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# Boundary stabilization and control of wave equations by means of a general multiplier method.

PIERRE CORNILLEAU\*, JEAN-PIERRE LOHÉAC\*†.

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## Abstract

We describe a general multiplier method to obtain boundary stabilization of the wave equation by means of a (linear or quasi-linear) Neumann feedback. This also enables us to get Dirichlet boundary control of the wave equation. This method leads to new geometrical cases concerning the "active" part of the boundary where the feedback (or control) is applied.

Due to mixed boundary conditions, the Neumann feedback case generate singularities. Under a simple geometrical condition concerning the orientation of the boundary, we obtain a stabilization result in linear or quasi-linear cases.

## Introduction

In this paper we are concerned with control and stabilization of the wave equation in a multi-dimensional body  $\Omega \subset \mathbb{R}^n$ .

Stabilization is obtained using a feedback law given by some part of the boundary of the spacial domain and some function defined on this part. The problem can be written as follows

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+^*, \\ \partial_\nu u = F & \text{on } \partial\Omega_N \times \mathbb{R}_+^*, \\ u(0) = u_0 & \text{in } \Omega, \\ u'(0) = u_1 & \text{in } \Omega, \end{cases}$$

where we denote by  $u'$ ,  $u''$ ,  $\Delta u$  and  $\partial_\nu u$  the first time-derivative of  $u$ , the second time-derivative of the scalar function  $u$ , the standard Laplacian of  $u$  and the normal outward derivative of  $u$  on  $\partial\Omega$ , respectively;  $(\partial\Omega_D, \partial\Omega_N)$  is a partition of  $\partial\Omega$  and  $F$  is the feedback function which may depend on state  $(u, u')$ , position  $x$  and time  $t$ .

Our purpose here is to choose the feedback law, that is to say the feedback function  $F$  and the "active" part of the boundary,  $\partial\Omega_N$ , so that for every initial data, the energy function

$$E(t) = \frac{1}{2} \int_{\Omega} (|u'(t)|^2 + |\nabla u(t)|^2) d\mathbf{x},$$

is decreasing with respect to time  $t$ , and vanishes as  $t \rightarrow \infty$ .

Formally, we can write the time-derivative of  $E$  as follows

$$E'(t) = \int_{\partial\Omega_N} F u' d\sigma,$$

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\*Université de Lyon, École centrale de Lyon, D.M.I., Institut Camille-Jordan (C.N.R.S. U.M.R. 5208), 36 avenue Guy-de-Collongue, 69134 Écully cedex, France.

†Independent Moscow University, Laboratoire J.-V. Poncelet (C.N.R.S. U.M.I. 2615), Bol. Vlasievsky Per. 11, 119002 Moscow, Russia.

and a sufficient condition for  $E$  to be non-increasing would be:  $Fu' \leq 0$  on  $\partial\Omega_N$ .

Thanks to the multiplier method introduced by L.F. Ho [12] in the framework of Hilbert Uniqueness Method [13], it can be shown that the energy function is uniformly decreasing as time  $t$  tends to  $\infty$  by choosing  $\mathbf{x} \mapsto \mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$ , where  $\mathbf{x}_0$  is some given point in  $\mathbb{R}^n$  and

$$\partial\Omega_N = \{\mathbf{x} \in \partial\Omega / \mathbf{m}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) > 0\}, \quad F = -\mathbf{m} \cdot \boldsymbol{\nu} u',$$

where  $\boldsymbol{\nu}$  is the normal unit vector pointing outward of  $\Omega$ .

This method has been performed by many authors, see for instance Komornik and Zuazua [11], Komornik [10] and the references therein. Here we extend the above result for rotated multipliers defined in [16] and we follow the analysis of singularities initiated by Grisvard [7, 8] and extended by Bey, Lohéac and Moussaoui [4]. This last work leads to results in case of higher dimensional domains with a non-empty boundary interface  $\Gamma = \overline{\partial\Omega_N} \cap \overline{\partial\Omega_D}$  under an additional geometrical assumption concerning the orientation of the boundary.

Concerning the control problem, our goal is to find  $v$  such that the solution of

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega_D \times (0, T), \\ u = v & \text{on } \partial\Omega_N \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \\ u'(0) = u_1 & \text{in } \Omega, \end{cases}$$

reaches an equilibrium at  $t = T$ .

We here follow [12]: in this work, Ho used the multiplier technique. His main purpose was to prove an inverse inequality for the linear wave equation implying its exact controllability. He introduced the so-called *exit condition*: the control region must contain a subset of the boundary where the scalar product between the outward normal and the vector pointing from some origin towards the normal is positive. By varying the origin, a family of boundary controls satisfying the condition is obtained.

In the last decades, micro-local techniques and geometric optics analysis allowed to find geometrical characterization of control and minimal control time in the exact controllability of waves. This condition has been introduced in [3] under the name of *Geometric Control Condition (GCC)*. It generalized the previous exit condition.

There is a certain balance: with *GCC*, control time is optimal but the observability constant is not explicit. With *exit condition*, time is not optimal but observability constants can be explicit, which is very useful in theoretical and numerical estimations.

In this paper we extend the family of multipliers recently introduced by Osses [16].

## 1 Notations and main results

Let  $\Omega$  be a bounded open connected set of  $\mathbb{R}^n (n \geq 2)$  such that

$$\partial\Omega \text{ is } \mathcal{C}^2 \text{ in the sense of Nečas [15].} \quad (1)$$

In the sequel, we denote by  $\mathbf{I}$  the  $n \times n$  identity matrix and by  $\mathbf{A}^s$  the symmetric part of a matrix  $\mathbf{A}$ . Let  $\mathbf{m}$  be a  $W^{1,\infty}(\Omega)$  vector-field such that

$$\text{ess inf}_{\overline{\Omega}}(\text{div}(\mathbf{m})) > \text{ess sup}_{\overline{\Omega}}(\text{div}(\mathbf{m}) - 2\lambda_{\mathbf{m}}) \quad (2)$$

where  $\text{div}$  is the usual divergence operator and  $\lambda_{\mathbf{m}}(\mathbf{x})$  is the smallest eigenvalue of the real symmetric matrix  $\nabla \mathbf{m}(\mathbf{x})^s$ . Using Sobolev embedding, one may also assume that  $\mathbf{m} \in \mathcal{C}(\overline{\Omega})$ .

**Remark 1** *The set of all  $W^{1,\infty}(\Omega)$  vector-fields such that (2) holds is an open cone. If  $\mathbf{m}$  belongs to this set, we denote*

$$c(\mathbf{m}) = \frac{1}{2} \left( \text{ess inf}_{\overline{\Omega}}(\text{div}(\mathbf{m})) - \text{ess sup}_{\overline{\Omega}}(\text{div}(\mathbf{m}) - 2\lambda_{\mathbf{m}}) \right).$$

## Examples

- An affine example is given by

$$\mathbf{m}(\mathbf{x}) = (A_1 + A_2)(\mathbf{x} - \mathbf{x}_0),$$

where  $A_1$  is a definite positive matrix,  $A_2$  a skew-symmetric matrix and  $\mathbf{x}_0$  any point in  $\mathbb{R}^n$ .

- A non linear example is

$$\mathbf{m}(\mathbf{x}) = (dI + A)(\mathbf{x} - \mathbf{x}_0) + \mathbf{F}(\mathbf{x}),$$

where  $d > 0$ ,  $A$  is a skew-symmetric matrix,  $\mathbf{x}_0$  any point in  $\mathbb{R}^n$  and  $\mathbf{F}$  is a  $W^{1,\infty}(\Omega)$  vector field such that

$$\operatorname{ess\,sup}_{\overline{\Omega}} \|\nabla \mathbf{F}^s\| < \frac{d}{n},$$

where  $\|\cdot\|$  stands for the usual 2-norm of matrices.

We consider a partition  $(\partial\Omega_N, \partial\Omega_D)$  of  $\partial\Omega$  such that

$$\left\{ \begin{array}{l} \Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} \text{ is a } \mathcal{C}^3\text{-manifold of dimension } n-2 \text{ such that } \mathbf{m} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma, \\ \exists \omega \text{ neighborhood of } \Gamma \text{ such that } \partial\Omega \cap \omega \text{ is a } \mathcal{C}^3\text{-manifold of dimension } n-1, \\ \mathcal{H}^{n-1}(\partial\Omega_D) > 0 \text{ (} \mathcal{H}^{n-1} \text{ is the } (n-1)\text{-dimensional Hausdorff measure).} \end{array} \right. \quad (3)$$

Furthermore, we assume

$$\partial\Omega_N \subset \{\mathbf{x} \in \partial\Omega / \mathbf{m}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) \geq 0\}, \quad \partial\Omega_D \subset \{\mathbf{x} \in \partial\Omega / \mathbf{m}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) \leq 0\}. \quad (4)$$

This assumption clearly implies:  $\mathbf{m} \cdot \boldsymbol{\nu} = 0$  on  $\Gamma$ .

## Boundary stabilization

Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function such that

$$g \text{ is non-decreasing and } \exists k_+ > 0 : |g(s)| \leq k_+ |s| \text{ a.e.} \quad (5)$$

Let us now consider the following wave problem,

$$(S) \quad \begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+^*, \\ \partial_\nu u = -\mathbf{m} \cdot \boldsymbol{\nu} g(u') & \text{on } \partial\Omega_N \times \mathbb{R}_+^*, \\ u(0) = u_0 & \text{in } \Omega, \\ u'(0) = u_1 & \text{in } \Omega, \end{cases}$$

where initial data satisfy

$$(u_0, u_1) \in H_D^1(\Omega) \times L^2(\Omega)$$

with  $H_D^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$ .

Problem (S) is well-posed in this space. Indeed, following Komornik [10], we define the non-linear operator  $\mathcal{W}$  on  $H_D^1(\Omega) \times L^2(\Omega)$  by

$$\begin{aligned} \mathcal{W}(u, v) &= (-v, -\Delta u), \\ D(\mathcal{W}) &= \{(u, v) \in H_D^1(\Omega) \times H_D^1(\Omega) / \Delta u \in L^2(\Omega) \text{ and } \partial_\nu u = -\mathbf{m} \cdot \boldsymbol{\nu} g(v) \text{ on } \partial\Omega_N\}, \end{aligned}$$

so that (S) can be written as follows,

$$\begin{cases} (u, v)' + \mathcal{W}(u, v) = 0, \\ (u, v)(0) = (u_0, u_1). \end{cases}$$

It is classical that  $\mathcal{W}$  is a maximal-monotone operator on  $H_D^1(\Omega) \times L^2(\Omega)$  and that  $D(\mathcal{W})$  is dense in  $H_D^1(\Omega) \times L^2(\Omega)$  for the usual norm. Following Brézis [1], we can deduce that for any initial data  $(u_0, v_0)$  in

$D(\mathcal{W})$  there is a unique strong solution  $(u, v)$  such that  $u \in W^{1,\infty}(\mathbb{R}; H_D^1(\Omega))$  and  $\Delta u \in L^\infty(\mathbb{R}_+; L^2(\Omega))$ . Moreover, for two initial data, the corresponding solutions satisfy

$$\forall t \geq 0, \quad \|(u^1(t), v^1(t)) - (u^2(t), v^2(t))\|_{H_D^1(\Omega) \times L^2(\Omega)} \leq \|(u_0^1, v_0^1) - (u_0^2, v_0^2)\|_{H_D^1(\Omega) \times L^2(\Omega)}.$$

Using the density of  $D(\mathcal{W})$ , one can extend the map

$$\begin{aligned} D(\mathcal{W}) &\longrightarrow H_D^1(\Omega) \times L^2(\Omega) \\ (u_0, v_0) &\longmapsto (u(t), v(t)) \end{aligned}$$

to a strongly continuous semi-group of contractions  $(S(t))_{t \geq 0}$  and define for  $(u_0, v_0) \in H_D^1(\Omega) \times L^2(\Omega)$  the weak solution  $(u(t), v(t)) = S(t)(u_0, v_0)$  with the regularity  $u \in \mathcal{C}(\mathbb{R}_+; H_D^1(\Omega)) \cap \mathcal{C}^1(\mathbb{R}_+; L^2(\Omega))$ . We hence define the energy function of solutions by

$$E(0) = \frac{1}{2} \int_{\Omega} (|u_1|^2 + |\nabla u_0|^2) d\mathbf{x} \quad \text{and} \quad E(t) = \frac{1}{2} \int_{\Omega} (|u'(t)|^2 + |\nabla u(t)|^2) d\mathbf{x}, \quad \text{for } t > 0.$$

In order to get stabilization results, we need further assumptions concerning the feedback function  $g$

$$\exists p \geq 1, \exists k_- > 0 : \quad |g(s)| \geq k_- \min\{|s|, |s|^p\}, \quad \text{a.e.}, \quad (6)$$

and the additional geometric assumption

$$\mathbf{m} \cdot \boldsymbol{\tau} \leq 0 \quad \text{on } \Gamma, \quad (7)$$

where  $\boldsymbol{\tau}(\mathbf{x})$  is the normal unit vector pointing outward of  $\partial\Omega_N$  at a point  $\mathbf{x} \in \Gamma$  when considering  $\partial\Omega_N$  as a sub-manifold of  $\partial\Omega$ .

**Remark 2** *It is not necessary to assume that*

$$\mathcal{H}^{n-1}(\{\mathbf{x} \in \partial\Omega_N / \mathbf{m}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) > 0\}) > 0$$

*to get stabilization. In fact, our choices of  $\mathbf{m}$  imply such properties (see examples in Section 5) whether the energy tends to zero.*

A main tool in this work is Rellich type relations [17]. They lead to results of controllability and stabilization for the wave problem (see [11] and [12]). When the interface  $\Gamma$  is not empty, the key-problem is to show the existence of a decomposition of the solution in a regular and a singular parts (see [7, 9]) in any dimension. The first results towards this direction are due to Moussaoui [14], and Bey-Lohéac-Moussaoui [4].

In this new case, our goal is to generalize those Rellich relations. This will lead us to get a stabilization result about (S) under (4), (7). As well as in [10], we shall prove here two results of uniform boundary stabilization.

### Exponential boundary stabilization

We here consider the case when  $p = 1$  in (6). This is satisfied when  $g$  is linear,

$$\exists \alpha > 0 : \quad \forall s \in \mathbb{R}, \quad g(s) = \alpha s.$$

In this case, the energy function is exponentially decreasing.

**Theorem 1** *Assume that conditions (1), (2), (3) and (4) hold and that the feedback function  $g$  satisfies (5) and (6) with  $p = 1$ .*

*Then under the further geometrical assumption (7), there exist  $C > 0$  and  $T > 0$  such that for every initial data in  $H_D^1(\Omega) \times L^2(\Omega)$ , the energy of the solution  $u$  of (S) satisfies*

$$\forall t > T, \quad E(t) \leq E(0) \exp\left(1 - \frac{t}{C}\right).$$

*The above constants  $C$  and  $T$  do not depend on initial data.*

## Rational boundary stabilization

We here consider the case  $p > 1$  and we get rational boundary stabilization.

**Theorem 2** *Assume that conditions (1), (2), (3) and (4) hold and that the feedback function  $g$  satisfies (5) and (6) with  $p > 1$ .*

*Then under the further geometrical assumption (7), there exist  $C > 0$  and  $T > 0$  such that for every initial data in  $H_D^1(\Omega) \times L^2(\Omega)$ , the energy of the solution  $u$  of (S) satisfies*

$$\forall t > T, \quad E(t) \leq C t^{2/(1-p)}.$$

where  $C$  depends on the initial energy  $E(0)$ .

**Remark 3** *Taking advantage of the work by Banasiak and Roach [2] who generalized Grisvard's results [7] in the piecewise regular case, we will see that Theorems 1 and 2 remain true in the bi-dimensional case when assumption (1) is replaced by following one,*

$$\begin{aligned} \partial\Omega \text{ is a curvilinear polygon of class } \mathcal{C}^2, \\ \text{each component of } \partial\Omega \setminus \Gamma \text{ is a } \mathcal{C}^2\text{-manifold of dimension 1,} \end{aligned} \quad (8)$$

and when condition (7) is replaced by

$$\forall \mathbf{x} \in \Gamma, \quad 0 \leq \varpi_{\mathbf{x}} \leq \pi \quad \text{and} \quad \text{if } \varpi_{\mathbf{x}} = \pi, \quad \mathbf{m}(\mathbf{x}) \cdot \boldsymbol{\tau}(\mathbf{x}) \leq 0. \quad (9)$$

where  $\varpi_{\mathbf{x}}$  is the angle of the boundary at point  $\mathbf{x}$ .

## Boundary control problem

Our problem consists in finding  $T_0$  such that for each  $T > T_0$  and for every  $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ , there exists  $v \in L^2(\partial\Omega_N \times (0, T))$  in such a way that the solution of the wave equation

$$(\Sigma) \quad \begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega_D \times (0, T), \\ u = v & \text{on } \partial\Omega_N \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \\ u'(0) = u_1 & \text{in } \Omega. \end{cases}$$

satisfies

$$u(T) = u'(T) = 0 \quad \text{in } \Omega. \quad (10)$$

**Theorem 3** *Assume that (1), (2), (3) and (4) hold.*

*Then if  $T > 2 \frac{\|\mathbf{m}\|_{\infty}}{c(\mathbf{m})}$ , for every initial data  $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ , there exists a control function  $v \in L^2(\partial\Omega_N \times (0, T))$  such that the corresponding solution of (Σ) satisfies final condition (10).*

Our paper is organized as follows.

In Section 2, we extend Rellich relations (Theorems 5 and 6) for elliptic problems with mixed boundary conditions.

In Section 3, we apply these relations to prove some stabilization results with linear or quasi-linear Neumann feedback (Theorems 1 and 2).

In Section 4, we extend some observability and controllability results for the wave equation (Proposition 11 and Theorem 3).

In Section 5, we detail affine examples in the case of a square domain.

## 2 Rellich relation

Here, we briefly extend Rellich relation obtained in [4], [5] to our framework.

## 2.1 A regular case

We can easily build a Rellich relation corresponding to the above vector-field  $\mathbf{m}$  when considered functions are smooth enough.

**Proposition 4** *Assume that  $\Omega$  is a open set of  $\mathbb{R}^n$  with boundary of class  $\mathcal{C}^2$  in the sense of Nečas. If  $u$  belongs to  $H^2(\Omega)$  then*

$$2 \int_{\Omega} \Delta u \mathbf{m} \cdot \nabla u \, d\mathbf{x} = \int_{\Omega} (\operatorname{div}(\mathbf{m})I - 2(\nabla \mathbf{m})^s)(\nabla u, \nabla u) \, d\mathbf{x} + \int_{\partial\Omega} (2\partial_{\nu} u \mathbf{m} \cdot \nabla u - \mathbf{m} \cdot \boldsymbol{\nu} |\nabla u|^2) \, d\sigma.$$

**Proof.** Using Green-Riemann identity we get

$$\int_{\Omega_{\varepsilon}} \Delta u \mathbf{m} \cdot \nabla u \, d\mathbf{x} = \int_{\partial\Omega_{\varepsilon}} \partial_{\nu} u \mathbf{m} \cdot \nabla u \, d\sigma - \int_{\Omega_{\varepsilon}} \nabla u \cdot \nabla(\mathbf{m} \cdot \nabla u) \, d\mathbf{x}.$$

So, observing that  $\nabla u \cdot \nabla(\mathbf{m} \cdot \nabla u) = \frac{1}{2} \mathbf{m} \cdot \nabla(|\nabla u|^2) + \nabla u \cdot (\nabla \mathbf{m}) \nabla u = \frac{1}{2} \mathbf{m} \cdot \nabla(|\nabla u|^2) + (\nabla \mathbf{m})^s(\nabla u, \nabla u)$ , for smooth functions  $u$ , we get

$$2 \int_{\Omega_{\varepsilon}} \Delta u \mathbf{m} \cdot \nabla u \, d\mathbf{x} = \int_{\partial\Omega_{\varepsilon}} 2\partial_{\nu} u \mathbf{m} \cdot \nabla u \, d\sigma - 2 \int_{\Omega_{\varepsilon}} (\nabla \mathbf{m})^s(\nabla u, \nabla u) \, d\mathbf{x} - \int_{\Omega_{\varepsilon}} \mathbf{m} \cdot \nabla(|\nabla u|^2) \, d\mathbf{x}.$$

With another use of Green-Riemann formula, we obtain the required formula thanks to a classical approximation. ■

We will now try to extend this result to the case of a less regular element  $u$  when  $\Omega$  is smooth enough.

## 2.2 Bi-dimensional case

We begin by the plane case: it is the simplest case from the point of view of singularity theory, and its understanding dates from Shamir [18].

**Theorem 5** *Assume  $n = 2$ . Under the geometrical conditions (8) and (3), let  $u \in H^1(\Omega)$  such that*

$$\Delta u \in L^2(\Omega), \quad u|_{\partial\Omega_D} \in H^{3/2}(\partial\Omega_D), \quad \partial_{\nu} u|_{\partial\Omega_N} \in H^{1/2}(\partial\Omega_N).$$

*Then  $2\partial_{\nu} u(\mathbf{m} \cdot \nabla u) - (\mathbf{m} \cdot \boldsymbol{\nu})|\nabla u|^2 \in L^1(\partial\Omega)$  and there exist some coefficients  $(c_{\mathbf{x}})_{\mathbf{x} \in \Gamma}$  such that*

$$\begin{aligned} 2 \int_{\Omega} \Delta u \mathbf{m} \cdot \nabla u \, d\mathbf{x} &= \int_{\Omega} (\operatorname{div}(\mathbf{m})I - 2(\nabla \mathbf{m})^s)(\nabla u, \nabla u) \, d\mathbf{x} + \int_{\partial\Omega} (2\partial_{\nu} u \mathbf{m} \cdot \nabla u - \mathbf{m} \cdot \boldsymbol{\nu} |\nabla u|^2) \, d\sigma \\ &+ \frac{\pi}{4} \sum_{\mathbf{x}/\varpi_{\mathbf{x}}=\pi} c_{\mathbf{x}}^2(\mathbf{m} \cdot \boldsymbol{\tau})(\mathbf{x}). \end{aligned}$$

**Proof.** We follow the proof of Theorem 4 in [5] to get this result. ■

**Remark 4** *As in Theorem 4 of [5], the assumption  $\mathcal{H}^1(\partial\Omega_D) > 0$  is not necessary in the above proof.*

## 2.3 General case

We now state the result in higher dimension.

**Theorem 6** *Assume  $n \geq 3$ . Under geometrical conditions (1) and (3), let  $u \in H^1(\Omega)$  such that*

$$\Delta u \in L^2(\Omega), \quad u|_{\partial\Omega_D} \in H^{3/2}(\partial\Omega_D), \quad \partial_{\nu} u|_{\partial\Omega_N} \in H^{1/2}(\partial\Omega_N).$$

*Then  $2\partial_{\nu} u(\mathbf{m} \cdot \nabla u) - (\mathbf{m} \cdot \boldsymbol{\nu})|\nabla u|^2 \in L^1(\partial\Omega)$  and there exists  $\zeta \in H^{1/2}(\Gamma)$  such that*

$$\begin{aligned} 2 \int_{\Omega} \Delta u \mathbf{m} \cdot \nabla u \, d\mathbf{x} &= \int_{\Omega} (\operatorname{div}(\mathbf{m})I - 2(\nabla \mathbf{m})^s)(\nabla u, \nabla u) \, d\mathbf{x} + \int_{\partial\Omega} (2\partial_{\nu} u \mathbf{m} \cdot \nabla u - \mathbf{m} \cdot \boldsymbol{\nu} |\nabla u|^2) \, d\sigma \\ &+ \int_{\Gamma} \mathbf{m} \cdot \boldsymbol{\tau} |\zeta|^2 \, d\gamma. \end{aligned}$$

**Proof.** We exactly follow the proof of Theorem 5 in [5] to get this result. ■

### 3 Linear and quasi-linear stabilization

We begin by writing the following consequence of results of Section 2.

**Corollary 7** *Assume that  $t \mapsto (u(t), u'(t))$  is a strong solution of (S) and that geometrical additional assumption (7) if  $n \geq 3$  (or (9) if  $n = 2$ ) holds, then, for every time  $t$ ,  $u(t)$  satisfies*

$$2 \int_{\Omega} \Delta u \mathbf{m} \cdot \nabla u \, d\mathbf{x} \leq \int_{\Omega} (\operatorname{div}(\mathbf{m})I - 2(\nabla \mathbf{m})^s)(\nabla u, \nabla u) \, d\mathbf{x} + \int_{\partial\Omega} (2\partial_{\nu} u \mathbf{m} \cdot \nabla u - \mathbf{m} \cdot \boldsymbol{\nu} |\nabla u|^2) \, d\sigma.$$

**Proof.** Indeed, under these hypotheses, for each time  $t$ ,  $(u(t), u'(t)) \in D(\mathcal{W})$  so that  $u(t)$  satisfies hypotheses of Theorems 5 or 6. The result follows immediately from (7) or (9). ■

The main tool in the proof of Theorems 1, 2 is the following result (see proof in [10]) which will be applied with  $\alpha = \frac{p-1}{2}$ .

**Proposition 8** *Let  $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a non-increasing function such that there exist  $\alpha \geq 0$  and  $C > 0$  which fulfill*

$$\forall t \geq 0, \quad \int_t^{\infty} E^{\alpha+1}(s) \, ds \leq CE(t).$$

Then, setting  $T = CE^{\alpha}(0)$ , one gets

$$\begin{aligned} \text{if } \alpha = 0, \quad \forall t \geq T, \quad E(t) &\leq E(0) \exp\left(1 - \frac{t}{T}\right), \\ \text{if } \alpha > 0, \quad \forall t \geq T, \quad E(t) &\leq E(0) \left(\frac{T + \alpha T}{T + \alpha t}\right)^{1/\alpha}. \end{aligned}$$

As usual in this context, we will perform the multiplier method to a suitable  $\mathbf{m}$ . Putting  $Mu = 2\mathbf{m} \cdot \nabla u + au$  with  $a$  a constant to be defined later, we prove the following result.

**Lemma 9** *For any  $0 \leq S < T < \infty$ , the following inequality holds*

$$\begin{aligned} &\int_S^T E^{\frac{p-1}{2}} \int_{\Omega} \left( (\operatorname{div}(\mathbf{m}) - a)(u')^2 + ((a - \operatorname{div}(\mathbf{m}))I + 2(\nabla \mathbf{m})^s)(\nabla u, \nabla u) \right) \, d\mathbf{x} \, dt \\ &\leq - \left[ E^{\frac{p-1}{2}} \int_{\Omega} u' Mu \, d\mathbf{x} \right]_S^T + \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \int_{\Omega} u' Mu \, d\mathbf{x} \, dt \\ &\quad + \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} \mathbf{m} \cdot \boldsymbol{\nu} \left( (u')^2 - |\nabla u|^2 - g(u') Mu \right) \, d\sigma \, dt. \end{aligned}$$

**Proof.** We here follow [6].

We Use the fact that  $u$  is solution of (S) and we observe that  $u'' Mu = (u' Mu)' - u' Mu'$ . Then an integration by parts gives

$$\begin{aligned} 0 &= \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} (u'' - \Delta u) Mu \, d\mathbf{x} \, dt \\ &= \left[ E^{\frac{p-1}{2}} \int_{\Omega} u' Mu \, d\mathbf{x} \right]_S^T - \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \int_{\Omega} u' Mu \, d\mathbf{x} \, dt - \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} (u' Mu' + \Delta u Mu) \, d\mathbf{x} \, dt. \end{aligned}$$

Corollary 7 now gives

$$\int_{\Omega} \Delta u Mu \, d\mathbf{x} \leq a \int_{\Omega} \Delta u u \, d\mathbf{x} + \int_{\Omega} (\operatorname{div}(\mathbf{m})I - 2(\nabla \mathbf{m})^s)(\nabla u, \nabla u) \, d\mathbf{x} + \int_{\partial\Omega} (2\partial_{\nu} u \mathbf{m} \cdot \nabla u - \mathbf{m} \cdot \boldsymbol{\nu} |\nabla u|^2) \, d\sigma.$$

Consequently, Green-Riemann formula leads to

$$\int_{\Omega} \Delta u Mu \, d\mathbf{x} \leq \int_{\Omega} ((\operatorname{div}(\mathbf{m}) - a)I - 2(\nabla \mathbf{m})^s)(\nabla u, \nabla u) \, d\mathbf{x} + \int_{\partial\Omega} (\partial_{\nu} u Mu - \mathbf{m} \cdot \boldsymbol{\nu} |\nabla u|^2) \, d\sigma.$$



Using boundary conditions and the fact that  $\nabla u = \partial_\nu u \boldsymbol{\nu}$  on  $\partial\Omega_D$ , we then get

$$\int_{\Omega} \Delta u M u \, d\mathbf{x} \leq \int_{\Omega} ((\operatorname{div}(\mathbf{m}) - a)I - 2(\nabla \mathbf{m})^s)(\nabla u, \nabla u) \, d\mathbf{x} - \int_{\partial\Omega_N} \mathbf{m} \cdot \boldsymbol{\nu} (g(u') M u + |\nabla u|^2) \, d\sigma.$$

On the other hand, another use of Green formula gives us

$$\int_{\Omega} u' M u' \, d\mathbf{x} = \int_{\Omega} (a - \operatorname{div}(\mathbf{m}))(u')^2 \, d\mathbf{x} + \int_{\partial\Omega_N} \mathbf{m} \cdot \boldsymbol{\nu} |u'|^2 \, d\sigma.$$

We complete the proof by summing up above estimates.  $\blacksquare$

Let us now prove Theorems 1 and 2.

**Proof.** Following [10] and [6], we will prove the estimates for  $(u_0, u_1) \in D(\mathcal{W})$  which will be sufficient thanks to a density argument.

Using Lemma 9, we have to find  $a$  such that  $\operatorname{div}(\mathbf{m}) - a$  and  $(a - \operatorname{div}(\mathbf{m}))I + 2(\nabla \mathbf{m})^s$  are uniformly minorized on  $\Omega$ , that is, almost everywhere on  $\Omega$

$$\begin{cases} \operatorname{div}(\mathbf{m}) - a \geq c, \\ 2\lambda_{\mathbf{m}} + (a - \operatorname{div}(\mathbf{m})) \geq c, \end{cases} \quad (11)$$

for some positive constant  $c$ . The latter condition is then equivalent to find  $a$  which fulfills

$$\operatorname{ess\,inf}_{\Omega}(\operatorname{div}(\mathbf{m})) > a > \operatorname{ess\,sup}_{\Omega}(\operatorname{div}(\mathbf{m}) - 2\lambda_{\mathbf{m}}),$$

and its existence is now garanted by (2). Moreover, it is straightforward to see that the greatest value of  $c$  such that (11) holds is

$$\frac{1}{2} \left( \operatorname{ess\,inf}_{\Omega}(\operatorname{div}(\mathbf{m})) - \operatorname{ess\,sup}_{\Omega}(\operatorname{div}(\mathbf{m}) - 2\lambda_{\mathbf{m}}) \right) = c(\mathbf{m}),$$

and obtained for  $a = a_0 := \frac{1}{2} \left( \operatorname{ess\,inf}_{\Omega}(\operatorname{div}(\mathbf{m})) + \operatorname{ess\,sup}_{\Omega}(\operatorname{div}(\mathbf{m}) - 2\lambda_{\mathbf{m}}) \right)$ .

With this value  $a_0$ , we apply Lemma 9 and get

$$\begin{aligned} 2c(\mathbf{m}) \int_S^T E^{\frac{p+1}{2}} \, dt &\leq - \left[ E^{\frac{p-1}{2}} \int_{\Omega} u' M u \, d\mathbf{x} \right]_S^T + \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \int_{\Omega} u' M u \, d\mathbf{x} \, dt \\ &\quad + \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} \mathbf{m} \cdot \boldsymbol{\nu} ((u')^2 - |\nabla u|^2 - g(u') M u) \, d\sigma \, dt. \end{aligned}$$

Young and Poincaré inequality gives

$$\left| \int_{\Omega} u' M u \, d\mathbf{x} \right| \leq C E(t).$$

It follows then

$$\begin{aligned} 2c(\mathbf{m}) \int_S^T E^{\frac{p+1}{2}} \, dt &\leq C(E^{\frac{p+1}{2}}(T) + E^{\frac{p+1}{2}}(S)) + C \int_S^T E^{\frac{p-1}{2}} E' \, dt \\ &\quad + \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} \mathbf{m} \cdot \boldsymbol{\nu} ((u')^2 - |\nabla u|^2 - g(u') M u) \, d\sigma \, dt. \end{aligned}$$

Let  $d\sigma_{\mathbf{m}} = \mathbf{m} \cdot \boldsymbol{\nu} \, d\sigma$ . If we observe that  $E'(t) = - \int_{\partial\Omega_N} g(u') u' \, d\sigma_{\mathbf{m}} \leq 0$ , we get, for a constant  $C > 0$  independent of  $E(0)$  if  $p = 1$ ,

$$2c(\mathbf{m}) \int_S^T E^{\frac{p+1}{2}} \, dt \leq C E(S) + \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} ((u')^2 - |\nabla u|^2 - g(u') M u) \, d\sigma_{\mathbf{m}} \, dt.$$

Using the definition of  $Mu$  and Young inequality, we get for any  $\varepsilon_0 > 0$

$$2c(\mathbf{m}) \int_S^T E^{\frac{p+1}{2}} dt \leq CE(S) + \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} ((u')^2(1 + \|\mathbf{m}\|_\infty^2) + \frac{\alpha^2}{4\varepsilon_0}g(u')^2 + \varepsilon_0 u^2) d\sigma_{\mathbf{m}} dt.$$

Now, using Poincaré inequality, we can choose  $\varepsilon_0 > 0$  such that

$$\varepsilon_0 \int_{\partial\Omega_N} u^2 d\sigma_{\mathbf{m}} \leq \frac{c(\mathbf{m})}{2} \int_{\Omega} |\nabla u|^2 d\mathbf{x} \leq c(\mathbf{m})E.$$

So we conclude

$$c(\mathbf{m}) \int_S^T E^{\frac{p+1}{2}} dt \leq CE(S) + C \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} ((u')^2 + g(u')^2) d\sigma_{\mathbf{m}} dt.$$

We split  $\partial\Omega_N$  to bound the last term of this estimate

$$\partial\Omega_N^1 = \{\mathbf{x} \in \partial\Omega_N / |u'(\mathbf{x})| > 1\}, \quad \partial\Omega_N^2 = \{\mathbf{x} \in \partial\Omega_N / |u'(\mathbf{x})| \leq 1\}.$$

Using (5) and (6), we get

$$\int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N^1} ((u')^2 + g(u')^2) d\sigma_{\mathbf{m}} dt \leq C \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} u'g(u') d\sigma_{\mathbf{m}} dt \leq CE(S),$$

where  $C$  depends on  $E(0)$  if  $p = 1$ .

On the other hand, using (5), (6), Jensen inequality and boundedness of  $\mathbf{m}$ , one successively obtains

$$\int_{\partial\Omega_N^2} ((u')^2 + g(u')^2) d\sigma_{\mathbf{m}} \leq C \int_{\partial\Omega_N^2} (u'g(u'))^{2/(p+1)} d\sigma_{\mathbf{m}} \leq C \left( \int_{\partial\Omega_N^2} u'g(u') d\sigma_{\mathbf{m}} \right)^{\frac{2}{p+1}} \leq C(-E')^{\frac{2}{p+1}}.$$

Hence, using Young inequality again, we get for every  $\varepsilon > 0$

$$\int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N^2} ((u')^2 + g(u')^2) d\sigma_{\mathbf{m}} dt \leq \int_S^T (\varepsilon E^{\frac{p+1}{2}} - C(\varepsilon)E') dt \leq \varepsilon \int_S^T E^{\frac{p+1}{2}} dt + C(\varepsilon)E(S).$$

Finally we get, for some  $C(\varepsilon)$  and  $C$  independent of  $E(0)$  if  $p = 1$

$$c(\mathbf{m}) \int_S^T E^{\frac{p+1}{2}} dt \leq C(\varepsilon)E(S) + \varepsilon C \int_S^T E^{\frac{p+1}{2}} dt.$$

Choosing now  $\varepsilon C \leq \frac{c(\mathbf{m})}{2}$ , one obtains

$$c(\mathbf{m}) \int_S^T E^{\frac{p+1}{2}} dt \leq CE(S),$$

and Theorems can be deduced from Lemma 8. ■

**Remark 5** As stated before, we can replace  $\mathbf{m}$  by  $\lambda\mathbf{m}$  for any positive  $\lambda$ . One can wonder what happens to the speed of stabilization  $\theta = \frac{c(\mathbf{m})}{C}$  found in Theorem 3. In fact, a careful estimation of all terms shows that one can obtain

$$C = k_- + k_+\lambda^2 + k_+ \frac{a_0^2}{4}(1 + C_P)C_{Tr}\lambda^3,$$

where  $C_P$  denotes the Poincaré constant and  $C_{Tr}$  the norm of the trace application  $Tr : H^1(\Omega) \rightarrow L^2(\partial\Omega)$ . The speed found in our proof is consequently

$$\theta = c(\mathbf{m}) \left( \frac{k_-}{\lambda} + k_+\lambda + k_+ \frac{a_0^2}{4}(1 + C_P)C_{Tr}\lambda^2 \right)^{-1}.$$

It can be shown that  $\theta$  reaches a maximum at some point

$$\lambda_0 \in \left[ \min \left( \left( \frac{k_-}{k_+ a_0^2 (1 + C_P) C_{Tr}} \right)^{1/3}, \frac{2}{a_0^2 (1 + C_P) C_{Tr}} \right), \sqrt{\frac{k_-}{k_+}} \right].$$

Besides,  $\theta$  tends to 0 when  $\lambda \rightarrow 0$  or  $\infty$ .

**Remark 6** In fact, one can replace the feedback law  $\mathbf{m} \cdot \boldsymbol{\nu} g(u')$  by a more general one  $g(\mathbf{x}, u')$  provided that, for some constant  $c > 1$ ,

$$\begin{aligned} c^{-1}(\mathbf{m} \cdot \boldsymbol{\nu})^{\frac{1}{p}} |s|^{\frac{1}{2} + \frac{1}{p}} &\leq |g(\mathbf{x}, s)| \leq c(\mathbf{m} \cdot \boldsymbol{\nu})^{\frac{1}{p}} |s|^{\frac{1}{2} + \frac{1}{p}} \quad \text{for a.e. } \mathbf{x} \in \partial\Omega_N \text{ and } |s| \leq 1, \\ c^{-1}(\mathbf{m} \cdot \boldsymbol{\nu}) |s| &\leq |g(\mathbf{x}, s)| \leq c(\mathbf{m} \cdot \boldsymbol{\nu}) |s| \quad \text{for a.e. } \mathbf{x} \in \partial\Omega_N \text{ and } |s| \geq 1. \end{aligned}$$

The details are left to the reader but the previous proof works also in this case.

## 4 Observability and controllability results

It is well-known that micro-local techniques [3] characterize all partitions of the boundary such that this result holds, but constants are not explicit. Thus, using this new choice of multiplier, we will enlarge the set of geometric examples with explicit knowledge of constants. We here follow [16].

### 4.1 Preliminary settings

Following HUM method [13], controllability of problem  $(\Sigma)$  is equivalent to observability of its adjoint problem. the solution of the control problem is equivalent to studying the observability properties of the adjoint problem. For each pair of initial conditions  $(\varphi_0, \varphi_1) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$ , let us consider the solution  $\varphi$  of the following wave problem,

$$(\Sigma') \quad \begin{cases} \varphi'' - \Delta\varphi = 0 & \text{in } \Omega \times (0, T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T), \\ \varphi(0) = \varphi_0 & \text{in } \Omega, \\ \varphi'(0) = \varphi_1 & \text{in } \Omega. \end{cases}$$

Observability of  $(\Sigma')$  is equivalent to the existence of a constant  $C < \infty$  independent of  $(\varphi_0, \varphi_1)$  such that

$$E_0 = \frac{1}{2} \int_{\Omega} (|\varphi_1|^2 + |\nabla\varphi_0|^2) \, d\mathbf{x} \leq C \int_{\partial\Omega_N \times (0, T)} |\partial_{\nu}\varphi|^2 \, d\sigma \, dt.$$

Let us define the operator  $\mathcal{W}_0$  on  $\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$  by

$$\begin{aligned} \mathcal{W}_0(\varphi, \psi) &= (-\psi, -\Delta\varphi), \\ D(\mathcal{W}_0) &= \{(\varphi, \psi) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) / \Delta\varphi \in \mathbf{L}^2(\Omega)\}, \end{aligned}$$

so that  $(\Sigma')$  can be written as follows,

$$\begin{cases} (\varphi, \psi)' + \mathcal{W}_0(\varphi, \psi) = 0, \\ (\varphi, \psi)(0) = (\varphi_0, \varphi_1). \end{cases}$$

**Remark 7** If  $(\varphi, \psi) \in D(\mathcal{W}_0)$ ,  $\varphi$  is the solution of some Dirichlet Laplace problem and hence regular (that is  $\varphi \in \mathbf{H}^2(\Omega)$ ).

$\mathcal{W}_0$  is a maximal-monotone operator on  $\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$  and  $D(\mathcal{W}_0)$  is dense in  $\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$  for the usual norm. Using Hille-Yosida Theorem, it generates a unitary semi-group on  $\mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$ , we denote its value applied at  $(\varphi_0, \varphi_1)$  at time  $t$  by  $(\varphi(t), \varphi'(t))$ .

As a consequence, we get conservation of energy.

**Proposition 10** If  $t \geq 0$  and  $\varphi$  is a weak solution of  $(\Sigma')$ , then

$$E(t) = \frac{1}{2} \int_{\Omega} (|\varphi'(t)|^2 + |\nabla\varphi(t)|^2) \, d\mathbf{x} = E_0.$$

A weak solution of  $(\Sigma')$  hence belongs to  $\mathcal{C}(\mathbb{R}_+; \mathbf{H}_D^1(\Omega)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathbf{L}^2(\Omega))$ .

A solution with  $(\varphi_0, \varphi_1) \in D(\mathcal{W}_0)$  is called a strong solution and satisfies  $(\varphi, \varphi') \in \mathcal{C}(\mathbb{R}_+; D(\mathcal{W}_0))$ .

## 4.2 Inverse inequality and exact controllability

We keep similar notations as in Section 2:  $a_0 = \frac{1}{2} \left( \operatorname{ess\,inf}_{\Omega} (\operatorname{div}(\mathbf{m})) + \operatorname{ess\,sup}_{\Omega} (\operatorname{div}(\mathbf{m}) - 2\lambda_{\mathbf{m}}) \right)$ .

**Proposition 11** *If  $T > 2 \frac{\|\mathbf{m}\|_{\infty}}{c(\mathbf{m})}$ , for each weak solution  $\varphi$  of  $(\Sigma')$ , the following inequality holds*

$$E_0 \leq \frac{\operatorname{ess\,sup}_{\partial\Omega_N} |\mathbf{m} \cdot \boldsymbol{\nu}|}{2(c(\mathbf{m})T - 2\|\mathbf{m}\|_{\infty})} \int_{\partial\Omega_N \times (0, T)} |\partial_{\nu} \varphi|^2 d\sigma dt.$$

**Remark 8** *In the case  $\mathbf{m}(x) = (dI + A)(\mathbf{x} - \mathbf{x}_0)$  with  $A$  skew-symmetric matrix, we recover classical results (see [11],[16]).*

**Proof.** Let  $(\varphi_0, \varphi_1) \in D(W_0)$ . We use again  $M\varphi = 2\mathbf{m} \cdot \nabla \varphi + a_0 \varphi$ . Using the fact that  $\varphi$  is solution of  $(\Sigma')$  and observing that  $\varphi'' M\varphi = (\varphi' M\varphi)' - \varphi' M\varphi'$ , we get

$$0 = \int_0^T \int_{\Omega} (-\varphi'' + \Delta \varphi) M\varphi d\mathbf{x} dt = - \left[ \int_{\Omega} \varphi' M\varphi d\mathbf{x} \right]_0^T + \int_0^T \int_{\Omega} (\varphi' M\varphi' + \Delta \varphi M\varphi) d\mathbf{x} dt.$$

As well as in the proof of Theorems 1 and 2, one uses Green-Riemann formula and Proposition 4 to get

$$\int_{\Omega} \Delta \varphi M\varphi d\mathbf{x} = \int_{\Omega} ((\operatorname{div}(\mathbf{m}) - a_0)I - 2(\nabla \mathbf{m})^s)(\nabla \varphi, \nabla \varphi) d\mathbf{x} + \int_{\partial\Omega} (\partial_{\nu} \varphi M\varphi - \mathbf{m} \cdot \boldsymbol{\nu} |\nabla \varphi|^2) d\sigma.$$

Dirichlet boundary conditions lead to

$$\int_{\Omega} \Delta \varphi M\varphi d\mathbf{x} = \int_{\Omega} ((\operatorname{div}(\mathbf{m}) - a_0)I - 2(\nabla \mathbf{m})^s)(\nabla \varphi, \nabla \varphi) d\mathbf{x} + \int_{\partial\Omega_N} \mathbf{m} \cdot \boldsymbol{\nu} |\partial_{\nu} \varphi|^2 d\sigma.$$

On the other hand, another use of Green formula gives us

$$\int_{\Omega} \varphi' M\varphi' d\mathbf{x} = \int_{\Omega} (a_0 - \operatorname{div}(\mathbf{m})) |\varphi'|^2 d\mathbf{x},$$

so, we finally get, using the same minoration as in proof of Theorems 1 and 2

$$c(\mathbf{m}) \int_0^T \int_{\Omega} |\varphi'|^2 + |\nabla \varphi|^2 d\mathbf{x} dt \leq - \left[ \int_{\Omega} \varphi' M\varphi d\mathbf{x} \right]_0^T + \int_{\partial\Omega_N} \mathbf{m} \cdot \boldsymbol{\nu} |\partial_{\nu} \varphi|^2 d\sigma. \quad (12)$$

Using the conservation of the energy, the left hand side in (12) is  $2cTE_0$ . It only remains to estimate the term  $\left[ - \int_{\Omega} \varphi' M\varphi d\mathbf{x} \right]_0^T$  to end the proof.

Let us fix a time  $t \in \{0, T\}$ . Cauchy-Schwarz inequality leads to

$$- \int_{\Omega} \varphi' M\varphi d\mathbf{x} \leq \left( \int_{\Omega} |\varphi'|^2 \right)^{1/2} \left( \int_{\Omega} |M\varphi|^2 \right)^{1/2}$$

Denoting by  $\|\cdot\|$  the  $L^2(\Omega)$ -norm, we get the following splitting

$$\|M\varphi\|^2 = \|2\mathbf{m} \cdot \nabla \varphi\|^2 + a_0^2 \|\varphi\|^2 + 4a_0 \int_{\Omega} \varphi \mathbf{m} \cdot \nabla \varphi d\mathbf{x}.$$

Green-Riemann formula and Dirichlet boundary conditions give

$$\int_{\Omega} \varphi \mathbf{m} \cdot \nabla \varphi d\mathbf{x} = -\frac{1}{2} \int_{\Omega} \operatorname{div}(\mathbf{m}) |\varphi|^2 d\mathbf{x},$$

and since  $a_0 - 2\operatorname{div}(\mathbf{m}) \leq a_0 - \operatorname{div}(\mathbf{m}) \leq -c(\mathbf{m})$  a.e., we finally get that  $\|M\varphi\| \leq 2\|\mathbf{m}\|_{\infty} \|\nabla \varphi\|$ .

Consequently, with Young inequality, we get the following estimate

$$-\int_{\Omega} \varphi' M \varphi d\mathbf{x} \leq 2\|\mathbf{m}\|_{\infty} \left( \int_{\Omega} |\varphi'|^2 \right)^{1/2} \left( \int_{\Omega} |\nabla \varphi|^2 \right)^{1/2} \leq 2\|\mathbf{m}\|_{\infty} E_0.$$

So (12) becomes

$$2(c(\mathbf{m})T - 2\|\mathbf{m}\|_{\infty})E_0 \leq \int_{\partial\Omega_N} \mathbf{m} \cdot \boldsymbol{\nu} |\partial_{\nu} \varphi|^2 d\sigma,$$

which ends the proof of Proposition 11, using the density of the domain. ■

Now we can deduce our exact controllability result (Theorem 3) from Proposition 11 following HUM method (see [13], Chapter IV).

## 5 Example

Let us consider here the case of a square domain  $\Omega = (0, 1)^2$  with the following affine multiplier,

$$\mathbf{m}(\mathbf{x}) = \begin{pmatrix} \cot \theta_1 & -1 \\ 1 & \cot \theta_2 \end{pmatrix} (\mathbf{x} - \mathbf{x}_0) \quad (13)$$

where  $\theta_1$  and  $\theta_2$  belong to  $(0, \frac{\pi}{2})$ .

We will discuss the dependence of  $\partial\Omega_N$  and  $\partial\Omega_D$  on  $\mathbf{x}_0$ .

First let us consider one edge  $[\mathbf{ab}]$  of  $\Omega$  with its normal unit vector  $\boldsymbol{\nu}$ . One can easily see that

$$\mathbf{m}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) = \frac{1}{\sin \theta} (\mathbf{x} - \mathbf{x}_0) \cdot \boldsymbol{\nu}_{\theta},$$

where  $\theta = \theta_1$  (resp.  $\theta_2$ ) if  $[\mathbf{ab}] \subset [0, 1] \times \{0, 1\}$  (resp.  $\{0, 1\} \times [0, 1]$ ) and  $\boldsymbol{\nu}_{\theta}$  is deduced from  $\boldsymbol{\nu}$  by rotation of angle  $-\theta$ . Without any restriction, we suppose  $\mathbf{a} \cdot \boldsymbol{\nu}_{\theta} < \mathbf{b} \cdot \boldsymbol{\nu}_{\theta}$ .

Then there exists an interface point along  $[\mathbf{ab}]$  if and only if  $\mathbf{x}_0$  belongs to the belt

$$\mathcal{B}_{\theta} = \{\mathbf{x} \in \mathbb{R}^2 / \mathbf{a} \cdot \boldsymbol{\nu}_{\theta} < \mathbf{x} \cdot \boldsymbol{\nu}_{\theta} < \mathbf{b} \cdot \boldsymbol{\nu}_{\theta}\}.$$

In this case, at this interface point  $\mathbf{x}_1$ , we get with similar notations,

$$\mathbf{m}(\mathbf{x}_1) \cdot \boldsymbol{\tau}(\mathbf{x}_1) = \frac{1}{\sin \theta} (\mathbf{x}_1 - \mathbf{x}_0) \cdot \boldsymbol{\tau}_{\theta}.$$

Then additional geometric assumption (7) is not satisfied if  $\mathbf{x}_0$  belongs to half-belt  $\mathcal{B}_{\theta}^+$  (see Fig. 1).

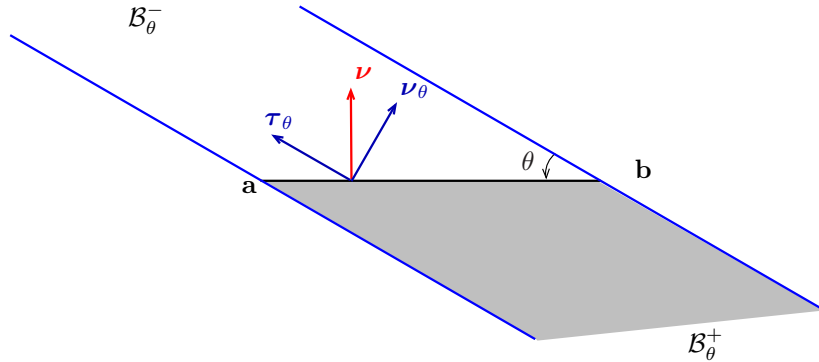


Figure 1: If  $\mathbf{x}_0$  belongs to  $\mathcal{B}_{\theta}^-$ , we get mixed boundary conditions along  $[\mathbf{ab}]$  and condition (7) is satisfied.

We now can describe every situation by considering only three following cases (see Fig. 2),

$$(C1): \quad 0 < \theta_1 \leq \theta_2 < \frac{\pi}{4}, \quad (C2): \quad 0 < \theta_1 < \frac{\pi}{4} \leq \theta_2 < \frac{\pi}{2}, \quad (C3): \quad \frac{\pi}{4} \leq \theta_1 \leq \theta_2 < \frac{\pi}{2}.$$

We also show a fully detailed partition in some particular case corresponding to (C2) (see Fig. 3).

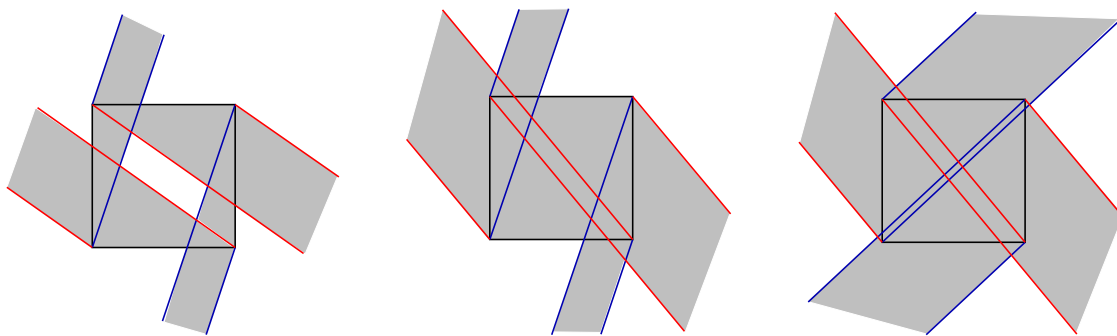


Figure 2: Cases (C1), (C2), (C3). Condition (7) is not satisfied in colored regions.

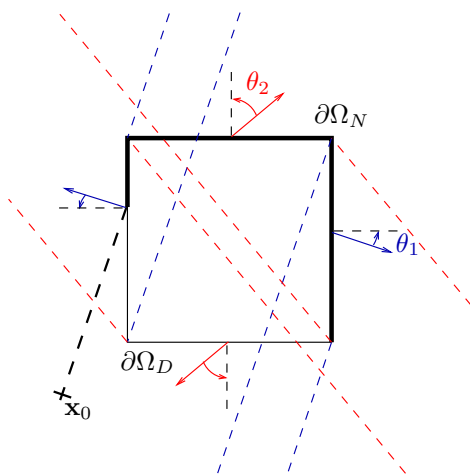


Figure 3: Example of Dirichlet and Neumann parts of the boundary in a case of (C2)-type.

## References

- [1] BRÉZIS, H. 1973, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, *Math. studies 5*, North Holland
- [2] BANASIAK, J., ROACH, G.-F., 1989, On mixed boundary value problems of Dirichlet oblique-derivative type in plane domains with piecewise differentiable boundary. *J. Diff. Equations*, **79**, no 1, 111-131.
- [3] BARDOS, C., LEBEAU, G., RAUCH, J., 1992, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. *SIAM J. Control Optim.*, **30**, no 5, 1024-1065.
- [4] BEY, R., LOHÉAC, J.-P., MOUSSAOUI, M., 1999, Singularities of the solution of a mixed problem for a general second order elliptic equation and boundary stabilization of the wave equation. *J. Math. pures et appl.*, **78**, 1043-1067.
- [5] CORNILLEAU, P., LOHÉAC, J.-P., OSSES, A., 2008, Nonlinear Neumann boundary stabilization of the wave equation using rotated multipliers. *J. of Dynamical and Control Systems*, **15**, no 4 (to appear).
- [6] CONRAD, F., RAO, B. 1993, Decay of solutions of the wave equation in a star-shaped domain with non linear boundary feedback. *Asymptotic Analysis*, **7**, no 1, 159-177.
- [7] GRISVARD, P., 1985, Elliptic problems in nonsmooth domains. *Pitman, London*.
- [8] GRISVARD, P., 1989, Contrôlabilité exacte des solutions de l'équation des ondes en présence de singularités. *J. Math. pures et appl.*, **68**, 215-259.

- [9] KOZLOV, V. A., MAZ'YA, V. G., ROSSMANN, J., 1997, Elliptic Boundary Value Problems in Domains with Point Singularities, *AMS, Providence*.
- [10] KOMORNIK, V., 1994, Exact controllability and stabilization ; the multiplier method. *Masson-John Wiley, Paris*.
- [11] KOMORNIK, V., ZUAZUA, E., 1990, A direct method for the boundary stabilization of the wave equation. *J. Math. pures et appl.*, **69**, 33-54.
- [12] HO, L.F., 1986, Observabilité frontière de l'équation des ondes. *C. R. Acad. Sci. Paris, Sér. I Math.* 302, 443-446.
- [13] LIONS, J.-L., 1988, Contrôlabilité exacte, stabilisation et perturbation des systèmes distribués, 1, coll. RMA, *Masson, Paris*.
- [14] MOUSSAOUI M., 1996, Singularités des solutions du problème mêlé, contrôlabilité exacte et stabilisation frontière. *ESAIM Proceedings, Élasticité, Viscoélasticité et Contrôle optimal, Huitièmes Entretiens du Centre Jacques Cartier*, 157-168.
- [15] NEČAS, J., 1967, Les méthodes directes en théorie des équations elliptiques. *Masson, Paris*.
- [16] OSSES, A., 2001, A rotated multiplier applied to the controllability of waves, elasticity and tangential Stokes control. *SIAM J. Control Optim.*, **40**, no 3, 777-800.
- [17] RELICH F., 1940, Darstellung der Eigenwerte von  $\Delta u + \lambda u$  durch ein Randintegral. *Math. Zeitschrift*, 46, 635-636.
- [18] SHAMIR, E., 1968, Regularity of mixed second order elliptic problems. *Israel Math. Journal*, 6, 150-168.