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# DISLOCATION MEASURE OF THE FRAGMENTATION OF A GENERAL LÉVY TREE

GUILLAUME VOISIN

ABSTRACT. Given a general critical or sub-critical branching mechanism and its associated Lévy continuum random tree, we consider a pruning procedure on this tree using a Poisson snake. It defines a fragmentation process on the tree. We compute the family of dislocation measures associated with this fragmentation. This work generalizes the work made for a Brownian tree [3] and for a tree without Brownian part [1].

## 1. INTRODUCTION

Continuous state branching processes (CSBP) have been introduced by Jirina [10] and it is known since Lamperti [11] that these processes are the scaling limits of Galton-Watson processes. They model the evolution of a large population on a long time interval. The law of a CSBP is characterized by the so-called branching mechanism  $\psi$ . When the CSBP is critical or sub-critical, one can associate a continuum random tree (CRT) which describes the genealogy of the CSBP. The construction of fragmentation processes from CRTs have already been studied by Abraham and Serlet [3] for the Brownian CRT (case where the Lévy measure of  $\psi$  is null) and by Abraham and Delmas [1] for the CRT without Brownian part (case where  $\psi$  has no quadratic part). In these works, Lévy Poisson snakes are used to create marks on the CRT and to obtain a fragmentation process. In the first case, the marks are built on the skeleton of the CRT, in the second, they are placed on the nodes. Abraham, Delmas and Voisin [2] constructed a general pruning of a CRT where the marks are placed on the whole CRT, skeleton and nodes. In this work, they study the law of the sub-tree obtained after the pruning according to the marks.

The aim of this article is to study the fragmentation process associated with a general CRT and more precisely the dislocation measure associated with. Note that this measure has been studied in the Brownian case and in the case without Brownian part (see [3] and [1]).

The three following parts give a brief presentation of the mathematical objects and give the main results.

**1.1. The exploration process.** The coding of the CRT by its height process is well known. The height process of Aldous' CRT [4] is a normalized Brownian excursion. In [13], Le Gall and Le Jan associated with a Lévy process with no negative jumps that does not drift to infinity,  $X = (X_t, t \geq 0)$ , a CSBP and a Lévy CRT which keeps track of the genealogy of the CSBP. Let  $\psi$  be the Laplace exponent of the general process  $X$ . By the Lévy-Khintchine formula,  $\psi$  is such that  $\mathbb{E}[e^{-\lambda X_t}] = e^{t\psi(\lambda)}$  and can be expressed by

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda l} - 1 + \lambda l)\pi(dl)$$

with  $\alpha \geq 0$ ,  $\beta \geq 0$  and the Lévy measure  $\pi$  is a positive  $\sigma$ -finite measure on  $(0, \infty)$  such that  $\int_{(0, \infty)} (l \wedge l^2) \pi(dl) < \infty$ . Following [8], we assume that  $X$  is of infinite variation, which implies that  $\beta > 0$  or  $\int_{(0, 1)} l \pi(dl) = +\infty$ .

The height process  $H = (H_t, t \geq 0)$  associated with the process  $X$  gives a distance (which corresponds to the number of generations) between each individual labeled  $t$  and the root of the CRT. This process is an important object but is not a Markov process in general. Thus we introduce, the exploration process  $\rho = (\rho_t, t \geq 0)$  which is a càd-làg, strong Markov process taking values in  $\mathcal{M}_f(\mathbb{R}_+)$ , the set of finite measures on  $\mathbb{R}_+$  endowed with the topology of weak convergence. The height process can easily be recovered from the exploration process as  $H_t = H(\rho_t)$  where  $H(\mu)$  is the supremum of the closed support of the measure  $\mu$  (with the convention that  $H(0) = 0$ ).

The definitions of these processes are recalled in Section 2.

**1.2. The fragmentation.** A fragmentation process is a Markov process which describes how an object with given total mass evolves as it breaks into several fragments randomly as time passes. This kind of processes has been widely studied in [6]. To be more precise, the state space of a fragmentation process is the space of the non-increasing sequences of masses with finite total mass

$$\mathcal{S}^\downarrow = \{s = (s_1, s_2, \dots); s_1 \geq s_2 \geq \dots \geq 0 \text{ and } \sum_{k=1}^{\infty} s_k < \infty\}.$$

If we denote by  $P_s$  the law of a  $\mathcal{S}^\downarrow$ -valued process  $\Lambda = (\Lambda^\theta, \theta \geq 0)$  starting at  $s = (s_1, s_2, \dots) \in \mathcal{S}^\downarrow$ , we say that  $\Lambda$  is a fragmentation process if it is a Markov process such that  $\theta \mapsto \sum(\Lambda^\theta)$  is decreasing and if it fulfills the fragmentation property : the law of  $(\Lambda^\theta, \theta \geq 0)$  under  $P_s$  is the non-increasing reordering of the fragments of independent processes of respective laws  $P_{(s_1, 0, \dots)}, P_{(s_2, 0, \dots)}, \dots$ . In other words, each fragment behaves independently of the others, and its evolution depends only of its initial mass. Hence, it suffices to study the laws  $P_r := P_{(r, 0, \dots)}$  that is the law of the fragmentation process starting with a single mass  $r \in (0, \infty)$ .

In order to construct our fragmentation process, we need to place marks on the nodes of the CRT and on the skeleton of the CRT. We follow the construction of [2]. However we have to adapt this construction to obtain a process which depends on time  $\theta$ .

Recall that the nodes of the CRT are represented by the Lévy measure  $\pi$  of the process  $X$ . To construct a measure which marks these nodes, we use the jumps part of the definition of the exploration process  $\rho$ . We also use a Poisson point measure (see Section 3.2) to increase the number of marks according to the parameter  $\theta$ .

For the marks on the skeleton of the CRT, we recall the construction of a Lévy Poisson snake made in [8] but we introduce the new parameter  $\theta$ . The marks will be attached by a Poisson process with intensity  $2\beta\theta$  (see Section 3.1).

By cutting according to these marks, we obtain a set of fragments. Let  $s_1, s_2, \dots$  be the "sizes" of these fragments ranked by non-increasing order completed with 0 if necessary so that  $(s_1, s_2, \dots) \in \mathcal{S}^\downarrow$ . When time  $\theta$  increases, the number of marks increases and the fragments break again. Thus we obtain a process  $(\Lambda^\theta, \theta \geq 0)$ , Theorem 4.1 checks that this process is a fragmentation.

**1.3. The dislocation measure.** The evolution of the process  $\Lambda$  is described by a family  $(\nu_r, r \geq 0)$  of  $\sigma$ -finite measures called dislocation measures.  $\nu_r$  describes how a fragment of

size  $r$  breaks into smaller fragments. The measure  $\nu_r$  can be written as

$$\nu_r = \nu_r^{nod} + \nu_r^{ske}.$$

$\nu_r^{nod}$  corresponds to a mark that appears on the node whereas  $\nu_r^{ske}$  to a mark on the skeleton.

The expression of the measure  $\nu_r^{ske}$  is the main result of this article :

**Theorem 1.1.** *Let  $S$  be a subordinator with Laplace exponent  $\psi^{-1}$ , let  $\pi_*$  be its Lévy measure.*

(1) *For all non negative measurable function  $F$  on  $\mathcal{S}^\downarrow$ ,*

$$\int_{\mathbb{R}_+ \times \mathcal{S}^\downarrow} F(x) \nu_r^{nod}(dx) \pi_*(dr) = \int \pi(dv) \mathbb{E}[S_v F((\Delta S_u, u \leq v))]$$

*where  $(\Delta S_u, u \leq v)$  represents the jumps of  $S$  before time  $v$ , ranked by decreasing order.*

(2) *The measure  $\nu_r^{ske}$  charges only the set of elements of  $\mathcal{S}^\downarrow$  of the form  $(x_1, x_2, 0, \dots)$  with  $x_1 \geq x_2$  and  $x_1 + x_2 = r$ . It is the distribution of the non-increasing reordering of the lengths given by the measure  $\hat{\nu}_r^{ske}$  defined by*

$$\int_{\mathbb{R}_+ \times \mathcal{S}^\downarrow} \frac{1}{x_2} (1 - e^{-\lambda_1 x_1}) (1 - e^{-\lambda_2 x_2}) \hat{\nu}_r^{ske}(dx) \pi_*(dr) = 2\beta \psi^{-1}(\lambda_1) \psi^{-1}(\lambda_2).$$

*Remark 1.2.* Under  $\hat{\nu}_r^{ske}(dx) \pi_*(dr)$ , the lengths of the two fragments are independent.

*Remark 1.3.* The proof of Part 1 is the same as in [1]. Only Part 2 needs a proof.

## 2. THE LÉVY SNAKE : NOTATIONS AND PROPERTIES

**2.1. The Lévy process.** We consider a  $\mathbb{R}$ -valued Lévy process  $(X_t, t \geq 0)$  with no negative jumps, starting from 0 characterized by its Laplace exponent  $\psi$  given by

$$\psi(\lambda) = \alpha_0 \lambda + \beta \lambda^2 + \int_{(0, +\infty)} \pi(d\ell) \left( e^{-\lambda \ell} - 1 + \mathbf{1}_{\ell < 1} \lambda \ell \right),$$

with  $\beta \geq 0$  and the Lévy measure  $\pi$  is a positive,  $\sigma$ -finite measure on  $(0, +\infty)$  such that  $\int_{(0, +\infty)} (1 \wedge \ell^2) \pi(d\ell) < \infty$ . We also assume that  $X$

- has first moments (i.e.  $\int_{(0, +\infty)} (\ell \wedge \ell^2) \pi(d\ell) < \infty$ ),
- is of infinite variation (i.e.  $\beta > 0$  or  $\int_{(0, 1)} \ell \pi(d\ell) = +\infty$ ),
- does not drift to  $+\infty$ .

With the first assumption, the Lévy exponent can be written as

$$\psi(\lambda) = \alpha \lambda + \beta \lambda^2 + \int_{(0, +\infty)} \pi(d\ell) \left( e^{-\lambda \ell} - 1 + \lambda \ell \right),$$

with  $\alpha \geq 0$  thanks to the third assumption.

Let  $\mathcal{J} = \{t \geq 0; X_t \neq X_{t-}\}$  be the set of jumping times of the process  $X$ .

For  $\lambda \geq \frac{1}{\epsilon} > 0$ , we have  $e^{-\lambda \ell} - 1 + \lambda \ell \geq \frac{1}{2} \lambda \ell \mathbf{1}_{\ell \geq 2\epsilon}$  this implies that  $\lambda^{-1} \psi(\lambda) \geq \alpha + \beta \frac{1}{\epsilon} + \int_{(2\epsilon, \infty)} \ell \pi(d\ell)$ . We deduce that

$$\lim_{\lambda \rightarrow \infty} \frac{\lambda}{\psi(\lambda)} = 0.$$

Let  $I = (I_t, t \geq 0)$  be the infimum process of  $X$ ,  $I_t = \inf_{0 \leq s \leq t} X_s$ . We also denote for all  $0 \leq s \leq t$ , the minimum of  $X$  on  $[s, t]$  :

$$I_t^s = \inf_{s \leq r \leq t} X_r.$$

The point 0 is regular for the Markov process  $X - I$ , and  $-I$  is the local time of  $X - I$  at 0 (see [5], Chap. VII). Let  $\mathbb{N}$  be the excursion measure of the process  $X - I$  away from 0, and let  $\sigma = \inf\{t > 0; X_t - I_t = 0\}$  be the lengths of the excursions of  $X - I$  under  $\mathbb{N}$ . Notice that, under  $\mathbb{N}$ ,  $X_0 = I_0 = 0$ .

Thanks to [5], Theorem VII.1, the right-continuous inverse of the process  $-I$  is a subordinator with Laplace exponent  $\psi^{-1}$ . We have already seen that this exponent has no drift, because  $\lim_{\lambda \rightarrow \infty} \lambda \psi(\lambda)^{-1} = 0$ . We denote by  $\pi_*$  its Lévy measure : for all  $\lambda \geq 0$

$$\psi^{-1}(\lambda) = \int_{(0, \infty)} \pi_*(dl)(1 - e^{-\lambda l}).$$

Under  $\mathbb{N}$ ,  $\pi_*$  is the law of the length of the excursions,  $\sigma$ . By decomposing the measure  $\mathbb{N}$  w.r.t. the distribution of  $\sigma$ , we get that  $\mathbb{N}(d\mathcal{E}) = \int_{(0, \infty)} \pi_*(dr) \mathbb{N}_r(d\mathcal{E})$ , where  $(\mathbb{N}_r, r \in (0, \infty))$  is a measurable family of probability measures on the set of excursions such that  $\mathbb{N}_r[\sigma = r] = 1$  for  $\pi_*$ -a.e.  $r > 0$ .

**2.2. The height process and the Lévy CRT.** For all  $t \geq 0$ , we consider the reversed process at time  $t$ ,  $\hat{X}^{(t)} = (\hat{X}_s^{(t)}, 0 \leq s \leq t)$  defined by :

$$\hat{X}_s^{(t)} = X_t - X_{(t-s)-} \quad \text{if } 0 \leq s < t,$$

and  $\hat{X}_t^{(t)} = X_t$ . We denote by  $\hat{S}^{(t)}$  the supremum process of  $\hat{X}^{(t)}$  and  $\hat{L}^{(t)}$  the local time at 0 of  $\hat{S}^{(t)} - \hat{X}^{(t)}$  with the same normalization as in [1].

**Definition 2.1.** *There exists a  $[0, \infty]$ -valued lower semi-continuous process, called the height process such that  $H_0 = 0$  and for all  $t \geq 0$ , a.s.  $H_t = \hat{L}_t^{(t)}$ . And a.s. for all  $s < t$  such that  $X_{s-} \leq I_t^s$  and for  $s = t$ , if  $\Delta_t > 0$  then  $H_s < \infty$  and for all  $t' > t \geq 0$ , the process  $H$  takes all the values between  $H_t$  and  $H_{t'}$  on the time interval  $[t, t']$ .*

The height process  $(H_t, t \in [0, \sigma])$  under  $\mathbb{N}$ , codes a continuum genealogical tree, the Lévy CRT (see [2], Section 2.2).

**2.3. The exploration process.** The height process is not a Markov process. But in general it is a very simple function of a measure-valued Markov process, the exploration process. If  $E$  is a polish space, we denote by  $\mathcal{B}(E)$  (resp.  $\mathcal{B}_+(E)$ ) the set of  $\mathbb{R}$ -valued measurable (resp. and non-negative) functions defined on  $E$  endowed with its Borel  $\sigma$ -field, and by  $\mathcal{M}(E)$  (resp.  $\mathcal{M}_f(E)$ ) the set of  $\sigma$ -finite (resp. finite) measures on  $E$ , endowed with the topology of vague (resp. weak) convergence. For any measure  $\mu \in \mathcal{M}(E)$ , and any function  $f \in \mathcal{B}_+(E)$ , we write

$$\langle \mu, f \rangle = \int f(x) \mu(dx).$$

The exploration process  $\rho = (\rho_t, t \geq 0)$  is a  $\mathcal{M}_f(\mathbb{R}_+)$ -valued process defined by, for every  $f \in \mathcal{B}_+(\mathbb{R}_+)$ ,  $\langle \rho_t, f \rangle = \int_{[0, t]} d_s I_t^s f(H_s)$ , or equivalently

$$\rho_t(dr) = \beta \mathbf{1}_{[0, H_t]}(r) dr + \sum_{\substack{0 < s \leq t \\ X_{s-} < I_t^s}} (I_t^s - X_{s-}) \delta_{H_s}(dr).$$

In particular, the total mass of  $\rho_t$  is  $\langle \rho_t, 1 \rangle = X_t - I_t$ .

For  $\mu \in \mathcal{M}(\mathbb{R}_+)$ , we put

$$H(\mu) = \sup \text{Supp } \mu,$$

where  $\text{Supp } \mu$  is the closed support of  $\mu$  with the convention  $H(0) = 0$ . We have

**Proposition 2.2.** ([8], Lemma 1.2.2 and Formula (1.12))

Almost surely, for every  $t > 0$ ,

- $H(\rho_t) = H_t$ ,
- $\rho_t = 0$  if and only if  $H_t = 0$ ,
- if  $\rho_t \neq 0$ , then  $\text{Supp } \rho_t = [0, H_t]$ ,
- $\rho_t = \rho_{t-} + \Delta_t \delta_{H_t}$ , where  $\Delta_t = 0$  if  $t \notin \mathcal{J}$ .

*Remark 2.3.*  $\langle \rho_t, 1 \rangle = X_t - I_t$ .

In the definition of the exploration process, as  $X$  starts from 0, we obtain  $\rho_0 = 0$  a.s. To state the Markov property of  $\rho$ , we must first define the process  $\rho$  starting at any initial measure  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ . We recall the notations given in [8].

For  $a \in [0, \langle \mu, 1 \rangle]$ , we write  $k_a \mu$  for the erased measure which is the measure  $\mu$  erased by a mass  $a$  backward from  $H(\mu)$ . In particular,  $\langle k_a \mu, 1 \rangle = \langle \mu, 1 \rangle - a$ .

For  $\nu, \mu \in \mathcal{M}_f(\mathbb{R}_+)$ , and  $\mu$  with compact support, we write  $[\mu, \nu] \in \mathcal{M}_f(\mathbb{R}_+)$  for the concatenation of the two measures.

Finally, we put for all  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$  and for all  $t > 0$ ,

$$\rho_t^\mu = [k_{-I_t} \mu, \rho_t].$$

We say that  $(\rho_t^\mu, t \geq 0)$  is the process  $\rho$  starting from  $\rho_0^\mu = \mu$ , and write  $\mathbb{P}_\mu$  for its law. Unless there is an ambiguity, we shall write  $\rho_t$  for  $\rho_t^\mu$ . We also denote by  $\mathbb{P}_\mu^*$  the law of  $\rho^\mu$  killed when it reaches 0.

**Proposition 2.4.** ([8], Proposition 1.2.3)

The process  $(\rho_t, t \geq 0)$  is a càd-làg strong Markov process in  $\mathcal{M}_f(\mathbb{R}_+)$ .

*Remark 2.5.* As in [1], 0 is also a regular point for  $\rho$ . Notice that  $\mathbb{N}$  is also the excursion measure of the process  $\rho$  away from 0, and that  $\sigma$ , the length of the excursion, is  $\mathbb{N}$ -a.e. equal to  $\inf\{t > 0; \rho_t = 0\}$ .

Exponential formula for the Poisson point process of jumps of the inverse subordinator of  $-I$  gives (see also the beginning of the Section 3.2.2 [8]) that for  $\lambda > 0$

$$\mathbb{N} \left[ 1 - e^{-\lambda \sigma} \right] = \psi^{-1}(\lambda).$$

**2.4. The dual process and the representation formula.** We shall need the  $\mathcal{M}_f(\mathbb{R}_+)$ -valued process  $\eta = (\eta_t, t \geq 0)$  defined by

$$\eta_t(dr) = \beta \mathbf{1}_{[0, H_t]}(r) dr + \sum_{\substack{0 < s \leq t \\ X_s - < I_t^s}} (X_s - I_t^s) \delta_{H_s}(dr).$$

This process is called the dual process of  $\rho$  under  $\mathbb{N}$  (see Corollary 3.1.6 of [8]). We also denote, for  $s \in [0, \sigma]$  fixed,  $\kappa_s = \rho_s + \eta_s$ . Recall the Poisson representation of  $(\rho, \eta)$  under  $\mathbb{N}$ . Let  $\mathcal{N}(dx dl du)$  be a point Poisson measure on  $[0, +\infty)^3$  with intensity

$$dx l \pi(dl) \mathbf{1}_{[0,1]}(u) du.$$

For all  $a > 0$ , we denote by  $\mathbb{M}_a$  the law of the pair  $(\mu_a, \nu_a)$  of measures on  $\mathbb{R}_+$  with finite mass defined by, for any  $f \in \mathcal{B}_+(\mathbb{R}_+)$

$$\begin{aligned} \langle \mu_a, f \rangle &= \int \mathcal{N}(dx dl du) \mathbf{1}_{[0,a]}(x) u l f(x) + \beta \int_0^a f(r) dr, \\ \langle \nu_a, f \rangle &= \int \mathcal{N}(dx dl du) \mathbf{1}_{[0,a]}(x) (1-u) l f(x) + \beta \int_0^a f(r) dr. \end{aligned}$$

We also put  $\mathbb{M} = \int_0^\infty dae^{-\alpha a}\mathbb{M}_a$ .

**Proposition 2.6.** ([8], Proposition 3.1.3) *For every non-negative measurable function  $F$  on  $\mathcal{M}_f(\mathbb{R}_+)^2$*

$$\mathbb{N} \left[ \int_0^\sigma F(\rho_t, \eta_t) dt \right] = \int \mathbb{M}(d\mu d\nu) F(\mu, \nu)$$

where we recall that  $\sigma = \inf\{s > 0; \rho_s = 0\}$  is the length of the excursion.

We also give the Bismut formula for the height process of the Lévy process.

**Proposition 2.7.** ([9], Lemma 3.4.)

*For every non negative function  $F$  defined on  $\mathcal{B}_+([0, \infty])^2$*

$$\mathbb{N} \left[ \int_0^\sigma ds F((H_{(s-t)_+}, t \geq 0), (H_{(s+t) \wedge \sigma}, t \geq 0)) \right] = \int \mathbb{M}(d\mu d\nu) \int \mathbb{P}_\mu^*(d\rho) \mathbb{P}_\nu^*(d\tilde{\rho}) F(H(\rho), H(\tilde{\rho})).$$

### 3. THE LÉVY POISON SNAKE

As in [2], we construct a Lévy Poisson snake which marks the Lévy CRT on its nodes and on its skeleton. The aim is to fragment the CRT in several fragments using point processes whose intensities depend on a parameter  $\theta$  such that, if  $\theta = 0$ , there is no marks on the CRT and the number of marks increases with  $\theta$ .

**3.1. Marks on the skeleton.** In order to mark the continuous part of the CRT and to keep track of marks along the lineage of each individual, we construct a snake on  $E = \mathcal{M}(\mathbb{R}_+^2)$  where the parameter  $\theta$  appears. To obtain a Polish space, we separate the space of the parameter  $\theta$  in bounded intervals.

We fix  $i \in \mathbb{N}$ , thanks to [7] Section 3.1,  $E_i = \mathcal{M}_f(\mathbb{R}_+ \times [i, i+1])$  the set of finite measures on  $\mathbb{R}_+ \times [i, i+1)$  is a Polish space for the topology of weak convergence.

Thanks to [8], Chap. 4, there exists a  $E_i$ -valued process  $(W_t^i, t \geq 0)$  such that conditionally on  $X$ ,

- (1) For each  $s \in [0, \sigma]$ ,  $W_s^i$  is a Poisson measure on  $[0, H_s] \times [i, i+1)$  with intensity  $2\beta \mathbf{1}_{[0, H_t]}(dr) \mathbf{1}_{[i, i+1)}(\theta) d\theta$ ,
- (2) For every  $s < s'$ ,  $W_{s'}^i(dr, d\theta) \mathbf{1}_{[0, H_{s, s'}]}(r) = W_s^i(dr, d\theta) \mathbf{1}_{[0, H_{s, s'}]}(r)$ .

We take the processes  $W^i$  independently and we set  $m_t^{ske} = \sum_{i \in \mathbb{N}} W_t^i$ .

If  $\beta = 0$ , the CRT has no Brownian part, in this case, there is no mark on the skeleton and we set  $m^{ske} = 0$ .

For  $t \geq 0$  fixed, conditionnally on  $H_t$ ,  $m_t^{ske}$  is Poisson point process with intensity

$$2\beta \mathbf{1}_{[0, H_t]}(r) dr d\theta.$$

The process  $(\rho, m^{ske})$  takes values in the space  $\tilde{\mathcal{M}}_f := \mathcal{M}_f(\mathbb{R}_+) \times \mathcal{M}(\mathbb{R}_+^2)$ . We denote by  $(\mathcal{F}_s, s \geq 0)$  the canonical filtration on the space of càd-làg trajectories on the space  $\tilde{\mathcal{M}}_f$ .

Using Theorem 4.1.2 of [8] when  $H$  is continuous or the adapted result when  $H$  is not continuous (Prop. 7.2, [2]), we get the following result

**Proposition 3.1.**  *$(\rho, m^{ske})$  is a strong Markov process with respect to the filtration  $(\mathcal{F}_{s+}, s \geq 0)$ .*

**3.2. Mark on the nodes.** When the Lévy measure of  $X$  is non trivial, we define the mark process on the nodes of the CRT as in [1]. We use a Poisson point measure to introduce the parameter  $\theta$ . Conditionally on  $X$ , we set

$$m_t^{\text{nod}}(dr, d\theta) = \sum_{\substack{0 < s \leq t \\ X_{s-} < I_t^s}} (I_t^s - X_{s-}) \left( \sum_{u>0} \delta_{V_{s,u}}(d\theta) \right) \delta_{H_s}(dr).$$

with  $\sum_{u>0} \delta_{V_{s,u}}$  is a point Poisson measure with intensity  $\Delta_s \mathbf{1}_{u>0} du$ .

If  $\pi = 0$ , it is the Brownian case and there is no mark on the nodes, thus we set  $m^{\text{nod}} = 0$ .

**3.3. The snake.** We join the marks on the skeleton and the marks on the nodes of the CRT in a mark process  $m = (m^{\text{nod}}, m^{\text{ske}})$ . We write  $\mathcal{S} = (\rho, m)$  the marked snake starting from  $\rho_0 = 0$  and  $m_0 = 0$ .

Let us recall the construction made in [2] to obtain a snake starting from an initial value and then to write a strong Markov property for the snake. We consider the set  $\mathbb{S}$  of triplets  $(\mu, \Pi^{\text{nod}}, \Pi^{\text{ske}})$  such that

- $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ ,
- $\Pi^{\text{nod}}$  can be written as  $\Pi^{\text{nod}}(dr, dx) = \mu(dr) \Pi_r^{\text{nod}}(dx)$  where  $(\Pi_r^{\text{nod}}, r > 0)$  is a family of  $\sigma$ -finite measures on  $\mathbb{R}_+$  and for every  $\theta > 0$ ,  $\Pi^{\text{nod}}(\mathbb{R}_+ \times [0, \theta]) < \infty$ ,
- $\Pi^{\text{ske}} \in \mathcal{M}(\mathbb{R}_+^2)$  and
  - $\text{Supp}(\Pi^{\text{ske}}(\cdot, \mathbb{R}_+)) \subset \text{Supp}(\mu)$
  - for every  $x < H(\mu)$  and every  $\theta > 0$ ,  $\Pi^{\text{ske}}([0, x] \times [0, \theta]) < \infty$ ,
  - if  $\mu(H(\mu)) > 0$ , then for every  $\theta > 0$ ,  $\Pi^{\text{ske}}(\mathbb{R}_+ \times [0, \theta]) < \infty$

Then we define the snake  $\mathcal{S}$  starting from an initial value  $(\mu, \Pi) \in \mathbb{S}$ , where  $\Pi = (\Pi^{\text{nod}}, \Pi^{\text{ske}})$ , we write  $H_t^\mu = H(k_{-I_t} \mu)$ ,

$$(m^{\text{nod}})_t^{(\mu, \Pi)} = \left[ \Pi^{\text{nod}} \mathbf{1}_{[0, H_t^\mu]} + \mathbf{1}_{\mu(\{H_t^\mu\}) > 0} \frac{k_{-I_t} \mu(\{H_t^\mu\}) \Pi^{\text{nod}}(\{H_t^\mu\}, \cdot)}{\mu(\{H_t^\mu\})} \delta_{H_t^\mu} \Pi_{H_t^\mu}^{\text{nod}}, m_t^{\text{nod}} \right]$$

$$(m^{\text{ske}})_t^{\mu, \Pi} = \left[ \Pi^{\text{ske}} \mathbf{1}_{[0, H_0, t)}, m_t^{\text{ske}} \right]$$

Notice that these definitions are coherent with the previous definitions of the processes  $m^{\text{nod}}$  and  $m^{\text{ske}}$ .

By using the strong Markov property for the process  $m^{\text{nod}}$  and Proposition 3.1, we obtain

**Proposition 3.2.** *The snake  $\mathcal{S}$  is strong Markov.*

We write  $m^{(\theta)}(dr) = m^{\text{ske}}(dr, [0, \theta]) + m^{\text{nod}}(dr, [0, \theta])$ . Due to the properties of the Poisson point measures, we obtain the following result.

**Proposition 3.3.**  $m_t^{(\theta+a)} - m_t^{(\theta)}$  is independent of  $m_t^{(\theta)}$  and has the same law as  $m_t^{(a)}$ .

We still denote by  $\mathbb{P}_\mu$  (resp.  $\mathbb{P}_\mu^*$ ) the law of the snake  $(\rho, m^{\text{nod}}, m^{\text{ske}})$  starting from  $(\mu, 0, 0)$  (resp. and killed when it reaches 0). We also denote by  $\mathbb{N}$  the law of the snake  $\mathcal{S}$  when  $\rho$  is distributed under  $\mathbb{N}$ .

We define  $\psi^{(\theta)}$  by, for any  $\theta \in \mathbb{R}$ ,



$$\begin{aligned}
\psi^{(\theta)}(\lambda) &= \psi(\theta + \lambda) - \psi(\theta) \\
&= \alpha^{(\theta)}\lambda + \beta^{(\theta)}\lambda^2 + \int_{(0,+\infty)} (e^{-\lambda l} - 1 + \lambda l)\pi^{(\theta)}(dl) \\
\text{with } \begin{cases} \alpha^{(\theta)} &= \alpha + 2\beta\theta + \int_{(0,+\infty)} (1 - e^{-\theta l})l\pi(dl) \\ \beta^{(\theta)} &= \beta \\ \pi^{(\theta)}(dl) &= e^{-\theta l}\pi(dl). \end{cases}
\end{aligned}$$

Let  $\rho^{(\theta)}$  be the exploration process under the first marks given by  $m^{(\theta)}$ . We can write its key property :

**Proposition 3.4.** *The exploration process  $\rho^{(\theta)}$  is associated to a Lévy process with Laplace exponent  $\psi^{(\theta)}$ .*

**3.4. Poisson representation of the snake.** We decompose the process  $\rho$  under  $\mathbb{P}_\mu^*$  according to excursions of the total mass of  $\rho$  above its past minimum. More precisely, let  $(\alpha_i, \beta_i), i \in J$  be the excursion intervals of  $X - I$  above 0 under  $\mathbb{P}_\mu^*$ . For  $i \in J$ , we define  $h_i = H_{\alpha_i}$  and  $\rho^i$  by the formula : for  $t \geq 0$  and  $f \in \mathcal{B}_+(\mathbb{R}_+)$ ,

$$\langle \rho_t^i, f \rangle = \int_{(h_i, +\infty[} f(x - h_i)\rho_{(\alpha_i+t) \wedge \beta_i}(dx).$$

We write  $\sigma^i = \inf\{s > 0; \langle \rho_s^i, 1 \rangle = 0\}$ .

We also define the mark process  $m$  above the intervals  $(\alpha_i, \beta_i)$ . For every  $t \geq 0$  and  $f \in \mathcal{B}_+(\mathbb{R}_+^2)$ , we set

$$\langle m_t^{i,a}, f \rangle = \int_{(h_i, +\infty[} f(x - h_i, \theta)m_{(\alpha_i+t) \wedge \beta_i}^a(dx, \theta)$$

with  $a = ske, nod$ . We set for all  $i \in J$ ,  $m^i = (m^{i,nod}, m^{i,ske})$ .

**Lemma 3.5.** (Lemma 4.2.4, [8])

Let  $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ . The point measure  $\sum_{i \in J} \delta_{(h_i, \mathcal{S}^i)}$  is under  $\mathbb{P}_\mu^*$  a Poisson point measure with intensity  $\mu(dr)\mathbb{N}(d\mathcal{S})$ .

#### 4. LINKS BETWEEN THE SNAKE AND THE FRAGMENTATION

**4.1. Construction of the fragmentation process.** We are interested in the fragments of the tree given by the marks process. We do the same construction as in [1], Section 4.1.

For fixed  $\theta \geq 0$ , we first construct an equivalence relation,  $\mathcal{R}_\theta$ , on  $[0, \sigma]$  under  $\mathbb{N}$  or under  $\mathbb{N}_\sigma$  by :

$$s\mathcal{R}_\theta t \Leftrightarrow m_s^{(\theta)}([H_{s,t}, H_s]) = m_t^{(\theta)}([H_{s,t}, H_t]) = 0.$$

Two individuals,  $s$  and  $t$ , belong to the same equivalence class if they belong to the same fragment, that is to say if there is no mark on their lineage down to their most recent common ancestor. For each fixed  $j \in \mathbb{N}$ , we get equivalence classes  $(R^{j,k})_{k \in J_j}$  such that  $R^{j,k}$  has positive Lebesgue measure and  $R^{j,k}$  represents the  $k$ -th fragment which has  $j$  marks on its lineage.

For  $j \in \mathbb{N}$  and  $k \in J_j$ , we set

$$A_t^{j,k} = \int_0^t \mathbf{1}_{s \in R^{j,k}} ds \quad \text{and} \quad C_t^{k,j} = \inf\{u \geq 0; A_u^{j,k} > t\},$$

with the convention  $\inf \emptyset = \sigma$ . We also construct the process  $\tilde{\mathcal{S}}^{j,k} = (\tilde{\rho}^{j,k}, \tilde{m}^{j,k})$  by : for every  $f \in \mathcal{B}_+(\mathbb{R}_+)$ ,  $\varphi \in \mathcal{B}_+(\mathbb{R}_+, \mathbb{R}_+)$  and  $t \geq 0$ ,

$$\begin{aligned} \langle \tilde{\rho}_t^{j,k}, f \rangle &= \int_{(H_{C_0^{j,k}}, +\infty)} f(x - H_{C_0^{j,k}}) \rho_{C_t^{j,k}}(dx) \\ \langle \tilde{m}_t^{j,k}, \varphi \rangle &= \int_{(H_{C_0^{j,k}}, +\infty) \times (\theta, +\infty)} \varphi(x - H_{C_0^{j,k}}, v - \theta) m_{C_t^{j,k}}(dx, dv) \end{aligned}$$

$\tilde{\sigma}^{j,k}$  corresponds to the Lebesgue measure of  $R^{j,k}$ .

We denote  $\mathcal{L}^{(\theta)} = (\tilde{\rho}^{j,k}; j \in \mathbb{N}, k \in J_j) = (\rho^i; i \in I^{(\theta)})$ . We also define  $\mathcal{L}^{(\theta-)} = (\rho^i; i \in I^{(\theta-)})$  the set defined similarly but using the equivalence relation  $\mathcal{R}_{\theta-}$  which gives the fragments just before time  $\theta$ .

We now define our fragmentation process  $\Lambda^\theta = (\Lambda_1^\theta, \Lambda_2^\theta, \dots)$  as the sequence of non trivial Lebesgue measure of the equivalence classes of  $\mathcal{R}_\theta$ ,  $(\tilde{\sigma}^{j,k}, j \in \mathbb{N}, k \in J_j)$ , ranked in decreasing order. Notice that this sequence is at most countable. If it is finite, we complete the sequence with zeros, such that  $\mathbb{N}$ -a.s. and  $\mathbb{N}_\sigma$ -a.e.

$$\Lambda^\theta \in \mathcal{S}^\downarrow = \{(x_1, x_2, \dots), x_1 \geq x_2 \geq \dots \geq 0, \sum x_i < \infty\}.$$

We write  $P_\sigma$  the law of  $(\Lambda^\theta, \theta \geq 0)$  under  $\mathbb{N}_\sigma$  and by convention  $P_0$  is the Dirac mass at  $(0, 0, \dots) \in \mathcal{S}^\downarrow$ . We get the following result which is similar to [1], Theorem 1.1

**Theorem 4.1.** *For  $\pi_*(dr)$ -almost every  $r$ , under  $P_r$ ,  $(\Lambda^\theta, \theta \geq 0)$  is a  $\mathcal{S}^\downarrow$ -valued fragmentation process.*

This Theorem is a direct consequence of the following Lemma

**Lemma 4.2.** ([1], Lemma 4.3)

*Under  $\mathbb{N}$ , the law of the family  $(\tilde{\mathcal{S}}^{j,k}, j \in \mathbb{N}, k \in J_j)$ , conditionally on  $(\tilde{\sigma}^{j,k}, j \in \mathbb{N}, k \in J_j)$ , is the law of Lévy Poisson snakes distributed respectively as  $\mathbb{N}_{\tilde{\sigma}^{j,k}}$ .*

*Proof.* It is based on Proposition 3.4 (which gives the law of two lowest fragment) and on the special Markov property ([2], Theorem 4.2) which gives the law of the exploration process above the first marks.  $\square$

**4.2. Another representation of the fragmentation.** We give an another representation of the fragmentation by using a Poisson point measure under the epigraph of the height process. Recall that for every  $t \in [0, \sigma]$ ,

$$\kappa_t(dr) = 2\beta \mathbf{1}_{[0, H_t]}(r)(dr) + \sum_{\substack{0 < s \leq t \\ X_{s-} < I_t^s}} (X_s - X_{s-}) \delta_{H_s}(dr).$$

Conditionally on the process  $H$  (or equivalently on  $\rho$ ), we set a Poisson point process  $\mathcal{Q}(d\theta, ds, da)$  under the epigraph of  $H$  with intensity  $d\theta q_\rho(ds, da)$  where

$$\begin{aligned} q_\rho(ds, da) &= \frac{ds \kappa_s(da)}{d_{s,a} - g_{s,a}} \\ &= q_\rho^{ske}(ds, da) + q_\rho^{nod}(ds, da) \end{aligned}$$

$$\text{with } \begin{cases} q_\rho^{\text{nod}}(ds, da) = \frac{ds}{d_{s,a} - g_{s,a}} \sum_{\substack{0 < u \leq s \\ X_{u-} < I_s^u}} (X_u - X_{u-}) \delta_{H_u}(da) \\ q_\rho^{\text{ske}}(ds, da) = \frac{2\beta ds \mathbf{1}_{[0, H_s]}(a) da}{d_{s,a} - g_{s,a}} \end{cases}$$

with  $d_{s,a} = \sup\{u \geq s, \min\{H_v, v \in [s, u]\} \geq a\}$  and  $g_{s,a} = \inf\{u \leq s, \min\{H_v, v \in [s, u]\} \geq a\}$ .  $[g_{s,a}, d_{s,a}]$  is the set of individuals of the CRT with common ancestor  $s$  after generation  $a$ .

We use a notation for the fragments of the CRT obtained from a mark  $(s, a)$  under the epigraph of  $H$ . For  $s$  and  $a$  such that  $s \in [0, \sigma]$  and  $0 \leq a \leq H_s$ , we denote the fragments of the Lévy snake  $(\rho^i, i \in \tilde{I})$  by :

- the open intervals of the excursion of  $H$  after  $s$  and above  $a$  :  $((\alpha_i, \beta_i), i \in \tilde{I}_+)$  which are such that  $\alpha_i > s$ ,  $H_{\alpha_i} = H_{\beta_i} = a$  and for every  $s' \in (\alpha_i, \beta_i)$ ,  $H_{s'} > a$  and  $H_{s,s'} = a$ .
- the open intervals of the excursion of  $H$  before  $s$  and above  $a$  :  $((\alpha_i, \beta_i), i \in \tilde{I}_-)$  which are such that  $\beta_i < s$ ,  $H_{\alpha_i} = H_{\beta_i} = a$  and for every  $s' \in (\alpha_i, \beta_i)$ ,  $H_{s'} > a$  and  $H_{s,s'} = a$ .
- the excursion  $i_s$ , of  $H$  above  $a$  and which contains  $s$  :  $(\alpha_{i_s}, \beta_{i_s})$  such that  $\alpha_{i_s} < s < \beta_{i_s}$ ,  $H_{\alpha_{i_s}} = H_{\beta_{i_s}} = a$  and for every  $s' \in (\alpha_{i_s}, \beta_{i_s})$ ,  $H_{s'} > a$  and  $H_{s,s'} = a$ .
- the excursion  $i_0$  of  $H$  under  $a$  :  $\{s \in [0, \sigma]; H_{s,s'} < a\} = [0, \alpha_{i_0}] \cup (\beta_{i_0}, \sigma]$

We write  $\tilde{I} = \tilde{I}_- \cup \tilde{I}_+ \cup \{i_s, i_0\}$ . For the mark on the skeleton  $(s, a)$ , the set  $\tilde{I}_- \cup \tilde{I}_+$  is empty. We are interested in the computation of  $\tilde{\nu}_\rho$ , the law of the process  $(\rho^i, i \in \tilde{I})$  under  $\mathbb{N}$ .

## 5. THE DISLOCATION PROCESS

Let  $\mathcal{T}$  be the set of jumping times of the Poisson process  $\mathcal{Q}$ . For  $\theta \in \mathcal{T}$ , we consider the processes  $\mathcal{L}^{(\theta)} = (\rho^i; i \in I^{(\theta)})$  and  $\mathcal{L}^{(\theta-)} = (\rho^i; i \in I^{(\theta-)})$  defined in the Section 4.1. The life times  $(\sigma(\rho^i); i \in I^{(\theta)})$  (resp.  $(\sigma(\rho^i); i \in I^{(\theta-)})$ ), ranked by decreasing order, of these Lévy snakes correspond to the "sizes" of the fragments at time  $\theta$  (resp. before time  $\theta$ ). Notice that, for  $\theta \in \mathcal{T}$  fixed, the families  $\mathcal{L}^{(\theta)}$  and  $\mathcal{L}^{(\theta-)}$  change in one family : the snake  $\rho^{i_\theta}$  breaks in one family  $(\rho^i, i \in \tilde{I}^{(\theta)}) \in \mathcal{L}^{(\theta)}$ . Thus we get

$$\mathcal{L}^{(\theta)} = \left( \mathcal{L}^{(\theta-)} \setminus \{\rho^{i_\theta}\} \right) \cup \{\rho^i; i \in \tilde{I}^{(\theta)}\}.$$

Let  $\nu_{\sigma(\rho)}$  be the distribution of the decreasing lengths of Lévy snakes under  $\tilde{\nu}_\rho$ , integrated w.r.t. the law of  $\rho$  conditionally on  $\sigma(\rho)$ , that is to say, for any non-negative measurable function  $F$  defined on  $\mathcal{S}^\downarrow$

$$\int_{\mathcal{S}^\downarrow} F(x) \nu_r(dx) = \mathbb{N}_r \left[ \int F((\sigma^i, i \in \tilde{I})) \tilde{\nu}_\rho(d(\rho^i, i \in \tilde{I})) \right]$$

where the  $(\sigma^i, i \in \tilde{I})$  are the lengths of the fragments  $(\rho^i, i \in \tilde{I})$  ranked in decreasing order.

**5.1. Computation of dislocation measure.** We are interested in the family of dislocation measures  $(\nu_r, r > 0)$ . Recall that  $\mathbb{N}(\cdot) = \int_{\mathbb{R}_+} \pi_*(dr) \mathbb{N}_r(\cdot)$ . The computation is easier under

$\mathbb{N}$ , then we compute for any  $\lambda \geq 0$  :

$$\begin{aligned} \int_{\mathbb{R}_+ \times \mathcal{S}^\downarrow} F(x) \nu_r(dx) \pi_*(dr) &= \mathbb{N} \left[ \int q_\rho(ds, da) F((\sigma^i, i \in \tilde{I})) \right] \\ &= \mathbb{N} \left[ \int q_\rho^{nod}(ds, da) F((\sigma^i, i \in \tilde{I})) \right] + \mathbb{N} \left[ \int q_\rho^{ske}(ds, da) F(\sigma^{i_0}, \sigma^{i_s}) \right] \end{aligned}$$

where we use the decomposition of  $q_\rho$  for the second equality. The first part has already been computed in [1]. Jumping times of the process  $\rho$  are represented by a subordinator  $W$  with Laplace exponent  $\psi' - \alpha$ . Then we construct the length of the excursions of the snake by  $S_W$  where  $S$  is a subordinator with exponent  $\psi^{-1}$ , independent of  $W$ . Then we have :

$$\mathbb{N} \left[ e^{-\lambda\sigma} \int q_\rho^{nod}(ds, da) F((\sigma^i, i \in \tilde{I})) \right] = \int \pi(dv) \mathbb{E} \left[ S_v e^{-\lambda S_v} F((\Delta S_u, u \leq v)) \right].$$

We now compute the second part. Thanks to the definition of the snake,  $\rho^{ske} = 0$  if and only if  $\beta = 0$  and in this case, we don't put mark on the skeleton of the tree. We assume that  $\beta > 0$  and we write the main theorem of this article :

**Theorem 5.1.** *We set  $\lambda_1 > 0$  and  $\lambda_2 > 0$ .*

$$\mathbb{N} \left[ \int q_\rho^{ske}(ds, da) \sigma^{i_s} e^{-\lambda_1 \sigma^{i_s} - \lambda_2 \sigma^{i_0}} \right] = \frac{2\beta}{\psi' \psi^{-1}(\lambda_1) \psi' \psi^{-1}(\lambda_2)}$$

We recall that the measure  $\hat{\nu}_r^{ske}$  gives the law of the non-reordering of the two lengths given by the fragmentation from  $\nu_r^{ske}$ .

**Corollary 5.2.** *For  $\lambda_1 > 0$  and  $\lambda_2 > 0$ , we get*

$$\int_{\mathbb{R}_+ \times \mathcal{S}^\downarrow} \frac{1}{x_2} (1 - e^{-\lambda_1 x_1}) (1 - e^{-\lambda_2 x_2}) \hat{\nu}_r^{ske}(dx) \pi_*(dr) = 2\beta \psi^{-1}(\lambda_1) \psi^{-1}(\lambda_2)$$

Thus, under  $\hat{\nu}_r^{ske}(dx) \pi_*(dr)$ , the lengths of the two fragments are independent.

*Proof of Corollary 5.2.* We use Theorem 5.1, let  $x_1$  and  $x_2$  be the lengths of the fragments from  $\nu_\rho^{ske}$  ranked by decreasing order among the elements of  $x \in \mathcal{S}^\downarrow$ , we get

$$\int_{\mathbb{R}_+ \times \mathcal{S}^\downarrow} x_1 e^{-\lambda_1 x_1 - \lambda_2 x_2} \hat{\nu}_r^{ske}(dx) \pi_*(dr) = \frac{2\beta}{\psi' \psi^{-1}(\lambda_1) \psi' \psi^{-1}(\lambda_2)}.$$

We integrate w.r.t.  $\lambda_1$  and we take the primitive which vanishes in 0, and we do the same with  $\lambda_2$ , we get the result.  $\square$

*Proof of Theorem 5.1.* In order to prove the theorem, we compute  $A_2 := \mathbb{N} \left[ \int q_\rho^{ske}(ds, da) G(\sigma^{i_s}, \sigma) \right]$  where  $G(x, y) = x e^{-\lambda_1 x - \lambda_2 y}$ .

$$\begin{aligned} A_2 &= \mathbb{N} \left[ 2\beta \int_0^\sigma ds \int \frac{1}{d_{s,a} - g_{s,a}} G(\sigma^{i_s}, \sigma) \mathbf{1}_{(0 \leq a \leq H_s)} da \right] \\ &= \mathbb{N} \left[ 2\beta \int_0^\sigma ds \int \frac{1}{d_{s,a} - g_{s,a}} G(d_{s,a} - g_{s,a}, \sigma) \mathbf{1}_{(0 \leq a \leq H_s)} da \right]. \end{aligned}$$

We denote for  $0 \leq s \leq \sigma$  and  $0 \leq a \leq H_s$  fixed

$$\begin{aligned} d_{s,a} - s &= \inf\{t \geq 0, H_{(s+t) \wedge \sigma} \leq a\} = J_2(a) \\ s - g_{s,a} &= \inf\{t \geq 0, H_{(s-t)_+} \leq a\} = J_1(a). \end{aligned}$$

We get

$$\begin{aligned} d_{s,a} - g_{s,a} &= J_2(a) + J_1(a), \\ \sigma &= J_2(0) + J_1(0), \end{aligned}$$

$$A_2 = \mathbb{N} \left[ 2\beta \int_0^\sigma ds \int \mathbf{1}_{0 \leq a \leq H_s} da \frac{G(J_1(a) + J_2(a), J_1(0) + J_2(0))}{J_1(a) + J_2(a)} \right].$$

We use the generalization for Lévy processes of Bismut formula, Proposition 2.7.

$$\begin{aligned} A_2 &= 2\beta \int \mathbb{M}(d\mu d\nu) \mathbb{E} \left[ \int \mathbf{1}_{0 \leq a \leq H(\mu)} da \frac{G(J^\nu(a) + J^\mu(a), J^\nu(0) + J^\mu(0))}{J^\nu(a) + J^\mu(a)} \right] \\ &= 2\beta \int \mathbb{M}(d\mu d\nu) \mathbf{1}_{0 \leq a \leq H(\mu)} da \mathbb{E}_\mu^* \left[ e^{-\lambda_1 J^\mu(a) - \lambda_2 J^\mu(0)} \right] \mathbb{E}_\nu^* \left[ e^{-\lambda_1 J^\nu(a) - \lambda_2 J^\nu(0)} \right] \end{aligned}$$

where  $J^\mu(a)$  is the first passage time of the process  $H^{(\mu)}$  at level  $a$ . By the Poissonian decomposition of  $\rho$  under  $\mathbb{P}_\mu^*$  w.r.t. the excursions of  $\rho$  above its minimum, under  $\mathbb{P}_\mu^*$ , we replace respectively  $J^\mu(0)$  and  $J^\mu(a)$  by  $\sum_{i \in I} \sigma^i$  and  $\sum_{h_i \geq a} \sigma^i$ . We separate  $\sum_{i \in I} \sigma^i = \sum_{h_i \geq a} \sigma^i + \sum_{h_i < a} \sigma^i$ .

$$\begin{aligned} A_2 &= 2\beta \int \mathbb{M}(d\mu d\nu) \int \mathbf{1}_{0 \leq a \leq H(\mu)} da \mathbb{E}_\mu^* \left[ \exp \left( -(\lambda_1 + \lambda_2) \sum_{h_i \geq a} \sigma^i - \lambda_2 \sum_{h_i < a} \sigma^i \right) \right] \\ &\quad \mathbb{E}_\nu^* \left[ \exp \left( -(\lambda_1 + \lambda_2) \sum_{h_i \geq a} \sigma^i - \lambda_2 \sum_{h_i < a} \sigma^i \right) \right]. \end{aligned}$$

Using standard properties of Poisson point measures, the atoms above level  $a$  are independent of the atoms below, the expectations can be separated.

$$\mathbb{E}_\mu^* \left[ e^{-(\lambda_1 + \lambda_2) \sum_{h_i \geq a} \sigma^i - \lambda_2 \sum_{h_i < a} \sigma^i} \right] = \mathbb{E}_\mu^* \left[ e^{-(\lambda_1 + \lambda_2) \sum_{h_i \geq a} \sigma^i} \right] \mathbb{E}_\mu^* \left[ e^{-\lambda_2 \sum_{h_i < a} \sigma^i} \right].$$

We use Lemma 3.5, and the equality  $\psi^{-1}(\lambda) = \mathbb{N} [1 - e^{-\lambda\sigma}]$ , we get

$$\mathbb{E}_\mu^* \left[ e^{-(\lambda_1 + \lambda_2) \sum_{h_i \geq a} \sigma^i} \right] = e^{-\mu([a, H(\mu)]) \mathbb{N} [1 - e^{-(\lambda_1 + \lambda_2)\sigma}]} = e^{-\mu([a, b]) \psi^{-1}(\lambda_1 + \lambda_2)}.$$

And we do the same for the second expectation.

$$A_2 = 2\beta \int_0^\infty db e^{-\alpha b} \int_0^b da \mathbb{M}_b \left[ e^{-(\mu + \nu)([a, b]) \psi^{-1}(\lambda_1 + \lambda_2)} e^{-(\mu + \nu)([0, a]) \psi^{-1}(\lambda_2)} \right].$$

Then,

$$\begin{aligned} &\mathbb{M}_b \left[ e^{-(\mu + \nu)([a, b]) \psi^{-1}(\lambda_1 + \lambda_2)} e^{-(\mu + \nu)([0, a]) \psi^{-1}(\lambda_2)} \right] \\ &= \mathbb{M}_b \left[ e^{-(\mu + \nu)([a, b]) \psi^{-1}(\lambda_1 + \lambda_2)} \right] \mathbb{M}_b \left[ e^{-(\mu + \nu)([0, a]) \psi^{-1}(\lambda_2)} \right] \\ &= e^{-2(b-a)\beta \psi^{-1}(\lambda_1 + \lambda_2)} \exp \left( - \int_0^b dx \int_0^\infty l \pi(dl) (1 - e^{-l \psi^{-1}(\lambda_1 + \lambda_2)}) \right) \\ &\quad e^{-2a\beta \psi^{-1}(\lambda_2)} \exp \left( - \int_0^a dx \int_0^\infty l \pi(dl) (1 - e^{-l \psi^{-1}(\lambda_2)}) \right) \\ &= e^{\alpha b} e^{-(b-a)\beta' \psi^{-1}(\lambda_1 + \lambda_2) - a\beta' \psi^{-1}(\lambda_2)}. \end{aligned}$$

We recall the expression of  $A_2$

$$\begin{aligned} A_2 &= 2\beta \int_0^\infty db \frac{e^{-b\psi'\psi^{-1}(\lambda_2)} - e^{-b\psi'\psi^{-1}(\lambda_1+\lambda_2)}}{\psi'\psi^{-1}(\lambda_1+\lambda_2) - \psi'\psi^{-1}(\lambda_2)} \\ &= \frac{2\beta}{\psi'\psi^{-1}(\lambda_1+\lambda_2) - \psi'\psi^{-1}(\lambda_2)} \left( \frac{1}{\psi'\psi^{-1}(\lambda_2)} - \frac{1}{\psi'\psi^{-1}(\lambda_1+\lambda_2)} \right) \\ &= \frac{2\beta}{\psi'\psi^{-1}(\lambda_2)\psi'\psi^{-1}(\lambda_1+\lambda_2)}. \end{aligned}$$

We use the equality  $\mathbb{N} \left[ \int q_\rho^{ske}(ds, da) \sigma^{is} G(\sigma^{is}, \sigma) \right] = \mathbb{N} \left[ \int q_\rho^{ske}(ds, da) \sigma^{is} e^{-(\lambda_1+\lambda_2)\sigma^{is} - \lambda_2\sigma^{i0}} \right]$ , we finally get the result.  $\square$

**5.2. Brownian case.** A similar result has been obtained in [3] in the Brownian case and conditionally on  $\sigma = 1$ . We consider a standard Brownian motion with Laplace exponent  $\psi(\lambda) = \frac{\lambda^2}{2}$  and we denote by  $\Gamma(de)$  the law of the Brownian excursion  $e$ . Thanks to [12], Section VIII.3, the height process of the Brownian motion is given by  $H_t = 2(X_t - I_t)$ . We resume the computation of [3] by taking marks under the epigraph of  $H$ , we get

$$\int F(\sigma^{is}, \sigma) \nu(ds) = \int \Gamma(de) \int_0^\sigma ds \int_0^{2e(s)} dt \frac{F(\sigma^{is}, \sigma)}{\sigma^{is}}$$

where  $\nu$  is the dislocation measure of [3]. The computation of [3] uses the law the two independent 3-dimensional Bessel processes, then we get

$$\int F(\sigma^{is}, \sigma) \nu(ds) = \frac{1}{4\pi} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} \int_0^\infty d\sigma \frac{F(\sigma z, \sigma)}{\sigma z}.$$

As before, we compute with  $F(x, y) = xe^{-\lambda_1 x - \lambda_2 y}$ .

$$\begin{aligned} \int F(\sigma^{is}, \sigma) \nu(ds) &= \frac{1}{4\pi} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} \int_0^\infty d\sigma e^{-\lambda_1 \sigma z - \lambda_2 \sigma} \\ &= \frac{1}{4\pi} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} \frac{1}{\lambda_1 z + \lambda_2} \end{aligned}$$

For the end of this computation, we use the two changes of variable :  $z \leftrightarrow \sin^2 x$  and then  $t \leftrightarrow \tan x$ .

$$\begin{aligned} \int F(\sigma^{is}, \sigma) \nu(ds) &= \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \frac{dx}{\lambda_1 \sin^2 x + \lambda_2} \\ &= \frac{1}{2\pi} \int_0^\infty \frac{dt}{(\lambda_1 + \lambda_2)t^2 + \lambda_2} \end{aligned}$$

We integrate a last time, we get the same result as in Theorem 5.1 :

$$\int F(\sigma^{is}, \sigma) \nu(ds) = \frac{1}{4} \frac{1}{\sqrt{\lambda_2(\lambda_1 + \lambda_2)}}.$$

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GUILLAUME VOISIN, MAPMO CNRS UMR 6628, FÉDÉRATION DENIS POISSON FR 2964, UNIVERSITÉ D’ORLÉANS, B.P. 6759, 45067 ORLÉANS CEDEX 2 FRANCE.

*E-mail address:* guillaume.voisin@univ-orleans.fr