhood of every smooth point of $V(r)$. Since r has no multiple irreducible factor, the set of smooth points is dense. Since $V(r)$ is connected, h is constant along $V(r)$.

We may assume that the constant $c = 0$. Thus $h = r\alpha$, with $\alpha \in \mathbb{C}[z, t]$, and $\varphi(z) \equiv z + x(r\alpha)_t \mod (x^2)$ and $\varphi(t) \equiv t - x(r\alpha)_z \mod (x^2)$.

It is easy to check that $\theta : \mathcal{A}_1(r) \to \mathbb{C}[z, t]$, $\varphi \mapsto \alpha$ is a group homomorphism, whose kernel is $\mathcal{A}_2$. We now prove that it is surjective. For any $\alpha \in \mathbb{C}[z, t]$ define an automorphism $\overline{\varphi} \in \mathrm{Aut}_R R[z, t]$ where $R = \mathbb{C}[x]/(x^2)$, given by $\overline{\varphi}(z) = z + x(r\alpha)_t$ and $\overline{\varphi}(t) = t - x(r\alpha)_z$. Note that the inverse of $\overline{\varphi}$ is given by $\overline{\varphi}^{-1}(z) = z - x(r\alpha)_t$ and $\overline{\varphi}^{-1}(t) = t + x(r\alpha)_z$. Also, $\overline{\varphi}$ is indeed an automorphism, and its Jacobian is 1. By a result of van den Essen, Maubach and Vénéreau, [18], there exists an automorphism φ of $\mathbb{C}[x][z, t]$ which projects to $\overline{\varphi}$. By construction, $\varphi \in \mathcal{A}_1(r)$ and $\theta(\varphi) = \alpha$. $\square$

4. EXTENSIONS OF AUTOMORPHISMS

In this section, we will continue with the more general setting in order to prove the lemma 4.2 below. We then will apply the lemma to the hypersurface X.

Notation 4.1. Let $r \in \mathbb{C}[z,t]$ be a polynomial with no multiple irreducible factor, whose zero set is connected, and let F be any polynomial in $\mathbb{C}[x,z,t]$. We define $P_{r,F} = x^2 y + r + xF \in \mathbb{C}[x,y,z,t]$, and we let $X_{r,F} = V(P_{r,F})$. Thus, for example, $X = X_{z^2+t^3,1}$, and $Y = X_{z^2+t^3,(1+z^2+t^3+x)}$.

As in the proof of proposition 3.2, for any $\varphi \in \mathcal{A}_1(r)$, we can construct an endomorphism Φ of $\mathbb{C}[x,y,z,t] = \mathbb{C}^{[4]}$ which induces a unique automorphism $\tilde{\varphi}$ of $\mathbb{C}[X_{r,F}]$ as follows. Φ is an extension of φ where we determine $\Phi(y)$. Suppose that $\theta(\varphi) = \alpha$. Let $\beta = \{r, \alpha\}$. Then, it is easily checked that $\varphi(r + xF) \equiv (1 + x\beta)(r + xF) \mod (x^2)$. Therefore, there exists a unique $G \in \mathbb{C}[x,z,t]$ such that $\Phi(P_{r,F}) = (1 + x\beta)P_{r,F}$ if we pose $\Phi(y) = (1 + x\beta)y + G$. We will denote by $\tilde{\varphi}$ the induced automorphism on $\mathbb{C}[X_{r,F}]$. In this way, $\mathcal{A}_1(r)$ can be considered as a subgroup of $\mathrm{Aut}(\mathbb{C}[X_{r,F}])$. We will now show that any such automorphism of $X_{r,F}$ lifts to an automorphism of $\mathbb{A}^4$. This is clear for the case that $\beta = 0$, since in this case, Φ is an automorphism of $\mathbb{C}^{[4]}$. In particular, any automorphism of $\mathcal{A}_2$ induces an automorphism of $X_{r,F}$ which extends. However, even if $\beta \neq 0$, we will show that by adding an appropriate multiple of $P_{r,F}$, we can lift $\tilde{\varphi}$ to an automorphism of $\mathbb{C}[x,y,z,t]$.

Lemma 4.2. *Let $\varphi \in \mathcal{A}_1(r)$, then $\tilde{\varphi}$, the corresponding automorphism of $\mathbb{C}[X_{r,F}]$, lifts to an automorphism of $\mathbb{C}^{[4]}$.*

Proof. Let $\varphi \in \mathcal{A}_1(r)$, and suppose that $\theta(\varphi) = \alpha$. Similarly to the method used in [13], we create a family of endomorphisms of $\mathbb{A}^4$ each one restricting to an automorphism of a fiber of $P_{r,F}$. Consider c as a variable, and denote by R_c the ring $R_c = \mathbb{C}[x,c]/(x^2)$. Consider now the automorphism $\overline{\phi} \in \mathrm{Aut}_{R_c}(R_c[z,t])$ given by $\overline{\phi}(z) = \varphi(z) + xc\alpha_t$, and $\overline{\phi}(t) = \varphi(t) - xc\alpha_z$. One checks easily that the Jacobian of $\overline{\phi}$ is 1, and therefore, by the result of van den Essen, Maubach and Vénéreau, [18], there exists an automorphism $\phi \in \mathrm{Aut}_{\mathbb{C}[c,x]}(\mathbb{C}[x,c][z,t])$ which restricts to $\overline{\phi}$.

For each $c \in \mathbb{C}$, denote by $\varphi_c \in \mathrm{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x][z,t])$ the automorphism defined by $\varphi_c(z) = \phi(z)$ and $\varphi_c(t) = \phi(t)$, where $\phi(z)$ and $\phi(t)$ are viewed as polynomials with coefficients in $\mathbb{C}[c]$. Note that $\varphi_c \in \mathcal{A}_1(r+c)$ and that the expression for φ_c depends polynomially on c.

For each $c \in \mathbb{C}$, we now construct, similarly to above, an automorphism $\tilde{\varphi}_c$ on $\mathbb{C}[X_{r+c,F}]$. Note that the expression for $\tilde{\varphi}_c$ depends polynomially on c. By making a formal substitution of c by $-P_{r,F}$, we construct an automorphism $\Psi = \tilde{\varphi}_{(-P_{r,F})}$ of $\mathbb{C}[x,y,z,t]$ which preserves the ideal $(P_{r,F})$. (See [13], lemma 3.4). Note that Ψ is a lift of the automorphism $\tilde{\varphi}_0 \in \mathrm{Aut}(\mathbb{C}[X_{r,F}])$. Also, φ_0 and φ are equivalent modulo (x^2). More precisely, $\varphi_0^{-1} \circ \varphi$ is an element of $\mathcal{A}_2$. By the comment before the lemma, $\varphi_0^{-1} \circ \varphi$ induces an automorphism $\widetilde{\varphi_0^{-1} \circ \varphi}$ which lifts to an automorphism of $\mathbb{C}^{[4]}$. By the unicity of the extension of an element of $\mathcal{A}_1(r)$ to an automorphism of $\mathbb{C}[X_{r,F}]$, we have that $\widetilde{\varphi_0^{-1} \circ \varphi} = \tilde{\varphi}_0^{-1} \circ \tilde{\varphi}$. Since $\tilde{\varphi}_0$ also lifts to an automorphism of $\mathbb{C}^{[4]}$, the same is true for $\tilde{\varphi}$. $\square$

Theorem 4.3. *Every automorphism of $X = V(P)$ extends to an automorphism of $\mathbb{A}^4$.*

Proof. The automorphism group of X is isomorphic to $\mathcal{A} = \mathcal{A}_1 \rtimes \mathbb{C}^*$. The automorphisms in $\mathbb{C}^*$ extend, and, by lemma 4.2, the automorphisms in $\mathcal{A}_1$ extend. Therefore all automorphisms extend to automorphism of $\mathbb{A}^4$. $\square$

Example 4.4. Consider the automorphism φ of $\mathbb{C}[x,z,t]$ given by $\varphi(x) = x$, $\varphi(z) = z + 3xt^5$ and $\varphi(t) = t + 2x(z + 3xt^5)^3$. It is indeed an automorphism, since it is the composition of two triangular automorphisms.

Also we can check that φ is an element of $\mathcal{A}_1$. It is obvious that $\varphi(x) = x$ and $\varphi \equiv \mathrm{id}$ mod (x); we will now show that $\varphi(z^2+t^3+x)$ is in the ideal (x^2, z^2+t^3+x). Indeed, we find that

there exists an element $G \in \mathbb{C}[x, z, t]$ such that $\varphi(z^2 + t^3 + x) = (z^2 + t^3 + x) + x(6zt^5 + 6t^2 z^3) + x^2 G$. This yields : $\varphi(z^2 + t^3 + x) = (z^2 + t^3 + x)(1 + 6xzt^2) + x^2(G - 6zt^2)$.

Thus φ is an element of $\mathcal{A}_1$. To find the corresponding automorphism of $\mathbb{C}[X]$, we extend φ to the automorphism $\tilde{\varphi}$ of $\mathbb{C}[X]$ where $\tilde{\varphi}(y) = (1 + 6xzt^2)y - (G - 6zt^2)$.

In order to lift this automorphism to an automorphism of $\mathbb{C}^{[4]}$, we apply the procedure above.

We have that $\alpha = (t^3 - z^2)/2$. We define, for each $c \in \mathbb{C}$, an automorphism $\varphi_c \in \mathcal{A}_1(z^2 + t^3 + c)$ as follows. $\varphi_c(z) = z + 3xt^2(t^3 + c/2)$, and $\varphi_c(t) = t + 2x\varphi_c(z)(\varphi_c(z)^2 + c/2)$. More precisely, we have that $\varphi(z) \equiv z + x((z^2 + t^3 + c)\alpha)_t \mod (x^2)$, and $\varphi(t) \equiv t - x((z^2 + t^3 + c)\alpha)_z \mod (x^2)$.

Now, we can define for each $c \in \mathbb{C}$, an automorphism $\tilde{\varphi}_c$ of $\mathbb{C}[X_{z^2+t^3+c,1}]$ if we pose $\tilde{\varphi}_c(y) = (1 + 6xzt^2)y + G_c$, for a suitable polynomial $G_c \in \mathbb{C}[x, z, t, c]$. Finally, to find the automorphism of $\mathbb{C}[x, y, z, t]$ which is a lift of $\tilde{\varphi}$, we make a formal substitution of c by $-P$.

Corollary 4.5. *The origin $o = (0, 0, 0, 0) \in X$ is fixed by all automorphisms of X.*

Proof. By proposition 2.5 (ii), any automorphism of X which extends to $\mathbb{A}^4$ fixes the origin. By theorem 4.3, all automorphisms of X extend to $\mathbb{A}^4$. $\square$

Remark. This corollary was first proven in collaboration with G. Freudenburg using a different method.

5. Inequivalent hypersurfaces

Consider now the two hypersurfaces $X = V(P)$ and $Y = V(Q)$. (As before, $P = x^2 y + z^2 + x + t^3$), and $Q = x^2 y + (1 + x)(z^2 + x + t^3)$). We know from theorem 2.4 that as abstract varieties, X and Y are isomorphic. We now show the following result

Theorem 5.1. *X and Y are inequivalent as hypersurfaces of $\mathbb{C}^4$.*

Proof. Suppose there were an automorphism Ψ of $\mathbb{A}^4$ such that $\Psi(X) = Y$. Then $\Psi(o) \neq o$ by proposition 2.5 (i). Consider now the isomorphism ϕ defined in theorem 2.4 between X and Y. Then $(\Psi^{-1})|_Y \circ \phi$ is an automorphism of X which does not preserve o. This contradicts the corollary 4.5. $\square$

It should be noted that, as another consequence of the description of the automorphism group of X, we can show that all automorphisms of $\mathbb{C}[Y]$ which fix the variable x also extend to automorphisms of $\mathbb{A}^4$.

This is the case, since $\mathrm{Aut}(Y) \cong \mathrm{Aut}(X) \cong \mathcal{A} = \mathcal{A}_1 \rtimes \mathbb{C}^*$. The subgroup of automorphisms which fix x (for X or for Y) corresponds via this isomorphism to the subgroup $\mathcal{A}_1 \rtimes C_6 \subset \mathcal{A}_1 \rtimes \mathbb{C}^*$, where C_6 is the subgroup of the sixth roots of unity in $\mathbb{C}^*$. The automorphisms corresponding to elements of $\mathcal{A}_1$ extend to Y by lemma 4.2, and the automorphisms corresponding to elements in C_6 extend to linear automorphisms on $\mathbb{A}^4$.

However, the action of $\mathbb{C}^*$ on Y does not extend to an action on $\mathbb{A}^4$. More precisely, for any $\lambda \in \mathbb{C}$ such that $\lambda^6 \neq 1$, the action of $\lambda \in \mathbb{C}^*$ on X, conjugated by ϕ to give an automorphism of Y, does not extend to an automorphism of $\mathbb{A}^4$. To see this, note that the action of λ does not fix the line $x = z = t = 0$. The only fixed point on this line is the origin. However, by proposition 2.5, (iii), the point $(0, -1, 0, 0)$ must be fixed.

6. Remarks and some open questions

6.1. Locally nilpotent derivations on $\mathbb{C}[X]$.
We can give a complete description of the locally nilpotent derivations of $\mathbb{C}[X]$. We denote by $\mathrm{LND}(\mathbb{C}[X])$ the set of locally nilpotent derivations of $\mathbb{C}[X]$. Denote by $\mathrm{LND}_x(\mathbb{C}[x][z, t])$ the $\mathbb{C}[x]$-module of locally nilpotent derivations of $\mathbb{C}[x, z, t]$ having x in the kernel. If ∂ is a locally nilpotent derivation on $\mathbb{C}[X]$, it restricts to a unique locally nilpotent derivation of $\mathrm{LND}_x(\mathbb{C}[x][z, t])$.

Proposition 6.1. $\mathrm{LND}(\mathbb{C}[X]) = x^2(\mathrm{LND}_x(\mathbb{C}[x][z, t]))$.

Proof. If $\partial = x^2\partial_0$ is an element of $x^2(\mathrm{LND}_x(\mathbb{C}[x][z,t])$, then one can extend it to a locally nilpotent derivation on $\mathbb{C}[X]$ by posing $\partial(y) = -\partial_0(z^2 + t^3)$. For the converse, if ∂ is a locally nilpotent derivation on $\mathbb{C}[X]$, then $\partial(z^2+t^3) = 2z\partial(z)+3t^3\partial(t) \in (x^2)$. Consider the derivation $\overline{\partial}$ of $\mathbb{C}[z,t]$ defined by $\overline{\partial}(f) \equiv \partial(f) \mod (x)$. Then $\overline{\partial}$ induces an action of $(\mathbb{C},+)$ on $\mathbb{A}^2$ which stabilizes the cuspidal curve $z^2 + t^3 = 0$. This implies that the action is trivial, and therefore that $\overline{\partial} = 0$. In other words, there exists an element $\partial_1 \in \mathrm{LND}_x(\mathbb{C}[x][z,t])$ satisfying $\partial = x\partial_1$. Now since $\partial(z^2 + t^3) \in (x^2)$, we have that $\partial_1(z^2 + t^3)$ belongs to (x); and the same argument proves that there exists ∂_0 such that $\partial_1 = x\partial_0$. $\qquad\square$

6.2. **Non-zero fibers of P and Q.**

Proposition 6.2. *The following hold:*

a) *For every $c \in \mathbb{C}$, $V(Q - c)$ is isomorphic to the hypersurface $V(F_c) = Z_c$ of $\mathbb{A}^4$ defined by the equation*

$$F_c = x^2y + z^2 + (1 + c)x + t^3 - c = 0.$$

b) *For every $c \in \mathbb{C} \setminus \{-1,0\}$ and every $c' \in \mathbb{C} \setminus \{0\}$, Z_c and $V(P - c')$ are isomorphic as abstract affine varieties.*

Proof. Recall that by definition

$$V(Q-c) \simeq \mathrm{Spec}\left(\mathbb{C}[x,y,z,t]/(Q - c)\right) = \mathrm{Spec}\left(\mathbb{C}[x,y,z,t]/\left(x^2y + (1 + x)\left(z^2 + t^3 + x\right) - c\right)\right).$$

We claim that the endomorphisms

$$\begin{cases} \Phi_c: & (x,y,z,t) \mapsto (x, (1 - x)\,y - z^2 - x - t^3, z, t) \\ \Psi_c: & (x,y,z,t) \mapsto (x, (1 + x)\,y + c, z, t) \end{cases}$$

of $\mathbb{A}^4$ restricts respectively to isomorphisms $\phi_c : V(Q - c) \xrightarrow{\sim} Z_c$ and $\psi_c : Z_c \xrightarrow{\sim} V(Q - c)$ which are inverse to each other. Indeed, one checks that $\Phi_c^*(F_c) = (1 - x)\,(Q - c)$ whereas $\Psi_c^*(Q - c) = (1 + x)\,F_c$ so that Φ_c and Ψ_c induce morphisms $\phi_c : V(Q-c) \to Z_c$ and $\psi_c : Z_c \to V(Q - c)$ respectively. Since ϕ_c and ψ_c are morphisms of schemes over $\mathbb{A}^3 = \mathrm{Spec}\,(\mathbb{C}[x,z,t])$, the identities $(\Phi_c^* \circ \Psi_c^*)(y) = y - (Q - c)$ and $(\Psi_c^* \circ \Phi_c^*)(y) = y - F_c$ guarantee that their are inverse isomorphisms. This proves assertion a).

Now, note that $V(P - c) \simeq V(P - 1)$ for every $c \in \mathbb{C} \setminus \{0\}$. Indeed, if $c \in \mathbb{C}^*$, consider the automorphism of $\mathbb{A}^4$ defined by $(x,y,z,t) = (a^6x, a^{-6}y, a^3z, a^2t)$, for a suitable constant $a \in \mathbb{C}$ such that $a^{-6} = c$. This automorphism maps $V(P - c)$ isomorphically onto $V(P - 1)$.

Finally, assertion b) follows from the fact that for every $c \in \mathbb{C} \setminus \{-1\}$, the automorphism of $\mathbb{A}^4$ defined by $(x,y,z,t) \mapsto \left((1 + c)\,x, (1 + c)^{-2}\,y, z, t\right)$ maps $V(P - c)$ isomorphically onto Z_c. $\qquad\square$

Together with the above discussion, this result motivates the following question.

Question 1. *Are the subvarieties of $\mathbb{A}^4 = \mathrm{Spec}\,(\mathbb{C}[x,y,z,t])$ defined by the equations*

$$x^2y + z^2 + x + t^3 + 1 = 0 \qquad and \qquad x^2y + z^2 + t^3 + 1 = 0$$

isomorphic ?

Remark 6.3. This question has an affirmative answer in holomorphic category. Indeed, one can easily check that the analytic automorphism of $\mathbb{A}^4$ defined by

$$(x,y,z,t) \mapsto (x, y + 1 - \frac{e^x - 1 - x}{x^2}(z^2 + t^3), e^{x/2}z, e^{x/3}t)$$

induces an isomorphism between the hypersurfaces $V(Q + 1)$ and $V(P + 1)$.

6.3. **Holomorphic and stable equivalence.**

Recall that two closed algebraic smooth subvarieties X and Y of $\mathbb{A}^4$ isomorphic as abstract algebraic varieties are called holomorphically equivalent if there exists a biholomorphism of $\mathbb{A}^4$ restricting to a biholomorphism between X and Y considered as complex manifolds. Similarly, we say that X and Y are stably equivalent if there exist $n \in \mathbb{N}$ and an algebraic automorphism of $\mathbb{A}^{4+n}$ restricting to an isomorphism between $X \times \mathbb{A}^n$ and $Y \times \mathbb{A}^n$. Here we show the following.

Proposition 6.4. *The subvarieties X and Y of $\mathbb{A}^4 = \mathrm{Spec}\,(\mathbb{C}\,[x,y,z,t])$ defined by the equations*

$$P = x^2y + z^2 + x + t^3 = 0 \qquad and \qquad Q = x^2y + (1+x)\left(z^2 + x + t^3\right) = 0$$

are holomorphically equivalent and stably equivalent.

Proof. By virtue of theorem 2.4, X and Y are isomorphic as abstract algebraic varieties. Holomorphic equivalence follows from the observation that the map $\zeta : \mathbb{A}^4 \to \mathbb{A}^4$ defined by

$$\zeta\,(x,y,z,t) = \left(x, e^x y + x^{-2}\,(e^x - 1 - x)\,(z^2 + x + t^3)\,, z, t\right)$$

is a biholomorphism of $\mathbb{A}^4$ which maps X isomorphically onto Y. Moreover, remark that ζ can be viewed as a holomorphic extension of the isomorphism $\phi : X \to Y$ defined in theorem 2.4. Indeed, we have $\zeta(x,y,z,t) = (x, (1+x)\,y + x^{-2}\,(e^x - 1 - x)\,P, z, t)$.

For stable algebraic equivalence, we consider the $\mathbb{C}\,[x]$-endomorphism Ψ^* of $\mathbb{C}\,[x,y,z,t,w]$ defined by

$$\begin{cases} y & \mapsto y + 1 + x^{-2}\left(\left(\left(1+\tfrac{1}{2}x\right)z + x^2w\right)^2 - (1+x)\,z^2\right) + x^{-2}\left(\left(\left(1+\tfrac{1}{3}x\right)t + x^2w\right)^3 - (1+x)\,t^3\right) \\ z & \mapsto \left(1+\tfrac{1}{2}x\right)z + x^2w \\ t & \mapsto \left(1+\tfrac{1}{3}x\right)t + x^2w \\ w & \mapsto -\tfrac{3}{4}z + \tfrac{2}{9}t + \left(1 - \tfrac{5}{6}x\right)w \end{cases}$$

By definition, Ψ^* restricts to a linear $\mathbb{C}\,[x]$-endomorphism $\tilde{\Psi}^*$ of $\mathbb{C}\,[x,z,t,w]$, which is an automorphism since the Jacobian is invertible :

$$\begin{pmatrix} \left(1+\tfrac{1}{2}x\right) & 0 & x^2 \\ 0 & \left(1+\tfrac{1}{3}x\right) & x^2 \\ -\tfrac{3}{4} & \tfrac{2}{9} & \left(1-\tfrac{5}{6}x\right) \end{pmatrix} \in \mathrm{GL}_3\,(\mathbb{C}\,[x])\,.$$

Since $\Psi^*(y)$ depends triangularly on the variables x, z, t and w, this implies in turn that Ψ^* is a $\mathbb{C}\,[x]$-automorphism of $\mathbb{C}\,[x,y,z,t,w]$. Now one checks easily that $\Psi^*P = Q$, which completes the proof. $\qquad\square$

Remark 6.5. In the proof above, we have shown that the isomorphism $\phi : X \to Y$ – which cannot be extended to an algebraic automorphism of $\mathbb{A}^4$ (theorem 2.4) –, can be extended to a holomorphic automorphism of $\mathbb{A}^4$. In particular, there is no topological obstruction to extending the isomorphism ϕ to an automorphism.

7. Locally nilpotent derivations on the cylinder over the Koras-Russell threefold

Recently, the first author proved that the Makar-Limanov invariant of the cylinder over the Koras-Russell threefold is trivial [3]. The idea of the proof is as follows. Let $X^0 = X \setminus V(t)$. Now consider the polynomial $P_1 = xy + z^2 + t^3 + x$, and let $X_1 = V(P_1)$, and $X_1^0 = X_1 \setminus V(t)$. It is shown that the cylinders $X^0 \times \mathbb{C}$ and $X_1^0 \times \mathbb{C}$ are isomorphic. Then one uses the fact that the Makar-Limanov invariant of X_1^0 is trivial to show that the Makar-Limanov invariant of $X^0 \times \mathbb{C}$ is trivial, and this implies the result.

We denote by, $A = \mathbb{C}[X]$ is the coordinate ring of X, by $B = \mathbb{C}[X_1]$ the coordinate ring of X_1, by A_t the coordinate ring of X^0, and by B_t the coordinate ring of X_1^0.

In this section we will construct an explicit isomorphism, and then we will obtain a locally nilpotent derivation ∂ on the coordinate ring $A[w]$ of $X \times \mathbb{A}^1$ such that $\partial(x) \neq 0$.

Proposition 7.1. *The algebraic endomorphism Φ of $\mathbb{C}[x, y, z, t^{\pm 1}, w]$ defined by*

$$
\begin{cases}
\Phi(x) = x \\
\Phi(y) = xy - xw^2 - 2zw \\
\Phi(z) = z + xw \\
\Phi(t) = t \\
\Phi(w) = 2w + yz + 3xyw - 3zw^2 - xw^3
\end{cases}
$$

induces an isomorphism

$$\phi : A_t[w] = \mathbb{C}[x, y, z, t^{\pm 1}, w]/(x^2y + z^2 + x + t^3) \xrightarrow{\sim} B_t[w] = \mathbb{C}[x, y, z, t^{\pm 1}, w]/(xy + z^2 + x + t^3)$$

whose inverse isomorphism $\phi^{-1} : B_t[w] \to A_t[w]$ is induced by the endomorphism of $\mathbb{C}[x, y, z, t^{\pm 1}, w]$ defined by

$$
\begin{cases}
\Psi(x) = x \\
\Psi(y) = -\frac{1}{t^3}(y + y^2 + wz) - \frac{1}{4t^6}(yz - xw)^2 \\
\Psi(z) = z - \frac{1}{2t^3}x(yz - xw) \\
\Psi(t) = t \\
\Psi(w) = \frac{1}{2t^3}(yz - xw).
\end{cases}
$$

Proof. Recall that P denotes the polynomial $P = x^2y + z^2 + x + t^3$. Let S be the polynomial defined by $S = xy + z^2 + x + t^3$.

Φ and Ψ induce well-defined algebraic morphisms $\phi : A_t[w] \to B_t[w]$ and $\psi : B_t[w] \to A_t[w]$. Indeed one checks $\Phi(S) = P$ and $\Psi(P) = (1 - xyt^{-3})S$.

Since ϕ and ψ are $\mathbb{C}[x, t^{\pm 1}]$-morphisms, the following equalities prove that ϕ and ψ are inverse isomorphisms.

$$\Psi \circ \Phi(z) = z \quad ; \quad \Psi \circ \Phi(y) = y - \frac{y}{t^3}S \quad ; \quad \Psi \circ \Phi(w) = w + \frac{xyw - y^2z - t^3w}{t^6}S;$$

$$\Phi \circ \Psi(w) = w - \frac{1}{t^3}P \quad ; \quad \Phi \circ \Psi(z) = z + \frac{xw}{t^3}P \quad ; \quad \Phi \circ \Psi(y) = y - \frac{y - w^2}{t^3}P - \frac{w^2}{t^6}P^2.$$

$\square$

Proposition 7.2. *Let Δ be the locally nilpotent derivation on $\mathbb{C}[x, y, z, t^{\pm 1}, w]$ defined by*

$$\Delta = t^6 \left(-2z\frac{\partial}{\partial x} + (y + 1)\frac{\partial}{\partial z} \right).$$

Then, the derivation ∂ of $\mathbb{C}[x, y, z, t^{\pm 1}, w]$ defined by

$$\partial = (\Phi \circ \Delta \circ \Psi)(x)\frac{\partial}{\partial x} + (\Phi \circ \Delta \circ \Psi)(y)\frac{\partial}{\partial y} + (\Phi \circ \Delta \circ \Psi)(z)\frac{\partial}{\partial z} + (\Phi \circ \Delta \circ \Psi)(w)\frac{\partial}{\partial w}$$

induces a locally nilpotent derivation on $A[w] = \mathbb{C}[x, y, z, t, w]/(x^2y + z^2 + x + t^3)$ which does not contain the variable x in its kernel.

Proof. Note first that $\partial(x), \partial(y), \partial(z)$ and $\partial(w)$ are all elements of $\mathbb{C}[x, y, z, t, w]$ and so ∂ restricts to a well-defined derivation on $\mathbb{C}[x, y, z, t, w]$. (For example, one checks that $\partial(x) = -2t^6(z + xw)$).

Secondly, notice that, since $\Delta(xy + z^2 + x + t^3) = 0$, Δ induces a locally nilpotent derivation on $B_t[w] = \mathbb{C}[x, y, z, t^{\pm 1}, w]/(xy + z^2 + x + t^3)$. Therefore, in light of proposition 7.1, on can conclude that ∂ induces a locally nilpotent derivation $\tilde{\partial}$ on $A_t[w] = \mathbb{C}[x, y, z, t^{\pm 1}, w]/(x^2y + z^2 + x + t^3)$.

The inclusion $\mathbb{C}[x, y, z, t, w] \subset \mathbb{C}[x, y, z, t^{\pm 1}, w]$ induces an inclusion of $A[w]$ in $A_t[w]$. More precisely, let $\pi : \mathbb{C}[x, y, z, t^{\pm 1}, w] \to A_t[w]$ be the canonical projection. Since P is prime in $\mathbb{C}[x, y, z, t, w]$, one can identify $A[w]$ with the image $\pi(\mathbb{C}[x, y, z, t, w])$ of the subalgebra $\mathbb{C}[x, y, z, t, w]$. We have that $\pi \circ \tilde{\partial} = \partial \circ \pi$, and therefore $\tilde{\partial}$ restricts to a locally nilpotent deriviation on $A[w] = \pi(\mathbb{C}[x, y, z, t, w])$. $\square$

Corollary 7.3. *The Makar-Limanov invariant of $A[w]$ is trivial.*

Proof. Let ∂_1 and ∂_2 be the locally nilpotent derivations on $A[w]$ defined by

$$\partial_1 = 2z\frac{\partial}{\partial y} - x^2\frac{\partial}{\partial z} \quad \text{and} \quad \partial_2 = 3t^2\frac{\partial}{\partial y} - x^2\frac{\partial}{\partial t}.$$

We have $\mathrm{Ker}(\partial_1) \cap \mathrm{Ker}(\partial_2) = \mathbb{C}[x]$. Thus, $\mathrm{ML}(A[w]) \subset \mathbb{C}[x]$. Now, proposition 7.2 allows us to conclude that $\mathrm{ML}(A[w]) = \mathbb{C}$. $\square$

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