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► **To cite this version:**

Thierry Huillet. Random walk with long-range interaction with a barrier and its dual : Exact results. Journal of Computational and Applied Mathematics, 2010, 233, pp.2449-2467. <hal-00370353>

HAL Id: hal-00370353

<https://hal.science/hal-00370353v1>

Submitted on 24 Mar 2009

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RANDOM WALK WITH LONG-RANGE INTERACTION WITH A BARRIER AND ITS DUAL : EXACT RESULTS

THIERRY HUILLET

ABSTRACT. We consider the random walk on $\mathbb{Z}_+ = \{0, 1, \dots\}$, with up and down transition probabilities given the chain is in state $x \in \{1, 2, \dots\}$:

$$(1) \quad p_x = \frac{1}{2} \left(1 - \frac{\delta}{2x + \delta} \right) \text{ and } q_x = \frac{1}{2} \left(1 + \frac{\delta}{2x + \delta} \right).$$

Here $\delta \geq -1$ is a real tuning parameter. We assume that this random walk is reflected at the origin. For $\delta > 0$, the walker is attracted to the origin: The strength of the attraction goes like $\frac{\delta}{2x}$ for large x and so is long-ranged. For $\delta < 0$, the walker is repelled from the origin. This chain is irreducible and periodic; it is always recurrent, either positive or null recurrent.

Using Karlin-McGregor's spectral representations in terms of orthogonal polynomials and first associated orthogonal polynomials, exact expressions are obtained for first-return time probabilities to the origin (excursion length), eventual return (contact) probability, excursion height and spatial moments of the walker. All exhibit power-law decay in some range of the parameter δ . In the study, an important role is played by the Wall duality relation for birth and death chains with reflecting barrier. Some qualitative aspects of the dual random walk (obtained by interchanging p_x and q_x) are therefore also included.

Keywords: Random interfaces, birth and death random walk, long-range interaction, orthogonal polynomials, Wall duality.

MSC primary 60J10, secondary 42C05

1. INTRODUCTION

This work is the announced companion paper to the one [5] where some particular aspects of the same model were investigated, but only when $\delta \in (1, 2)$. It gives a lot more details on the qualitative behavior of this spatially inhomogeneous random walk (say RW), together with precise informations on its dual RW which is central in the understanding of the latter. Indeed, thanks to the Wall duality, the Karlin-McGregor spectral representation of this Markov chain and its dual can be made explicit, leading to exact or asymptotical results for both chains. In this birth and death chains context, the precise knowledge of the spectral measures is a remarkable fact as few explicit examples are known: The purpose of this work therefore is to take advantage of this situation and to extract more information from it.

The study includes first-return time probabilities to the origin (excursion length), eventual return (contact) probability to the origin, excursion height, time to failure and spatial moments of both direct and dual walkers.

2. PRELIMINARIES

2.1. The Model: A special random walk on \mathbb{Z}_+ . We shall consider the following discrete-time homogeneous Markov chain $(X_n; n \geq 0)$ with state-space $\mathbb{Z}_+ = \{0, 1, \dots\}$ and transition probabilities characterized by:

- given $X_n = x \in \{1, 2, \dots\}$, the increment of X_n is

$$+1 \text{ with probability : } p_x = \frac{1}{2} \left(1 - \frac{\delta}{2x + \delta} \right) = \frac{x}{2x + \delta}$$

$$-1 \text{ with probability : } q_x = \frac{1}{2} \left(1 + \frac{\delta}{2x + \delta} \right) = \frac{x + \delta}{2x + \delta}.$$

- given $X_n = 0$, the increment of X_n is $+1$ with probability $p_0 = 1$.

For this model to make sense, we impose $\left| \frac{\delta}{2x + \delta} \right| \leq 1$ for all $x \geq 1$, leading to $\delta \geq -1$.

Note that when $\delta = -1$, $p_1 = 1$ and $q_1 = 0$ and $(X_n; n \geq 0)$ is also reflected at $x = 1$. Thus, when $\delta = -1$, the return probability to the origin is zero and so the Markov chain should rather be considered on the state-space $\mathbb{Z}_+ \setminus \{0\}$.

This irreducible Markov chain is in the class of general RWs (whose transition probabilities are state-dependent), reflected at the origin. When $\delta = 0$, we get the classical fair RW reflected at 0. For $\delta > 0$, the walker is attracted to the origin: The strength of the attraction goes like $\frac{\delta}{2x}$ for large x . For $\delta < 0$, the walker is repelled from the origin correspondingly. When x approaches ∞ , the RW gets close to the familiar fair RW: this RW has 0 drift at ∞ .

One may interpret this RW as follows: Consider a length- n string of binary digits $\{0, 1\}$. Assume at time n there are $X_n = x$ active sites labeled $\{1\}$ in the n -string. We wish to decide whether site $n + 1$ is a $\{1\}$ or a $\{0\}$ so as to decide what is the number (state) X_{n+1} of type $\{1\}$ sites. Suppose that at each step, there is a probability p that one of the x active sites mutates, a probability $q = 1 - p$ that none of these sites mutates. If one of the x sites mutates, it is replaced by a random number of new auxiliary but contributing sites labeled say $\{+\}$, with expected value $\delta + 1 \geq 0$. The occurrence of a mutation is assumed to inhibit the ultimate occurrence of a $\{1\}$ in the $n + 1$ position and the transition probabilities are assumed to be proportional to the average number of contributing sites (a property, typical of urn models where balls are drawn at random). Then the new site $n + 1$ is labeled $\{1\}$ with probability proportional to qx ; it is labeled $\{0\}$ with probability proportional to $p(x - 1 + \delta + 1)$. In the latter case, the mutated site labeled $\{1\}$ has to be removed and replaced by a symbol $\{0\}$ before proceeding with the next step. Normalizing the transition probabilities $X_n \rightarrow X_{n+1} = X_n \pm 1$, we obtain

$$\begin{aligned} p_x &= qx / (qx + p(x + \delta)) = q(1 - p\delta / (x + p\delta)) \\ q_x &= p(x + \delta) / (qx + p(x + \delta)) = p(1 + q\delta / (x + p\delta)). \end{aligned}$$

This is a two-parameter version of the model (1) which is obtained in the fair mutation case: $p = 1/2$.

We shall refer to model (1) as the *special* one-parameter RW. The states $\{X_n\}$ can also represent a random polymer chain in the presence of long-range interactions with a wall or a random interface, in which case parameter δ represents an ‘‘affinity’’ constant for the wall. Although interesting, the two-parameter model ($p \neq 1/2, \delta$) is much more involved and deserves a special study which is postponed to another work.

2.2. First properties of the general RW: a reminder. Consider a general RW reflected at the origin ($p_0 = 1$ and $q_0 = 0$) for which both p_x and $q_x > 0$, for all $x \geq 1$, with $p_x + q_x = 1$. The associated stochastic transition matrix is $P = [P(x, y)]$, $(x, y) \in \mathbb{Z}_+^2$, with tri-diagonal structure: $P(0, 1) = 1$, $P(x, x+1) = p_x$, $P(x, x-1) = q_x$, $x \geq 1$ with $P(x, y) = 0$ if $y \neq (x-1, x+1)$, $x \geq 1$ and $P(0, 0) = 0$.

Let $\boldsymbol{\pi} \equiv (\pi_0, \pi_1, \dots)$ be the row-vector of the invariant measure, whenever it exists. Then $\boldsymbol{\pi}$ should solve $\boldsymbol{\pi} = \boldsymbol{\pi}P$, whose formal solution is:

$$(2) \quad \pi_x = \pi_0 \prod_{y=0}^{x-1} \frac{p_y}{q_{y+1}}, \quad x \geq 1.$$

If $S_1 \equiv \sum_{x \geq 1} \prod_{y=0}^{x-1} \frac{p_y}{q_{y+1}} < \infty$, then $\pi_0 = \frac{1}{1+S_1} \in (0, 1)$ and there is a unique proper invariant probability measure. When $S_1 = \infty$, the above measure exists but is not a probability measure as its total mass is $\pi_0(1+S_1)$ and so sums to infinity.

Consider now the same RW but assume that $p_0 = 0$, $q_0 = 1$. In this case, the state 0 is absorbing. Consider then the restriction \overline{P} of matrix P to the states $\{1, 2, \dots\}$. Let ϕ_x , $x \geq 1$ be the probabilities that state 0 is hit given the chain started originally at x . Let $\boldsymbol{\phi} \equiv (\phi_1, \phi_2, \dots)'$ be the column-vector of the absorption probabilities, with $\phi_0 = 1$. Let $\mathbf{e} \equiv (1, 0, 0, \dots)'$. Then $\boldsymbol{\phi}$ is the smallest non-negative solution to $\boldsymbol{\phi} = q_1 \mathbf{e} + \overline{P} \boldsymbol{\phi}$ whose formal solution is: $\boldsymbol{\phi} = q_1 (I - \overline{P})^{-1} \mathbf{e}$. All ϕ_x can therefore be expressed in terms of ϕ_1 , leading to:

$$(3) \quad \phi_x = 1 - (1 - \phi_1) \left(1 + \sum_{y=1}^{x-1} \prod_{z=1}^y \frac{q_z}{p_z} \right), \quad x \geq 1.$$

If $S_2 \equiv \sum_{x \geq 1} \prod_{y=1}^x \frac{q_y}{p_y} = \infty$, the restriction $\phi_x \in [0, 1]$ forces $\phi_1 = 1$ and so $\phi_x = 1$ for all $x < \infty$: State 0 is hit with probability 1, starting from x , for all $x < \infty$. The RW is recurrent.

But, if $S_2 < \infty$, then we can take $\phi_1 < 1$ so long as $\phi_x \geq 0$ for all $x \geq 1$. Then the minimal solution occurs when $1 - \phi_1 = (1 + S_2)^{-1}$, leading to:

$$(4) \quad \phi_x = \frac{\sum_{y=x}^{\infty} \prod_{z=1}^y \frac{q_z}{p_z}}{1 + \sum_{x \geq 1} \prod_{y=1}^x \frac{q_y}{p_y}}.$$

In this case, $\phi_x < 1$ for $x \geq 1$ and the absorbed random walker started at x avoids 0 with positive probability (a transience case for the original reflected RW).

To summarize, we have: (i) If $S_2 < \infty$ the RW is transient. (ii) If $S_2 = \infty$ the RW is recurrent. Moreover, it is: null recurrent if $S_1 = \infty$, positive recurrent if $S_1 < \infty$.

Due to irreducibility (because p_x and $q_x > 0$, for all $x \geq 1$), states are either all transient or recurrent.

When $S_2 = \infty$, the recurrent chain started at x first hits 0 with probability 1 and returns infinitely often to 0. Given $X_0 = x$, with $\mathcal{N}_{x,y} \equiv \sum_{n \geq 0} \mathbf{1}(X_n = y)$, the number of visits to state y , then $\mathcal{N}_{x,y} = \infty$, \mathbb{P}_x -almost surely. If $\tau_{x,x}$ is the first return time at x , then $\mathbb{P}(\tau_{x,x} < \infty) = 1$. Furthermore, with $\mathcal{N}_{x,y} \equiv \sum_{n=0}^{\tau_{x,x}} \mathbf{1}(X_n = y)$ the number of visits to state y before the first return time to state x , then: $\mathbb{E}(\mathcal{N}_{x,y}) = \frac{\pi_y}{\pi_x}$ and by the Chacon-Ornstein ergodic theorem:

$$\frac{\sum_{n=0}^N \mathbf{1}(X_n = y)}{\sum_{n=0}^N \mathbf{1}(X_n = x)} \xrightarrow{N \nearrow \infty} \frac{\pi_y}{\pi_x}, \quad \mathbb{P}_x - \text{almost surely,}$$

a result known as the limit ratio theorem.

Starting in particular from $x = 0$, a recurrent chain is made of infinitely many independent and identically distributed (iid) excursions which are the sample paths of $(X_n; n \geq 0)$ between consecutive visits to state 0. We have: $\mathbb{E}(\mathcal{N}_{0,x}) = \frac{\pi_x}{\pi_0} = \prod_{y=0}^{x-1} \frac{p_y}{q_{y+1}}$. When the chain is positive recurrent ($S_1 < \infty$) the expected time elapsed between consecutive visits to 0 is finite and equal to $\mathbb{E}(\tau_{0,0}) \equiv \mu = 1/\pi_0 = 1 + S_1$, whereas this expected time is infinite when the chain is null recurrent.

When $S_2 = \infty$, the state $x \geq 0$ is transient. Thus, $\mathcal{N}_{x,x} < \infty$, \mathbb{P}_x -almost surely and $\mathbb{P}(\mathcal{N}_{x,x} = k) = (1 - \rho_x) \rho_x^{k-1}$ where $\rho_x = \mathbb{P}(\tau_{x,x} < \infty) < 1$.

With $x \geq 1$, let $\tau_{x,0}$ be the time it takes to first hit 0, starting from $X_0 = x \geq 0$. With $x \geq 1$, we clearly have:

$$\tau_{x,0} \stackrel{d}{=} (1 - B_x)(1 + \tau_{x-1,0}) + B_x(1 + \tau_{x+1,0}),$$

where B_x is a Bernoulli random variable with $\mathbb{P}(B_x = 1) = p_x$. Therefore with $\phi_x(z) = \mathbb{E}(z^{\tau_{x,0}})$, $\phi_x(z) = q_x z \phi_{x-1}(z) + p_x z \phi_{x+1}(z)$, with initial condition $\phi_0(z) \equiv \mathbb{E}(z^{\tau_{0,0}}) = 1$. With $\phi(z) = (\phi_1(z), \phi_2(z), \dots)'$ the column-vector of the $\phi_x(z)$, $\phi(z)$ solves:

$$(5) \quad \phi(z) = q_1 z \mathbf{e} + z \bar{P} \phi(z),$$

whose formal solution is $\phi(z) = q_1 z (I - z \bar{P})^{-1} \mathbf{e}$, involving the resolvent of \bar{P} . When $z = 1$, $\phi_x(1) = \phi_x$ are the absorption times already computed.

3. THE SPECIAL RANDOM WALK

The special RW (1) deserves interest in particular because it is, to a large extent, amenable to exact analytic computations.

Assume $\delta \geq -1$ and consider then the RW determined by: $p_x = x/(2x + \delta)$ and $q_x = (x + \delta)/(2x + \delta)$, $x \geq 1$ satisfying $p_0 = 1$ and $q_0 = 0$ (reflection at the origin). Given $X_n = x$, the drift of this RW is $(x + 1)p_x + (x - 1)q_x - x = -\delta/(2x + \delta)$, showing that when $\delta > 0$, X_n is attracted by the wall where it is reflected. Although when $\delta > 0$, $\mathbb{E}(X_{n+1} | X_n = x) < x$ for all $x \geq 1$, X_n is not a convergent positive supermartingale because $\mathbb{E}(X_1 | X_0 = 0) = 1 > 0$ at the wall. For large x , the

shape of the drift goes like $-\delta/(2x)$ which is reminiscent of the one occurring in a Bessel diffusion process in continuous space-time with dimension $d = 1 - \delta$, [10]: One may indeed view the special RW as a discrete Bessel process, the scaling limit of which being the Bessel diffusion process.

Let us first consider the invariant measure of this RW: Observing that $p_y \sim \frac{1}{2}e^{-\frac{\delta}{2y}}$ and $q_y \sim \frac{1}{2}e^{\frac{\delta}{2y}}$ for large y , using (2), we get a power law behavior

$$\pi_x \sim 2\pi_0 e^{-\delta \log x} = 2\pi_0 x^{-\delta},$$

suggesting that $\boldsymbol{\pi} \equiv (\pi_x; x \in \mathbb{Z}_+)$ is a proper (summable) probability measure if and only if $\delta > 1$. Using this, we get:

Proposition 1. *When $\delta > 1$, the RW is positive recurrent; it has a unique and summable invariant measure satisfying $\pi_x \underset{x \nearrow \infty}{\sim} x^{-\delta}$, which is maximal at $x = 1$. It is given explicitly by*

$$(6) \quad \pi_0 = \frac{1}{1 + S_1} = \frac{\delta - 1}{2\delta},$$

and

$$(7) \quad \pi_x = \pi_0 \prod_{y=0}^{x-1} \frac{p_y}{q_{y+1}} = (\delta - 1) \frac{2x + \delta}{2\delta} \frac{\Gamma(\delta + 1) \Gamma(x)}{\Gamma(x + \delta + 1)}, \quad x \geq 1.$$

When $\delta > 2$,

$$(8) \quad \mathbb{E}(X_\infty) = \sum_{x \geq 1} x \pi_x = \frac{\delta}{2(\delta - 2)} \in (1/2, \infty),$$

so that the invariant measure has a finite explicit mean.

Proof: The only thing that remains to be proved is (6). The evaluation of π_0 can easily be derived from the Gauss identity involving two Gauss hypergeometric functions ${}_2F_1$. Indeed, it can easily be checked that

$$1 + S_1 = 1 + 2 \cdot ({}_2F_1(1, 1; \delta + 1; 1) - 1) + \frac{\delta}{\delta + 1} \cdot {}_2F_1(1, 1; \delta + 2; 1) = \frac{2\delta}{\delta - 1}.$$

The proof of (8) uses a similar argument. \triangle

From the general introductory reminder, when $\delta \in (-1, 1]$, the RW is null recurrent. It still is null recurrent when $\delta = -1$, but with 2 reflecting states, namely $\{0, 1\}$. In other words, considering the RW on $\mathbb{Z}_+ \setminus \{0\}$ rather than on \mathbb{Z}_+ , the critical special walker ($\delta = -1$) started at state 1 returns infinitely often to state 1. However, the expected return time is ∞ . When $\delta > 1$, the chain is positive recurrent. There is no transience case for the special RW.

3.1. Spectral theory of the special RW. We now show some computational issues making extensive use of spectral theory.

We shall first recall some aspects of the spectral theory for this RW ($\delta > -1$). Consider the polynomials in the variable $t \in [-1, 1]$, determined by the 3-term recurrence:

$$tQ_x(t) = p_x Q_{x+1}(t) + q_x Q_{x-1}(t), \quad x \geq 0; \quad Q_{-1}(t) = 0, \quad Q_0(t) = 1,$$

or, with $Q_0(t) = 1$, $Q_1(t) = t$, by:

$$(9) \quad (2x + \delta) t Q_x(t) = x Q_{x+1}(t) + (x + \delta) Q_{x-1}(t), \quad x \geq 1.$$

These polynomials satisfy $Q_x(1) = 1$, $x \geq 0$ and, as seen from the recurrence, they satisfy the parity property: $Q_x(-t) = (-1)^x Q_x(t)$, $x \geq 0$. They are often called the RW polynomials. They are important in view of the Karlin and McGregor spectral representation theorem [14]. Indeed, we have

$$(10) \quad \mathbb{P}_x(X_n = y) = \gamma_y \int_{-1}^1 t^n Q_x(t) Q_y(t) d\mu(t),$$

with weights: $\gamma_y = 1 / \int_{-1}^1 Q_y(t)^2 d\mu(t) = \prod_{z=0}^{y-1} \frac{p_z}{q_{z+1}}$, $y \geq 1$, ($\gamma_0 \equiv 1$). Here, $d\mu(t)$ is the symmetric probability measure on $[-1, 1]$ with respect to which $(Q_y(t), y \geq 1)$ are orthogonal: $\int_{-1}^1 Q_x(t) Q_y(t) d\mu(t) = \gamma_y^{-1} \delta_{x,y}$. Note that the weights read:

$$(11) \quad \gamma_y = (2y + \delta) \frac{\Gamma(\delta + 1) \Gamma(y)}{\Gamma(y + \delta + 1)}.$$

At this point, the polynomials $(Q_x(t); x \geq 0)$ are *not* well-known, *nor* is their orthogonality measure μ . We shall return to this crucial point later. Before that, let us compute the first return probability to the origin.

3.2. First return probability to the origin. To do this, we need to consider the first associated polynomials $(Q_x^1(t); x \geq 1)$ defined by the recurrence

$$t Q_x^1(t) = p_{x+1} Q_{x+1}^1(t) + q_{x+1} Q_{x-1}^1(t), \quad x \geq 0; \quad Q_{-1}^1(t) = 0, \quad Q_0^1(t) = 1,$$

or with $Q_{-1}^1(t) = 0$, $Q_0^1(t) = 1$, by:

$$(12) \quad (2x + \delta + 2) t Q_x^1(t) = (x + 1) Q_{x+1}^1(t) + (x + \delta + 1) Q_{x-1}^1(t), \quad x \geq 0.$$

The Gegenbauer polynomials $G_x^\lambda(t)$ satisfy the recurrence ($G_{-1}^\lambda(t) = 0$, $G_0^\lambda(t) = 1$):

$$2(\lambda + x) t G_x^\lambda(t) = (x + 1) G_{x+1}^\lambda(t) + (2\lambda + x - 1) G_{x-1}^\lambda(t), \quad x \geq 0.$$

Our first associated polynomials $Q_x^1(t)$ therefore are recognized to be the ultraspherical (Gegenbauer) polynomials, namely: $Q_x^1(t) = G_x^{\delta/2+1}(t)$. They are well-known orthogonal polynomials with respect to the spectral measure

$$(13) \quad d\mu^1(t) = \frac{\Gamma(\delta/2 + 2)}{\sqrt{\pi} \Gamma(\frac{\delta+3}{2})} (1 - t^2)^{\frac{\delta+1}{2}} dt,$$

which is a symmetric probability measure on $[-1, 1]$. It holds indeed that;

$$\int_{-1}^1 Q_x^1(t) Q_y^1(t) d\mu^1(t) = \frac{1}{p_0 q_1} \gamma_{y+1}^{-1} \delta_{x,y}.$$

Let $\tau_{0,0}$ stand for the first return time to the origin of the RW starting from $X_0 = 0$. Applying the results in [6], we get an exact and asymptotic expression of the law of $\tau_{0,0}$:

Proposition 2. (i) For odd $k = 2l + 1$, $\mathbb{P}_0(\tau_{0,0} = k) = 0$.

(ii) For even $k = 2l$:

$$(14) \quad \mathbb{P}(\tau_{0,0} = 2l) = \frac{\left(\frac{\delta+1}{2}\right) \Gamma\left(\frac{\delta}{2} + 1\right)}{\sqrt{\pi}} \frac{\Gamma\left(\frac{2l-1}{2}\right)}{\Gamma\left(\frac{2l+\delta+2}{2}\right)}.$$

(iii) The generating function $\phi_0(z) \equiv \mathbb{E}(z^{\tau_{0,0}/2})$ of $\tau_{0,0}/2$ is given by:

$$(15) \quad \phi_0(z) = 1 - (1-z)_2 F_1(1, 1/2; \delta/2 + 1; z).$$

(iii) The tails of $\tau_{0,0}$ are given by:

$$(16) \quad \mathbb{P}(\tau_{0,0} = 2l) \underset{l \nearrow \infty}{\sim} \frac{\binom{\delta+1}{2} \Gamma(\frac{\delta}{2} + 1)}{\sqrt{\pi}} l^{-\binom{\delta+3}{2}}.$$

Proof: To obtain (14), using Beta integrals and the Legendre duplication formula for gamma functions (see below for a reminder):

$$\begin{aligned} \mathbb{P}(\tau_{0,0} = 2l) &= p_0 q_1 \int_{-1}^1 t^{2l-2} d\mu^1(t) \\ &= \frac{\delta+1}{\delta+2} \frac{\Gamma(\frac{\delta}{2} + 2)}{\sqrt{\pi} \Gamma(\frac{\delta+3}{2})} \int_{-1}^1 t^{2l-2} (1-t^2)^{\frac{\delta+1}{2}} dt = \frac{\Gamma(\frac{\delta}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{\delta+1}{2})} \int_{-1}^1 t^{2l-2} (1-t^2)^{\frac{\delta+1}{2}} dt \\ &= \frac{\Gamma(\frac{\delta}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{\delta+1}{2})} \int_0^1 u^{\frac{2l-1}{2}-1} (1-u)^{\frac{\delta+3}{2}-1} du = \frac{\binom{\delta+1}{2} \Gamma(\frac{\delta}{2} + 1)}{\sqrt{\pi}} \frac{\Gamma(\frac{2l-1}{2})}{\Gamma(\frac{2l+\delta+2}{2})}. \end{aligned}$$

Eq. (15) will be shown later in (32) and (41). The asymptotic result (16) follows from (14) using Stirling formula. \triangle

We now draw some conclusions from this result:

- When $\frac{\delta+3}{2} > 1$ (equivalently $\delta > -1$), (16) corresponds to a proper probability distribution with $\mathbb{P}(\tau_{0,0} < \infty) = 1$.
- When $\frac{\delta+3}{2} > 2$ (equivalently $\delta > 1$), (16) corresponds to a proper probability distribution with $\mathbb{E}(\tau_{0,0}) \equiv \mu = \frac{1}{\pi_0} = \frac{2\delta}{\delta-1} < \infty$.

Thus the RW is recurrent when $\delta > -1$ ($\tau_{0,0} < \infty$ with probability 1), null recurrent when $\delta \in (-1, 1]$ ($\mathbb{E}(\tau_{0,0}) = \mu = \infty$), positive recurrent when $\delta > 1$.

- When $\delta = -1$, as already underlined, the RW has 2 reflecting states, namely $\{0, 1\}$. Starting from state 0, the walker never returns to this state. However, considering the RW on $\mathbb{Z}_+ \setminus \{0\}$ rather than on \mathbb{Z}_+ , the critical special walker ($\delta = -1$) started at state 1 returns infinitely often to state 1. The expected return time is ∞ and so the RW is null recurrent but on state-space $\{1, 2, \dots\}$.

When $\delta > 1$, given $X_0 = x$, $\frac{1}{N} \sum_{n=0}^N \mathbf{1}(X_n = 0) \xrightarrow[N \nearrow \infty]{} \pi_0 = \frac{\delta-1}{2\delta}$, \mathbb{P}_x -almost surely.

The RW (polymer chain) is pinned at the origin in that there exists a limiting positive contact fraction π_0 at 0. The point $\delta = 1$ separating the null recurrent phase from the positive recurrent one therefore is a pinning transition point.

3.3. KMG representation and first passage times. Let $\tau_{x,y}$ be the first passage time at y when the process is started at x . We have:

$$\tau_{x,y} = \sum_{z=x}^{y-1} \tau_{z,z+1} \text{ if } y > x \text{ and } \tau_{x,y} = \sum_{z=x}^{y+1} \tau_{z,z-1} \text{ if } y < x$$

where the random times $\tau_{z,z\pm 1}$ are mutually independent.

The KMG representation is also useful to compute the law of these random times. Indeed, let

$$\phi_{x,y}(z) \equiv \sum_{k=1}^{\infty} z^k \mathbb{P}(\tau_{x,y} = k)$$

be the generating function of the law of $\tau_{x,y}$, with $x \neq y$. Then, with

$$P_{x,y}(z) \equiv \sum_{n=0}^{\infty} z^n \mathbb{P}_x(X_n = y) = \gamma_y \int_{-1}^1 \frac{Q_x(t) Q_y(t)}{1-zt} d\mu(t)$$

the generating function of $P^n(x,y)$ (the Green potential function of the chain), using (10), we easily get the expression:

$$\phi_{x,y}(z) = \frac{P_{x,y}(z)}{P_{y,y}(z)} = \frac{\int_{-1}^1 \frac{Q_x(t) Q_y(t)}{1-zt} d\mu(t)}{\int_{-1}^1 \frac{Q_y(t)^2}{1-zt} d\mu(t)},$$

in terms of Stieltjes transforms. In particular,

$$\phi_{x,0}(z) = \frac{\int_{-1}^1 \frac{Q_x(t)}{1-zt} d\mu(t)}{\int_{-1}^1 \frac{1}{1-zt} d\mu(t)} \text{ and } \phi_{0,x}(z) = \frac{\int_{-1}^1 \frac{Q_x(t)}{1-zt} d\mu(t)}{\int_{-1}^1 \frac{Q_x(t)^2}{1-zt} d\mu(t)}$$

are the generating functions of $\tau_{x,0}$ and $\tau_{0,x}$ satisfying

$$\frac{\phi_{x,0}(z)}{\phi_{0,x}(z)} = \frac{\int_{-1}^1 \frac{Q_x(t)^2}{1-zt} d\mu(t)}{\int_{-1}^1 \frac{1}{1-zt} d\mu(t)} = \frac{P_{x,x}(z)}{P_{0,0}(z)}.$$

Note that, with $\mathcal{Q}_t(s) \equiv \sum_{x \geq 0} s^x Q_x(t)$ the generating function of the sequence $(Q_x(t); x \geq 0)$ of orthogonal polynomials, we have

$$\Phi(s, z) \equiv \sum_{x \geq 0} s^x \phi_{x,0}(z) = \frac{\int_{-1}^1 \frac{\mathcal{Q}_t(s)}{1-zt} d\mu(t)}{\int_{-1}^1 \frac{1}{1-zt} d\mu(t)}.$$

It can be useful to estimate the large x behavior of $\phi_{x,0}(z)$.

4. THE DUAL RANDOM WALK

We shall learn much on the special RW $(X_n; n \geq 0)$ by looking at its “dual” Markov chain. Consider indeed the dual RW $(X_n^*; n \geq 0)$ whose probability transitions are $p_x^* = q_x = \frac{x+\delta}{2x+\delta}$ and $q_x^* = p_x = \frac{x}{2x+\delta}$ (switching p_x and q_x), $x \geq 1$, also satisfying: $p_0^* = 1$ and $q_0^* = 0$ (keeping the reflection at the origin condition). Assume $\delta > -1$. Consider the dual polynomials determined by the 3-term recurrence:

$$tQ_x^*(t) = p_x^* Q_{x+1}^*(t) + q_x^* Q_{x-1}^*(t), \quad x \geq 0; \quad Q_{-1}^*(t) = 0, \quad Q_0^*(t) = 1,$$

or, with $Q_{-1}^*(t) = 0$, $Q_0^*(t) = 1$, by:

$$(17) \quad (2x + \delta) t Q_x^*(t) = (x + \delta) Q_{x+1}^*(t) + x Q_{x-1}^*(t), \quad x \geq 0.$$

These polynomials satisfy $Q_x^*(1) = 1$, $x \geq 0$. They are now important in view of the spectral representation of $(X_n^*; n \geq 0)$ itself, namely:

$$(18) \quad \mathbb{P}_x(X_n^* = y) = \gamma_y^* \int_{-1}^1 t^n Q_x^*(t) Q_y^*(t) d\mu^*(t),$$

where $\gamma_y^* = 1 / \int_{-1}^1 t^n Q_y^*(t)^2 d\mu(t) = \prod_{z=0}^{y-1} \frac{q_z}{p_{z+1}}$ and $d\mu^*(t)$ is the probability measure on $[-1, 1]$ with respect to which $(Q_y^*(t), y \geq 1)$ are orthogonal; to wit

$$\int_{-1}^1 Q_x^*(t) Q_y^*(t) d\mu^*(t) = \gamma_y^{*-1} \delta_{x,y}.$$

We clearly have:

$$(19) \quad Q_x^*(t) = \frac{x! \Gamma(\delta)}{\Gamma(x+\delta)} G_x^{\delta/2}(t),$$

where $G_x^\lambda(t)$ are the ultraspherical (Gegenbauer) polynomials already introduced. As a result, the spectral measure of $(X_n^*; n \geq 0)$ is directly identified to be:

$$(20) \quad d\mu^*(t) = \frac{\Gamma(\frac{\delta}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{\delta+1}{2})} (1-t^2)^{\frac{\delta-1}{2}} dt.$$

Note that the dual weights satisfy

$$(21) \quad \gamma_y^* = \prod_{z=0}^{y-1} \frac{q_z}{p_{z+1}} = (2y+\delta) \frac{\Gamma(y+\delta)}{\Gamma(\delta+1) \cdot y!} \underset{y \text{ large}}{\sim} \frac{2}{\Gamma(\delta+1)} y^\delta.$$

This model was also studied in [1] and [2] in the context of a RW on a hypersphere with non-integral dimension; to switch to the model of these authors, one should simply relate their (non-integral) hypersphere dimension parameter D to our parameter δ through: $D = 1 + \delta$.

Consider then the first associated polynomials $(Q_x^{*,1}(t); x \geq 1)$ defined by

$$tQ_x^{*,1}(t) = p_{x+1}^* Q_{x+1}^{*,1}(t) + q_{x+1}^* Q_{x-1}^{*,1}(t), \quad x \geq 0; \quad Q_{-1}^{*,1}(t) = 0, \quad Q_0^{*,1}(t) = 1,$$

or, with $Q_{-1}^{*,1}(t) = 0, Q_0^{*,1}(t) = 1$, by

$$(22) \quad (2x+\delta+2)tQ_x^{*,1}(t) = (x+\delta+1)Q_{x+1}^{*,1}(t) + (x+1)Q_{x-1}^{*,1}(t), \quad x \geq 0.$$

We have:

$$(23) \quad Q_x^{*,1}(t) = \frac{(x+1)! \Gamma(\delta+1)}{\Gamma(x+\delta+1)} G_x^{\frac{\delta}{2},1}(t),$$

where $G_x^{\frac{\delta}{2},1}(t)$ are the first associated ultraspherical polynomials satisfying:

$$(2x+\delta+2)tG_x^{\frac{\delta}{2},1}(t) = (x+2)G_{x+1}^{\frac{\delta}{2},1}(t) + (x+\delta)G_{x-1}^{\frac{\delta}{2},1}(t), \quad x \geq 1,$$

with $G_{-1}^{\frac{\delta}{2},1}(t) = 0$ and $G_0^{\frac{\delta}{2},1}(t) = 1$.

With $(\alpha)_x \equiv \alpha(\alpha+1)\dots(\alpha+x-1)$ the rising factorials, these polynomials can be expressed as: $G_x^{\frac{\delta}{2},1}(t) = \frac{(\delta+1)_x}{(\frac{\delta+1}{2}+1)_x} P_x^{\frac{\delta-1}{2}, \frac{\delta-1}{2}}(t, 1)$ where $P_x^{\alpha,\beta}(t, c)$ are the c -associated Jacobi polynomials (see [11] and [15]). Putting all this together, we get:

Proposition 3. (i) For $x \geq 1$

$$Q_x^{*,1}(t) = \frac{(x+1)! \Gamma(\frac{\delta+1}{2} + 1)}{\Gamma(x + \frac{\delta+1}{2} + 1)} P_x^{\frac{\delta-1}{2}, \frac{\delta-1}{2}}(t, 1).$$

(ii) The spectral measure on $[-1, 1]$ with respect to which $(Q_x^{*,1}(t), x \geq 1)$ are orthogonal is:

$$(24) \quad d\mu^{*,1}(t) = \frac{1}{Z_1} \frac{(1-t^2)^{\frac{\delta-1}{2}}}{|F(t)|^2} dt.$$

With $K \equiv \frac{\Gamma(\delta)\Gamma(\frac{1-\delta}{2})}{\Gamma(\frac{\delta-1}{2})}$, the shape factor $F(t)$ is given by:

$$(25) \quad F(t) = {}_2F_1\left(1, 1-\delta; \frac{3-\delta}{2}; \frac{1+t}{2}\right) + K e^{i\pi\frac{\delta-1}{2}} \left(\frac{1+t}{2}\right)^{\frac{\delta-1}{2}} \cdot {}_2F_1\left(\frac{1+\delta}{2}, \frac{1-\delta}{2}; \frac{1+\delta}{2}; \frac{1+t}{2}\right).$$

It satisfies the skew-symmetry property: $F(-t) = -F^*(t)$.

Proof: The orthogonality measure (24) is the one of the first associated Jacobi polynomials (see [15], Theorem 3, page 996 and [13], Theorem 5.7.1). The constant Z_1 in (24) is the normalization constant which renders $\mu^{*,1}$ a probability measure with mass 1. Δ

If $\frac{\delta-1}{2}$ is an integer ($\delta = 1, 3, 5, \dots$) one should be cautious with this expression of $d\mu^{*,1}(t)$, as limits must be taken; logarithms arise in this case (see Wimp, page 997, for a discussion).

4.1. The orthogonality measure $d\mu(t)$. Assume $\delta \neq 1, 3, 5, \dots$. It was shown in (see [6], Theorem 2.1 and [7]) that the spectral measure of the first associated polynomials of a RW is related to the spectral probability measure of the dual RW through

$$(26) \quad d\mu^1(t) = \frac{1}{p_0 q_1} (1-t^2) d\mu^*(t).$$

In our case, (from (13) and (26))

$$d\mu^*(t) = \frac{\Gamma(\frac{\delta}{2}+1)}{\sqrt{\pi}\Gamma(\frac{\delta+1}{2})} (1-t^2)^{\frac{\delta-1}{2}} dt,$$

which indeed is (20), and also, by duality

$$d\mu^{*,1}(t) = \frac{1}{p_0^* q_1^*} (1-t^2) d\mu(t).$$

From (24), we get:

Proposition 4. The spectral orthogonality measure of the $(Q_x(t); x \geq 1)$ is:

$$(27) \quad d\mu(t) = \frac{1}{Z} \frac{(1-t^2)^{\frac{\delta-3}{2}}}{|F(t)|^2} dt, \text{ if } \delta \in (-1, 1),$$

$$(28) \quad d\mu(t) = \pi_0 (\delta_{-1} + \delta_1) + \frac{1}{\delta Z} \frac{(1-t^2)^{\frac{\delta-3}{2}}}{|F(t)|^2} dt, \text{ if } \delta > 1,$$

where $Z = Z_1/q_1^* = (2+\delta)Z_1$.

Proof: When $\delta > 1$, $d\mu(t)$ has two atomic charges at points $t = \pm 1$ translating the fact that the 2-periodic chain X_n has two ergodic components: $\pi_{\text{even}} = (2\pi_0, 0, 2\pi_2, 0, \dots)$ and $\pi_{\text{odd}} = (0, 2\pi_1, 0, 2\pi_3, 0, \dots)$, depending on the evenness or oddness of the chain starting point. The number Z in (27) is the normalization constant that makes μ a probability measure. The normalization constant in (28) uses $\pi_0 = \frac{\delta-1}{2\delta}$. Δ

We shall call $d\mu_c(t)$ the absolutely continuous part of $d\mu(t)$ in (28). When $\delta \in (1, 2)$, the partial result (28) also appears in [5]. The measures μ given by (27) and (28) are probability measures. We will now check that they are integrable near the endpoint $t = 1$ of the support. As we shall see in the process, the shape of the function $F(t)$ near $t = 1$ can be of a very different nature, depending on whether $\delta \in (-1, 1)$ or $\delta > 1$.

• We first focus on the parameter range $\delta \in (-1, 1)$ (null recurrence for X_n). In this range, ${}_2F_1\left(1, 1-\delta; \frac{3-\delta}{2}; 1\right) = \infty$ and, by Euler's integral formula

$${}_2F_1\left(1, 1-\delta; \frac{3-\delta}{2}; t\right) \underset{t \nearrow 1^-}{\sim} K_0 (1-t)^{-\frac{1-\delta}{2}}, \quad K_0 = \frac{\Gamma\left(\frac{3-\delta}{2}\right) \Gamma\left(\frac{1-\delta}{2}\right)}{\Gamma(1-\delta)}.$$

Further,

$${}_2F_1\left(\frac{1+\delta}{2}, \frac{1-\delta}{2}; \frac{1+\delta}{2}; \frac{1+t}{2}\right) = \left(\frac{1-t}{2}\right)^{-\frac{1-\delta}{2}}.$$

As a result: $F(t) \underset{t \nearrow 1^-}{\sim} \left(K_0 + K e^{i\pi \frac{\delta-1}{2}}\right) \left(\frac{1-t}{2}\right)^{-\frac{1-\delta}{2}}$ and, when $\delta \in (-1, 1)$

$$d\mu(t) \underset{t \nearrow 1^-}{\sim} \frac{1}{Z \left|K_0 + K e^{i\pi \frac{\delta-1}{2}}\right|^2 2^{1-\delta}} (1-t)^{-\left(\frac{\delta+1}{2}\right)} dt,$$

which is integrable at $t = 1$.

- When $\delta = 1$, logarithmic singularity effects should be considered.
- In the parameter range $\delta \in (1, 3)$, due to well-known results on special values of the Gauss hypergeometric function at $t = 1$,

$${}_2F_1\left(1, 1-\delta; \frac{3-\delta}{2}; 1\right) = \frac{\Gamma\left(\frac{3-\delta}{2}\right) \Gamma\left(\frac{\delta-1}{2}\right)}{\Gamma\left(\frac{1-\delta}{2}\right) \Gamma\left(\frac{1+\delta}{2}\right)} = -1.$$

Further, one can easily show that ${}_2F_1\left(1, 1-\delta; \frac{3-\delta}{2}; 1\right) = -1 + O\left((1-t)^{\frac{\delta-1}{2}}\right)$.

Finally, $F(t) \underset{t \nearrow 1^-}{=} -1 + O\left((1-t)^{\frac{\delta-1}{2}}\right)$ so that $F(t)^2 \underset{t \nearrow 1^-}{\sim} 1$ which is not singular at $t = 1$ and so when $\delta \in (1, 3)$

$$d\mu(t) \underset{t \nearrow 1^-}{\sim} \frac{\delta-1}{2\delta} \delta_1 + \frac{1}{\delta Z} (1-t)^{\frac{\delta-3}{2}} dt,$$

which is integrable at $t = 1$. Integrability in the domains $(k, k+2)$ for $k = \{3, 5, \dots\}$ also holds. We skip the special logarithmic singular behaviors at points $\delta \in \{1, 3, 5, \dots\}$.

To be complete, we briefly recall some facts pertaining to the Euler integral representation of the Gauss hypergeometric function that we used. When $c > b > 0$:

$${}_2F_1(a, b; c; t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1} (1-u)^{c-b-1} (1-tu)^{-a} du.$$

Two cases arise: - (Gauss) If $\alpha \equiv c - (b + a) > 0$, then ${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(\alpha)}{\Gamma(c-a)\Gamma(c-b)} < \infty$. - If $\alpha \equiv c - (b + a) < 0$, then (by Euler-Kummer transformation identity) ${}_2F_1(a, b; c; t) = (1-t)^\alpha \cdot {}_2F_1(c-a, c-b; c; t)$ with ${}_2F_1(c-a, c-b; c; 1) = \frac{\Gamma(c)\Gamma(-\alpha)}{\Gamma(a)\Gamma(b)} < \infty$. Thus, if $\alpha < 0$, ${}_2F_1(a, b; c; t) \underset{t \nearrow 1^-}{\sim} \frac{\Gamma(c)\Gamma(-\alpha)}{\Gamma(a)\Gamma(b)} (1-t)^\alpha$ showing that this function has an algebraic singularity at $t = 1$.

When $t = 0$, the well-known expression of Beta integrals follows from the Euler formula ($\alpha, \beta > 0$)

$$\int_0^1 u^{\alpha-1} (1-u)^{\beta-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Here $\Gamma(\alpha)$ is the Euler gamma function satisfying the Legendre duplication formula: $\Gamma(\alpha)\Gamma(\alpha+1/2) = \sqrt{\pi}2^{1-2\alpha}\Gamma(2\alpha)$.

5. MORE WITH DUALITY: FIRST RETURN TIMES TO 0 AND CONTACT PROBABILITY AT 0

The duality relation between the two RWs allowed us to identify the orthogonality measure μ with respect to which our RW polynomials $(Q_x(t); x \geq 1)$ were orthogonal. We wish here to continue in this direction and see what more one can learn on the special RW from its dual.

5.1. Generating functions and Stieltjes transforms. With $|z| < 1$, let

$$(29) \quad \phi_0(z) = \sum_{l \geq 1} z^l \mathbb{P}(\tau_{0,0} = 2l)$$

be the generating function of the first return time to zero probability. Let also

$$(30) \quad u_0(z) = 1 + \sum_{m \geq 1} z^m \mathbb{P}_0(X_{2m} = 0)$$

be the Green potential function of the chain at state $x = 0$. As can easily be checked by renewal arguments, $u_0(z) = 1 + u_0(z)\phi_0(z)$, showing that

$$(31) \quad u_0(z) = \frac{1}{1 - \phi_0(z)} \text{ and } \phi_0(z) = 1 - \frac{1}{u_0(z)}.$$

Further, $\phi_0(z)$ admits the continued fraction representation (see [8])

$$\phi_0(z) = q_1 z / (1 - (p_1 q_2 z / (1 - p_2 q_3 z (1 - \dots))))),$$

and consequently so does $u_0(z)$:

$$u_0(z) = 1 - (1 - q_1 z / (1 - (p_1 q_2 z / (1 - p_2 q_3 z (1 - \dots))))).$$

Random walk and first associated polynomials (as from (9) and (12)) appear in the numerator and denominator of the rational approximations of all order for these quantities.

Next, by the Dette-Karlin-McGregor representation theorem stating that $\mathbb{P}(\tau_{0,0} = k) = p_0 q_1 \int_{-1}^1 t^{k-2} d\mu^1(t)$, we get

$$(32) \quad \phi_0(z) = p_0 q_1 \sum_{l \geq 1} z^l \int_{-1}^1 t^{2(l-1)} d\mu^1(t) = p_0 q_1 z \int_{-1}^1 \frac{d\mu^1(t)}{1-zt^2},$$

and so $\phi_0(z)$ is related to the Stieltjes transform of the measure μ^1 , $S_{\mu^1}(z) \equiv \int_{-1}^1 \frac{d\mu^1(t)}{1-zt^2}$.

Recalling also $\mathbb{P}_0(X_n = 0) = \int_{-1}^1 t^n d\mu(t)$, we get

$$(33) \quad u_0(z) = 1 + \sum_{m \geq 1} z^m \int_{-1}^1 t^{2m} d\mu(t) = \int_{-1}^1 \frac{d\mu(t)}{1-zt^2} =: S_{\mu}(z),$$

showing that the Stieltjes transform of the measures μ and μ^1 are related by:

$$(34) \quad S_{\mu}(z) = \frac{1}{1 - p_0 q_1 z S_{\mu^1}(z)}.$$

5.2. Dual random walk: Stieltjes transforms. Define similarly $\phi_0^*(z)$ and $u_0^*(z)$ for the dual RW with

$$(35) \quad u_0^*(z) = \frac{1}{1 - \phi_0^*(z)} \text{ and } \phi_0^*(z) = 1 - \frac{1}{u_0^*(z)}.$$

Now, the generating functions $u_0(z)$ and $u_0^*(z)$ are related by the Wall identity (see [7] for a review and a simple proof),

$$(36) \quad u_0(z) u_0^*(z) = (1-z)^{-1}, \quad |z| < 1.$$

This expresses the fact that: $\mathbb{P}_0(X_{2m}^* = 0) = \mathbb{P}(\infty \geq \tau_{0,0} > 2m)$, relating the tail probability of the first return time to 0 to the probability that the dual RW stays at the origin at some time. In terms of generating function, this is indeed: $u_0^*(z) = \frac{1 - \phi_0(z)}{1-z}$ which is the Wall identity. Thus, in terms of Stieltjes transforms

$$(37) \quad u_0(z) = \int_{-1}^1 \frac{d\mu(t)}{1-zt^2} = \frac{1}{(1-z) \int_{-1}^1 \frac{d\mu^*(t)}{1-zt^2}} =: \frac{1}{(1-z) S_{\mu^*}(z)}.$$

Recalling $d\mu^*(t) = C(1-t^2)^{\frac{\delta-1}{2}} dt$, where $C = \frac{\Gamma(\frac{\delta+1}{2})}{\sqrt{\pi}\Gamma(\frac{\delta+1}{2})}$, we have

$$(38) \quad \int_{-1}^1 \frac{d\mu^*(t)}{1-zt^2} = C \int_{-1}^1 \frac{(1-t^2)^{\frac{\delta-1}{2}}}{1-zt^2} dt = C \int_0^1 \frac{u^{-1/2}(1-u)^{\frac{\delta-1}{2}}}{1-zu} du.$$

By Euler representation theorem, we get:

$$(39) \quad u_0^*(z) = \int_{-1}^1 \frac{d\mu^*(t)}{1-zt^2} = {}_2F_1(1, 1/2; \delta/2 + 1; z)$$

and so the Stieltjes transform of μ^* is a known specific Gauss hypergeometric function. Exploiting the duality formulas, we obtain:

Proposition 5. *With $u_0^*(z)$ given in terms of the Gauss hypergeometric function (39), the Stieltjes transform*

$$(40) \quad S_{\mu}(z) = \int_{-1}^1 \frac{d\mu(t)}{1-zt^2} = 1/((1-z)u_0^*(z)),$$

together with,

$$(41) \quad S_{\mu^1}(z) = \int_{-1}^1 \frac{d\mu^1(t)}{1-zt^2} = \frac{1}{p_0 q_1 z} (1 - ((1-z)u_0^*(z))),$$

and

$$(42) \quad S_{\mu^{*,1}}(z) \equiv \int_{-1}^1 \frac{d\mu^{*,1}(t)}{1-zt^2} = \frac{1}{p_1 z} (1 - 1/u_0^*(z)),$$

are explicitly known.

Now, the singular behavior of the Gauss hypergeometric function u_0^* in (39) near $z = 1$ is easily seen to be:

$$(43) \quad u_0^*(z) \underset{z \nearrow 1^-}{\sim} \kappa_1 \text{ if } \delta > 1 \text{ and } u_0^*(z) \underset{z \nearrow 1^-}{\sim} \kappa_2 (1-z)^{-\left(\frac{1-\delta}{2}\right)} \text{ if } -1 < \delta < 1,$$

for some constants $\kappa_1 = u_0^*(1) = \frac{\delta}{\delta-1} > 1$ and $\kappa_2 = \frac{1}{\sqrt{\pi}} \Gamma(\delta/2 + 1) \Gamma((1-\delta)/2)$. This asymptotic behavior of $u_0^*(z)$, together with its relationship with the quantities of interest displayed in Eqs. (37 - 42), allows one to obtain the large time behaviors of the desired probabilities from singularity analysis.

5.3. Asymptotics and singularity analysis. Before we come into this, we first briefly recall a general transfer result of singularity analysis (see [9]) of generating functions.

Let $H(z)$ be any analytic function in the indented domain defined by

$$D = \{z : |z| \leq z_1, |\text{Arg}(z - z_0)| > \pi/2 - \eta\}$$

where $z_0, z_1 > z_0$, and η are positive real numbers. Assume that, with $\sigma(x) = x^\alpha \log^\beta x$, α and β any real number (the singularity exponents of H), we have

$$(44) \quad H(z) \sim \kappa_1 + \kappa_2 \sigma\left(\frac{1}{1-z/z_0}\right) \text{ as } z \rightarrow z_0 \text{ in } D,$$

for some real constants κ_1 and κ_2 . Then:

- if $\alpha \notin \{0, -1, -2, \dots\}$ the coefficients in the expansion of $H(z)$ satisfy

$$(45) \quad [z^n] H(z) \sim \kappa_1 + \kappa_2 z_0^{-n} \cdot \frac{\sigma(n)}{n} \frac{1}{\Gamma(\alpha)} \text{ as } n \nearrow \infty,$$

where $\Gamma(\alpha)$ is the Euler function. $H(z)$ presents an algebraic-logarithmic singularity at $z = z_0$.

- if $\alpha \in \{0, -1, -2, \dots\}$, the singularity $z = z_0$ is purely logarithmic and

$$(46) \quad [z^n] H(z) \sim \kappa_1 + \kappa_2 \cdot \beta \cdot z_0^{-n} \cdot \frac{\sigma(n)}{n \cdot \log n} \left(\frac{1}{\Gamma}\right)'(\alpha) \text{ as } n \nearrow \infty,$$

involving the derivative of the reciprocal Euler function at α .

Thus, for algebraic-logarithmic singularities, the asymptotics of the coefficients can be read from the singular behavior of the partition function under study and vice-versa. Let us derive precise conclusions from this, concerning the special RW and its dual.

5.4. The special RW asymptotics. • Assume $-1 < \delta < 1$. (Recall $\kappa_1 = \frac{\delta}{\delta-1}$ and $\kappa_2 = \frac{1}{\sqrt{\pi}}\Gamma(\delta/2 + 1)\Gamma((1-\delta)/2)$).

Observing $\phi_0(z) = 1 - (1-z)u_0^*(z)$, we have: $\phi_0(z) \underset{z \nearrow 1^-}{\sim} 1 - \kappa_2(1-z)^{\frac{\delta+1}{2}}$. This leads by singularity analysis arguments to (16) which we already know using a different approach. Next,

$$u_0(z) = \frac{1}{(1-z)u_0^*(z)} \underset{z \nearrow 1^-}{\sim} \kappa_2^{-1}(1-z)^{-\left(\frac{\delta+1}{2}\right)},$$

so that,

$$(47) \quad \mathbb{P}_0(X_{2m} = 0) \underset{m \nearrow \infty}{\sim} \frac{\kappa_2^{-1}}{\Gamma\left(\frac{\delta+1}{2}\right)} m^{-\left(\frac{1-\delta}{2}\right)} = \frac{2^\delta}{\Gamma((1-\delta)/2)\Gamma(\delta+1)} m^{-\left(\frac{1-\delta}{2}\right)},$$

where, in the last equality, we used the Legendre duplication formula.

Proposition 6. *Assume $-1 < \delta < 1$. The probability that the special random walker hits 0 at time $2m$ decays algebraically with exponent $(1-\delta)/2$ as in (47).*

• When $\delta = 1$ (critical case), we have:

$$u_0^*(z) = {}_2F_1(1, 1/2; 3/2; z) = 1 + \frac{1}{2} \sum_{m \geq 1} z^m / (m+1/2) \underset{z \nearrow 1^-}{\sim} 1 - 1/2 \log(1-z).$$

Thus,

$$\phi_0(z) = 1 - (1-z)u_0^*(z) \underset{z \nearrow 1^-}{\sim} 1 + 1/2(1-z)\log(1-z) \underset{z \nearrow 1^-}{\rightarrow} 1.$$

Next, $u_0(z) \underset{z \nearrow 1^-}{\sim} -2/[(1-z)\log(1-z)]$ and so, by (44 - 45)

$$(48) \quad \mathbb{P}_0(X_{2m} = 0) \underset{m \nearrow \infty}{\sim} \frac{2}{\log m},$$

with slow logarithmic decay of the probability to be in state 0 at time $2m$.

• When $\delta > 1$, the probability $\mathbb{P}_0(X_{2m} = 0)$ tends to the constant $\kappa_1^{-1} = \frac{\delta-1}{\delta}$ (which is *twice* the probability mass $\pi_0 = \frac{\delta-1}{2\delta}$ at 0 of the invariant measure). This is in accordance with the fact that $u_0(z) \underset{z \nearrow 1^-}{\sim} \kappa_1^{-1}(1-z)^{-1}$ and singularity arguments. Next, $\phi_0(z) \underset{z \nearrow 1^-}{\sim} 1 - \kappa_1(1-z)$, showing that

$$(49) \quad \mathbb{P}(\tau_{0,0} < \infty) = \lim_{z \nearrow 1^-} \phi_0(z) = 1.$$

The return time to the origin only occurs in finite time ($\tau_{0,0} < \infty$ with probability 1). Note also that $\kappa_1 = \mathbb{E}(\tau_{0,0})/2 = \frac{\delta}{\delta-1} < \infty$ in accordance with the fact that when $\delta > 1$, the special RW is positive recurrent. A more detailed study of the singularities of $\phi_0(z)$ would again give (16) showing that $\tau_{0,0}$ has no moment of order larger or equal than $\frac{\delta+1}{2}$.

• When $\delta = -1$, $u_0^*(z) = {}_2F_1(1, 1/2; 1/2; z) = (1-z)^{-1}$. Thus $\phi_0(z) = 0$ and $u_0(z) = 1$ and so $\mathbb{P}(\tau_{0,0} = 2m) = \mathbb{P}_0(X_{2m} = 0) = 0$, $m \geq 1$. This is because the special walker started at 0 moves to 1 with probability 1 in the first step and then moves to 2 with probability 1 with no possibility to return to 0 if ever in state 1 again ($p_1 = 1$ and $q_1 = 0$).

5.5. The dual RW asymptotics. Let us now consider the dual RW. Recalling $\phi_0^*(z) = 1 - \frac{1}{u_0^*(z)}$, with:

$$u_0^*(z) \underset{z \nearrow 1^-}{\sim} \kappa_1 \text{ if } \delta > 1 \text{ and } u_0^*(z) \underset{z \nearrow 1^-}{\sim} \kappa_2 (1-t)^{\frac{\delta-1}{2}} \text{ if } \delta < 1,$$

we get:

$$\phi_0^*(z) \underset{z \nearrow 1^-}{\sim} 1 - \kappa_1^{-1} \text{ if } \delta > 1 \text{ and } \phi_0^*(z) \underset{z \nearrow 1^-}{\sim} 1 - \kappa_2^{-1} (1-t)^{-\left(\frac{\delta-1}{2}\right)} \text{ if } \delta < 1.$$

This shows that,

Proposition 7. (i) *To the leading order,*

$$(50) \quad \sum_{m \geq 0} \mathbb{P}_0(X_{2m}^* = 0) \underset{m \nearrow \infty}{\rightarrow} \kappa_1 < \infty \text{ if } \delta > 1 \text{ and,}$$

$$(51) \quad \mathbb{P}_0(X_{2m}^* = 0) \underset{m \nearrow \infty}{\sim} \frac{\kappa_2}{\Gamma((1-\delta)/2)} m^{-\left(\frac{\delta+1}{2}\right)} \text{ if } -1 < \delta < 1.$$

(ii)

$$(52) \quad \sum_{l \geq 1} \mathbb{P}(\tau_{0,0}^* = 2l) \underset{l \nearrow \infty}{\rightarrow} 1 - \kappa_1^{-1} < 1 \text{ if } \delta > 1,$$

and, if $-1 < \delta < 1$, ($\delta \neq 0$)

$$(53) \quad \mathbb{P}(\tau_{0,0}^* = 2l) \underset{l \nearrow \infty}{\sim} -\frac{2^{\delta-1}}{\delta \Gamma((1-\delta)/2) \Gamma(\delta-1)} l^{-\left(\frac{3-\delta}{2}\right)}.$$

The case $\delta = 0$ is well-known (the fair RW) and should be treated slightly differently. When $-1 < \delta < 1$, the dual RW is null recurrent.

Proposition 8. *When $\delta > 1$, the dual RW is transient and the number of passages to state 0 by time N , namely $\mathcal{N}_0(N) \equiv \sum_{n=0}^N \mathbf{1}(X_n = 0)$, satisfies from (50)*

$$\mathcal{N}_0(N) \xrightarrow{d} \text{Geometric} \left(\frac{1}{\delta} \right) \text{ as } N \nearrow \infty,$$

a limiting geometric-distributed random variable on $\{1, 2, \dots\}$ with success probability $\frac{1}{\delta}$.

Proof: The eventual probability of return to 0, which is $\mathbb{P}(\tau_{0,0}^* < \infty) = \phi_0^*(1)$ indeed reads

$$(54) \quad \mathbb{P}(\tau_{0,0}^* < \infty) = 1 - \frac{1}{u_0^*(1)} = \frac{1}{\delta} < 1.$$

The dual RW being transient, there is a probability $1 - 1/\delta$ never to return to the origin in finite time and after a finite time, it quits state 0 for ever. Δ

6. EXCURSION STATISTICS OF THE SPECIAL RANDOM WALK AND ITS DUAL:
EXTREME VALUE ANALYSIS HEURISTIC

We start with the special RW itself. When $\delta > -1$, the excursion lengths $\tau_{0,0}$ are now well understood. We would like also to have some information on the excursion height, call it H . Assume the height $H = h \geq 1$ for some excursion. This event will be realized if and only if (i) downward paths started from h hit state 0 before hitting state $h + 1$ and (ii) upward paths started at 1 hit h without returning to 0 again in the intervening time. These two events are independent. Therefore

$$\mathbb{P}(H = h) = \mathbb{P}(\tau_{1,h} < \tau_{1,0}) \mathbb{P}(\tau_{h,0} < \tau_{h,h+1}).$$

Assume $X_0 = x$. Let $X_{n \wedge \tau_{x,0}}$ be the special RW stopped when it first hits 0. Let us define the scale (or harmonic) function φ of this RW as the function which makes $Y_n \equiv \varphi(X_{n \wedge \tau_{x,0}})$ a martingale. The function φ is important because, as is well-known, for all $0 < x < x_*$, with τ the first hitting time of $\{0, x_*\}$

$$\mathbb{P}_x(X_\tau = x_*) = \frac{\varphi(x)}{\varphi(x_*)}.$$

(55)

$$\mathbb{P}(H = h) = \mathbb{P}(\tau_{1,h} < \tau_{1,0}) \mathbb{P}(\tau_{h,0} < \tau_{h,h+1}) = \frac{\varphi(1)}{\varphi(h)} \left(1 - \frac{\varphi(h)}{\varphi(h+1)}\right), \quad h \geq 1.$$

We clearly have $\sum_{h \geq 1} \mathbb{P}(H = h) = 1$ because partial sums are part of a telescoping series. It remains to compute φ . We wish to have: $\mathbb{E}_x(Y_{n+1} | Y_n = y) = y$, leading to

$$\varphi(x) = q_x \varphi(x-1) + p_x \varphi(x+1),$$

where $p_x = x/(2x + \delta)$ and $q_x = (x + \delta)/(2x + \delta)$, $x \geq 1$. Thus, the searched 'harmonic' function is $\varphi(x) = \sum_{y=1}^x \psi(y)$ where $\psi(y)$ satisfies: $(y + \delta)\psi(y) = y\psi(y+1)$, with $\psi(1) \equiv 1$. Thus

$$(56) \quad \varphi(x) = 1 + \sum_{y=1}^{x-1} \prod_{z=1}^y \frac{z + \delta}{z}, \quad x \geq 1, \quad \varphi(0) \equiv 0.$$

Note $\varphi(1) = 1$. Equations (55) and (56) characterize the law of the excursion height of the special random walker. Note that if $\delta = 0$, $\varphi(x) = x$, as required, and $\mathbb{P}(H = h) = 1/[h(h+1)]$, $h \geq 1$ as can be shown for the fair simple RW, using different techniques.

We note that for all $\delta > -1$, $\varphi(x) \nearrow \infty$ as $x \nearrow \infty$. It can also easily be checked that $\varphi(h) \asymp h^{\delta+1}$ for large h (with \asymp meaning that the ratio of the two quantities tends to some constant). Therefore

$$(57) \quad \mathbb{P}(H = h) \asymp h^{-(\delta+2)}$$

and H has power-law tails with $\mathbb{E}(H) = \infty$ when $\delta \in (-1, 0)$, $\mathbb{E}(H) < \infty$ as soon as $\delta > 0$ and $\mathbb{E}(H^2) < \infty$ as soon as $\delta > 1$. We will summarize these results as follows:

Proposition 9. *With the scale function given by (56), the law of the excursion height of the special walker is exactly given by:*

$$(58) \quad \mathbb{P}(H \geq h) = 1/\varphi(h),$$

satisfying $\mathbb{P}(H \geq h) \asymp h^{-(\delta+1)}$.

(i) If $\delta = -1$, $\varphi(h) \asymp \log h$ and $\mathbb{P}(H \geq h) \asymp 1/\log h$.

(ii) when $\delta \in (-1, 0)$, the RW is null recurrent with both $\mathbb{E}(\tau_{0,0})$ and $\mathbb{E}(H) = \infty$.

(iii) when $\delta \in (0, 1)$, the RW is null recurrent with $\mathbb{E}(\tau_{0,0}) = \infty$ but with $\mathbb{E}(H) < \infty$.

(iv) In the positive recurrent case ($\delta > 1$) the special RW sample paths are made of infinitely many iid excursions with both $\mathbb{E}(\tau_{0,0})$ and $\mathbb{E}(H) < \infty$.

We note from (16) and (57) that $\tau_{0,0}$ and H^2 are both heavy-tailed and are tail-equivalent in that the ratio $\mathbb{P}(\tau_{0,0} \geq k)/\mathbb{P}(H^2 \geq k)$ tends to a constant when $k \nearrow \infty$.

6.1. Height and length of the largest excursion: the positive recurrent case. These informations allow to derive the following qualitative result about the maximal height \mathcal{H}_N reached by time N in the positive recurrent case ($\delta > 1$): by time N , with $\mu = \mathbb{E}(\tau_{0,0}) < \infty$ there are indeed N/μ iid excursions on average. Thus,

$$\mathcal{H}_N = \max_{n=1, \dots, [N/\mu]} H^{(n)}$$

where $H^{(n)} \stackrel{d}{=} H$ are iid with law governed by (55). Due to (56), there exists a sequence h_N such that $\frac{N}{\mu} \mathbb{P}(H > h_N) \rightarrow_{N \nearrow \infty} \alpha$ for some $\alpha > 0$, say $\alpha = \log 2/\mu$. From (57), we obtain:

$$(59) \quad h_N \asymp N^{1/(\delta+1)}.$$

We have

$$\mathbb{P}(\mathcal{H}_N \leq h_N) \rightarrow_{N \nearrow \infty} e^{-\mu\alpha} = 1/2,$$

and therefore (when $\delta > 1$), the typical (median) maximal height that the RW reaches by time N , grows like $N^{1/(\delta+1)}$. With \mathbb{M} denoting the median value, we therefore have: $\mathbb{M}(\max(X_1, \dots, X_N)) = h_N \asymp N^{1/(\delta+1)}$.

Next, the number of H_n , $n = 1, \dots, [N/\mu]$ exceeding h_N converges to a Poisson($\mu\alpha$) distributed random variable.

One therefore expects that $\mathcal{H}_N/h_N \xrightarrow[N \nearrow \infty]{d} F$ where F is Fréchet ($\delta + 1$) distributed.

Similarly, let

$$\tau_N = \max_{n=1, \dots, [N/\mu]} \tau_{0,0}^{(n)}$$

be the length of the largest excursion, with $\tau_{0,0}^{(n)} \stackrel{d}{=} \tau_{0,0}$ iid with law governed by (14). Due to (16), there exists a sequence of time lags k_N such that $\frac{N}{\mu} \mathbb{P}(\tau_N > k_N) \rightarrow_{N \nearrow \infty} \alpha > 0$. Since $\mathbb{P}(\tau_{0,0} > k) \asymp k^{-(\delta+1)/2}$, we get:

$$(60) \quad k_N \asymp N^{2/(\delta+1)}.$$

Therefore, when $\delta > 1$, the typical length of the excursion with maximal length by time N grows like $N^{2/(\delta+1)}$. Note that

$$(61) \quad h_N \asymp k_N^{1/2},$$

so that the typical height of the largest excursion scales like the square-root of its length.

Let $\tau_{0,h}$ be the first time at which some excursion height exceeds the level h (the time between failure at h). We have

$$\mathbb{P}(\tau_{0,h} > N) = \mathbb{P}(\mathcal{H}_N \leq h) = (1 - \mathbb{P}(H > h))^{[N/\mu]}.$$

Due to (57), for all $\alpha > 0$, assuming h large, we get:

$$(62) \quad \mathbb{P}\left(h^{-(\delta+1)}\tau_{0,h} > \alpha\right) \underset{h \text{ large}}{\sim} \left(1 - h^{-(\delta+1)}\right)^{\frac{\alpha}{\mu}h^{\delta+1}} \rightarrow e^{-\frac{\alpha}{\mu}},$$

showing that $\tau_{0,h}$ is of order $\mathbb{M}(\tau_{0,h}) \asymp h^{\delta+1}$ with an exponential limit law. This point is in accordance with (59). We shall summarize these results as follows:

Proposition 10. *Assume $\delta > 1$ (positive recurrence of the special walker). Then,*

- (i) *the typical (median) height h_N of its largest excursion satisfies $h_N \asymp N^{1/(\delta+1)}$.*
- (ii) *the typical (median) length k_N of its largest excursion satisfies $k_N \asymp N^{2/(\delta+1)}$, so with $h_N \asymp k_N^{1/2}$.*
- (iii) *the typical (median) time to failure at level h satisfies: $\mathbb{M}(\tau_{0,h}) \asymp h^{\delta+1}$.*

6.2. The null recurrent case: To some extent, this situation extends to the range $\delta \in (-1, 1)$, although it deserves a special treatment. In this null recurrent case indeed, $\mu = \infty$ and so one deals with very large and therefore rare excursions (see [12]). By renewal arguments, the expected number of such excursions by time N (large) now is of order $N^{(1+\delta)/2}/c$, much smaller than N , where $c = \frac{(\frac{\delta+1}{2})\Gamma(\frac{\delta}{2}+1)}{\sqrt{\pi}}$ is the constant appearing in (16). The typical length k_N of the largest excursion by time N is now given by:

$$\frac{N^{(1+\delta)/2}}{c} \mathbb{P}(\tau_N > k_N) \xrightarrow{N \nearrow \infty} \alpha > 0,$$

leading to $k_N \asymp N$ (with no δ -dependence of the scaling exponent): In this regime, the size of a typical excursion is the largest possible, corresponding to a single big excursion (or perhaps a few of them).

Similarly, the maximal height \mathcal{H}_N reached by time N in this null recurrent case is now given by:

$$\mathcal{H}_N = \max_{n=1, \dots, [N^{(1+\delta)/2}/c]} H^{(n)},$$

where $H^{(n)}$ are iid with law governed by (55). Due to (56), there exists a sequence h_N such that $\frac{1}{c}N^{(1+\delta)/2}\mathbb{P}(H > h_N) \xrightarrow{N \nearrow \infty} \alpha$ for some $\alpha > 0$, leading to

$$(63) \quad h_N \asymp N^{1/2}.$$

Thus, for all $\delta \in (-1, 1)$, $h_N \asymp k_N^{1/2}$. Although when $\delta \in (-1, 1)$, h_N and k_N are not individually of the same order of magnitude as when $\delta > 1$, the typical height of the largest excursion continues to scale like the square-root of its length. When $\delta = 0$, these results are confirmed by the more detailed study of these questions for the fair simple RW developed in [3].

In the null recurrent case, we also have

$$\mathbb{P}(\tau_{0,h} > N) = \mathbb{P}(\mathcal{H}_N \leq h) = (1 - \mathbb{P}(H > h))^{[N^{(1+\delta)/2}/c]}.$$

Due to (57), for all $\alpha > 0$, when h is large, we get:

$$(64) \quad \mathbb{P}(h^{-2}\tau_{0,h} > \alpha) \underset{h \text{ large}}{\sim} \left(1 - h^{-(\delta+1)}\right)^{\alpha^{(1+\delta)/2} h^{\delta+1}/c} \rightarrow e^{-\frac{1}{c}\alpha^{\frac{1+\delta}{2}}},$$

showing that the time to failure $\tau_{0,h}$ at h is now of order $\mathbb{M}(\tau_{0,h}) \asymp h^2$ with a Weibull limit law. We shall summarize these results as follows:

Proposition 11. *Assume $\delta \in (-1, 1)$ (null recurrence of the special walker). Then,*

- (i) *the typical (median) height h_N of its largest excursion satisfies $h_N \asymp N^{1/2}$.*
- (ii) *the typical (median) length k_N of its largest excursion satisfies $k_N \asymp N$, so still with $h_N \asymp k_N^{1/2}$.*
- (iii) *the typical (median) time to failure at level h satisfies: $\mathbb{M}(\tau_{0,h}) \asymp h^2$.*

From this result concerning the null recurrent case and the previous one in the positive recurrent case, we observe that the typical height of the largest excursion always scales like the square-root of its length.

6.3. Expected height of a length k excursion. The previous study in the positive recurrent case, suggests that, at least when $\delta \in (1, 2)$, it could always be true that the expected height of any excursion (not only the largest) should scale like the square-root of its length. If this were to be the case, defining the empirical mean of the walk as

$$(65) \quad \bar{X}_N \equiv \frac{1}{N} \sum_{n=1}^N X_n,$$

then,

$$\mathbb{E}(\bar{X}_N) = \frac{1}{N} \sum_{k=1}^N k \frac{N}{\mu} \mathbb{P}(\tau_{0,0} = k) k^{1/2},$$

because there are approximately $\frac{N}{\mu} \mathbb{P}(\tau_{0,0} = k)$ excursions of length k whose contribution to the empirical mean is $k \frac{N}{\mu} \mathbb{P}(\tau_{0,0} = k)$, each with supposed expected height $k^{1/2}$. Due to (14) and (16), we therefore expect that in the range $\delta \in (1, 2)$

$$(66) \quad \mathbb{E}(\bar{X}_N) \asymp \sum_{k=1}^N k^{3/2} \mathbb{P}(\tau_{0,0} = k) = \sum_{k=1}^N k^{-\delta/2} \asymp N^{1-\delta/2}.$$

If the conjecture is to hold true, then the expected height of the RW should grow like $N^{1-\delta/2}$, which is a sub-diffusive regime. We shall see in the next Section that this is indeed the case, supporting this conjecture, see (77). Rather considering the median value $\mathbb{M}(\bar{X}_N)$ of \bar{X}_N , when $\delta \in (1, 2)$, we get

$$\mathbb{M}(\bar{X}_N) \asymp \sum_{k=1}^{k_N} k^{3/2} \mathbb{P}(\tau_{0,0} = k) = \sum_{k=1}^{k_N} k^{-\delta/2} \asymp k_N^{1-\delta/2} \asymp N^{(2-\delta)/(\delta+1)},$$

with $\mathbb{M}(\bar{X}_N) \ll \mathbb{E}(\bar{X}_N) \ll h_N = \mathbb{M}(\max(X_1, \dots, X_N))$ for the parameter range $\delta \in (1, 2)$ under concern. The estimated median height $\mathbb{M}(\bar{X}_N)$ of \bar{X}_N can also be

obtained in the following way:

$$(67) \quad \mathbb{M}(\bar{X}_N) \asymp \sum_{x=1}^{h_N} x \mathbb{P}(\bar{X}_N = x) \asymp \sum_{x=1}^{h_N} x x^{-\delta} \asymp h_N^{2-\delta} \asymp N^{(2-\delta)/(\delta+1)},$$

where we used that when N is large, $\mathbb{P}(\bar{X}_N = x)$ is ‘close’ to the invariant measure π which scales for large x like $x^{-\delta}$. Indeed, with $\|P - Q\|_{TV} \equiv \sup_A \{P(A) - Q(A)\}$ defining the total variation distance between probability measures P and Q ,

$$\|\mathbb{P}(\bar{X}_N = \cdot) - \pi\|_{TV} \asymp \mathbb{P}(X_N \geq h_N) \asymp h_N^{1-\delta} \asymp N^{-(\delta-1)/(\delta+1)},$$

from (59). This heuristic approach also suggests that the convergence to equilibrium of $\mathbb{P}(\bar{X}_N = \cdot)$ should be algebraically slow when $\delta \in (1, 2)$. To summarize:

Proposition 12. *Let \bar{X}_N in (65) define the empirical mean height of the walker by time N .*

(i) *Assume $\delta \in (1, 2)$. Then, the expected value $\mathbb{E}(\bar{X}_N)$ of \bar{X}_N satisfies: $\mathbb{E}(\bar{X}_N) \asymp N^{1-\delta/2}$. Its typical (median) value $\mathbb{M}(\bar{X}_N)$ satisfies: $\mathbb{M}(\bar{X}_N) \asymp N^{(2-\delta)/(\delta+1)}$, with $\mathbb{M}(\bar{X}_N) \ll \mathbb{E}(\bar{X}_N) \ll \mathbb{M}(\max(X_1, \dots, X_N)) \asymp h_N \asymp N^{1/(\delta+1)}$.*

(ii) *When $\delta > 2$, $\mathbb{E}(\bar{X}_N) \rightarrow_{N \nearrow \infty} \mathbb{E}(X_\infty) = \frac{\delta}{2(\delta-2)}$, the finite mean of the invariant measure π .*

6.4. Excursion height and length for the dual random walk in the null recurrent regime ($\delta \in (-1, 1)$). The above analysis can also be performed in this case. Indeed, the dual harmonic function is

$$\varphi^*(x) = 1 + \sum_{y=1}^{x-1} \prod_{z=1}^y \frac{z}{z+\delta}, \quad x \geq 1, \quad \varphi^*(0) \equiv 0$$

with $\varphi^*(x) \asymp x^{1-\delta}$. The dual height’s law is given by $\mathbb{P}(H^* \geq h) = \frac{\varphi^*(1)}{\varphi^*(h)}$, $h \geq 1$ so that $\mathbb{P}(H^* \geq h) \asymp h^{-(1-\delta)}$. Recalling from (53) that $\mathbb{P}(\tau_{0,0}^* > k) \sim c_* \cdot k^{-(1-\delta)/2}$ where $c_* = -\frac{2^{\delta-1}}{\delta\Gamma((1-\delta)/2)\Gamma(\delta-1)}$, there are on average $N^{(1-\delta)/2}/c_*$ excursions by time N , leading to the typical height and length of the largest excursion: $h_N^* \asymp N^{1/2}$ and $k_N^* \asymp N$ satisfying: $h_N^* = k_N^{*1/2}$. Next, when $\delta \in (-1, 1)$, the time to failure $\tau_{0,h}^*$ of the dual RW at h is also of order h^2 .

7. MOMENTS OF THE SPECIAL RANDOM WALK AND ITS DUAL

The objective of this paragraph is to understand the large n behavior of the spatial moments of the RW $(X_n; n \geq 0)$, specifically $\mathbb{E}_x(X_n^k)$, given $X_0 = x$. Without loss of generality, we shall limit ourselves to the simpler case $X_0 = 0$ because, to the leading order, $\mathbb{E}_x(X_n^k)$ does not depend on x . We shall give the general ingredients on how to compute these moments and only detailed computations for the mean height ($k = 1$). We shall also consider the same problem for the dual RW. Thanks to the knowledge of the spectral measures for both chains, precise informations on the mean heights reached by time n can be evaluated for large n .

7.1. Dual random walk. We start with the simpler computation of the spatial moments of the dual RW $(X_n^*; n \geq 0)$, namely $\mathbb{E}_x(X_n^{*k})$, $k = 1, 2, \dots$. First, using the spectral representation of the dual RW, we get

$$(68) \quad \mathbb{E}_x(X_n^{*k}) = \sum_{y \geq 0} y^k \mathbb{P}_x(X_n^* = y) = \sum_{y=0}^n y^k \gamma_y^* \int_{-1}^1 t^n Q_x^*(t) Q_y^*(t) d\mu^*(t).$$

For $|z| < 1$, defining the generating function of the moments

$$g_{k,x}(z) = \sum_n z^n \mathbb{E}_x(X_n^{*k}),$$

we get:

$$g_{k,x}(z) = \int_{-1}^1 d\mu^*(t) \frac{Q_x^*(t)}{1-zt} \Psi_t^{*(k)}(zt)$$

where $\Psi_t^{*(k)}(zt) \equiv (u \partial_u)^k \Psi_t^*(zt)$ is the k -th derivative of the weighted generating function of the orthogonal polynomials

$$\Psi_t^*(u) \equiv \sum_{y \geq 0} u^y \gamma_y^* Q_y^*(t),$$

evaluated at $u = zt$. Note that

$$\mathbb{E}_x(X_n^{*k}) = [z^n] g_{k,x}(z),$$

and so the order k -moment is the z^n -coefficient of the Stieltjes transform of the measure

$$t \rightarrow \frac{Q_x^*(t)}{1-zt} \Psi_t^{*(k)}(zt) d\mu^*(t)$$

on the interval $[-1, 1]$. Large n behavior of $\mathbb{E}_x(X_n^{*k})$ is dictated by the singularity type of $g_{k,x}(z)$ at $z = 1$.

Under this form, it is clear that, to the leading order, $\mathbb{E}_x(X_n^{*k})$ does not depend on the starting point $X_0^* = x$. Indeed, the integral giving $\mathbb{E}_x(X_n^{*k})$, for large n , is dominated by the values of t close to 1 where $Q_x^*(1) = 1$, independently of x . We can therefore limit ourselves to the simpler case $X_0^* = 0$ (for which $Q_0^*(t) = 1$).

Next, concentrating on $k = 1$ and assuming the times n are even numbers, we get $\mathbb{E}_0(X_n^*) = [z^n] g_{1,0}^{even}(z)$ where:

$$g_{1,0}^{even}(z) = \int_0^1 d\mu^*(t) \left(\frac{\Psi_t^{*(1)}(zt)}{1-z^2 t^2} - \frac{\Psi_t^{*(1)}(z)}{1-z^2} \right)$$

is the leading contribution of the even part of $g_{1,0}(z)$. Recall the expression of the spectral measure of the dual RW in (20).

Generating function: Thus, the computation of the mean value $\mathbb{E}_0(X_n^*)$ requires the preliminary computation of the generating function: $\Psi_t^*(u)$. To do this, we can proceed as follows: Recall $\gamma_y^* = (2y + \delta) \frac{\Gamma(y + \delta)}{\Gamma(\delta + 1) \cdot y!}$; Let first $G_y^{(\delta/2)}(t) = \frac{\Gamma(y + \delta)}{\Gamma(\delta) \cdot y!} Q_y^*(t)$, $y \geq 0$ be a new sequence of polynomials, satisfying $G_0(t) = 1$. We

can identify these polynomials as being the Gegenbauer polynomials with parameter $\frac{\delta}{2}$. We shall first compute the generating function $\Phi_t^{(\delta/2)}(u) = \sum_{y \geq 0} u^y G_y(t)$. Next, we clearly have:

$$(69) \quad \Psi_t^*(u) = \frac{1}{\delta} \sum_{y \geq 0} (2y + \delta) u^y G_y(t) = \frac{2}{\delta} u \partial_u \Phi_t^{(\delta/2)}(u) + \Phi_t^{(\delta/2)}(u).$$

The function $\Phi_t^{(\delta/2)}(u)$ being the generating function of the Gegenbauer polynomials with parameter $\frac{\delta}{2}$, with $q_t(u) \equiv 1 - 2ut + u^2$, it solves the first order differential equation

$$q_t(u) \partial_u \Phi_t^{(\delta/2)}(u) = \delta(t - u) \Phi_t^{(\delta/2)}(u), \quad \Phi_t^{(\delta/2)}(0) = 1,$$

with well-known solution: $\Phi_t^{(\delta/2)}(u) = q_t(u)^{-\delta/2}$. Finally, we obtain

Proposition 13. *The weighted generating function of the $Q_y^*(t)$, $y \geq 0$ is:*

$$(70) \quad \Psi_t^*(u) = (1 - u^2) q_t(u)^{-(\delta/2+1)},$$

with first derivative is

$$\begin{aligned} \Psi_t^{*(1)}(u) &= (u \partial_u) \Psi_t^*(u) = \\ &= u q_t(u)^{-(\delta/2+2)} (-2u q_t(u) + (\delta + 2)(t - u)(1 - u^2)). \end{aligned}$$

Singularity analysis (sketch): When both z and t get close to 1 with $1 - t = x(1 - z)$ for some $x > 0$, $1 - zt \sim (1 - z)(1 + x)$ and $q_t(zt) = (1 - zt)^2 + 2zt(1 - t) \sim 2x(1 - z)$. Next

$$\Psi_t^{*(1)}(zt) \sim -2q_t(zt)^{-(\delta/2+1)} \sim -2(2x)^{-(\frac{\delta}{2}+1)} (1 - z)^{-(\delta/2+1)}.$$

Therefore, performing the change of variables $t \rightarrow x$ in the integral giving $g_{1,0}^{even}(z)$, we obtain

$$g_{1,0}^{even}(z) \sim \frac{\Gamma(\frac{\delta}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{\delta+1}{2})} \frac{1}{\sqrt{2}} (1 - z)^{-3/2} \int_0^\infty \frac{x^{-\frac{1}{2}} dx}{1 + x} = \frac{\Gamma(\frac{\delta}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{\delta+1}{2})} \frac{\pi}{\sqrt{2}} (1 - z)^{-3/2}.$$

This shows (see (45)) by singularity analysis at $z = 1$ that

Proposition 14. *The expected height of the dual random walk satisfies:*

$$(71) \quad \mathbb{E}_0(X_n^*) \underset{n \nearrow \infty}{\sim} \frac{\Gamma(\frac{\delta}{2} + 1)}{\Gamma(\frac{\delta+1}{2})} (2n)^{1/2},$$

for all values of $\delta > -1$.

More generally, $\mathbb{E}_0(X_n^{*k}) \asymp n^{k/2}$. These results were first stated (without much detailed proofs) in [1] and [2]. For more details on the proportionality constants and correcting terms, see these authors. The striking feature of the dual RW is that the scaling exponents of the spatial moments do not depend on δ : From the point of view of the moments, the dual walker does not ‘see’ the passage from recurrence to transience when δ crosses 1. As we shall now sketch, this is not the case in general for the special RW under study: the random walker is sensitive to the passage from null to positive recurrence as δ crosses 1.

7.2. Direct random walk. Assume $\delta > -1$. Let us now come to our problem of estimating the large n behavior of $\mathbb{E}_0(X_n)$ for the special RW. We shall proceed similarly. First, we have:

$$(72) \quad \mathbb{E}_0(X_n) = [z^n] g_{1,0}(z) \text{ where } g_{1,0}(z) \equiv \int_{-1}^1 \frac{\Psi_t^{(1)}(zt)}{1-zt} d\mu(t).$$

Here $\Psi_t(u) = \sum_{y \geq 0} u^y \gamma_y Q_y(t)$ now is the weighted generating function of the polynomials $(\gamma_y Q_y(t); y \geq 0)$ with $Q_0(t) = 1$ and $Q_1(t) = t$. It has to be computed explicitly. Recalling $\gamma_y = (2y + \delta) \frac{\Gamma(\delta+1)\Gamma(y)}{\Gamma(y+\delta+1)}$, we first introduce

$$(73) \quad H_y(t) = \frac{\Gamma(\delta+1)\Gamma(y)}{\Gamma(y+\delta+1)} Q_y(t), y \geq 1,$$

a new sequence of polynomials. We shall first compute the generating function $\Phi_t(u) = \sum_{y \geq 1} u^y H_y(t)$; Then, with $\Phi'_t(u) = \partial_u \Phi_t(u)$, we will deduce:

$$\Psi_t(u) = \sum_{y \geq 1} (2y + \delta) u^y H_y(t) = 2u\Phi'_t(u) + \delta\Phi_t(u).$$

Let us then compute $\Phi_t(u)$. The polynomials sequence $H_y(t)$ satisfies the 3-term recurrence:

$$(2y + \delta)tH_y(t) = (y + \delta + 1)H_{y+1}(t) + (y - 1)H_{y-1}(t), y \geq 2.$$

From this and the initial conditions $H_1(t) = \frac{t}{\delta+1}$ and $H_2(t) = \frac{(\delta+2)t^2 - (\delta+1)}{(\delta+1)(\delta+2)}$, $\Phi_t(u)$ solves the first-order differential equation:

$$uq_t(u)\Phi'_t(u) = u(t-u) - \delta(1-ut)\Phi_t(u), \Phi_t(0) = 0,$$

whose solution is obtained (by the method of variation of the parameter of the fundamental system) as:

$$(74) \quad \Phi_t(u) = \frac{1}{\delta} - \left(\frac{\sqrt{q_t(u)}}{u} \right)^\delta \int_0^u v^{\delta-1} q_t(v)^{-\delta/2} dv.$$

Consequently,

$$(75) \quad \Psi_t(u) = -1 + \delta \frac{1-u^2}{q_t(u)} \left[\frac{1}{\delta} - \Phi_t(u) \right]$$

follows, together with $\Psi_t^{(1)}(u)$. Next, assuming the n are even numbers, we get $\mathbb{E}_0(X_n) = [z^n] g_{1,0}^{even}(z)$ where:

$$g_{1,0}^{even}(z) = \int_0^1 d\mu_c(t) \left(\frac{\Psi_t^{(1)}(zt)}{1-z^2t^2} - \frac{\Psi_t^{(1)}(z)}{1-z^2} \right)$$

is the leading contribution of the even part of $g_{1,0}(z)$ (see [5]). Note that only $d\mu_c(t)$ appears in $g_{1,0}^{even}(z)$. As for the dual RW, we are then in the position to perform the singularity analysis of $g_{1,0}^{even}(z)$. The question of evaluating the large n scaling behavior of $\mathbb{E}_0(X_n)$ only makes sense when $\delta \in (-1, 2]$ because when $\delta > 2$, the RW is positive recurrent and has finite mean in the limit: $\mathbb{E}_0(X_n) \rightarrow \mathbb{E}(X_\infty) = \sum_x x\pi_x$, with $(\pi_x; x \geq 0)$ the invariant probability measure of the chain given by (6) and (7).

From the previous study indeed, three cases arise:

- $\delta \in (-1, 1)$. (null recurrence). In this parameter range, Laplace method gives:

$$(76) \quad \mathbb{E}_0(X_n) \underset{n \nearrow \infty}{\asymp} n^{1/2},$$

just like for the simple fair RW ($\delta = 0$). Thus, for the whole range $\delta \in (-1, 1)$, the expected height of the random walker grows like $n^{1/2}$ with a dependency on parameter δ only in the proportionality constant.

- $\delta \in (1, 2)$. (a case of positive recurrence). By singularity analysis, we get the result in [5] to which we refer for additional technical details

Proposition 15. *When $\delta \in (1, 2)$, the expected height of the special walker satisfies:*

$$(77) \quad \mathbb{E}_0(X_n) \underset{n \nearrow \infty}{\sim} K_\delta \cdot n^{1-\delta/2},$$

where

$$K_\delta = 2^{\frac{\delta-5}{2}} \sqrt{\frac{\pi}{2}} \frac{\Gamma(\frac{\delta-1}{2})}{1-\frac{\delta}{2}} / \int_{-1}^1 dt \frac{(1-t^2)^{\frac{\delta-3}{2}}}{|F(t)|^2}.$$

- $\delta > 2$, $\mathbb{E}_0(X_n) \underset{n \nearrow \infty}{\rightarrow} \mathbb{E}(X_\infty) = \frac{\delta}{2(\delta-2)}$, from (8).

As discussed before, when δ passes 1, the system becomes pinned in that it spends a positive fraction of time at the origin. Still, when $\delta \in (1, 2)$, the expected position by time n of the walker diverges.

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