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THE INTEGRAL HOMOLOGY OF PSL_2 OF IMAGINARY QUADRATIC INTEGERS WITH NON-TRIVIAL CLASS GROUP

ALEXANDER D. RAHM AND MATHIAS FUCHS

ABSTRACT. We show that a cellular complex defined by Flöge allows to determine the integral homology of the *Bianchi groups* $\mathrm{PSL}_2(\mathcal{O}_{-m})$, where \mathcal{O}_{-m} is the ring of integers of an imaginary quadratic number field $\mathbb{Q}[\sqrt{-m}]$ for a square-free natural number m .

In the cases of non-trivial class group, we handle the difficulties arising from the cusps associated to the non-trivial ideal classes of \mathcal{O}_{-m} . We use this to compute in the cases $m = 5, 6, 10, 13$ and 15 the integral homology of $\mathrm{PSL}_2(\mathcal{O}_{-m})$, which before was known only in the cases $m = 1, 2, 3, 7$ and 11 with trivial class group.

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1. INTRODUCTION

The objects of study of this paper are the PSL_2 -groups Γ of the ring of integers $\mathcal{O}_{-m} := \mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}$ of an imaginary quadratic number field $\mathbb{Q}[\sqrt{-m}]$, where m is a square-free positive integer. We have $\mathcal{O}_{-m} = \mathbb{Z}[\omega]$ with $\omega = \sqrt{-m}$ for m congruent to 1 or 2 modulo 4, and $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-m}$ for m congruent to 3 modulo 4.

The arithmetic groups under study have often been called Bianchi groups, because Luigi Bianchi [66, 6] computed fundamental domains for them as early as in 1892. They act on $\mathrm{PSL}_2(\mathbb{C})$'s symmetric

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Figure 1. Results in the group homology with simple integer coefficients

$$\begin{aligned}
\mathbb{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-5}); \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2, & q = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 2, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q, & q \geq 3; \end{cases} \\
\mathbb{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-10}); \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2, & q = 1, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 2, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q, & q \geq 3; \end{cases} \\
\mathbb{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-15}); \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 2, \\ \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q \geq 3; \end{cases} \\
\mathbb{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-13}); \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2, & q = 1, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/2, & q = 2, \\ (\mathbb{Z}/2)^q \oplus (\mathbb{Z}/3)^2, & q = 4k + 3, \quad k \geq 0, \\ (\mathbb{Z}/2)^q, & q = 4k + 4, \quad k \geq 0, \\ (\mathbb{Z}/2)^q, & q = 4k + 1, \quad k \geq 1, \\ (\mathbb{Z}/2)^q \oplus (\mathbb{Z}/3)^2, & q = 4k + 2, \quad k \geq 1; \end{cases} \\
\mathbb{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-6}); \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2, & q = 2, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+2}, & q = 6k + 3, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+1}, & q = 6k + 4, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+4}, & q = 6k + 5, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+3}, & q = 6k + 6, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+2}, & q = 6k + 7, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+5}, & q = 6k + 8. \end{cases}
\end{aligned}$$

space, the hyperbolic three-space \mathcal{H} . Interest in this action first arose when Felix Klein and Henri Poincaré studied certain groups of Möbius transformations with complex coefficients [1414, 1418, 18], laying the groundwork for the study of *Kleinian groups*. The latter are nowadays defined as discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$. Each non-cocompact arithmetic Kleinian group is commensurable with some Bianchi group [1515, 15]. Thus, the Bianchi groups play a key role in the study of arithmetic Kleinian groups. A wealth of information on the Bianchi groups can be found in the pertinent monographs [99, 988, 81515, 15].

Poincaré gave an explicit formula for their action on \mathcal{H} . However, the virtual cohomological dimension of arithmetic groups which are lattices in $\mathrm{SL}_2(\mathbb{C})$ is two, so it is desirable to restrict this proper action on \mathcal{H} to a contractible cellular two-dimensional space. Moreover, this space should be cofinite. In principle, this has been achieved by Mendoza [1616, 16] and also by Flöge [1010, 10], using reduction theory of Minkowski, Humbert, Harder and others. Their two approaches have in common that they consider two-dimensional Γ -equivariant retracts which are cocompact and are endowed with a natural CW-structure such that the action of Γ is cellular and the quotient is a finite CW-complex.

Using Mendoza's complex, Schwermer and Vogtmann [2020, 20] calculated the integral group homology in the cases of trivial class group $m = 1, 2, 3, 7, 11$, and Vogtmann [2424, 24] computed the rational homology as the homology of the quotient space in many cases of non-trivial class group. The integral cohomology in the cases $m = 2, 3, 5, 6, 7, 10, 11$ has been determined by Berkove [55, 5], based on Flöge's presentation of the groups with generators and relations. A completely different method to obtain group presentations has been chosen by Yasaki [2626, 26], who has implemented an algorithm of Gunnells [1212, 12] to compute the perfect forms modulo the action of $\mathrm{GL}_2(\mathcal{O}_{-m})$ and obtain the facets of the Voronoï polyhedron arising from a construction of Ash [33, 3].

It is the purpose of the present paper to show how Flöge's complex can be used to obtain the integral homology of Bianchi groups also when the class group is non-trivial. We obtain the results displayed in figure 1. Thus for $q \geq 2$, the torsion in $H_*(\mathrm{PSL}_2(\mathcal{O}_{-5}); \mathbb{Z})$ is the same as that in $H_*(\mathrm{PSL}_2(\mathcal{O}_{-10}); \mathbb{Z})$, analogous to the cohomology results of Berkove [55, 5]. The free part of these homology groups is in accordance with the rational homology results of Vogtmann [2424, 24].

In the cases of non-trivial ideal class group, there is a difference between the approaches of Mendoza and Flöge. We use the upper-half-space model of \mathcal{H} and identify its boundary with $\mathbb{C} \cup \infty \cong \mathbb{C}P^1$. The elements of the class group of the number field are in bijection with the Γ -orbits of the cusps, where the cusps are ∞ and the elements of the number field $\mathbb{Q}[\sqrt{-d}]$, thought of as elements of the canonical boundary $\mathbb{C}P^1$. The cusps which represent a non-trivial element of the class group are commonly called *singular* points. Whilst Mendoza retracts away from all cusps, Flöge retracts away only from the non-singular ones. Rather than the space \mathcal{H} itself, he considers the space $\hat{\mathcal{H}}$ obtained from \mathcal{H} by adjoining the Γ -orbits of the singular points. We consider an analogous equivariant retraction of $\hat{\mathcal{H}}$ such that its retract X contains the singular points. Now it turns out that the quotient space of X by Γ is compact, and X is a suitable contractible 2-dimensional Γ -complex also in the case of non-trivial class group.

With an implementation in Pari/GP [22, 2], due to the first named author, of Swan's algorithm [2323, 23] we obtain a fundamental polyhedron for Γ in \mathcal{H} . In the cases considered, Bianchi has already computed this polyhedron, so we have a control of the correctness of the implementation.

In the cases $m = 5, 6$ and 10 , Flöge has computed the cell stabilizers and cell identifications; and with our Pari/GP program, we redo Flöge's computations and do the same computation in the cases $m = 13$ and 15 . We use the equivariant Euler characteristic to check our computations. Then we follow the lines of Schwermer and Vogtmann [2020, 20], encountering a spectral sequence which degenerates on the E^3 -page, in contrast to the cases of trivial class group where it does so already on the E^2 -page. This is because of the singular points in our cell complex X , which have infinite stabilizers. So we have some additional use of homological algebra to obtain the homology of the Bianchi group. We give the full details for our homology computation in the case $m = 13$. We then give slightly fewer details in the cases $m = 5, 6, 10$ and 15 .

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This article is dedicated to Fritz Grunewald (1949 - 2010).

2. FLÖGE'S COMPLEX, CONTRACTIBILITY AND A SPECTRAL SEQUENCE

Denote the hyperbolic three-space by $\mathcal{H} \cong \mathbb{C} \times \mathbb{R}_+^*$. We will not use its smooth structure, only its structure as a homogeneous $\mathrm{SL}_2(\mathbb{C})$ -space. The action is given by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) := \left(\frac{(\bar{d} - \bar{c}z)(az - b) - r^2 \bar{c}a}{|cz - d|^2 + r^2 |c|^2}, \frac{r}{|cz - d|^2 + r^2 |c|^2} \right);$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. As usual, we extend the action of $\mathrm{SL}_2(\mathbb{C})$ to the boundary $\mathbb{C}P^1$ which we identify with $\{r = 0\} \cup \infty \cong \mathbb{C} \cup \infty$. The action passes continuously to the boundary, where it reduces

to the usual action by Möbius transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az-b}{-cz+d}$. As $-1 \in \mathrm{SL}_2(\mathbb{C})$ acts trivially, the action passes to $\mathrm{PSL}_2(\mathbb{C})$. Now, fix a square-free $m \in \mathbb{N}$, let \mathcal{O}_{-m} be the ring of integers in $\mathbb{Q}[\sqrt{-m}]$, and define $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$. When the class number of $\mathbb{Q}[\sqrt{-m}]$ is one, then classical reduction theory provides a natural equivariant deformation retract of \mathcal{H} which is a CW-complex. This complex is defined as follows. One first considers the union of all hemispheres

$$S_{\mu,\lambda} := \left\{ (z, r) : \left| z - \frac{\lambda}{\mu} \right|^2 + r^2 = \frac{1}{|\mu|^2} \right\} \subset \mathcal{H},$$

for any two μ, λ with $\mu\mathcal{O}_{-m} + \lambda\mathcal{O}_{-m} = \mathcal{O}_{-m}$. Then one considers the “space above the hemispheres”

$$B := \left\{ (z, r) : |cz - d|^2 + r^2 |c|^2 \geq 1 \text{ for all } c, d \in \mathcal{O}_{-m}, c \neq 0 \text{ such that } c\mathcal{O}_{-m} + d\mathcal{O}_{-m} = \mathcal{O}_{-m} \right\}$$

and its boundary ∂B inside \mathcal{H} . For nontrivial class group, the following definition comes to work.

Definition 1. A point $s \in \mathbb{C}P^1 - \{\infty\}$ is called a singular point if for all $c, d \in \mathcal{O}_{-m}$, $c \neq 0$, $c\mathcal{O}_{-m} + d\mathcal{O}_{-m} = \mathcal{O}_{-m}$ we have $|cs - d| \geq 1$.

The singular points modulo the action of Γ on $\mathbb{C}P^1$ are in bijection with the nontrivial elements of the class group [2222, 22]. In [1010, 10], Flöge extends the hyperbolic space \mathcal{H} to a larger space $\widehat{\mathcal{H}}$ as follows.

Definition 2. As a set, $\widehat{\mathcal{H}} \subset \mathbb{C} \times \mathbb{R}^{\geq 0}$ is the closure under the Γ -action of the union $\widehat{B} := B \cup \{\text{singular points}\}$. The topology is generated by the topology of \mathcal{H} together with the following neighborhoods of the translates s of singular points:

$$\widehat{U}_\epsilon(s) := \{s\} \cup \begin{pmatrix} s & 0 \\ -1 & s^{-1} \end{pmatrix} \cdot \{(z, r) \in \mathcal{H} : r > \epsilon^{-1}\}.$$

Remark 3. The matrix $\begin{pmatrix} s & 0 \\ -1 & s^{-1} \end{pmatrix}$ maps the point at infinity into s , thus giving the point s the topology of ∞ . The neighborhood $\widehat{U}_\epsilon(s)$ is sometimes called a “horoball” because in the upper-half space model it is a Euclidean ball, but with the hyperbolic metric it has “infinite radius”.

The space $\widehat{\mathcal{H}}$ is endowed with the natural Γ -action. Now the essential aspect of Flöge’s construction is the following consequence of Flöge’s theorem [1111; 11, 6.6], which we append as theorem 28.

Corollary 4. There is a retraction ρ from $\widehat{\mathcal{H}}$ onto the set $X \subset \widehat{\mathcal{H}}$ of all Γ -translates of $\partial\widehat{B}$, i. e. there is a continuous map $\rho : \widehat{\mathcal{H}} \rightarrow X$ such that $\rho(p) = p$ for all $p \in X$. The set X admits a natural structure as a cellular complex X^\bullet on which Γ acts cellularly.

Remark 5. (1) We show with the lemma below that ρ is a homotopy equivalence, without giving a continuous path of maps $\widehat{\mathcal{H}} \rightarrow \widehat{\mathcal{H}}$ connecting ρ to the identity on $\widehat{\mathcal{H}}$.
(2) The map ρ is Γ -equivariant because its fibers are geodesics. But we do not make use of this fact, as we do not need to show that the homotopy type of $\Gamma \backslash \widehat{\mathcal{H}}$ is the same as that of $\Gamma \backslash X$. This would be useful in the case of trivial class group, i. e. the case of a proper action, to compute the rational homology $H_*(\Gamma; \mathbb{Q}) \cong H_*(\Gamma \backslash \mathcal{H}; \mathbb{Q})$.
(3) We will provide X^\bullet with a cellular structure which is fine enough to make the cell stabilizers fix the cells pointwise.

Lemma 6. Let Y be a CW-complex which admits an inclusion i into a contractible topological space A , such that i is a homeomorphism between Y with its cellular topology and the image $i(Y)$ with the subset topology of A . Let $p : A \rightarrow Y$ be a continuous map with $p \circ i = \mathrm{id}_Y$. Then p is a homotopy equivalence.

Proof. For all $n \in \mathbb{N}$, the induced maps on the homotopy groups $(\mathrm{id}_Y)_* = (p \circ i)_* : \pi_n(Y) \rightarrow \pi_n(Y)$ factor through $\pi_n(A) = 0$, hence are the zero map; and $\pi_n(Y) = 0$. Denote by c the constant map from A to the one-point space. Then $c \circ i$ is a morphism of CW-complexes, and the zero maps it induces on the homotopy groups are isomorphisms. Thus by Whitehead's Theorem, $c \circ i$ is a homotopy equivalence. As A is contractible, the composition $(c \circ i) \circ p = c$ is a homotopy equivalence, so the same holds already for p . \square

Taking $Y = X$, $A = \widehat{\mathcal{H}}$, $p = \rho$, and using lemma 8, we obtain a crucial fact for our computations.

Corollary 7. *X^\bullet is contractible.*

The following is an observation on Flöge's construction.

Lemma 8. *The space $\widehat{\mathcal{H}}$ is contractible.*

Proof. One can identify the boundary of $\mathcal{H} \cong \{(z, r) \in \mathbb{C} \times \mathbb{R} \mid r > 0\}$ with $\mathbb{C}P^1 \cong \mathbb{C}U_\infty \cong \{r = 0\} \cup \infty$. By viewing the singular points as part of the boundary, we arrive at an upper half-space model of $\widehat{\mathcal{H}}$. Now consider $\mathcal{H}_1 := \{(z, r) \in \widehat{\mathcal{H}} : r \geq 1\}$ with the subspace topology of $\widehat{\mathcal{H}}$. The idea of the proof is to consider a vertical retraction onto \mathcal{H}_1 , and to show by an explicit argument that preimages of open sets are open. Flöge [1111; 11, Korollar 5.8] suggests using the map

$$\phi : \widehat{\mathcal{H}} \times [0, 1] \rightarrow \widehat{\mathcal{H}}, ((z, r), t) \mapsto \begin{cases} (z, r) & \text{for all } t \in [0, 1], & \text{if } r \geq 1 \\ (z, r + t(1 - r)), & & \text{if } r < 1. \end{cases}$$

Let us now check that this is a continuous family of continuous maps. Consider the collection of open balls with respect to the Euclidean metric on $\mathbb{C} \times \mathbb{R}_+$ as soon as they are either contained in $\mathbb{C} \times \mathbb{R}_+^*$, or touch the boundary $\mathbb{C} \times \{0\}$ in a cusp in $\widehat{\mathcal{H}} - \mathcal{H}$. This is a basis for the topology of $\widehat{\mathcal{H}}$. Consider one such open ball \mathcal{B} , and its preimage under some ϕ_t , $t \in [0, 1)$. This either lies entirely in \mathcal{H} , and is open, or it has boundary points. In the latter case, consider the inverse of ϕ_t on $\widehat{\mathcal{H}} - \mathcal{H}_1$, given by

$$\phi_t^{-1} = \left(z, \frac{r-t}{1-t} \right)$$

if this is in $\widehat{\mathcal{H}}$. Suppose there is a cusp s with $s \in \widehat{\mathcal{H}} - \mathcal{H}$ and $\phi_t(s, 0) \in \mathcal{B}$. As \mathcal{B} is open, we find $\beta > 0$ and $\delta > 0$ such that $(s, t + \beta)$ and $(s + \delta, t)$ are in \mathcal{B} . Since

$$\begin{cases} \phi_t \left(s, \frac{\beta}{1-t} \right) = (s, t + \beta) \in \mathcal{B} \\ \phi_t(s + \delta, 0) = (s + \delta, t) \in \mathcal{B}, \end{cases}$$

we know that $(s, \frac{\beta}{1-t})$ and $(s + \delta, 0)$ are in the preimage of \mathcal{B} under ϕ_t . We deduce that the whole horosphere of Euclidean diameter $\min\{\beta, \delta\}$ touching at the cusp s is included in the preimage of \mathcal{B} . Thus each point of the preimage has a neighborhood entirely contained in the preimage, which therefore also is open. The continuity at $t = 1$ as well as the continuity in the variable t follow from very similar arguments. The space \mathcal{H}_1 is homeomorphic to $\mathbb{C} \times \mathbb{R}_+$, thus contractible. \square

2.1. The equivariant spectral sequence in group homology.

Corollary 7 gives us a contractible complex X^\bullet on which Γ acts cellularly. As a consequence, the integral homology $H_*(\Gamma; \mathbb{Z})$ can be computed as the hyperhomology $\mathbb{H}_*(\Gamma; C_\bullet(X))$ of Γ with coefficients in the cellular chain complex $C_\bullet(X)$ associated to X . This hyperhomology is computable because there is a spectral sequence as in [77; 7, VII] which is also the one used in [2020, 20]. It is the spectral sequence associated to the double complex $\Theta_\bullet^\Gamma \otimes_{\mathbb{Z}\Gamma} C_\bullet(X)$ computing the hyperhomology, where we denote by Θ_\bullet^Γ the bar resolution of the group Γ . This spectral sequence can be rewritten (see [2020; 20, 1.1]) to yield

$$E_{p,q}^1 = \bigoplus_{\sigma \in \Gamma \backslash X^p} H_q(\Gamma_\sigma; \mathbb{Z}) \implies H_{p+q}(\Gamma; \mathbb{Z}),$$

where Γ_σ denotes the stabilizer of (the chosen representative for) the p -cell σ . We have stated the above E^1 -term with trivial \mathbb{Z} -coefficients in $H_q(\Gamma_\sigma; \mathbb{Z})$, because we use a fundamental domain which is strict enough to give X a cell structure on which Γ acts without inversion of cells. We shall also make extensive use of the description of the d^1 -differential given in [2020, 20].

The technical difference to the cases of trivial class group, treated by [2020, 20], is that the stabilizers of the singular points are free abelian groups of rank two. In particular, the Γ -action on our complex X^\bullet is not a *proper action* in the sense that all stabilizers are finite. As a consequence, the resulting spectral sequence does not degenerate on the E^2 -level as it does in Schwermer and Vogtmann's cases. So we compute a nontrivial differential d^2 , making some additional use of homological algebra, in particular the below lemma and its corollary.

Remark 9. It would be possible to shift the technical difficulty away from homological algebra, using a topological modification of our complex. In our cases of class number two, there is one singular point in the fundamental domain, representing the nontrivial element of the class group. Its stabilizer is free abelian of rank two, and contributes the homology of a torus to the zeroth column of the E^2 -term of our spectral sequence: $H_1(\mathbb{Z}^2; \mathbb{Z}) \cong \mathbb{Z}^2$, $H_2(\mathbb{Z}^2; \mathbb{Z}) \cong \mathbb{Z}$ and $H_q(\mathbb{Z}^2; \mathbb{Z}) = 0$ for $q > 2$. One could modify our complex in order to make the Γ -action on it proper, by replacing each singular point by an \mathbb{R}^2 with the former stabilizer \mathbb{Z}^2 now acting properly. Then the nontriviality of our differential is equivalent to the existence of a nontrivial homology relation induced by adjoining the torus $\mathbb{R}^2/\mathbb{Z}^2$ to the fundamental domain.

The following lemma will be useful for computing our d^2 -differential in the situations where cycles for Γ_σ are given in terms of the standard resolution of Γ instead of Γ_σ . In order to state it, let Γ_σ be a finite subgroup of Γ , let M be a $\mathbb{Z}\Gamma_\sigma$ -module, and $\ell : \Gamma/\Gamma_\sigma \rightarrow \Gamma$ a set-theoretical section of the quotient map $\pi : \Gamma \rightarrow \Gamma/\Gamma_\sigma$. Further, denote the standard bar resolution of a discrete group Γ by Θ_\bullet^Γ . It will be convenient to view Θ_\bullet^Γ as a complex of $\mathbb{Z}\Gamma$ -right modules resp. $\mathbb{Z}\Gamma_\sigma$ -right modules. Thus, Θ_q^Γ is defined as the free \mathbb{Z} -module generated by the $(q+1)$ -tuples $(\gamma_0, \dots, \gamma_q)$ of elements of Γ with the action given by $(\gamma_0, \dots, \gamma_q) \cdot \gamma = (\gamma_0\gamma, \dots, \gamma_q\gamma)$ and the same boundary operator as in the left module case, namely $\partial = \sum_{i=0}^q (-1)^i d_i$ where $d_i(\gamma_0, \dots, \gamma_q) = (\gamma_0, \dots, \widehat{\gamma}_i, \dots, \gamma_q)$.

Lemma 10. *The section ℓ defines a map of $\mathbb{Z}\Gamma_\sigma$ -complexes*

$$\hat{\varepsilon}_\ell : \Theta_\bullet^\Gamma \longrightarrow \Theta_\bullet^{\Gamma_\sigma}$$

of degree zero which is a retraction of the resolution Θ_\bullet^Γ of the group Γ to the resolution $\Theta_\bullet^{\Gamma_\sigma}$ of Γ_σ . For each $\gamma \in \Gamma$, $\ell(\pi(\gamma))$ is in the same orbit of the Γ_σ -right-action on Γ as γ , so $(\ell(\pi(\gamma)))^{-1}\gamma \in \Gamma_\sigma$. The map $\hat{\varepsilon}_\ell$ is induced on $\Theta_0^\Gamma = \mathbb{Z}\Gamma$ by

$$\begin{aligned} \Gamma &\xrightarrow{\varepsilon_\ell} \Gamma_\sigma \rightarrow \mathbb{Z}\Gamma_\sigma, \\ \gamma &\mapsto (\ell(\pi(\gamma)))^{-1}\gamma \end{aligned}$$

and is continued as a tensor product $\hat{\varepsilon}_\ell = \varepsilon_\ell \otimes \dots \otimes \varepsilon_\ell = \varepsilon_\ell^{\otimes(n+1)}$ on Θ_n^Γ .

- Remark 11.**
- (1) Since the group Γ_σ acts from the right, the map ε_ℓ is $\mathbb{Z}\Gamma_\sigma$ -linear.
 - (2) Note that the resulting isomorphism in homology from $H_*(\Theta_\bullet^\Gamma \otimes_{\mathbb{Z}\Gamma_\sigma} M)$ to $H_*(\Theta_\bullet^{\Gamma_\sigma} \otimes_{\mathbb{Z}\Gamma_\sigma} M)$ is independent of the choice of ℓ , and consistent with the canonical isomorphisms of both sides with $H_*(\Gamma_\sigma; M)$.
 - (3) Note that in the above lemma, it is not necessary to require $\ell(\pi(1)) = 1$. This would imply that ε_ℓ is the identity on $\Theta_\bullet^{\Gamma_\sigma}$. However, we will choose $\ell(\pi(1)) = 1$ for simplicity.
 - (4) In explicit terms, the map ε_ℓ is described as follows:

$$\varepsilon_\ell : \mathbb{Z}\Gamma \rightarrow \mathbb{Z}\Gamma_\sigma,$$

$$\sum_{\gamma \in \Gamma} a_\gamma \gamma = \sum_{\gamma_\sigma \in \Gamma_\sigma} \sum_{\rho \in \Gamma/\Gamma_\sigma} a_{\gamma_\sigma \ell(\rho)} \gamma_\sigma \ell(\rho) \mapsto \sum_{\gamma_\sigma \in \Gamma_\sigma} \left(\sum_{\rho \in \Gamma/\Gamma_\sigma} a_{\gamma_\sigma \ell(\rho)} \right) \gamma_\sigma,$$

where the a_γ are coefficients from \mathbb{Z} . The map ε_ℓ restricts to the identity on $\mathbb{Z}\Gamma_\sigma$ and gives an isomorphism of \mathbb{Z} -modules from $\mathbb{Z}[\ell(\rho)\Gamma_\sigma]$ to $\mathbb{Z}\Gamma_\sigma$ for every Γ_σ -orbit $\ell(\rho)\Gamma_\sigma$.

Proof (of the lemma). In fact, the statement holds for any chain map $\hat{\varepsilon}$ in the place of ε_ℓ that satisfies the following conditions. They are easily checked to hold for the maps $\hat{\varepsilon}_\ell$.

(1) $\hat{\varepsilon}$ is $\mathbb{Z}\Gamma_\sigma$ -linear.

(2) The augmentation $\Theta_0^\Gamma \rightarrow \mathbb{Z}$ is the composition of $\hat{\varepsilon}$ with the augmentation $\Theta_0^{\Gamma_\sigma} \rightarrow \mathbb{Z}$.

Then the statement follows from the comparison theorem [2525; 25, 2.2.6] of fundamental homological algebra. In fact, the properties imply that $\hat{\varepsilon}$ is a chain map of resolutions lifting the identity on \mathbb{Z} . An inverse is given by the canonical inclusion $\Theta_\bullet^\Gamma \rightarrow \Theta_\bullet^{\Gamma_\sigma}$, and since the composition is unique up to chain homotopy equivalence, it must be homotopic to the identity. \square

The group Γ_σ acts diagonally from the right on $\Theta_1^\Gamma \cong \mathbb{Z}\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma$, and trivially on \mathbb{Z} , so we can consider $\Theta_1^\Gamma \otimes_{\mathbb{Z}\Gamma_\sigma} \mathbb{Z}$. Denote the commutator quotient map $\Gamma_\sigma \rightarrow (\Gamma_\sigma)^{\mathrm{ab}} \cong \mathrm{H}_1(\Gamma_\sigma)$ by $a \mapsto \bar{a}$.

Corollary 12. *Consider a cycle for $\mathrm{H}_1(\Gamma_\sigma; \mathbb{Z})$ of the form $\sum_i (a_i \otimes_{\mathbb{Z}} b_i) \otimes_{\mathbb{Z}\Gamma_\sigma} 1 \in \Theta_1^\Gamma \otimes_{\mathbb{Z}\Gamma_\sigma} \mathbb{Z}$ where $a_i, b_i \in \mathbb{Z}\Gamma$. Assume that all a_i and b_i are elements of Γ . The ensuing homology class is then given by*

$$\sum_i \overline{\varepsilon_\ell(b_i) \varepsilon_\ell(a_i)^{-1}} \in (\Gamma_\sigma)^{\mathrm{ab}}.$$

By the linearity of the described map, this covers the general case as the cycles of the form appearing in the corollary generate the submodule of all cycles. Note that the cycle condition on $\sum_i (a_i \otimes_{\mathbb{Z}} b_i) \otimes_{\mathbb{Z}\Gamma_\sigma} 1$ says that $\sum_i (b_i - a_i) \otimes_{\mathbb{Z}\Gamma_\sigma} 1 = 0$, which means that $\sum_i a_i$ is equivalent to $\sum_i b_i$ modulo $\mathbb{Z}\Gamma_\sigma$.

Proof (of the corollary). Using lemma 10, we apply the map

$$(\varepsilon_\ell \otimes_{\mathbb{Z}} \varepsilon_\ell) \otimes_{\mathbb{Z}\Gamma_\sigma} 1 : (\mathbb{Z}\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma_\sigma} \mathbb{Z} \longrightarrow (\mathbb{Z}\Gamma_\sigma \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma_\sigma) \otimes_{\mathbb{Z}\Gamma_\sigma} \mathbb{Z}$$

to get

$$\sum (\varepsilon_\ell \otimes_{\mathbb{Z}} \varepsilon_\ell \otimes_{\mathbb{Z}\Gamma_\sigma} 1)(a_i \otimes_{\mathbb{Z}} b_i \otimes_{\mathbb{Z}\Gamma_\sigma} 1) = \sum (\varepsilon_\ell(a_i) \otimes_{\mathbb{Z}} \varepsilon_\ell(b_i)) \otimes_{\mathbb{Z}\Gamma_\sigma} 1.$$

Denote the augmentation $\mathbb{Z}\Gamma_\sigma \rightarrow \mathbb{Z}$ by ε . As $a_i \in \Gamma$, we have $\varepsilon_\ell(a_i) \in \Gamma_\sigma$ which is invertible in $\mathbb{Z}\Gamma_\sigma$, and $\varepsilon(\varepsilon_\ell(a_i)) = 1$. So the above term equals

$$\sum (1 \otimes_{\mathbb{Z}} \varepsilon_\ell(b_i) (\varepsilon_\ell(a_i))^{-1}) \otimes_{\mathbb{Z}\Gamma_\sigma} \varepsilon(\varepsilon_\ell(a_i)) = \sum (1 \otimes_{\mathbb{Z}} \varepsilon_\ell(b_i) (\varepsilon_\ell(a_i))^{-1}) \otimes_{\mathbb{Z}\Gamma_\sigma} 1,$$

where we take into account that the action of $\mathbb{Z}\Gamma_\sigma$ on $\mathbb{Z}\Gamma_\sigma \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma_\sigma$ is the diagonal right action, and that of $\mathbb{Z}\Gamma_\sigma$ on \mathbb{Z} is the trivial action $a \cdot 1 = \varepsilon(a)$ for $a \in \mathbb{Z}\Gamma_\sigma$. In bar notation, we thus obtain the cycle $\sum [\varepsilon_\ell(b_i) (\varepsilon_\ell(a_i))^{-1}] \otimes_{\mathbb{Z}\Gamma_\sigma} 1$, which is mapped to

$$\sum_i \overline{\varepsilon_\ell(b_i) \varepsilon_\ell(a_i)^{-1}} \in (\Gamma_\sigma)^{\mathrm{ab}}$$

by the map described in [77; 7, page 36]; it is easy to check that an isomorphism $\mathrm{H}_1(\Theta_\bullet^G \otimes_G \mathbb{Z}) \cong G^{\mathrm{ab}}$ is described by $(1 \otimes g) \otimes_G 1 = [g] \otimes_G 1 \mapsto \bar{g}$ also in the case where Θ_\bullet^G is acted on by G from the right. Moreover, this isomorphism is natural with respect to group inclusions. \square

Another property of the spectral sequence is that a part of it can be checked whenever the geometry of the fundamental domain and a presentation of Γ are known. As Flöge shows, an inspection of the complex X and the associated stabilizer groups and identifications yields, together with [11; 1, theorem 4.5], a presentation of Γ by means of generators and relations. We will use the presentation computed by Flöge for $m = 5, 6, 10$ and that computed by Swan [2323, 23] for $m = 15$.

Remark 13. Let us look at the low term short exact sequence

$$0 \longrightarrow E_{0,1}^\infty \longrightarrow \Gamma^{\text{ab}} \xrightarrow{\rho} E_{1,0}^\infty \longrightarrow 0$$

of the spectral sequence. We have $E_{1,0}^\infty = H_1(\Gamma \backslash X) = (\pi_1(\Gamma \backslash X))^{\text{ab}}$, and the projection ρ is the abelianization of the map $\Gamma \rightarrow \pi_1(\Gamma \backslash X)$ given as follows. Choose a fixed base point $x \in X$. For every $\gamma \in \Gamma$, choose a continuous path in X from x to γx . This gives a well-defined loop in $\Gamma \backslash X$ since X is contractible.

The abelianization of Γ can be immediately deduced from its presentation. Thus, we can compute the group $E_{0,1}^\infty = E_{0,1}^3$ as the kernel of the projection ρ and check this with the result obtained from detailed analysis of the d^2 -differential.

2.2. The homology of the finite subgroups in the Bianchi groups.

In order to compute the E^1 -term of the spectral sequence introduced in section 2.1, we will need the isomorphism classes of the homology groups of the stabilizers.

Lemma 14 (Schwermer/Vogtmann [2020, 20]). *The only isomorphism classes of finite subgroups in $\text{PSL}_2(\mathcal{O})$ are the cyclic groups of orders two and three, the trivial group, the Klein four-group $\mathcal{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the symmetric group \mathcal{S}_3 and the alternating group \mathcal{A}_4 .*

The homology with trivial \mathbb{Z} respectively \mathbb{Z}/n -coefficients, $n = 2$ or 3 , of these groups is

$$\begin{aligned} H_q(\mathbb{Z}/n; \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}/n, & q \text{ odd}, \\ 0, & q \text{ even}, q > 0; \end{cases} & H_q(\mathbb{Z}/n; \mathbb{Z}/n) &\cong \mathbb{Z}/n, q \in \mathbb{N} \cup \{0\}; \\ \\ H_q(\mathcal{D}_2; \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}, & q = 0, \\ (\mathbb{Z}/2)^{\frac{q+3}{2}}, & q \text{ odd}, \\ (\mathbb{Z}/2)^{\frac{q}{2}}, & q \text{ even}, q > 0; \end{cases} & H_q(\mathcal{D}_2; \mathbb{Z}/2) &\cong (\mathbb{Z}/2)^{q+1} & H_q(\mathcal{D}_2; \mathbb{Z}/3) &= 0, q \geq 1; \\ \\ H_q(\mathcal{S}_3; \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}/2, & q \equiv 1 \pmod{4}, \\ 0, & q \equiv 2 \pmod{4}, \\ \mathbb{Z}/6, & q \equiv 3 \pmod{4}, \\ 0, & q \equiv 0 \pmod{4}, q > 0; \end{cases} & H_q(\mathcal{S}_3; \mathbb{Z}/3) &\cong \begin{cases} \mathbb{Z}/3, & q = 0, \\ 0, & q \equiv 1 \pmod{4}, \\ 0, & q \equiv 2 \pmod{4}, \\ \mathbb{Z}/3, & q \equiv 3 \pmod{4}, \\ \mathbb{Z}/3, & q \equiv 0 \pmod{4}, q > 0; \end{cases} & H_q(\mathcal{S}_3; \mathbb{Z}/2) &\cong \mathbb{Z}/2, q \in \mathbb{N} \cup \{0\}; \\ \\ H_q(\mathcal{A}_4; \mathbb{Z}) &\cong \begin{cases} \mathbb{Z}, & q = 0, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/3, & q = 6k + 1, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2, & q = 6k + 2, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/6, & q = 6k + 3, \\ (\mathbb{Z}/2)^k, & q = 6k + 4, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6, & q = 6k + 5, \\ (\mathbb{Z}/2)^{k+1}, & q = 6k + 6. \end{cases} & H_q(\mathcal{A}_4; \mathbb{Z}/2) &\cong \begin{cases} \mathbb{Z}/2, & q = 0, \\ (\mathbb{Z}/2)^{2k}, & q = 6k + 1, \\ (\mathbb{Z}/2)^{2k+1}, & q = 6k + 2, \\ (\mathbb{Z}/2)^{2k+2}, & q = 6k + 3, \\ (\mathbb{Z}/2)^{2k+1}, & q = 6k + 4, \\ (\mathbb{Z}/2)^{2k+2}, & q = 6k + 5, \\ (\mathbb{Z}/2)^{2k+3}, & q = 6k + 6. \end{cases} & H_q(\mathcal{A}_4; \mathbb{Z}/3) &\cong \mathbb{Z}/3, q \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Using the Universal Coefficient Theorem, we see that in degrees $q \geq 1$, the homology with trivial $\mathbb{Z}/4$ -coefficients is isomorphic to the homology with trivial $\mathbb{Z}/2$ -coefficients for the groups listed above. The stabilizers of the points inside \mathcal{H} are finite and hence of the above listed types. The

stabilizers of the singular points are isomorphic to \mathbb{Z}^2 , which has homology $H_q(\mathbb{Z}^2; \mathbb{Z}) \cong \begin{cases} 0, & q \geq 3, \\ \mathbb{Z}, & q = 2, \\ \mathbb{Z}^2, & q = 1. \end{cases}$

The maps induced on homology by inclusions of the stabilizers determine the d^1 -differentials of the spectral sequence from section 2.1.

Observation 15. The three images in $H_2(\mathcal{D}_2; \mathbb{Z}/2)$ of the non-trivial element of $H_2(\mathbb{Z}/2; \mathbb{Z}/2)$ under the maps induced by the inclusions of the three order-2-subgroups of \mathcal{D}_2 are linearly independent, but

the three images of the non-trivial element of $H_2(\mathbb{Z}/2; \mathbb{Z}/4)$ are linearly dependent in $H_2(\mathcal{D}_2; \mathbb{Z}/4)$. More precisely, there is a canonical basis for $H_2(\mathcal{D}_2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^3$ coming from the resolution for \mathcal{D}_2 used by [2020, 20], associated to the decomposition $\mathcal{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. One checks by direct calculation that in this basis, the inclusions of the three order-2-subgroups in \mathcal{D}_2 induce the images

$$\left\{ \mathbf{0}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \left\{ \mathbf{0}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \text{ and } \left\{ \mathbf{0}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ in } H_2(\mathcal{D}_2; \mathbb{Z}/2);$$

and in the basis coming from the same resolution used for $\mathbb{Z}/4$ -coefficients these images are

$$\left\{ \mathbf{0}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \left\{ \mathbf{0}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}, \text{ and } \left\{ \mathbf{0}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ in } H_2(\mathcal{D}_2; \mathbb{Z}/4).$$

The difference between the cases $\mathbb{Z}/2$ and $\mathbb{Z}/4$ comes from the behavior of the kernels of the differential maps.

Lemma 16 (Schwermer/Vogtmann [2020, 20]). *Let $C \in \{\mathbb{Z}\} \cup \{\mathbb{Z}/n : n = 2, 3, 4\}$. Consider group homology with trivial C -coefficients. Then the following hold.*

- (1) *Any inclusion $\mathbb{Z}/2 \rightarrow \mathcal{S}_3$ induces an injection on homology.*
- (2) *An inclusion $\mathbb{Z}/3 \rightarrow \mathcal{S}_3$ induces an injection on homology in degrees congruent to 3 or 0 mod 4, and is zero otherwise.*
- (3) *Any inclusion $\mathbb{Z}/2 \rightarrow \mathcal{D}_2$ induces an injection on homology in all degrees.*
- (4) *An inclusion $\mathbb{Z}/3 \rightarrow \mathcal{A}_4$ induces injections on homology in all degrees.*
- (5) *In the case $C \in \{\mathbb{Z}\} \cup \{\mathbb{Z}/n : n = 2, 3\}$, an inclusion $\mathbb{Z}/2 \rightarrow \mathcal{A}_4$ induces injections on homology in degrees greater than 1, and is zero on H_1 .*
In the case $C = \mathbb{Z}/4$, the same holds in homology degrees $q \neq 2$.
An inclusion $\mathbb{Z}/2 \rightarrow \mathcal{A}_4$ induces the zero map on $H_2(-; \mathbb{Z}/4)$.

Sketch of proof. Schwermer and Vogtmann prove this for $C = \mathbb{Z}$, and leave it to the reader in the case $C = \mathbb{Z}/2$. Details for the latter case can be found in [1919, 19]. In the following, we are going to give the main arguments.

- (1) This follows for all coefficients from the fact that the group extension $1 \rightarrow \mathbb{Z}/3 \rightarrow \mathcal{S}_3 \rightarrow \mathbb{Z}/2 \rightarrow 1$ has the property that any inclusion, $\mathbb{Z}/2 \rightarrow \mathcal{S}_3$, composed with its quotient map is the identity on $\mathbb{Z}/2$.
- (2) The assertion is trivial for $C \in \{\mathbb{Z}/2, \mathbb{Z}/4\}$ because then $H_q(\mathbb{Z}/3; C) = 0$ for $q \geq 1$ by the Universal Coefficient Theorem. For $C = \mathbb{Z}/3$, one computes the Lyndon/Hochschild/Serre spectral sequence with $\mathbb{Z}/3$ -coefficients associated to the extension $1 \rightarrow \mathbb{Z}/3 \rightarrow \mathcal{S}_3 \rightarrow \mathbb{Z}/2 \rightarrow 1$. Its E^2 -term $E_{p,q}^2 = H_p(\mathbb{Z}/2; H_q(\mathbb{Z}/3; \mathbb{Z}/3))$ is concentrated in the column $p = 0$; special care has to be taken with the action of $\mathbb{Z}/2$ on $H_q(\mathbb{Z}/3; \mathbb{Z}/3)$. So $E_{p,q}^2 \cong E_{p,q}^\infty$, and the assertion follows from a computation of the map $H_q(\mathbb{Z}/3; \mathbb{Z}/3) \rightarrow H_0(\mathbb{Z}/2; H_q(\mathbb{Z}/3; \mathbb{Z}/3))$, i. e. the projection onto the coinvariants.
- (3) Similar to (1), this is an immediate consequence of the fact that $1 \rightarrow \mathbb{Z}/2 \rightarrow \mathcal{D}_2 \rightarrow \mathbb{Z}/2 \rightarrow 1$ splits.
- (4) Similar to (1) and (3), this follows for all coefficients from the fact that any inclusion $\mathbb{Z}/3 \rightarrow \mathcal{A}_4$ composed with the unique quotient map $\mathcal{A}_4 \rightarrow \mathbb{Z}/3$ is an isomorphism, hence induces an isomorphism in homology.
- (5) The assertion is trivial for $C = \mathbb{Z}/3$ because then $H_q(\mathbb{Z}/2; C) = 0$ for $q \geq 1$. For $C \in \{\mathbb{Z}/2, \mathbb{Z}/4\}$, one considers the factorization of the inclusion $\mathbb{Z}/2 \rightarrow \mathcal{A}_4$ as $\mathbb{Z}/2 \rightarrow \mathcal{D}_2 \rightarrow \mathcal{A}_4$ where the first map is one out of three possible inclusions $\mathbb{Z}/2 \rightarrow \mathcal{D}_2$, denoted by α, β, γ . By (3), α, β and γ induce injections. Furthermore, one considers the spectral sequence with C -coefficients associated to the extension $1 \rightarrow \mathcal{D}_2 \rightarrow \mathcal{A}_4 \rightarrow \mathbb{Z}/3 \rightarrow 1$. Similar to the case considered in (2), the

E^2 -term $E_{p,q}^2 = H_p(\mathbb{Z}/3; H_q(\mathcal{D}_2; C))$ is concentrated in the column $p = 0$, thus $E_{p,q}^2 \cong E_{p,q}^\infty$, and the map $H_q(\mathbb{Z}/2; C) \rightarrow H_q(\mathcal{A}_4; C)$ is written as the composition

$$H_q(\mathbb{Z}/2; C) \rightarrow H_q(\mathcal{D}_2; C) \rightarrow H_0(\mathbb{Z}/3; H_q(\mathcal{D}_2; C)) \cong H_q(\mathcal{A}_4; C)$$

where the first map is α_*, β_* or γ_* and the second one is the projection onto the $\mathbb{Z}/3$ -coinvariants. From this, the statement can be directly deduced for $q \neq 2$. For the case $q = 2$, denote the generator of $H_2(\mathbb{Z}/2; C)$ by x . The action of $\mathbb{Z}/3$ on \mathcal{D}_2 comes from conjugation within \mathcal{A}_4 and permutes the three non-trivial elements. There is an automorphism ϕ of \mathcal{D}_2 given by the action of a generator of $\mathbb{Z}/3$, such that $\phi \circ \alpha = \beta$. Then $\phi_*(\alpha_*(x)) = (\phi \circ \alpha)_*(x) = \beta_*(x)$ and $\phi_*(\beta_*(x)) = \gamma_*(x)$. For $C = \mathbb{Z}/4$, observation 15 implies that $\alpha_*(x) = \gamma_*(x) - \beta_*(x) = \gamma_*(x) - (\phi^2)_*(\gamma_*(x))$, and thus $\alpha_*(x)$ is in $\text{Im}(1 - \phi_*) = \text{Im}(1 - (\phi^2)_*)$, hence is zero in the coinvariants. Therefore, the same holds for $\beta_*(x)$ and $\gamma_*(x)$, and the assertion follows. For $C = \mathbb{Z}/2$, one computes with the help of observation 15 that $\alpha_*(x) \notin \text{Im}(1 - \phi_*)$; thus, the same holds for $\beta_*(x)$ and $\gamma_*(x)$ and the assertion follows. \square

2.3. The mass formula for the equivariant Euler characteristic.

We will use the Euler characteristic to check the geometry of the quotient $\Gamma \backslash X$. Recall the following definitions and proposition, which we include for the reader's convenience.

Definition 17 (Euler characteristic). *Suppose Γ' is a torsion-free group. Then we define its Euler characteristic as*

$$\chi(\Gamma') = \sum_i (-1)^i \dim H_i(\Gamma'; \mathbb{Q}).$$

Suppose further that Γ' is a torsion-free subgroup of finite index in a group Γ . Then we define the Euler characteristic of Γ as

$$\chi(\Gamma) = \frac{\chi(\Gamma')}{[\Gamma : \Gamma']}.$$

This is well-defined because of [77; 7, IX.6.3].

Definition 18 (Equivariant Euler characteristic). *Suppose X is a Γ -complex such that*

- (1) *every isotropy group Γ_σ is of finite homological type;*
- (2) *X has only finitely many cells mod Γ .*

Then we define the Γ -equivariant Euler characteristic of X as

$$\chi_\Gamma(X) := \sum_\sigma (-1)^{\dim \sigma} \chi(\Gamma_\sigma),$$

where σ runs over the orbit representatives of cells of X .

Proposition 19 ([77; 7, IX.7.3 e']). *Suppose X is a Γ -complex such that $\chi_\Gamma(X)$ is defined. If Γ is virtually torsion-free, then Γ is of finite homological type and $\chi(\Gamma) = \chi_\Gamma(X)$.*

Let now Γ be $\text{PSL}_2(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]})$. Then the above proposition applies to X taken to be Flöge's (or still, Mendoza's) Γ -equivariant deformation retract of \mathcal{H} , because Γ is virtually torsion-free by Selberg's lemma. Using $\chi(\Gamma_\sigma) = \frac{1}{\text{card}(\Gamma_\sigma)}$ for Γ_σ finite, the fact that the singular points have stabilizer \mathbb{Z}^2 , and the torsion-free Euler characteristic

$$\chi(\mathbb{Z}^2) = \sum_i (-1)^i \text{rank}_{\mathbb{Z}}(H_i \mathbb{Z}^2) = 1 - 2 + 1 = 0,$$

we get the formula

$$\chi(\Gamma) = \sum_{\sigma} (-1)^{\dim \sigma} \frac{1}{\mathrm{card}(\Gamma_{\sigma})},$$

where σ runs over the orbit representatives of cells of X with finite stabilizers.

Proposition 20. *The Euler characteristic $\chi(\Gamma)$ vanishes.*

Remark 21. Thus, the formula

$$0 = \sum_{\sigma} (-1)^{\dim \sigma} \frac{1}{\mathrm{card}(\Gamma_{\sigma})},$$

allows to check the joint data of the geometry of the fundamental domain, cell stabilizers and cell identifications.

Proof of proposition 20. Denote by ζ_K the Dedekind ζ -function associated to the number field $K := \mathbb{Q}[\sqrt{-m}]$. Brown [77; 7, below (IX.8.7)] deduces the following from Harder's result [1313; 13, p. 453]:

$$\chi(SL_n(\mathcal{O}_K)) = \prod_{j=2}^n \zeta_K(1-j),$$

so especially we have $\chi(SL_2(\mathcal{O}_K)) = \zeta_K(-1)$. As Γ is a quotient of $SL_2(\mathcal{O}_K)$ by a group of order two, it follows [44, 4] that

$$\chi(\Gamma) = 2 \cdot \chi(SL_2(\mathcal{O}_K)) = 2 \cdot \zeta_K(-1).$$

Using the functional equation of ζ_K [1717, 17] and the fact that K has no real embeddings because it is imaginary quadratic, we get $\zeta_K(-1) = 0$. \square

Remark 22. One can prove the above proposition without using the Dedekind zeta function. This alternative proof applies to *any* cofinite arithmetically defined subgroup Γ of $\mathrm{PSL}(2, \mathbb{C})$. Let Γ' denote a torsion-free subgroup of Γ of finite index. It is the main theorem of Harder's article on the Gauss-Bonnet theorem [1313, 13] that the Euler characteristic of Γ' is its covolume with respect to the Euler-Poincaré form μ on \mathcal{H} , i. e. $\chi(\Gamma') = \int_Y d\mu$, where Y is a fundamental domain for the action of Γ' on \mathcal{H} . This extends the classical Gauss-Bonnet theorem from the theory of the Euler-Poincaré form, see [2121; 21, paragraph 3] (where the theorem is hidden as the existence assertion of the Euler-Poincaré measure) to non-cocompact but cofinite discrete subgroups. The measure μ is a fundamental datum associated to the symmetric space, without reference to any discrete group. In [2121; 21, paragraph 3,2a] it is shown that $\mu = 0$ on any odd-dimensional space. Since $\dim \mathcal{H} = 3$, we have $\chi(\Gamma') = \chi(\Gamma) = 0$.

3. COMPUTATIONS OF THE INTEGRAL HOMOLOGY OF $\mathrm{PSL}_2(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]})$

Throughout this section, the action on the homology coefficients is trivial because the stabilizers fix the cells pointwise. We mean \mathbb{Z} -coefficients wherever we do not mention the coefficients. Throughout, we label the singular point in the fundamental domain by s and use the notation

$$\otimes_{\sigma} := \otimes_{\mathbb{Z}[\Gamma_{\sigma}]}.$$

We write \mathcal{D}_2 for the Klein four group, \mathcal{S}_3 for the permutation group on three objects and \mathcal{A}_4 for the alternating group on four objects.

We have $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}) = \mathrm{PSL}_2(\mathbb{Z}[\omega])$ with $\omega := \sqrt{-m}$ in the cases $m = 5, 6, 10, 13$. The coordinates in hyperbolic space of the vertices of the fundamental domains have been computed by Bianchi [66, 6]. There, they are listed up to complex conjugation for $m = 5, 6, 15$; and for $m = 10, 13$, the reader has to divide out the reflection called *riflessione impropria* by Bianchi.

3.1. $m = 13$. We make the following definitions.

$$\begin{aligned}
A &:= \pm \begin{pmatrix} 9 & 7\omega \\ \omega & -10 \end{pmatrix}, & B &:= \pm \begin{pmatrix} -2-\omega & 2-\omega \\ 4 & 2+1\omega \end{pmatrix}, & C &:= \pm \begin{pmatrix} -1-\omega & 8-\omega \\ 3 & 1+2\omega \end{pmatrix}, \\
D &:= \pm \begin{pmatrix} 5 & 2\omega \\ \omega & -5 \end{pmatrix}, & E &:= \pm \begin{pmatrix} -\omega & 6 \\ 2 & \omega \end{pmatrix}, & J &:= \pm \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \\
S &:= \pm \begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix}, & K &:= \pm \begin{pmatrix} 11+4\omega & -17+7\omega \\ -8+\omega & -10-3\omega \end{pmatrix}, & M &:= \pm \begin{pmatrix} 4-2\omega & 12+\omega \\ 4+\omega & -4+2\omega \end{pmatrix}, \\
U &:= \pm \begin{pmatrix} 1 & \omega \\ & 1 \end{pmatrix}, & V &:= \pm \begin{pmatrix} -\omega & 6-\omega \\ 2 & 2+\omega \end{pmatrix}, & W &:= \pm \begin{pmatrix} 14-\omega & 13+6\omega \\ 2\omega & -12+\omega \end{pmatrix}, \\
P &:= V^{-1}D, & T &:= P^{-1}S^2, & R &:= TU^{-1}S^2U.
\end{aligned}$$

Vertices with the same letter in the fundamental domain displayed in figure 2 are identified by the action of Γ , for instance, y is identified with y' and y'' and so on. This yields relations between the matrices in the same way as shown by [1010, 10]. Amongst these relations, we will use $T = CKCA(CKC)^{-1}$, $V^{-1} = CAC^{-1}M$ and $S^2 = BS^{-1}BS$ in our further calculations, in particular in the computation of the d^2 -differential. Note that the 2-cell $(y'', a'', u', x', b'', v'')$ is identified with the 2-cell (y, a, u, x, b, v) , hence only one of them can be in the fundamental domain. The matrix U acts as a vertical translation by $-\omega$. Furthermore, we will use the identifications $C \cdot x' = x$, $U \cdot j' = j$, $C \cdot y' = y$ and $K \cdot z = z'$.

Amongst the edge identifications, we will use $CAC^{-1} \cdot (c, z) = (c', z)$, $V^{-1} \cdot (s, c) = (s, c')$, $CAC^{-1} \cdot (b, x) = (b', x)$, $V^{-1} \cdot (b, v) = (b', v')$, $P \cdot (y, v) = (y', v')$, $S^2 \cdot (a, y) = (a', y')$, and $B \cdot (a, u) = (a', u)$. There are seventeen orbits of vertices, which have the following stabilizers.

$$\begin{aligned}
\Gamma_o &= \langle J|J^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_a &= \langle S^{-1}BS|(S^{-1}BS)^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_b &= \Gamma_c = \langle M|M^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_u &= \langle B|B^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_v &= \langle D|D^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_f &= \langle D, E|D^2 = E^2 = (DE)^2 = 1 \rangle \cong \mathcal{D}_2, \\
\Gamma_h &= \langle E, AU^{-1}JU|E^2 = (AU^{-1}JU)^2 = (EAU^{-1}JU)^2 = 1 \rangle \cong \mathcal{D}_2, \\
\Gamma_e &= \langle A, U^{-1}JU|A^3 = (U^{-1}JU)^2 = (AU^{-1}JU)^2 = 1 \rangle \cong \mathcal{S}_3, \\
\Gamma_g &= \langle J, T|J^2 = T^3 = (JT)^2 = 1 \rangle \cong \mathcal{S}_3, \\
\Gamma_t &= \langle R, U^{-1}SU|R^2 = (U^{-1}SU)^3 = (RU^{-1}SU)^2 = 1 \rangle \cong \mathcal{S}_3, \\
\Gamma_w &= \langle B, S|B^2 = S^3 = (BS)^2 = 1 \rangle \cong \mathcal{S}_3, \\
\Gamma_j &= \langle S|S^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_x &= \Gamma_z = \langle CAC^{-1}|(CAC^{-1})^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_y &= \langle T|T^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_s &= \langle V, W|VW = WV \rangle \cong \mathbb{Z}^2.
\end{aligned}$$

There are twenty-eight orbits of edges.

The edge stabilizers isomorphic to $\mathbb{Z}/3$ are given on the chosen representatives as

$$\begin{aligned}
\Gamma_{(e, x')} &= \langle A|A^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(x, z)} &= \langle CAC^{-1}|(CAC^{-1})^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(g, y)} &= \langle T|T^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(j, w)} &= \langle S|S^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(t, j')} &= \langle U^{-1}SU|(U^{-1}SU)^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(y', z')} &= \langle KCA(KC)^{-1}|(KCA(KC)^{-1})^3 = 1 \rangle \cong \mathbb{Z}/3,
\end{aligned}$$

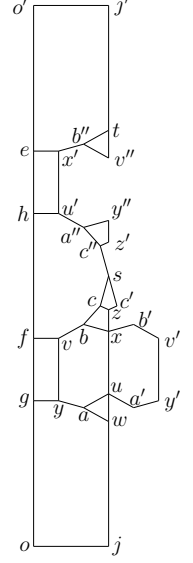


Figure 2. The fundamental domain for $m = 13$

It is given by the matrix displayed in figure 4, where $\alpha = 1$ for $q \equiv 3 \pmod{4}$ and $\alpha = 0$ else. This matrix has full rank 6 (injectivity) for $q \equiv 3 \pmod{4}$, and rank 4 (surjectivity) for $q \equiv 1 \pmod{4}$. For $q = 1$, there is an additional module $H_1(\Gamma_s) \cong \mathbb{Z}^2$ on the target side, which can not be hit because the edge stabilizers are only torsion.

Remark 23. So, the 3-torsion in $H_1(\Gamma)$ has already been killed by the d^1 differential. This is useful for showing that the map

$$H_1(\mathrm{PSL}_2(\mathbb{Z})) \rightarrow H_1(\Gamma)$$

is not injective. In fact, the matrix S of order 3 defines a non-zero element in the abelianization of $\mathrm{PSL}_2(\mathbb{Z})$ but becomes subject to the relation $S^2 = BS^{-1}BS$ in Γ where B is the matrix of order two defined above. Thus, the class of S is zero in Γ^{ab} .

3.1.3. The even rows of the E^1 -term.

There is a zero map arriving at $\bigoplus_{\sigma \in \Gamma \backslash X^0} H_q(\Gamma_\sigma) \cong (\mathbb{Z}/2)^q$ for q bigger than 2, and respectively at

$$\bigoplus_{\sigma \in \Gamma \backslash X^0} H_2(\Gamma_\sigma) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2.$$

3.1.4. The E^2 -term.

In the rows with $q \geq 2$, $E_{p,q}^2$ is concentrated in the columns $p = 0$ and $p = 1$ given as follows:

$$\begin{array}{l|ll} q = 4k + 1, & q \geq 5 & (\mathbb{Z}/2)^q & (\mathbb{Z}/3)^2 \\ q \text{ even,} & q \geq 4 & (\mathbb{Z}/2)^q & 0 \\ q = 4k + 3, & q \geq 3 & (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/2)^q & 0 \\ \dots & & \dots & \dots \\ q = 2 & & \mathbb{Z} \oplus (\mathbb{Z}/2)^2 & 0 \end{array}$$

In the rows $q = 0$ and $q = 1$, $E_{p,q}^2$ is concentrated in the columns $p = 0, 1, 2$:

$$\begin{array}{ccc} q = 1 & \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2 & (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/2 & 0 \\ & \swarrow d^2 & & \\ q = 0 & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z}^2 \end{array}$$

3.1.5. The differential d^2 .

The only nontrivial d^2 -arrow is determined on the E^0 -level by the following maps connecting $E_{2,0}^0$ with $E_{0,1}^0$:

$$\begin{array}{ccc} \bigoplus_{\sigma \in \Gamma \backslash X^0} \Theta_1 \otimes_{\sigma} \mathbb{Z} & \xleftarrow{1 \otimes \delta} & \bigoplus_{\sigma \in \Gamma \backslash X^1} \Theta_1 \otimes_{\sigma} \mathbb{Z} \\ & & \downarrow d_{\Theta} \otimes 1 \\ & & \bigoplus_{\sigma \in \Gamma \backslash X^1} \Theta_0 \otimes_{\sigma} \mathbb{Z} \xleftarrow{1 \otimes \delta} \bigoplus_{\sigma \in \Gamma \backslash X^2} \Theta_0 \otimes_{\sigma} \mathbb{Z} \end{array}$$

where d_{Θ} is the differential of the bar resolution Θ_{\bullet} for Γ , and δ is the differential of Flöge's cellular complex. The generators of the abelian group $E_{2,0}^2 \cong \mathbb{Z}^2$ are represented by the face (c, s, c', z) and the union of two faces $(b, x, b', v', y', a', u, a, y, v) =: F$, whose quotients by Γ are homeomorphic to 2-spheres.

Using the identifications stated in 3.1, we compute that the above d^2 -arrow is induced by

$$\delta((c, s, c', z)) = (CAC^{-1} - 1) \cdot (c, z) + (V^{-1} - 1) \cdot (s, c)$$

Figure 4. $(d_{1,q}^1)_{(3)}$

| | (e, x') | (g, y) | (x, z) | (y', z') | (j, w) | (t, j') |
|-----|-----------|-----------|----------|------------|----------|-----------|
| e | $-\alpha$ | 0 | 0 | 0 | 0 | 0 |
| x | 1 | 0 | -1 | 0 | 0 | 0 |
| g | 0 | $-\alpha$ | 0 | 0 | 0 | 0 |
| y | 0 | 1 | 0 | -1 | 0 | 0 |
| z | 0 | 0 | 1 | 1 | 0 | 0 |
| j | 0 | 0 | 0 | 0 | -1 | 1 |
| w | 0 | 0 | 0 | 0 | α | 0 |
| t | 0 | 0 | 0 | 0 | 0 | $-\alpha$ |

and

$$\delta((b, x, b', v', y', a', u, a, y, v)) = (CAC^{-1} - 1) \cdot (x, b) + (V^{-1} - 1) \cdot (b, v) + (P - 1) \cdot (v, y) + (S^2 - 1) \cdot (y, a) + (B - 1) \cdot (a, u).$$

The lift $1 \otimes_F 1$ in $E_{2,0}^0$ of the generator of $E_{2,0}^2$ represented by $F = (b, x, b', v', y', a', u, a, y, v)$ is mapped as follows:

$$\begin{array}{ccc} \begin{array}{l} (1, CAC^{-1}) \otimes_b 1 - (1, CAC^{-1}) \otimes_x 1 \\ + (1, V^{-1}) \otimes_v 1 - (1, V^{-1}) \otimes_b 1 \\ + (1, P) \otimes_y 1 - (1, P) \otimes_v 1 \\ + (1, S^2) \otimes_a 1 - (1, S^2) \otimes_y 1 \\ + (1, B) \otimes_u 1 - (1, B) \otimes_a 1 \end{array} & \xleftarrow{1 \otimes \delta} & \begin{array}{l} (1, CAC^{-1}) \otimes_{(x,b)} 1 \\ + (1, V^{-1}) \otimes_{(b,v)} 1 \\ + (1, P) \otimes_{(v,y)} 1 \\ + (1, S^2) \otimes_{(y,a)} 1 \\ + (1, B) \otimes_{(a,u)} 1 \end{array} \\ & & \downarrow d_{\Theta} \otimes 1 \\ & & \begin{array}{l} (CAC^{-1} - 1) \otimes_{(x,b)} 1 \\ + (V^{-1} - 1) \otimes_{(b,v)} 1 \\ + (P - 1) \otimes_{(v,y)} 1 \\ + (S^2 - 1) \otimes_{(y,a)} 1 \\ + (B - 1) \otimes_{(a,u)} 1 \end{array} \xleftarrow{1 \otimes \delta} 1 \otimes_F 1 \end{array}$$

The passage to E^1 .

We attribute the symbols t_σ to the part of this sum lying in $\Theta_1 \otimes_\sigma \mathbb{Z}$:

$$\begin{aligned} t_x &:= -(1, CAC^{-1}) \otimes_x 1, \\ t_b &:= (1, CAC^{-1}) \otimes_b 1 - (1, V^{-1}) \otimes_b 1, \\ t_v &:= (1, V^{-1}) \otimes_v 1 - (1, P) \otimes_v 1, \\ t_y &:= (1, P) \otimes_y 1 - (1, S^2) \otimes_y 1, \\ t_a &:= (1, S^2) \otimes_a 1 - (1, B) \otimes_a 1, \\ t_u &:= (1, B) \otimes_u 1. \end{aligned}$$

With the formula in our corollary 12, we find the classes \bar{t}_σ in $H_1(\Theta_* \otimes_\sigma \mathbb{Z})$ as follows:
As $V^{-1}M = CAC^{-1}$ and $\Gamma_b = \langle M \mid M^2 = 1 \rangle$,

$$t_b = [CAC^{-1}] \otimes_b 1 - [V^{-1}] \otimes_b 1 = [V^{-1}M] \otimes_b 1 - [V^{-1}] \otimes_b 1$$

gives the cycle

$$\overline{VV^{-1}M} - \overline{VV^{-1}} = \bar{M} \in \langle \bar{M} \mid 2\bar{M} = 0 \rangle \cong H_1(\Gamma_b; \mathbb{Z}).$$

As $V^{-1} = PD$ and $\Gamma_v = \langle D \mid D^2 = 1 \rangle$,

$$t_v = [V^{-1}] \otimes_v 1 - [P] \otimes_v 1 = [PD] \otimes_v 1 - [P] \otimes_v 1$$

gives the cycle

$$\overline{P^{-1}PD} - \overline{P^{-1}P} = \bar{D} \in \langle \bar{D} \mid 2\bar{D} = 0 \rangle \cong H_1(\Gamma_v; \mathbb{Z}).$$

As $S^2 = BS^{-1}BS$ and $\Gamma_a = \langle S^{-1}BS \mid (S^{-1}BS)^2 = 1 \rangle$,

$$t_a = [S^2] \otimes_a 1 - [B] \otimes_a 1 = [BS^{-1}BS] \otimes_a 1 - [B] \otimes_a 1$$

gives the cycle

$$\overline{B^{-1}BS^{-1}BS} - \overline{B^{-1}B} = \overline{S^{-1}BS} \in \langle \overline{S^{-1}BS} \mid 2\overline{S^{-1}BS} = 0 \rangle \cong H_1(\Gamma_a; \mathbb{Z}).$$

Finally, $t_u = [B] \otimes_u 1$ gives the cycle

$$\bar{B} \in \langle \bar{B} \mid 2\bar{B} = 0 \rangle \cong H_1(\Gamma_u; \mathbb{Z}).$$

The term $E_{0,1}^2$ has no 3-torsion, so the 3-torsion part $\bar{t}_x + \bar{t}_y$ of the above sum makes no contribution to the image of d^2 .

The 2-torsion part, $\bar{t}_b + \bar{t}_a + \bar{t}_v + \bar{t}_u$, equals the image

$$d_{1,1}^1(\overline{t_{(b,c)}} + \overline{t_{(c',a'')}} + \overline{t_{(v,f)}} + \overline{t_{(f,h)}} + \overline{t_{(h,u')}}),$$

where \bar{t}_σ stands for the generator of $H_1(\Gamma_\sigma; \mathbb{Z}) \cong \mathbb{Z}/2$. Thus it is a boundary and is quotiented to zero on the E^2 -page. Hence it makes no contribution either to the image of d^2 , so we obtain that $d^2(F) = 0$.

The lift $1 \otimes_{(c,s,c',z)} 1$ of the generator (c, s, c', z) is mapped as follows:

$$\begin{array}{ccc}
\begin{array}{l} (1, CAC^{-1}) \otimes_z 1 \\ -(1, CAC^{-1}) \otimes_c 1 \\ +(1, V^{-1}) \otimes_c 1 \\ -(1, V^{-1}) \otimes_s 1 \end{array} & \xleftarrow{1 \otimes \delta} & \begin{array}{l} (1, CAC^{-1}) \otimes_{(c,z)} 1 \\ +(1, V^{-1}) \otimes_{(s,c)} 1 \end{array} \\
& & \downarrow d_{\Theta \otimes 1} \\
& & \begin{array}{l} (CAC^{-1} - 1) \otimes_{(c,z)} 1 \\ +(V^{-1} - 1) \otimes_{(s,c)} 1 \end{array} \xleftarrow{1 \otimes \delta} 1 \otimes_{(c,s,c',z)} 1
\end{array}$$

The passage to E^1 .

We attribute the symbols t_σ to the part of this sum lying in $\Theta_1 \otimes_\sigma \mathbb{Z}$:

$$\begin{aligned}
t_z &:= (1, CAC^{-1}) \otimes_z 1, \\
t_c &:= (1, V^{-1}) \otimes_c 1 - (1, CAC^{-1}) \otimes_c 1, \\
t_s &:= -(1, V^{-1}) \otimes_s 1.
\end{aligned}$$

With the formula in our corollary 12, we find the classes \bar{t}_σ in $H_1(\Theta_* \otimes_\sigma \mathbb{Z})$ as follows: As $V^{-1}M = CAC^{-1}$ and $\Gamma_c = \langle M \mid M^2 = 1 \rangle$,

$$t_c = [V^{-1}] \otimes_c 1 - [CAC^{-1}] \otimes_c 1 = [V^{-1}] \otimes_c 1 - [V^{-1}M] \otimes_c 1$$

gives the cycle

$$\overline{VV^{-1}} - \overline{VV^{-1}M} = -\overline{M} \in \langle \overline{M} \mid 2\overline{M} = 0 \rangle \cong H_1(\Gamma_c; \mathbb{Z}).$$

Finally,

$$t_s = -[V^{-1}] \otimes_s 1$$

gives the cycle

$$\overline{V} \in \langle \overline{V}, \overline{W} \rangle \cong H_1(\Gamma_s; \mathbb{Z}) \cong \mathbb{Z}^2.$$

The term $E_{0,1}^2$ has no 3-torsion, so the 3-torsion part \bar{t}_z of the above sum makes no contribution to the image of d^2 .

However the 2-torsion part, $\bar{t}_c = \overline{M}$, passes to the E^2 -page because no chain of edges can have the single point c as its boundary. Furthermore, \overline{V} is one of the generators of the free part of $E_{0,1}^2 \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2$, so we obtain $d^2((c, s, c', z)) = \overline{M} + \overline{V}$, which is of infinite order and has the following property: there is no element $\eta \in E_{0,1}^2$ with $k\eta = \overline{M} + \overline{V}$ for an integer $k > 1$. As we have seen that $d^2(F) = 0$, we obtain the quotient

$$E_{0,1}^3 \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2.$$

Hence we obtain for integral homology the following short exact sequences:

$$\left\{ \begin{array}{ll} 0 \rightarrow (\mathbb{Z}/2)^q \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow (\mathbb{Z}/3)^2 \rightarrow 0, & q = 4k + 2, \\ 0 \rightarrow (\mathbb{Z}/2)^q \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow 0, & q = 4k + 1, \\ 0 \rightarrow (\mathbb{Z}/2)^q \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow 0, & q = 4k + 4, \\ 0 \rightarrow (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/2)^q \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow 0, & q = 4k + 3, \\ 0 \rightarrow \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \rightarrow H_2(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z} \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/2 \rightarrow 0, & \\ 0 \rightarrow \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \rightarrow H_1(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z}^2 \rightarrow 0. & \end{array} \right.$$

We will resolve the ambiguity in the torsion part of the group extension $H_2(\Gamma; \mathbb{Z})$ by a reflection like the one on [2020; 20, page 587], for which we have to recompute the spectral sequence with $\mathbb{Z}/2$ -, $\mathbb{Z}/3$ - and $\mathbb{Z}/4$ -coefficients. The free part is unambiguous, as we can see from tensoring with \mathbb{Q} .

3.1.6. The E^1 -term with $\mathbb{Z}/2$ -coefficients.

We can apply the functor $-\otimes \mathbb{Z}/2$ to the row $q = 0$ and obtain in the columns $p = 0, 1, 2$:

$$(\mathbb{Z}/2)^{17} \xleftarrow{d_{1,0}^1} (\mathbb{Z}/2)^{28} \xleftarrow{d_{2,0}^1} (\mathbb{Z}/2)^{12}.$$

The rest of this row are zeroes. The matrix $d_{1,0}^1$ has rank 16 and the matrix $d_{2,0}^1$ has rank 10.

In the rows with $q > 0$, the differential d^1 is given by a single arrow $d_{1,q}^1$ from

$$E_{1,q}^1 \cong (\mathrm{H}_q(\mathbb{Z}/2; \mathbb{Z}/2))^{13} \oplus (\mathrm{H}_q(\mathbb{Z}/3; \mathbb{Z}/2))^6 \cong (\mathbb{Z}/2)^{13} \text{ to}$$

$$E_{0,q}^1 \cong \mathrm{H}_q(\mathbb{Z}^2; \mathbb{Z}/2) \oplus (\mathrm{H}_q(\mathbb{Z}/2; \mathbb{Z}/2))^6 \oplus (\mathrm{H}_q(\mathcal{D}_2; \mathbb{Z}/2))^2 \oplus (\mathrm{H}_q(\mathcal{S}_3; \mathbb{Z}/2))^4,$$

and the rest of these rows are zeroes. For $q = 1$, we have $d_{1,1}^1$ of rank 12 arriving at $E_{0,1}^1 \cong (\mathbb{Z}/2)^{16}$.

For $q \geq 3$, we have $d_{1,q}^1$ of rank 13 arriving at $E_{0,q}^1 \cong (\mathbb{Z}/2)^{12+2q}$. For $q = 2$, we have $d_{1,2}^1$ of rank 13 arriving at $E_{0,2}^1 \cong (\mathbb{Z}/2)^{17}$. The only difficulty in seeing this is to compare the maps from $\mathrm{H}_q(\mathbb{Z}/2; \mathbb{Z}/2)$ to $\mathrm{H}_q(\mathcal{D}_2; \mathbb{Z}/2)$ induced by the different inclusions $\mathbb{Z}/2 \rightarrow \mathcal{D}_2$; we use observation 15 for this purpose.

3.1.7. The E^2 -term with $\mathbb{Z}/2$ -coefficients.

We obtain in the rows with $q \geq 2$ the E^2 -term concentrated in the column $p = 0$,

$$\begin{array}{l|l} q \geq 3 & (\mathbb{Z}/2)^{2q-1} \\ q = 2 & (\mathbb{Z}/2)^4, \end{array}$$

and in the rows $q = 0, q = 1$ it is concentrated in the columns $p = 0, 1, 2$:

$$\begin{array}{ccccc} q = 1 & (\mathbb{Z}/2)^4 & \mathbb{Z}/2 & 0 & \\ & & \swarrow d_{2,0}^2 & & \\ q = 0 & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2 & \end{array}$$

The differential $d_{2,0}^2$ with $\mathbb{Z}/2$ -coefficients.

The basis $\{(c, s, c', z), F\}$ of $E_{2,0}^2$ with \mathbb{Z} -coefficients induces a basis of $E_{2,0}^2$ with $\mathbb{Z}/2$ -coefficients. The Universal Coefficient Theorem yields an isomorphism from $\mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2$ to $\mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}/2)$, which we will use to transfer the elements $\bar{t}_\sigma \in \mathrm{H}_1(\Gamma_\sigma; \mathbb{Z})$ computed in subsection 3.1.5 to $\mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}/2)$.

For $d_{2,0}^2((c, s, c', z))$ the computation is as follows. As \bar{t}_c generates $\mathrm{H}_1(\Gamma_c; \mathbb{Z}) \cong \mathbb{Z}/2$, it is transferred to the generator of $\mathrm{H}_1(\Gamma_c; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Since \bar{t}_s can be completed with a second element to a \mathbb{Z} -basis of $\mathrm{H}_1(\Gamma_s; \mathbb{Z}) \cong \mathbb{Z}^2$, it is transferred to a nontrivial element of $\mathrm{H}_1(\Gamma_s; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^2$. The element \bar{t}_z vanishes because $\mathrm{H}_1(\Gamma_z; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong \mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}/2 = 0$. The sum $\bar{t}_c + \bar{t}_s$ is quotiented to a nontrivial element on the E^2 -page because $\mathrm{H}_1(\Gamma_s; \mathbb{Z}/2)$ is not hit by the d^1 -differential. So $d_{2,0}^2((c, s, c', z)) \cong \mathbb{Z}/2$.

For $d_{2,0}^2(F)$, the computation is as follows. Since the 3-torsion vanishes when tensored with $\mathbb{Z}/2$, the 3-torsion part $\bar{t}_x + \bar{t}_y$ of the sum makes no contribution to the image of d^2 . The 2-torsion part, $\bar{t}_b + \bar{t}_a + \bar{t}_v + \bar{t}_u$, equals the image

$$d_{1,1}^1(\bar{t}_{(b,c)} + \bar{t}_{(c'',a'')} + \bar{t}_{(v,f)} + \bar{t}_{(f,h)} + \bar{t}_{(h,u')}),$$

where \bar{t}_σ , $\sigma \in \{b, a, v, u, (b, c), (c'', a''), (v, f), (f, h), (h, u')\}$ is the generator of $\mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Hence it makes no contribution neither, and we obtain $d^2(F) = 0$. Thus $d_{2,0}^2$ has rank 1.

As $\mathbb{Z}/2$ -modules are vector spaces over the field with two elements \mathbb{F}_2 , the $E^3 = E^\infty$ -page yields immediately the results. We do an analogous computation with $\mathbb{Z}/3$ - and $\mathbb{Z}/4$ -coefficients and obtain

$$\dim_{\mathbb{F}_2} \mathrm{H}_q(\Gamma; \mathbb{Z}/2) = \begin{cases} 2q-1, & q \geq 3, \\ 6, & q = 2, \\ 5, & q = 1, \end{cases} \quad \dim_{\mathbb{F}_3} \mathrm{H}_q(\Gamma; \mathbb{Z}/3) = \begin{cases} 2, & q \equiv 0 \text{ or } 2 \pmod{4}, q > 2, \\ 4, & q \equiv 3 \pmod{4}, \\ 0, & q \equiv 1 \pmod{4}, q > 2; \end{cases}$$

and the exact sequence $1 \rightarrow (\mathbb{Z}/2)^5 \rightarrow H_3(\Gamma; \mathbb{Z}/4) \rightarrow \mathbb{Z}/2 \rightarrow 1$. The short exact sequence

$$1 \rightarrow \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \rightarrow H_2(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z} \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/2 \rightarrow 1$$

tells us that $H_2(\Gamma; \mathbb{Z})$ is one of the group extensions $\begin{cases} \mathbb{Z}^2 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/2)^3, \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/2)^2, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^3, \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4. \end{cases}$

Using the Universal Coefficient Theorem in the form

$$H_q(\Gamma; \mathbb{Z}/n) \cong H_q(\Gamma; \mathbb{Z}) \otimes (\mathbb{Z}/n) \oplus \text{Tor}_1^{\mathbb{Z}}(H_{q-1}(\Gamma; \mathbb{Z}), \mathbb{Z}/n)$$

with $n = 2, 3$ and 4 , we can now eliminate all the wrong answers and retain

$$H_q(\text{PSL}_2(\mathcal{O}_{-13}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2, & q = 1, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/2, & q = 2, \\ (\mathbb{Z}/2)^q \oplus (\mathbb{Z}/3)^2, & q = 4k + 3, \quad k \geq 0, \\ (\mathbb{Z}/2)^q, & q = 4k + 4, \quad k \geq 0, \\ (\mathbb{Z}/2)^q, & q = 4k + 1, \quad k \geq 1, \\ (\mathbb{Z}/2)^q \oplus (\mathbb{Z}/3)^2, & q = 4k + 2, \quad k \geq 1. \end{cases}$$

3.2. $m = 5$. We will make use of the following matrices, which agree with those in [1010, 10]:

$$\begin{aligned} A &:= \pm \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, & B &:= \pm \begin{pmatrix} -\omega & 2 \\ 2 & \omega \end{pmatrix}, & M &:= \pm \begin{pmatrix} -\omega & 4 \\ 1 & \omega \end{pmatrix}, & S &:= \pm \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \\ U &:= \pm \begin{pmatrix} 1 & \omega \\ & 1 \end{pmatrix}, & V &:= \pm \begin{pmatrix} -\omega & 2 - \omega \\ 2 & 2 + \omega \end{pmatrix}, & W &:= \pm \begin{pmatrix} 6 - \omega & 5 + 2\omega \\ 2\omega & \omega - 4 \end{pmatrix}. \end{aligned}$$

These are subject to the relations $UMU^{-1} = A$, $UWS(UW)^{-1} = S$, $WABW^{-1} = MB$ and $S = ABV$. A fundamental domain is displayed in figure 5. There are five orbits of vertices, with stabilizers

$$\begin{aligned} \Gamma_b &= \langle A, B | A^2 = B^2 = 1 \rangle \cong \mathcal{D}_2, \\ \Gamma_u &= \langle B, M | B^2 = M^2 = 1 \rangle \cong \mathcal{D}_2, \\ \Gamma_a &= \langle AB | AB^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_v &= \langle S | S^3 = 1 \rangle \cong \mathbb{Z}/3, \\ \Gamma_s &= \langle V, W | VW = WV \rangle \cong \mathbb{Z}^2 \end{aligned}$$

As in the case $m = 13$, vertices with the same letter in the fundamental domain are identified by the action of Γ . Amongst the identifications of the vertices, we will use the following. $UW \cdot a = a_1$, $V^{-1} \cdot a = a_2$, $S^2 \cdot a = a_2$, $U \cdot u = u_1$ and $UW \cdot v = v_1$. There are seven orbits of edges, with stabilizers

$$\begin{aligned} \Gamma_{(b,a)} &= \langle AB | AB^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_{(v,v_1)} &= \langle S | S^3 = 1 \rangle \cong \mathbb{Z}/3, \\ \Gamma_{(a_3,u)} &= \langle MB | MB^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_{(u,b)} &= \langle B | B^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_{(u_1,b)} &= \langle A | A^2 = 1 \rangle \cong \mathbb{Z}/2; \end{aligned}$$

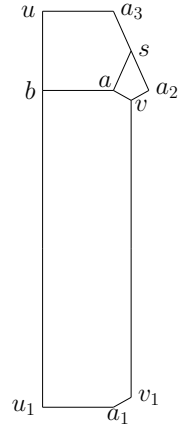


Figure 5. The fundamental domain for $m = 5$

(a, v) and (a, s) having the trivial stabilizer. There are three orbits of faces, with trivial stabilizers. The above data gives the Γ -equivariant Euler characteristic of X :

$$\chi_{\Gamma}(X) = \frac{1}{2} + \frac{1}{3} + \frac{2}{4} - 2 - \frac{4}{2} - \frac{1}{3} + 3 = 0,$$

in accordance with remark 21.

3.2.1. The bottom row of the E^1 -term.

This row identifies with the cellular chain complex of the quotient complex $\Gamma \backslash X$.

We obtain for the row $q = 0$ in the columns $p = 0, 1, 2$:

$$\mathbb{Z}^5 \xleftarrow{d_{1,0}^1} \mathbb{Z}^7 \xleftarrow{d_{2,0}^1} \mathbb{Z}^3$$

where 1 is the only elementary divisor of the differential matrices, with multiplicity four for $d_{1,0}^1$, and multiplicity two for $d_{2,0}^1$. The homology of $\Gamma \backslash X$ is generated in degree 1 by the loop represented by the edge (v, v_1) , and in degree 2 by the quotient of the face (a_2, s, a, v) , which is homeomorphic to a 2-sphere.

3.2.2. The odd rows of the E^1 -term.

We start by investigating the morphism

$$\mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^5 \xleftarrow{d_{1,1}^1} \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^4$$

and the morphism

$$\mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{q+4} \xleftarrow{d_{1,q}^1} \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^4$$

for $q \geq 3$. On the 3-torsion, $d_{1,q}^1$ is zero.

On the 2-torsion, $d_{1,q}^1$ is given by the matrix $(d_{1,q}^1)_{(2)}$ of figure 6, where we replace the $\frac{q-1}{2}$ dotted entries between the two 1's with 1's, and the $\frac{q-1}{2}$ dotted entries between the -1 's with -1 's. The rest is filled with zeroes. Thus $(d_{1,1}^1)_{(2)}$ has rank 3 and $(d_{1,q}^1)_{(2)}$ has rank 4 for $q \geq 3$.

Figure 6. $(d_{1,q}^1)_{(2)}$

| | (b, a) | (a_3, u) | (u, b) | (u_1, b) |
|----------|----------|------------|----------|------------|
| a | 1 | -1 | 0 | 0 |
| b | -1 | 0 | 0 | 1 |
| \vdots | \vdots | \vdots | \vdots | \vdots |
| b | -1 | 0 | 1 | 0 |
| u | 0 | 1 | -1 | 0 |
| \vdots | \vdots | \vdots | \vdots | \vdots |
| u | 0 | 1 | 0 | -1 |

3.2.3. The even rows of the E^1 -term.

There is a zero map arriving at $E_{0,2}^1 \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$.

For $q \geq 4$, there is a zero map arriving at $E_{0,q}^1 \cong (\mathbb{Z}/2)^q$.

The rest of the E^1 -page are zeroes.

3.2.4. The E^2 -term.

In the rows with $q \geq 2$, the E^2 -page is concentrated in the columns

$p = 0$ and $p = 1$:

$$\begin{array}{l|ll} q \geq 4 \text{ even} & (\mathbb{Z}/2)^q & 0 \\ q \geq 3 \text{ odd} & (\mathbb{Z}/2)^q \oplus \mathbb{Z}/3 & \mathbb{Z}/3 \\ q = 2 & \mathbb{Z} \oplus (\mathbb{Z}/2)^2 & 0 \end{array}$$

Its lowest two rows are concentrated in the columns $p = 0, 1, 2$:

$$\begin{array}{ccc} q = 1 & \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 & 0 \\ & \swarrow d^2 & & \\ q = 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{array}$$

Let us compute the only nontrivial d^2 -arrow. The generator of $E_{2,0}^2$ comes from the 2-cell (a_2, s, a, v) . Using the identifications listed in 3.2, we see that the lift $1 \otimes_{(a_2, s, a, v)} 1$ of the generator of $E_{2,0}^2$ is

mapped as follows in the E^0 -page:

$$\begin{array}{ccc}
\begin{array}{l} -(1, V^{-1}) \otimes_s 1 + (1, V^{-1}) \otimes_a 1 \\ +(1, S^2) \otimes_v 1 - (1, S^2) \otimes_a 1 \end{array} & \xleftarrow{1 \otimes \delta} & \begin{array}{l} -(1, V^{-1}) \otimes_{(a,s)} 1 \\ +(1, S^2) \otimes_{(a,v)} 1 \end{array} \\
& & \downarrow d_{\Theta} \otimes 1 \\
& & \begin{array}{l} 1 \otimes_{(a,s)} 1 - V^{-1} \otimes_{(a,s)} 1 \\ + S^2 \otimes_{(a,v)} 1 - 1 \otimes_{(a,v)} 1 \end{array} \xleftarrow{1 \otimes \delta} 1 \otimes_{(a_2, s, a, v)} 1
\end{array}$$

As $S = ABV$, the part lying in $\Theta_1 \otimes_a \mathbb{Z}$ is $[V^{-1}] \otimes_a 1 - [S^2] \otimes_a 1 = [S^2 AB] \otimes_a 1 - [S^2] \otimes_a 1$; and goes to $\overline{S^{-1}S^2AB} - \overline{S^{-1}S^2} = \overline{AB}$, the generator of $H_1(\Gamma_a; \mathbb{Z})$. So, our image in $E_{0,1}^0$ passes to

$$(\overline{V}, 2\overline{S}, \overline{AB}) \in \langle \overline{V}, \overline{W} \rangle \oplus \langle \overline{S} \mid 3\overline{S} = 0 \rangle \oplus (\mathbb{Z}/2)^2 \cong E_{0,1}^2,$$

which is of infinite order and has the following property: There is no element $\eta \in E_{0,1}^2$ with $k\eta = (\overline{V}, 2\overline{S}, \overline{AB})$ for an integer $k > 1$. So,

$$E_{0,1}^3 \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3.$$

Thus the E^∞ -page yields the following short exact sequences:

$$\begin{cases} 0 \rightarrow (\mathbb{Z}/2)^q \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z}/3 \rightarrow 0 & q \geq 4 \text{ even,} \\ 0 \rightarrow \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow 0 & q \geq 3 \text{ odd,} \\ 0 \rightarrow \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \rightarrow H_2(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z}/3 \oplus \mathbb{Z}/2 \rightarrow 0, \\ 0 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2 \rightarrow H_1(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow 0. \end{cases}$$

To resolve the ambiguity at the group extension $H_2(\Gamma; \mathbb{Z})$, we compute

$$\dim_{\mathbb{F}_2} H_q(\Gamma; \mathbb{Z}/2) = \begin{cases} 4 & q = 1, \\ 5 & q = 2, \\ 2q - 1 & q \geq 3 \end{cases} \quad H_q(\Gamma; \mathbb{Z}/3) \cong (\mathbb{Z}/3)^2, \quad q > 2, \quad \begin{cases} 1 \rightarrow (\mathbb{Z}/2)^5 \rightarrow H_3(\Gamma; \mathbb{Z}/4) \rightarrow \mathbb{Z}/2 \rightarrow 1, \\ 1 \rightarrow (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/4 \rightarrow H_2(\Gamma; \mathbb{Z}/4) \rightarrow \mathbb{Z}/2 \rightarrow 1; \end{cases}$$

where the last two sequences are exact; and get the result

$$H_q(\mathrm{PSL}_2(\mathcal{O}_{-5}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2 & q = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2 & q = 2, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q & q \geq 3. \end{cases}$$

Remark 24. For $m = 5$, the check introduced in remark 13 takes the following form.

The abelianization is $\Gamma^{\mathrm{ab}} \cong \langle \overline{A}, \overline{B}, \overline{S}, \overline{U}, \overline{V} : 2\overline{A} = 0, 2\overline{B} = 0, 3\overline{S} = 0 \rangle$. The fundamental group of the quotient space is free, so only the parabolic elements U and V can define nontrivial loops in the quotient space. The element U generates a nontrivial loop, while V generates a trivial loop. It follows that $E_{0,1}^\infty \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$, generated by $\overline{V}, \overline{A}, \overline{B}$ and \overline{S} . This is consistent with the computation above, involving the detailed analysis of the d^2 -differential.

3.3. $m = 10$. Let $\omega := \sqrt{-10}$. We will use the following definitions:

$$\begin{aligned}
A &:= \pm \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, & B &:= \pm \begin{pmatrix} -\omega & 3 \\ 3 & \omega \end{pmatrix}, & C &:= \pm \begin{pmatrix} -1-\omega & 4-\omega \\ 2 & 1+\omega \end{pmatrix}, & D &:= \pm \begin{pmatrix} \omega-1 & -4 \\ 3 & 1+\omega \end{pmatrix}, \\
L &:= \pm \begin{pmatrix} \omega & 3 \\ 3 & -\omega \end{pmatrix}, & R &:= \pm \begin{pmatrix} 5+\omega & 2\omega-3 \\ \omega-3 & -4-\omega \end{pmatrix}, & S &:= \pm \begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix}, & U &:= \pm \begin{pmatrix} 1 & \omega \\ & 1 \end{pmatrix}, \\
V &:= \pm \begin{pmatrix} 1-\omega & 5 \\ 2 & 1+\omega \end{pmatrix}, & W &:= \pm \begin{pmatrix} 11 & 5\omega \\ 2\omega & -9 \end{pmatrix}, & Y &:= \pm \begin{pmatrix} \omega-2 & -5 \\ 3 & 2+\omega \end{pmatrix}.
\end{aligned}$$

Vertices with the same letter are identified by the action. The matrix U acts as a vertical translation by $-\omega$ on the fundamental domain, which is shown in figure 7. There are nine orbits of vertices, labelled $a, b, r, u, v, w, x, y, s$. We have the following identifications: $UWa = a_1$, $Wa = a_2$, $Va = a_3$; $S^{-1}v = v_1$, $U^{-1}Dv = v_2$; $Dw = w_1$, $U^{-1}Dw = w_2$; $Db = b_1$, $Cb = b_2$; $Dr = r_1$;

$UWx = x_1$. On the vertices of (a, s, a_3, x) , we have the identifications $B \cdot a = a_3$ and $V \cdot a = a_3$, where the matrix B fixes x and the matrix V fixes s . For (v_1, b_2, r, b, v, w) , we have the identifications of vertices $Cb = b_2$, $Cr = r$, $S^2v = v_1$ and $S^2w = w$; and we pay particular attention to the matrix $CR = S^2AB$ identifying the edge $(b, v) \cong (b_2, v_1)$.

The stabilizers of the vertex orbit representatives are

$$\begin{aligned} \Gamma_a &= \Gamma_b = \langle R \mid R^3 = 1 \rangle \cong \mathbb{Z}/3, \\ \Gamma_w &= \langle S \mid S^3 = 1 \rangle \cong \mathbb{Z}/3, \\ \Gamma_y &= \langle A, L \mid A^2 = L^2 = (AL)^2 = 1 \rangle \cong \mathcal{D}_2, \\ \Gamma_u &= \langle A, B \mid A^2 = B^2 = (AB)^2 = 1 \rangle \cong \mathcal{D}_2, \\ \Gamma_r &= \langle C \mid C^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_v &= \langle AB \mid (AB)^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_x &= \langle B \mid B^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_s &= \langle V, W \mid VW = WV \rangle \cong \mathbb{Z}^2. \end{aligned}$$

There are fifteen orbits of edges, labelled $(b, v), (r, w), (b, r), (v, w), (a_2, w_2), (y, r_1), (x, a), (u, y), (a, b), (u, v), (a, s), (w, b_1), (r, v_2), (y, x_1), (x, u)$.

Amongst their stabilizers only

$$\begin{aligned} \Gamma_{(a_2, w_2)} &= \Gamma_{a_2} = W^{-1}\Gamma_a W = \langle W^{-1}RW \mid (W^{-1}RW)^3 = 1 \rangle \cong \mathbb{Z}/3, \\ \Gamma_{(a, b)} &= \Gamma_a = \Gamma_b = \langle R \mid R^3 = 1 \rangle \cong \mathbb{Z}/3, \\ \Gamma_{(w, b_1)} &= \Gamma_{b_1} = \Gamma_w = \langle S \mid S^3 = 1 \rangle \cong \mathbb{Z}/3, \\ \Gamma_{(y, r_1)} &= \Gamma_{r_1} = D\Gamma_r D^{-1} = \langle AL = DCD^{-1} \mid (DCD^{-1})^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_{(u, v)} &= \Gamma_v = \langle AB \mid (AB)^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_{(r, v_2)} &= \Gamma_{v_2} = \Gamma_r = \langle C \mid C^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_{(y, x_1)} &= \Gamma_{x_1} = UW\Gamma_x(UW)^{-1} = \langle L \mid L^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_{(x, u)} &= \Gamma_x = \langle B \mid B^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_{(u, y)} &= \Gamma_y = \langle A \mid A^2 = 1 \rangle \cong \mathbb{Z}/2 \end{aligned}$$

are nontrivial. Furthermore, there are seven orbits of faces, with trivial stabilizers. With the above information on the isomorphism types of the cell stabilizers, we get the Γ -equivariant Euler characteristic of X :

$$\chi_\Gamma(X) = \frac{3}{3} + \frac{2}{4} + \frac{3}{2} - \frac{3}{3} - \frac{6}{2} - 6 + 7 = 0,$$

in accordance with remark 21.

3.3.1. *The bottom row $q = 0$ of the E^1 -term.* We obtain for the row $q = 0$ in the columns $p = 0, 1, 2$:

$$\mathbb{Z}^9 \xleftarrow{d_{1,0}^1} \mathbb{Z}^{15} \xleftarrow{d_{2,0}^1} \mathbb{Z}^7,$$

where 1 is the only elementary divisor of the differential matrices, with multiplicity eight for $d_{1,0}^1$, and multiplicity five for $d_{2,0}^1$. The rest of this row is zero.

3.3.2. *The odd rows of the E^1 -term.*

For odd q , the morphism

$$\bigoplus_{\sigma \in \Gamma \backslash X^0} \mathrm{H}_q(\Gamma_\sigma) \xleftarrow{d_{1,q}^1} \bigoplus_{\sigma \in \Gamma \backslash X^1} \mathrm{H}_q(\Gamma_\sigma)$$

is for $q \geq 3$ of the form

$$(\mathbb{Z}/3)^3 \oplus (\mathbb{Z}/2)^{q+6} \leftarrow (\mathbb{Z}/3)^3 \oplus (\mathbb{Z}/2)^6.$$

For $q = 1$, we have to add $\mathrm{H}_1(\Gamma_s) \cong \mathbb{Z}^2$ on the target side of the morphism $d_{1,q}^1$, but the incoming torsion must reach it trivially. On the 3-primary part, $d_{1,q}^1$ is given by the matrix

$$(d_{1,q}^1)_{(3)} = \begin{array}{c|ccc} & (a, b) & (Db, w) & (Wa, U^{-1}Dw) \\ \hline a & -1 & 0 & -1 \\ w & 0 & 1 & 1 \\ b & 1 & -1 & 0. \end{array}$$

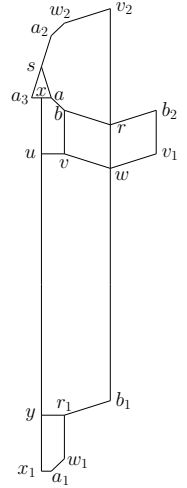


Figure 7. The fundamental domain for $m = 10$

This matrix has rank 2, so its image is isomorphic to $(\mathbb{Z}/3)^2$ and its kernel is $\mathbb{Z}/3$. On the 2-primary part, $d_{1,q}^1$ is for odd q given by the matrix

$$(d_{1,q}^1)_{(2)} = \begin{array}{c|cccccc} & (y, r_1) & (u, v) & (r, v_2) & (y, x_1) & (x, u) & (u, y) \\ \hline u & 0 & -1 & 0 & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u & 0 & -1 & 0 & 0 & 1 & 0 \\ y & -1 & 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y & -1 & 0 & 0 & -1 & 0 & 0 \\ x & 0 & 0 & 0 & 1 & -1 & 0 \\ r & 1 & 0 & -1 & 0 & 0 & 0 \\ v & 0 & 1 & 1 & 0 & 0 & 0, \end{array}$$

where we replace, as in the computation for $m = 13$, the $\frac{q-1}{2}$ dotted entries between the two 1's with 1's, and the $\frac{q-1}{2}$ dotted entries between the -1's with -1's. The rest is filled with zeroes. The resulting matrix $(d_{1,q}^1)_{(2)}$ has rank 5 for $q = 1$, and full rank 6 for $q \geq 3$.

3.3.3. The even rows of the E^1 -term.

These rows are given by zero maps into $\bigoplus_{\sigma \in \Gamma \setminus X^0} H_q(\Gamma_\sigma) \cong (\mathbb{Z}/2)^q$ for $q > 2$, respectively into

$$\bigoplus_{\sigma \in \Gamma \setminus X^0} H_2(\Gamma_\sigma) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \text{ for } q = 2.$$

3.3.4. The E^2 -term.

In the rows with $q \geq 2$, the E^2 -page is concentrated in the columns $p = 0$ and $p = 1$:

$$\begin{array}{l|ll} q \geq 4 \text{ even} & (\mathbb{Z}/2)^q & 0 \\ q \geq 3 \text{ odd} & (\mathbb{Z}/2)^q \oplus \mathbb{Z}/3 & \mathbb{Z}/3 \\ q = 2 & \mathbb{Z} \oplus (\mathbb{Z}/2)^2 & 0 \end{array}$$

Its lowest two rows are concentrated in the columns $p = 0, 1, 2$:

$$\begin{array}{ccc} q = 1 & \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 & 0 \\ & \swarrow d^2 & \searrow & \\ q = 0 & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z}^2 \end{array}$$

3.3.5. The differential d^2 .

The generators of the abelian group $E_{2,0}^2 \cong \mathbb{Z}^2$ are represented by the 2-cell (a, s, a_3, x) and the union of two 2-cells (v_1, b_2, r, b, v, w) , whose quotients by Γ are homeomorphic to 2-spheres. Using the identifications given in 3.3, we see that the only nontrivial d^2 -arrow is induced by

$$\delta((a, s, a_3, x)) = (a, s) + V \cdot (s, a) + B \cdot (a, x) + (x, a)$$

and

$$\delta((v_1, b_2, r, b, v, w)) = (b, r) - C \cdot (b, r) + CR \cdot (b, v) + S^2 \cdot (v, w) - (v, w) - (b, v).$$

The lift $1 \otimes_{(v_1, b_2, r, b, v, w)} 1$ of the generator obtained from (v_1, b_2, r, b, v, w) is mapped as follows:

$$\begin{array}{ccc} (C, 1) \otimes_r 1 - (C, 1) \otimes_b 1 & & (C, 1) \otimes_{(b, r)} 1 \\ + (1, CR) \otimes_v 1 - (1, CR) \otimes_b 1 & \xleftarrow{1 \otimes \delta} & + (1, CR) \otimes_{(b, v)} 1 \\ + (1, S^2) \otimes_w 1 - (1, S^2) \otimes_v 1 & & + (1, S^2) \otimes_{(v, w)} 1 \\ & & \downarrow d_{\Theta} \otimes 1 \\ & & 1 \otimes_{(b, r)} 1 - C \otimes_{(b, r)} 1 \\ & & + CR \otimes_{(b, v)} 1 - 1 \otimes_{(b, v)} 1 \\ & & + S^2 \otimes_{(v, w)} 1 - 1 \otimes_{(v, w)} 1 \end{array} \xleftarrow{1 \otimes \delta} 1 \otimes_{(v_1, b_2, r, b, v, w)} 1$$

We obtain $d_{2,0}^2(\langle\langle(v_1, b_2, r, b, v, w)\rangle\rangle) \cong \mathbb{Z}/3$.

The lift $1 \otimes_{(a,s,a_3,x)} 1$ of the generator obtained from (a, s, a_3, x) is mapped

$$\begin{array}{ccc} (V, 1) \otimes_s 1 - (V, 1) \otimes_a 1 & \xleftarrow{1 \otimes \delta} & (V, 1) \otimes_{(a,s)} 1 \\ + (1, B) \otimes_x 1 - (1, B) \otimes_a 1 & & + (1, B) \otimes_{(a,x)} 1 \\ & & \downarrow d_{\Theta \otimes 1} \\ & & 1 \otimes_{(a,s)} 1 - V \otimes_{(a,s)} 1 \\ & & + B \otimes_{(a,x)} 1 - 1 \otimes_{(a,x)} 1 \end{array} \xleftarrow{1 \otimes \delta} 1 \otimes_{(a,s,a_3,x)} 1$$

We attribute the symbols t_σ to the part of this sum lying in $\Theta_1 \otimes_\sigma \mathbb{Z}$,

$$\begin{aligned} t_s &:= (V, 1) \otimes_s 1, \\ t_x &:= (1, B) \otimes_x 1, \\ t_a &:= -(V, 1) \otimes_a 1 - (1, B) \otimes_a 1. \end{aligned}$$

We find the class $\bar{t}_s = -\bar{V} \in \langle\bar{V}, \bar{W}\rangle = \Gamma_s^{\mathrm{ab}} \cong H_1(\Gamma_s; \mathbb{Z}) \cong \mathbb{Z}^2$, which is a generator of the free part of $E_{0,1}^1$. It can not be the image of a torsion element from $E_{1,1}^1 = (\mathbb{Z}/3)^3 \oplus (\mathbb{Z}/2)^2$. Therefore, it is preserved when passing from $E_{0,1}^1$ to $E_{0,1}^2$. The cycles \bar{t}_x and \bar{t}_a are torsion, so the fact that \bar{t}_s is a generator of the free part determines that the image $d_{2,0}^2(\langle\langle(a, s, a_3, x)\rangle\rangle)$ is of infinite order and has the following property: There is no element $\eta \in E_{0,1}^2 \cong \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2$ with $k\eta = d_{2,0}^2(\langle\langle(a, s, a_3, x)\rangle\rangle)$ for an integer $k > 1$. Together with the isomorphism $d_{2,0}^2(\langle\langle(v_1, b_2, r, b, v, w)\rangle\rangle) \cong \mathbb{Z}/3$, we obtain

$$E_{0,1}^3 \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2.$$

Thus the E^∞ -page gives the following short exact sequences:

$$\begin{cases} 0 \rightarrow (\mathbb{Z}/2)^q \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z}/3 \rightarrow 0, & \text{for } q \geq 4 \text{ even,} \\ 0 \rightarrow \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow 0, & \text{for } q \geq 3 \text{ odd,} \\ 0 \rightarrow \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \rightarrow H_2(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2 \rightarrow 0, \\ 0 \rightarrow \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \rightarrow H_1(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z}^2 \rightarrow 0. \end{cases}$$

Therefore, there is ambiguity in the 3-torsion and the 2-torsion of the short exact sequence for $H_2(\Gamma; \mathbb{Z})$. To identify the correct group extension, we compute

$$\dim_{\mathbb{F}_2} H_q(\Gamma; \mathbb{Z}/2) \cong \begin{cases} 2q - 1, & q \geq 3 \\ 6, & q = 2, \\ 5, & q = 1. \end{cases}$$

Furthermore, we find $H_q(\Gamma; \mathbb{Z}/3) \cong (\mathbb{Z}/3)^2$ for all $q \geq 3$ and the exact sequence $1 \rightarrow (\mathbb{Z}/2)^5 \rightarrow H_3(\Gamma; \mathbb{Z}/4) \rightarrow \mathbb{Z}/2 \rightarrow 1$.

From here, we easily see the results,

$$H_q(\mathrm{PSL}_2(\mathcal{O}_{-10}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2, & q = 1, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 2, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q, & q \geq 3; \end{cases}$$

Remark 25. For $m = 10$, the check introduced in remark 13 takes the following form. The abelianization is the group $\Gamma^{\mathrm{ab}} \cong \langle\bar{A}, \bar{B}, \bar{D}, \bar{U}, \bar{W} : 2\bar{A} = 2\bar{B} = 0\rangle$. The elements of infinite order are D , U and W . The elements U and $U^{-1}D$ give the cycles generating $H_1(\Gamma \backslash X)$, while W generates a trivial loop. So it follows that $E_{0,1}^\infty = \mathbb{Z} \oplus (\mathbb{Z}/2)^2$, generated by \bar{W} , \bar{A} and \bar{B} . This is consistent with the computation above.

3.4. $m = 6$. We obtain the fundamental domain for $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-6}])$ displayed in figure 8. The matrix $U := \pm \begin{pmatrix} 1 & \omega \\ & 1 \end{pmatrix}$ performs a vertical translation by $-\omega$ of the fundamental domain. The following matrices occur in the cell stabilizers.

$$\begin{aligned} A &:= \pm \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, & B &:= \pm \begin{pmatrix} -1-\omega & 2-\omega \\ 2 & 1+\omega \end{pmatrix}, & R &:= \pm \begin{pmatrix} -\omega & 5-\omega \\ 1 & 1+\omega \end{pmatrix}, \\ S &:= \pm \begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix}, & V &:= \pm \begin{pmatrix} 1-\omega & 3 \\ 3 & 1+\omega \end{pmatrix}, & W &:= \pm \begin{pmatrix} 7 & 3\omega \\ 2\omega & -5 \end{pmatrix}. \end{aligned}$$

There are five orbits of vertices, labelled b, a, u, v, s , with stabilizers

$$\begin{aligned} \Gamma_u &= \langle B, S \mid B^2 = S^3 = (BS)^3 = 1 \rangle \cong \mathcal{A}_4, \\ \Gamma_v &= \langle B, R \mid B^2 = R^3 = (BR)^3 = 1 \rangle \cong \mathcal{A}_4, \\ \Gamma_a &= \langle SB \mid (SB)^3 = 1 \rangle \cong \mathbb{Z}/3, \\ \Gamma_b &= \langle A \mid A^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_s &= \langle V, W \mid VW = WV \rangle \cong \mathbb{Z}^2, \end{aligned}$$

and identifications $UW \cdot a = a_1$, $W \cdot a = a_2$, $V \cdot a = a_3$, $A \cdot a = a_3$, $UW \cdot b = b_1$ and $U \cdot v = v_1$. There are seven orbits of edges, labelled (b, a) , (a, s) , (a, u) , (u, v) , (a_2, v) , (b, b_1) and (u, v_1) , amongst whose stabilizers only

$$\begin{aligned} \Gamma_{(a_2, v)} &= \langle RB \mid (RB)^3 = 1 \rangle = \Gamma_{a_2} \cong \mathbb{Z}/3, \\ \Gamma_{(u, v_1)} &= \langle S \mid S^3 = 1 \rangle \cong \mathbb{Z}/3, \\ \Gamma_{(a, u)} &= \langle SB \mid (SB)^3 = 1 \rangle = \Gamma_a \cong \mathbb{Z}/3, \\ \Gamma_{(u, v)} &= \langle B \mid B^2 = 1 \rangle \cong \mathbb{Z}/2, \\ \Gamma_{(b, b_1)} &= \langle A \mid A^2 = 1 \rangle = \Gamma_b = \Gamma_{b_1} \cong \mathbb{Z}/2 \end{aligned}$$

are nontrivial; and three orbits of faces with trivial stabilizers. The above data gives the Γ -equivariant Euler characteristic of X :

$$\chi_\Gamma(X) = \frac{2}{12} + \frac{1}{3} + \frac{1}{2} - 2 - \frac{3}{3} - \frac{2}{2} + 3 = 0,$$

in accordance with remark 21.

3.4.1. The bottom row of the E^1 -term.

We obtain in the columns $p = 0, 1, 2$:

$$\mathbb{Z}^5 \xleftarrow{d_{1,0}^1} \mathbb{Z}^7 \xleftarrow{d_{2,0}^1} \mathbb{Z}^3$$

where 1 is the only occurring elementary divisor of the differential matrices, with multiplicity four for $d_{1,0}^1$, and multiplicity two for $d_{2,0}^1$. The homology of this sequence is generated by the cycle (b, b_1) in degree one and by the face (a, s, a_3, b) in degree two.

3.4.2. The odd rows of the E^1 -term.

The map $d_{1,q}^1$ is on the 2-primary part induced by the inclusion of $\Gamma_{(u,v)} \cong \mathbb{Z}/2$ into Γ_v and Γ_u which are isomorphic to \mathcal{A}_4 . By [2020; 20, lemma 4.5(2)], every inclusion of $\mathbb{Z}/2$ into \mathcal{A}_4 induces injections on homology in degrees greater than 1, and is zero on H_1 . So the morphism

$$\mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^3 \xleftarrow{d_{1,1}^1} (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$$

has $\mathbb{Z}/2$ -rank 0 on the 2-primary part, and

$$\mathbb{Z}/3 \oplus \mathbb{Z}/2 \oplus (H_q(\mathcal{A}_4))^2 \xleftarrow{d_{1,q}^1} (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$$

in the odd rows of degree $q \geq 3$ has $\mathbb{Z}/2$ -rank 1 on the 2-primary part.

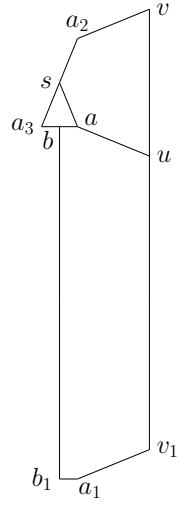


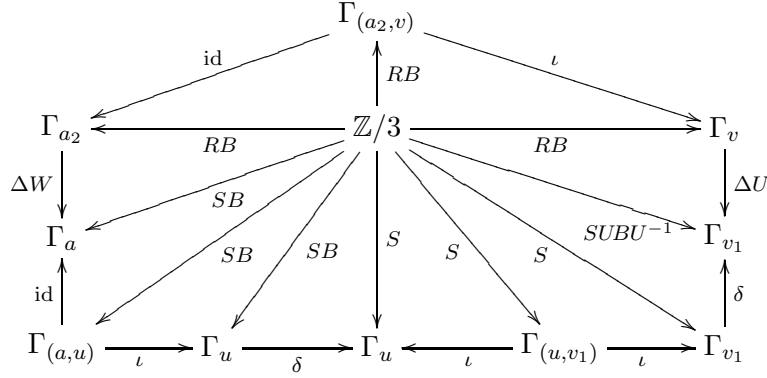
Figure 8. The fundamental domain for $m = 6$

On the 3-primary part, $d_{1,q}^1$ is for all odd q given by the following rank 2 matrix.

$$(d_{1,q}^1)_{(3)} = \begin{array}{c|ccc} & (a,u) & (a_2,v) & (u,v_1) \\ \hline a & -1 & -1 & 0 \\ u & 1 & 0 & -1 \\ v & 0 & 1 & 1. \end{array}$$

In order to determine its rank, we make use of the following facts.

First, by [2020; 20, lemma 4.5], each of the occurring group inclusions induces an injection in homology. So we have to determine the relative positions of the images coming from the edges in each direct summand over the points. In order to find out if cancellation occurs between terms with positive and negative signs, let us look at the following diagram. The symbol ΔW denotes the isomorphism given by conjugation with W , δ denotes an inner automorphism, ι denotes any canonical inclusion, and the arrows emanating from $\mathbb{Z}/3$ are labeled with the image of the canonical generator.



Applying homology H_q for odd q and taking into account that the fact that inner automorphisms act trivially on homology, we get a similar slightly smaller commutative diagram. One can then unambiguously identify all occurring groups $H_q(\mathbb{Z}/3) \cong \mathbb{Z}/3$ and its images in $H_q(\mathcal{A}_4)$ with the “abstract” $H_q(\mathbb{Z}/3) \cong \mathbb{Z}/3$ in the middle. This gives a basis for the 3-primary parts of the source and a subspace of the image. In this basis, the 3-primary map is given by the above matrix $(d_{1,q}^1)_{(3)}$, followed by an injection which does not influence the homology.

3.4.3. The even rows of the E^1 -term.

The even rows are the zero map to $E_{0,2}^1 \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$, and to $E_{0,q}^1 \cong (H_q(\mathcal{A}_4))^2$ for degree $q \geq 4$.

3.4.4. The E^2 -term.

In the rows with $q \geq 2$, the E^2 -page is concentrated in the columns $p = 0$ and $p = 1$:

$$\begin{array}{l|cc} q = 6k + 8 & (\mathbb{Z}/2)^{2k+4} & 0 \\ q = 6k + 7 & (\mathbb{Z}/2)^{2k+2} \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \\ q = 6k + 6 & (\mathbb{Z}/2)^{2k+2} & 0 \\ q = 6k + 5 & (\mathbb{Z}/2)^{2k+4} \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \\ q = 6k + 4 & (\mathbb{Z}/2)^{2k} & 0 \\ q = 6k + 3 & (\mathbb{Z}/2)^{2k+2} \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \\ q = 2 & \mathbb{Z} \oplus (\mathbb{Z}/2)^2 & 0 \end{array}$$

Its lowest two rows are concentrated in the columns $p = 0, 1, 2$:

$$\begin{array}{lccccc} q = 1 & \mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 & & (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 & & 0 \\ & & & \swarrow & & \\ q = 0 & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \end{array}$$

3.4.5. The $E^3 = E^\infty$ -term.

For the calculation of the d^2 -differential, we have

$$\begin{aligned}\delta(a, s, a_3, b) &= (a_3, s) + (s, a) + (a, b) + (b, a_3) \\ &= (V \cdot a, s) + (s, a) + (a, b) + (b, A \cdot a) \\ &= V \cdot (a, s) - (a, s) - (b, a) + A \cdot (b, a),\end{aligned}$$

$$\begin{aligned}(1 \otimes \delta)(1 \otimes_{(a,s,a_3,b)} 1) &= 1 \otimes_{V \cdot (a,s)} 1 - 1 \otimes_{(a,s)} 1 - 1 \otimes_{(b,a)} 1 + 1 \otimes_{A \cdot (b,a)} 1 \\ &= (V - 1) \otimes_{(a,s)} 1 + (A - 1) \otimes_{(b,a)} 1 \\ &= (d_\Theta \otimes 1) \left((1, V) \otimes_{(a,s)} 1 + (1, A) \otimes_{(b,a)} 1 \right) \\ &= (d_\Theta \otimes 1) \left([V] \otimes_{(a,s)} 1 + [A] \otimes_{(b,a)} 1 \right).\end{aligned}$$

We then get

$$(1 \otimes \delta) \left([V] \otimes_{(a,s)} 1 + [A] \otimes_{(b,a)} 1 \right) = [V] \otimes_s 1 - [V] \otimes_a 1 + [A] \otimes_a 1 - [A] \otimes_b 1.$$

As $[V] \otimes_s 1$ and $[W] \otimes_s 1$ represent the generators of the torsion-free part of $E_{0,1}^2 \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$, we see that the above computed element of $E_{0,1}^0$ represents an element $\nu \in E_{0,1}^2$ of infinite order with the following property: there is no element $\eta \in E_{0,1}^2$ with $k\eta = \nu$ for an integer $k > 1$. So, $E_{0,1}^3 \cong \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2$ and $E_{2,0}^3 = 0$.

3.4.6. The short exact sequences.

We thus obtain for integral homology the following short exact sequences:

$$\begin{cases} 0 \rightarrow (\mathbb{Z}/2)^{2k+4} \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z}/3 \oplus \mathbb{Z}/2 \rightarrow 0, & q = 6k + 8 \\ 0 \rightarrow (\mathbb{Z}/2)^{2k+2} \oplus \mathbb{Z}/3 \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow 0, & q = 6k + 7 \\ 0 \rightarrow (\mathbb{Z}/2)^{2k+2} \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z}/3 \oplus \mathbb{Z}/2 \rightarrow 0, & q = 6k + 6, \\ 0 \rightarrow (\mathbb{Z}/2)^{2k+4} \oplus \mathbb{Z}/3 \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow 0, & q = 6k + 5, \\ 0 \rightarrow (\mathbb{Z}/2)^{2k} \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z}/3 \oplus \mathbb{Z}/2 \rightarrow 0, & q = 6k + 4, \\ 0 \rightarrow (\mathbb{Z}/2)^{2k+2} \oplus \mathbb{Z}/3 \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow 0, & q = 6k + 3, \\ 0 \rightarrow \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \rightarrow H_2(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2 \rightarrow 0, \\ 0 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2 \rightarrow H_1(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow 0. \end{cases}$$

Thus, there is ambiguity similar to the case $m = 10$ in the 3-torsion of the short exact sequence for $H_2(\Gamma; \mathbb{Z})$ and in the 2-torsion for all even degrees. To resolve it, we compute

$$\dim_{\mathbb{F}_2} H_q(\Gamma; \mathbb{Z}/2) \cong \begin{cases} 3, & q = 1, \\ 5, & q = 2, \\ 4k + 5, & q = 6k + 3, \\ 4k + 3, & q = 6k + 4, \\ 4k + 5, & q = 6k + 5, \\ 4k + 7, & q = 6k + 6, \\ 4k + 5, & q = 6k + 7 \\ 4k + 7, & q = 6k + 8, \end{cases} \quad \begin{aligned} &H_q(\Gamma; \mathbb{Z}/3) \cong (\mathbb{Z}/3)^2 \text{ for all } q \geq 3, \\ &\text{and the exact sequences} \\ &\begin{cases} 1 \rightarrow (\mathbb{Z}/2)^4 \rightarrow H_3(\Gamma; \mathbb{Z}/4) \rightarrow (\mathbb{Z}/2)^2 \rightarrow 1, \\ 1 \rightarrow \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^3 \rightarrow H_2(\Gamma; \mathbb{Z}/4) \rightarrow (\mathbb{Z}/2)^2 \rightarrow 1. \end{cases} \end{aligned}$$

Summarizing, we have resolved the ambiguities and obtain:

$$H_q(\mathrm{PSL}_2(\mathcal{O}_{-6}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2, & q = 2, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+2}, & q = 6k + 3, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+1}, & q = 6k + 4, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+4}, & q = 6k + 5, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+3}, & q = 6k + 6, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+2}, & q = 6k + 7, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+5}, & q = 6k + 8, q \geq 8. \end{cases}$$

Remark 26. For $m = 6$, the check introduced in remark 13 takes the following form. The abelianization is $\Gamma^{\mathrm{ab}} \cong \langle \overline{A}, \overline{R}, \overline{U}, \overline{W} : 2\overline{A} = 0, 3\overline{R} = 0 \rangle$. The parabolic element U gives the cycle generating $H_1(\Gamma \backslash X)$, while the parabolic element W generates a trivial loop in the quotient space. So it follows that $E_{0,1}^\infty \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$, generated by $\overline{W}, \overline{A}$ and \overline{R} . This is consistent with the computation above.

3.5. $m = 15$. We have $\mathcal{O}_{\mathbb{Q}[\sqrt{-15}]} = \mathbb{Z}[\omega]$ with $\omega := -\frac{1}{2} + \frac{1}{2}\sqrt{-15}$. Writing $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[\omega])$ and

$$A := \pm \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad C := \pm \begin{pmatrix} 4 & -1-2\omega \\ 1+2\omega & 4 \end{pmatrix}, \quad T := \pm \begin{pmatrix} -3+\omega & -3-2\omega \\ -1-2\omega & 4 \end{pmatrix},$$

$$U := \pm \begin{pmatrix} 1 & 1+\omega \\ & 1 \end{pmatrix}, \quad V := \pm \begin{pmatrix} -1-2\omega & 3-\omega \\ 4 & 3+2\omega \end{pmatrix}, \quad W := \pm \begin{pmatrix} -1-2\omega & 4 \\ 4+\omega & -1+2\omega \end{pmatrix}, \quad S := \pm \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},$$

we have the identifications $U^{-1}A \cdot (o, c) = (o', c')$, $T \cdot (a, b') = (a', b)$, $W \cdot (s, b') = (s, b)$, and $V^{-1} \cdot (s, a) = (s, a')$ in the fundamental domain displayed in figure 9. There is no identification between the edges (b, c) and (b', c') , nor between the edges (a, o) and (a', o') . Thus the quotient by the Γ -action is homeomorphic to the sum of a Möbius band and a 2-sphere, with a disk amalgamated. There are five orbits of vertices, labelled o, a, b, c, s , with stabilizers

$$\begin{aligned} \Gamma_o &= \Gamma_a = \langle A \mid A^2 = 1 \rangle && \cong \mathbb{Z}/2, \\ \Gamma_c &= \Gamma_b = \langle S \mid S^3 = 1 \rangle && \cong \mathbb{Z}/3, \\ \Gamma_s &= \langle V, W \mid VW = WV \rangle && \cong \mathbb{Z}^2. \end{aligned}$$

There are eight orbits of edges, labelled (o, a) , (o', a') , (a, s) , (a, b') , (b, s) , (b, c) , (b', c') and (o, c) , amongst whose stabilizers only

$$\begin{aligned} \Gamma_{(o,a)} &= \langle A \mid A^2 = 1 \rangle &= \Gamma_o &= \Gamma_a \cong \mathbb{Z}/2, \\ \Gamma_{(o',a')} &= \langle V^{-1}AV \mid (V^{-1}AV)^2 = 1 \rangle &= \Gamma_{o'} &= \Gamma_{a'} \cong \mathbb{Z}/2, \\ \Gamma_{(b,c)} &= \langle S \mid S^3 = 1 \rangle &= \Gamma_b &= \Gamma_c \cong \mathbb{Z}/3, \\ \Gamma_{(b',c')} &= \langle U^{-1}ASA^{-1}U \mid (U^{-1}ASA^{-1}U)^3 = 1 \rangle &= \Gamma_{b'} &= \Gamma_{c'} \cong \mathbb{Z}/3 \end{aligned}$$

are nontrivial; and four orbits of faces with trivial stabilizers. The above data gives the Γ -equivariant Euler characteristic of X , in accordance with remark 21:

$$\chi_\Gamma(X) = \frac{2}{2} + \frac{2}{3} - 4 - \frac{2}{2} - \frac{2}{3} + 4 = 0.$$

3.5.1. *The bottom row of the E^1 -term.*

We obtain in the columns $p = 0, 1, 2$:

$$\mathbb{Z}^5 \xleftarrow{d_{1,0}^1} \mathbb{Z}^8 \xleftarrow{d_{2,0}^1} \mathbb{Z}^4$$

where 1 is the only occurring elementary divisor of the differential matrices, with multiplicity four for $d_{1,0}^1$, and multiplicity three for $d_{2,0}^1$. The homology of this sequence is generated by the cycle $(o, a) + (a, b') + (b', c') + (c', o')$ in degree one and by the cycle $(a, s, b') - (a', s, b)$ in degree two.

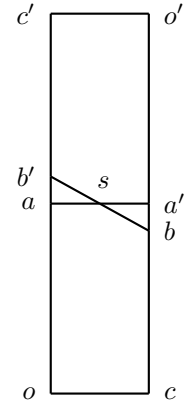


Figure 9. The fundamental domain for $m = 15$

3.5.2. The odd rows of the E^1 -term.

The maps

$$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \xleftarrow{d_{1,q}^1} (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$$

for $q \geq 3$, and

$$\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \xleftarrow{d_{1,1}^1} (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$$

are on the 2-primary part induced by the identity maps $\Gamma_{(o,a)} = \Gamma_o = \Gamma_a$ and $\Gamma_{(o',a')} = \Gamma_{o'} = \Gamma_{a'}$. So, we obtain the following rank 1 matrix for the 2-primary part:

$$(d_{1,q}^1)_{(2)} = \frac{\begin{array}{c|cc} & (o, a) & (o', a') \\ \hline a & -1 & -1 \\ o & 1 & 1 \end{array}}{}$$

On the 3-primary part, they are induced by the identity maps $\Gamma_{(b,c)} = \Gamma_b = \Gamma_c$ and $\Gamma_{(b',c')} = \Gamma_{b'} = \Gamma_{c'}$. So, we obtain the following rank 1 matrix for the 3-primary part:

$$(d_{1,q}^1)_{(3)} = \frac{\begin{array}{c|cc} & (b, c) & (b', c') \\ \hline b & -1 & -1 \\ c & 1 & 1 \end{array}}{}$$

3.5.3. The even rows of the E^1 -term.

The even rows are the zero map to $E_{0,2}^1 \cong \mathbb{Z}$, and to $E_{0,q}^1 = 0$ for $q \geq 4$.

3.5.4. The E^2 -term.

In the rows with $q \geq 2$, the E^2 -page is concentrated in the columns $p = 0$ and $p = 1$:

$$\begin{array}{l} q \geq 4 \text{ even} \\ q \geq 3 \text{ odd} \\ q = 2 \end{array} \left| \begin{array}{cc} 0 & 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \\ \mathbb{Z} & 0 \end{array} \right.$$

Its lowest two rows are concentrated in the columns $p = 0, 1, 2$:

$$\begin{array}{rcccl} q = 1 & \mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 & 0 & \\ & & \swarrow d_{2,0}^2 & & \\ q = 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \end{array}$$

3.5.5. The $E^3 = E^\infty$ -term.

For the calculation of the d^2 -differential, we have

$$\begin{aligned} \delta((a, s, b') - (a', s, b)) &= (a, s) + (s, b') + (b', a) - (a', s) - (s, b) - (b, a') \\ &= (a, s) + W^{-1} \cdot (s, b) + (b', a) - V^{-1} \cdot (a, s) - (s, b) - T \cdot (b', a), \end{aligned}$$

$$\begin{aligned} (1 \otimes \delta)(1 \otimes_{(a,s,b')-(a',s,b)} 1) &= -(V^{-1} - 1) \otimes_{(a,s)} 1 + (W^{-1} - 1) \otimes_{(s,b)} 1 - (T - 1) \otimes_{(b',a)} 1 \\ &= (d_\Theta \otimes 1) (-1, V^{-1}) \otimes_{(a,s)} 1 + (1, W^{-1}) \otimes_{(s,b)} 1 - (1, T) \otimes_{(b',a)} 1 \\ &= (d_\Theta \otimes 1) (-[V^{-1}] \otimes_{(a,s)} 1 + [W^{-1}] \otimes_{(s,b)} 1 - [T] \otimes_{(b',a)} 1). \end{aligned}$$

We then get

$$1 \otimes \delta (-[V^{-1}] \otimes_{(a,s)} 1 + [W^{-1}] \otimes_{(s,b)} 1 - [T] \otimes_{(b',a)} 1) = [V^{-1}] \otimes_a 1 - [V^{-1}] \otimes_s 1 + [W^{-1}] \otimes_b 1 - [W^{-1}] \otimes_s 1 + [T] \otimes_{b'} 1 - [T] \otimes_a 1.$$

As the generators of the torsion-free part of $E_{0,1}^2 \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$ are represented by $-[V^{-1}] \otimes_s 1$ and $-[W^{-1}] \otimes_s 1$, we see that the above computed element of $E_{0,1}^0$ represents an element $\nu \in E_{0,1}^2$ of infinite order with the following property: There is no element $\eta \in E_{0,1}^2$ with $k\eta = \nu$ for an integer $k > 1$. So, $E_{0,1}^3 \cong \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2$ and $E_{2,0}^3 = 0$.

3.5.6. The short exact sequences.

We thus obtain for integral homology the following short exact sequences:

$$\begin{cases} 0 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/3 \rightarrow H_q(\Gamma; \mathbb{Z}) \rightarrow 0, & q \geq 3, \\ 0 \rightarrow \mathbb{Z} \rightarrow H_2(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/3 \rightarrow 0, \\ 0 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 \rightarrow H_1(\Gamma; \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow 0. \end{cases}$$

Thus, there is ambiguity in the 2- and 3-torsion in $H_2(\Gamma; \mathbb{Z})$, similar to the cases $m = 10$ and $m = 6$. In order to resolve it, we only need to compute the homology with $\mathbb{Z}/2$ - and $\mathbb{Z}/3$ -coefficients,

$$H_q(\Gamma; \mathbb{Z}/2) \cong \begin{cases} (\mathbb{Z}/2)^3, & q \in \{1, 2\}, \\ (\mathbb{Z}/2)^2, & q \geq 3. \end{cases} \quad H_q(\Gamma; \mathbb{Z}/3) \cong \begin{cases} (\mathbb{Z}/3)^3, & q \in \{1, 2\}, \\ (\mathbb{Z}/3)^2, & q \geq 3. \end{cases}$$

and then use the Universal Coefficient Theorem to compare. This yields the result:

$$H_q(\mathrm{PSL}_2(\mathcal{O}_{-15}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 2, \\ \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q \geq 3. \end{cases}$$

Remark 27. For $m = 15$, the check introduced in remark 13 takes the following form. The abelianization is $\Gamma^{\mathrm{ab}} \cong \langle \overline{AS}, \overline{C}, \overline{U} : 6\overline{AS} = 0 \rangle$. The elements of infinite order U and C^{-1} give the same cycle, which generates $H_1(\Gamma \backslash X)$. However, the element $U^{-1}C^{-1}$ has infinite order as well, and generates a trivial loop in the quotient space. So it follows that $E_{0,1}^\infty \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$, generated by $\overline{U^{-1}C^{-1}}$ and \overline{AS} . This is consistent with the computation above.

4. APPENDIX: THE EQUIVARIANT RETRACTION

In this section, we give Flöge's proof of the existence of a retraction ρ from $\widehat{\mathcal{H}}$ to the cell complex X^\bullet . We do not show the fact that ρ is Γ -equivariant, which can be observed since the fibers of ρ are geodesic arcs.

Theorem 28 ([1111; 11, theorem 6.6]). *X is a retract of $\widehat{\mathcal{H}}$, i. e. there is a continuous map $\rho : \widehat{\mathcal{H}} \rightarrow X$ such that $\rho(p) = p$ for all $p \in X$.*

The map ρ is first defined as the orthogonal projection π from \widehat{B} to $\partial\widehat{B}$, and is then continued to the whole of $\widehat{\mathcal{H}}$ by Γ . Bianchi [66, 6] has shown that a nearly strict fundamental domain for the action of Γ on \mathcal{H} can be chosen in the form of a Euclidean vertical column D inside B . Define

$$\widehat{D} := \{(z, r) \in \widehat{B} \mid 0 \leq \mathrm{Re}(z) \leq 1, \quad 0 \leq \mathrm{Im}(z) \leq \sqrt{m}\},$$

and denote by S the set of singular points in \widehat{D} . Finally, $D := \widehat{D} - S$.

Remark 29 ([1111, 11], D is Γ -normal). For every $p \in \mathcal{H}$, there exists a neighborhood U of p in \mathcal{H} such that there are at most finitely many $g \in \Gamma$ with $gD \cap U \neq \emptyset$.

We will use the following lemmas to prove theorem 28.

Lemma 30 ([1111; 11, lemma 6.5]). *For any subset $A \subset D$ which is closed in \mathcal{H} and any $p \in \mathcal{H}$, there exists an open neighborhood U_p of p such that we have for all $g \in \Gamma$: $gA \cap U_p \neq \emptyset$ if and only if $p \in gA$.*

Proof. By remark 29, there is a neighborhood U of p in \mathcal{H} for which $\{g \in \Gamma \mid gD \cap U \neq \emptyset\}$ is finite. In particular, its subset

$$\Gamma_o := \{g \in \Gamma \mid gA \cap U \neq \emptyset \text{ and } p \notin gA\}$$

is finite. Therefore, A being closed, $\bigcup_{g \in \Gamma_o} gA$ is closed in \mathcal{H} . Thus $U_p := U - (\bigcup_{g \in \Gamma_o} gA)$ is open in \mathcal{H} and satisfies to the requested condition. \square

Lemma 31 ([1111; 11, lemma 6.3]). *There is an $\varepsilon_0 > 0$ such that for all singular points $s, s' \in S$, for all $\varepsilon \leq \varepsilon_0$ and $g \in \Gamma$ we have the following statement: $g\widehat{U}_\varepsilon(s) \cap \widehat{U}_\varepsilon(s') \neq \emptyset$ implies $gs = s'$.*

For class number two, as we obtain a fundamental domain for the action of Γ on $\widehat{\mathcal{H}}$ (stricter than \widehat{D}) containing just one singular point, this lemma states only that Γ acts discontinuously on $\widehat{\mathcal{H}}$ (with respect to its topology which is finer than the subset topology of \mathbb{R}^3); and we skip Flöge's proof which is useful for class number three or greater.

Lemma 32 ([1111; 11, lemma 6.4]). *There exists an $\varepsilon_1 > 0$ with the following property: If $\varepsilon \leq \varepsilon_1$ and $(z, r) \in \widehat{D}$ with $r < \varepsilon$, then there is an $s' \in S$ such that $(z, r) \in \widehat{U}_{2\varepsilon}(s')$.*

Flöge draws the sketch of the situation in a vertical half-plane, which we reproduce in figure 10 with his kind permission. He gives only some hints on the proof, which we want to make slightly more explicit here.

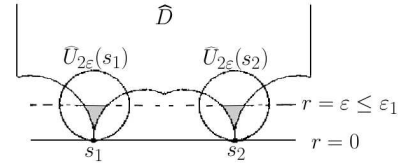


Figure 10. Flöge's sketch

Sketch of proof. We consider the Euclidean geometry of the upper-half space model for $\widehat{\mathcal{H}}$ and write coordinates in $\mathbb{C} \times \mathbb{R}^{\geq 0}$. Denote by ε_1 the “height of the lowest non-singular vertex”, more precisely the minimum of the values $r > 0$ occurring as the real coordinate of the non-singular vertices $(z, r) \in \mathcal{H}$ of the fundamental domain $\rho(\widehat{D})$ for Γ . Then $\{(z, r) \in \widehat{D} \mid r < \varepsilon_1\}$ consists of one connected component for each singular point $s' \in S$. We will denote by $\widehat{D}_{s'}$ the connected component containing s' . Now fix $s' \in S$. There are finitely many hemispheres limiting \widehat{D} from below and touching s' . We will consider the situation in a vertical half-plane containing s' . The most critical vertical half-planes for our assertion contain the intersection arc of two such hemispheres, because the other vertical half-planes contain circle segments of $\partial\widehat{D}$ of greater radius. The intersection of two non-identical Euclidean 2-spheres which have more than one point in common, is a circle with center on the line segment connecting the two 2-sphere centers. Thus the intersection of the two hemispheres mentioned above is a semicircle with center in the plane $r = 0$. Denote by ζ the radius of this semicircle. Then $\varepsilon_1 \leq \zeta$, because an edge of our fundamental domain, connecting s' with a non-singular vertex, lies on this semicircle. Now it is easy to see that $\widehat{D}_{s'}$ is a subset of the truncated cone obtained as the convex envelope of s' and the horizontal disk with radius ζ and center (s', ζ) . We conclude that for all $\varepsilon < \varepsilon_1$, $\varepsilon > 0$, the set $\{(z, r) \in \widehat{D}_{s'} \mid r < \varepsilon\}$ is a subset of the horoball $\widehat{U}_{2\varepsilon}(s')$. So we have seen that ε_1 has the property claimed in the lemma. \square

Proof of theorem 28. For any $(z, r) \in \widehat{D}$ there is a unique $r_z \geq 0$ such that $(z, r_z) \in \widehat{D} \cap \partial\widehat{B} =: \widehat{G}$, in fact $r_z = \min \{r' : (z, r') \in \widehat{D}\}$. We can thus define the map $\pi : \widehat{D} \rightarrow \widehat{G}$ by $\pi(z, r) := (z, r_z)$. The map π is continuous with respect to the subset topology of \mathbb{R}^3 , and by [1111; 11, corollary 5.10] also with respect to the topology of $\widehat{\mathcal{H}}$. Furthermore, we have $\pi(p) = p$ for all $p \in \widehat{G}$. We now extend π to a map $\rho : \widehat{\mathcal{H}} \rightarrow X$ as follows. Because of $\left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} : b \in \mathbb{R} \right\} \cdot \widehat{D} = \widehat{G}$, we find for any $p \in \widehat{\mathcal{H}}$ a $\gamma \in \Gamma$ such that $\gamma(p) \in \widehat{D}$. We set $\rho(p) := \gamma^{-1} \circ \pi \circ \gamma(p)$. In order to show that this makes sense, we have to show that $p \in \gamma^{-1}\widehat{D} \cap \xi^{-1}\widehat{D}$ implies $\gamma^{-1} \circ \pi \circ \gamma(p) = \xi^{-1} \circ \pi \circ \xi(p)$, where $\gamma, \xi \in \Gamma$. We have

$\xi(p) \in \xi\gamma^{-1}\widehat{D} \cap \widehat{D}$, then $\gamma\xi^{-1}(\xi(p)) = \gamma(p) \in \widehat{D} \cap \gamma\xi^{-1}\widehat{D}$, and either $\xi(p), \gamma(p)$ are both from \widehat{G} , or both from $\widehat{D} \cap B^\circ$. In the first case, it immediately follows that $\gamma^{-1} \circ \pi \circ \gamma(p) = \xi^{-1} \circ \pi \circ \xi(p) = p$, and $\xi^{-1} \circ \xi(p) = p$. In the second case, we have by [1111; 11, lemma 3.4] that if $\gamma\xi^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the entry c must vanish. So $\gamma\xi^{-1}$ is the product $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & db \\ 0 & 1 \end{pmatrix}$. Both of the latter two matrices commute with π since any such element ζ satisfies $\zeta(\partial\widehat{B}) = \partial\widehat{B}$, and ζ maps vertical half-lines to vertical half-lines. So we have $(\gamma\xi^{-1} \circ \pi \circ \xi\gamma^{-1})p' = \pi p'$ for all $p' \in \widehat{D}$ with $\xi\gamma^{-1}p' \in \widehat{D}$, and then it follows that

$$\xi^{-1} \circ \pi \circ \xi(p) = \gamma \in \gamma(\xi^{-1} \circ \pi \circ \xi)\gamma^{-1}\gamma(p) = \gamma^{-1} \circ \pi \circ \gamma(p) = \gamma^{-1} \circ \pi \circ \gamma(p).$$

Thus, ρ is well-defined. Furthermore, $\pi(p) = p$ for all $p \in \widehat{G}$ implies $\rho(p) = p$ for all $p \in X$. It remains to show that ρ is continuous at any $p \in \widehat{\mathcal{H}}$.

1st case. In the case $p \in \mathcal{H}$, by lemma 30, p has an open neighborhood U_p such that: for any $\gamma \in \Gamma$, we have $\gamma U_p \cap D \neq \emptyset \iff \gamma(p) \in D$. Furthermore, the set $\{\gamma \in \Gamma : \gamma(p) \in D\}$ is finite [1111; 11, remark 3.6], say $\gamma_1, \dots, \gamma_n$. Let now V be an open neighborhood of $\rho(p)$. Because of the continuity of all γ_i, γ_i^{-1} and the continuity of $\pi : \widehat{D} \rightarrow \widehat{G}$, there exist neighborhoods U_i of p such that $\gamma_i^{-1} \circ \pi \circ \gamma_i(U_i) \subset V$. Note that for all γ_i we have $\gamma_i^{-1} \circ \pi \circ \gamma_i(p) = \rho(p)$. Setting $U := U_p \cap (\bigcap_{i=1}^n U_i)$, we have $\rho(U) \subset V$, i. e. ρ is continuous at the point p .

2nd case. In the case $p \in \widehat{\mathcal{H}} \cap \mathbb{C}$, let ϵ_0, ϵ_1 and ϵ_s for $s \in S$ be positive real numbers as in lemma 31, lemma 32 and [1111; 11, lemma 5.9]; and let $\epsilon > 0$ be less than the minimum of $\frac{\epsilon_0}{2}, \epsilon_1, \epsilon_s$ for $s \in S$. Because of $\left\{ \begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix} : b \in R \right\} \cdot \widehat{D} = \widehat{G}$, there exist $s \in S, \xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\xi s = p$ and by [1111; 11, remark 5.5(a)], we have $\xi\widehat{U}_\epsilon(s) = \widehat{U}_{\frac{\epsilon}{|cs-d|^2}}(p)$. Let us now show that $\rho(\widehat{U}_{\frac{\epsilon}{|cs-d|^2}}(p)) \subset \widehat{U}_{2\epsilon}(p)$. Let $p' \in \widehat{U}_{\frac{\epsilon}{|cs-d|^2}}(p)$, and let $\gamma \in \Gamma$ with $\gamma p' \in \widehat{D}$. Then $\rho(p') = \gamma^{-1} \circ \pi \circ \gamma(p')$. By [1111; 11, remark 5.5(b)], applied to s and $\gamma\xi$ it follows that $\gamma p' = \gamma\xi(\xi^{-1}p') \in \widehat{U}_\epsilon(\gamma\xi s) = \widehat{U}_\epsilon(\gamma p)$, and by [1111; 11, remark 5.6] all conditions of lemma 32 are satisfied. So there is an $s' \in S$ such that $\gamma p' \in \widehat{U}_{2\epsilon}(s')$. This means that $\gamma\xi(\widehat{U}_{2\epsilon}(s)) \cap \widehat{U}_{2\epsilon}(s') \neq \emptyset$, and by lemma 31 it follows that $s' = \gamma\xi s = \gamma p$. Let us now consider $\gamma p'$ again.

Since $\gamma p' \in \widehat{U}_\epsilon(\gamma p) = \widehat{U}_\epsilon(s') = U_\epsilon(s')$ and $\pi(U_\epsilon(s')) \subset U_\epsilon(s')$; and by [1111; 11, lemma 5.9] we have $U_\epsilon(s') \cap \widehat{B} \subset \widehat{U}_{2\epsilon}(s')$. So $\pi \circ \gamma p' \in \widehat{U}_{2\epsilon}(s')$. By [1111; 11, remark 5.5(b)] it finally follows that

$$\rho(p') = \gamma^{-1} \circ \pi \circ \gamma p' \in \gamma^{-1}\widehat{U}_{2\epsilon}(s') \subset \widehat{U}_{2\epsilon}(\gamma^{-1}s') = \widehat{U}_{2\epsilon}(p),$$

and we are done. □

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