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Abstract. In this paper, we study the Gevrey regularity of weak solution for a class of linear and quasilinear Fokker-Planck equations.

1. Introduction

Recently, a lot of progress has been made on the study for the spatially homogeneous Boltzmann equation without angular cutoff, cf. [2, 3, 8, 22] and references therein, which shows that the singularity of collision cross-section yields some gain of regularity in the Sobolev space frame on weak solutions for Cauchy problem. That means, this gives the C^∞ regularity of weak solution for the spatially homogeneous Boltzmann operator without angular cutoff. The local solutions having the Gevrey regularity have been constructed in [21] for initial data having the same Gevrey regularity, and a genearal Gevrey regularity results have given in [17] for spatially homogeneous and linear Boltzmann equation of Cauchy problem for any initial data. In the other word, there is the smoothness effet similary to heat equation.

However, there is no general theory for the spatially inhomogeneous problems. It is now a kinetic equation in which the diffusion part is nonlinear operator of velocity variable. In [1], by using the uncertainty principle and microlocal analysis, they obtain a C^∞ regularity results for linear spatially inhomogeneous Boltzmann equation without angular cutoff.

In this paper, we will study the Gevrey regularity of weak solution for the the following Fokker-Planck operator in $\mathbb{R}^{2n+1}$

$$\mathcal{L} = \partial_t + v \cdot \partial_x - a(t,x,v)\triangle_v, \tag{1.1}$$

where $\triangle_v$ is Laplace operator of velocity variables v.

The motivation of study for this class of operator is , as in [15], attempt to study inhomogenous Boltzmann equations without angular cutoff and non linear Vlasov-Fokker-Planck equation (see [11, 12]).

Before stating the result, we recall the definition of Gevrey class function. Let U be an open subset of $\mathbb{R}^N$ and f be a real function defined in U. We say $f \in G^s(U)(s \geq 1)$ if $f \in C^\infty(U)$ and for any

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compact subset K of U, there exists a constant $C = C_K$, depending only on K, such that for all multi-indices $\alpha \in \mathbb{N}^N$ and for all $x \in K$

$$|\partial^\alpha f(x)| \leq C_K^{|\alpha|+1}(|\alpha|!)^s. \tag{1.2}$$

Denote by $\bar{U}$ the closure of U in $\mathbb{R}^N$. we say $f \in G^s(\bar{U})$ if $f \in G^s(W)$ for some open neighborhood W of $\bar{U}$. The estimate (1.2) for $x \in K$ is valid if and only if the following one is valid (cf.Chen hua-Rodino[5] or Rodino[18]):

$$\|\partial^\alpha f\|_{L^2(K)} \leq C_K^{|\alpha|+1}(|\alpha|)^{s|\alpha|}.$$

In this paper, we use the above estimate in L^2.

We say an operator P is G^s hypoelliptic in U if $u \in \mathcal{D}', Pu \in G^s(U)$ implies $u \in G^s(U)$. Likewise, we say an operator P is C^∞ hypoelliptic in U if $u \in \mathcal{D}', Pu \in C^\infty(U)$ implies $u \in C^\infty(U)$.

The operator $\mathcal{L}$ satisfies the Hörmander' condition. By virtue of the results of Hörmander [5], we know that $\mathcal{L}$ is C^∞ hypoelliptic. In the aspect of Gevrey class, Derridj-Zuily [7] proved that $\mathcal{L}$ is G^s-hypoelliptic for $s > 6$ in a general form of Hörmander's operators.

In this paper, we improve firstly the results of [7] for Fokker-Planck operators as the following theorems.

Theorem 1.1. *For any $s \geq 3$, if the coefficient a is in $G^s(\mathbb{R}^{2n+1})$ and $a(t,x,v) \geq c_0 > 0$, then the operator $\mathcal{L}$ given in (1.1) is G^s hypoelliptic in $\mathbb{R}^{2n+1}$,*

Of course, Theorem 1.1 is also true for the following general operators,

$$\tilde{\mathcal{L}} = \partial_t + A(v)\cdot\partial_x - \sum_{j,k=1}^{n} a_{jk}(t,x,v)\partial^2_{v_j v_k}$$

in an open domain U of $\mathbb{R}^{2n+1}$, where A is a non singular $n \times n$ constant matrix, $\big(a_{jk}(t,x,v)\big)$ is positive defined on U and belongs to $G^s(U)$.

Remark Our results is a local and interior regularity results, that means if there exists a weak solution in $\mathcal{D}'$, then this solution is in Gevrey class in interior of domain. So that if the weak solution is a solution of the Cauchy problem, we don't need the regularity of initial data.

Secondly, we consider the quasi-linear equation

$$\partial_t u + v\cdot\nabla_x u - a\triangle_v u = F(t,x,v,u,\nabla_v u) \tag{1.3}$$

where F is nonlinear function of real variable (t,x,v,w,p).

Theorem 1.2. *Let u be a weak solution of equation (1.3) such that $u, \nabla_v u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$, then*

$$u \in G^s(\mathbb{R}^{2n+1})$$

for any $s \geq 3$, provided the coefficients a is in $G^s(\mathbb{R}^{2n+1})$, $a(t,x,v) \geq c_0 > 0$ and nonlinear function $F(t,x,v,w,p)$ is in $G^s(\mathbb{R}^{2n+2+n})$.

Remark : If the nonlinear term F is independent of p or in the form of

$$\nabla_v(F(t,x,v,u)),$$

we can suppose that the weak solution $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$.

The plan of this paper is as follows : In section 2, we obtain a sharp subelliptic estimate for the Fokker-Planck operator $\mathcal{L}$ via direct computation, and then prove the Gevrey hypoellipticity of $\mathcal{L}$. In section 3, we prove the Gevrey regularity for the weak solutions of the quasi-linear Fokker-Planck equation.

2. Subelliptic estimate

We recall firstly some notations, $\|\cdot\|_\kappa, \kappa \in \mathbb{R}$, is the classical Sobolev norm in $H^\kappa(\mathbb{R}^{2n+1})$, and $(h,\ k)$ is the inner product of $h, k \in L^2(\mathbb{R}^{2n+1})$. Moreover if $f, g \in C_0^\infty(\mathbb{R}^{2n+1})$, from Hölder inequality and Young inequality, for any $\varepsilon > 0$,

$$|(f,\ g)| \leq \|h\|_\kappa \|g\|_{-\kappa} \leq \frac{\varepsilon \|h\|_\kappa^2}{2} + \frac{\|g\|_{-\kappa}^2}{2\varepsilon}. \tag{2.1}$$

We have also the interpolation inequality for Sobolev space, for any $\varepsilon > 0$ and any $r_1 < r_2 < r_3$,

$$\|h\|_{r_2} \leq \varepsilon \|h\|_{r_3} + \varepsilon^{-(r_2-r_1)/(r_3-r_2)} \|h\|_{r_1}. \tag{2.2}$$

Let Ω be an open subset of $\mathbb{R}^{2n+1}$. We denote by $S^m = S^m(\Omega), m \in \mathbb{R}$, the symbol space of classical pseudo-differential operator and $P = P(t, x, v, D_t, D_x, D_v) \in \mathrm{Op}(S^m)$ a pseudo-differential operator of symbol $p(t, x, v; \tau, \xi, \eta) \in S^m$. If $P \in \mathrm{Op}(S^m)$, then P is a continuous operator from $H_c^\kappa(\Omega)$ to $H_{loc}^{\kappa-m}(\Omega)$. Here $H_c^\kappa(\Omega)$ is the subspace of $H^\kappa(R^{2n+1})$ consisting of the distributions having their compact support in Ω, and $H_{loc}^{\kappa-m}(\Omega)$ consists of the distributions h such that $\phi h \in H^{\kappa-m}(\mathbb{R}^{2n+1})$ for any $\phi \in C_0^\infty(\Omega)$. The more properties can be found in the Treves' book [20]. Remark that if $P_1 \in \mathrm{Op}(S^{m_1})$, $P_2 \in \mathrm{Op}(S^{m_2})$, then $[P_1,\ P_2] \in \mathrm{Op}(S^{m_1+m_2-1})$.

Now we show a sharp subelliptic estimate for the operator $\mathcal{L}$, our proof bases on the work of Bouchut [4] and Morimoto-Xu [15].

Proposition 2.1. *Let K be a compact subset of $\mathbb{R}^{2n+1}$. Then for any $r \geq 0$, there exists a constant $C_{K,r}$, depending only on K and r, such that for any $f \in C_0^\infty(K)$,*

$$\|f\|_r \leq C_{K,r}\{\ \|\mathcal{L}f\|_{r-2/3} + \|f\|_0\ \}. \tag{2.3}$$

To simplify the notation, in this section we will denote by C_K the different suitable constants depending only on K. We have firstly the following three lemmas, which establish the gain of regularity in the velocity variable v, in the space variable x and in the time variable t, respectively.

Lemma 2.2. *There exists a constant C_K such that for any $f \in C_0^\infty(K)$,*

$$\|\nabla_v f\|_0 \leq C_K\big\{\ |(\mathcal{L}f,\ f)| + \|f\|_0\big\}.$$

Moreover, for any $r \geq 0$, for any $\varepsilon > 0$,

$$\|\nabla_v f\|_r \leq \varepsilon\, \|\mathcal{L}f\|_r + C_{K,\varepsilon}\, \|f\|_r\,.$$

We get a gain of regularity of order 1 for v variable. This is obtained directly by the positivity of coefficient a and compact support of f. For the space variable x, we have also the following subelliptic estimate.

Lemma 2.3. *There exists a constant C_K such that for any $f \in C_0^\infty(K)$,*

$$\|D_x^{2/3} f\|_0 \leq C_K(\|\mathcal{L}f\|_0 + \|f\|_0),$$

where $D_x^{2/3} = (-\triangle_x)^{1/3}$.

This is a result of [4], and it is deduced by following two estimates

$$\|D_x^{2/3} f\|_0 \leq C_K\ \|\triangle_v f\|_0^{1/3} \|\partial_t f + v \cdot \partial_x f\|_0^{2/3},$$

and

$$\|\triangle_v f\|_0 \leq C_K(\ \|\mathcal{L}f\|_0 + \|f\|_0\ ).$$

For the time variable t, we have also a gain of regularity of order $2/3$.

Lemma 2.4. *There exists a constant C_K such that for any $f \in C_0^\infty(K)$,*

$$\|\partial_t f\|_{-1/3} \le C_K(\|\mathcal{L}f\|_0 + \|f\|_0).$$

In fact, we have

$$\|\partial_t f\|_{-1/3} = \|\Lambda^{-1/3}\partial_t f\|_0 \le \|\Lambda^{-1/3}(\partial_t + v\cdot\partial_x)f\|_0 + \|\Lambda^{-1/3}v\cdot\partial_x f\|_0,$$

where $\Lambda = (1+|D_t|^2+|D_x|^2+|D_v|^2)^{1/2}$. From Lemma 2.3, we have

$$\|\Lambda^{-1/3}v\cdot\partial_x f\|_0 \le C_K\|D_x^{2/3}f\|_0 \le C_K(\|\mathcal{L}f\|_0 + \|f\|_0).$$

The estimation for the term $\|\Lambda^{-1/3}(\partial_t + v\cdot\partial_x)f\|_0$ can be obtained by direct calculus as in [15].

Proof of Proposition 2.1. The Lemma 2.2, Lemma 2.3 and Lemma 2.4 deduce immediately

$$\|f\|_{2/3} \quad \le \quad C_K\{\ \|\mathcal{L}f\|_0 + \|f\|_0\ \}. \tag{2.4}$$

Moreover, choose a function $\psi \in C_0^\infty(\mathbb{R}^{2n+1})$ such that $\psi|_K \equiv 1$, Supp ψ is a neighborhood of K. Then for any $f \in C_0^\infty(K)$ and any $r \ge 0$,

$$\|f\|_r \quad = \quad \|\psi f\|_r \le C_K\{\ \|\psi\Lambda^{r-2/3}f\|_{2/3} + \|[\Lambda^{r-2/3},\ \psi]f\|_{2/3}\ \}.$$

By virtue of (2.4) and the interpolation inequality (2.2), we have

$$\begin{aligned}\|f\|_r &\le C_K\{\ \|\mathcal{L}\psi\Lambda^{r-2/3}f\|_0 + \|f\|_{r-2/3}\ \}\\ &\le C_{\varepsilon,K}\{\ \|\mathcal{L}\psi\Lambda^{r-2/3}f\|_0 + \|f\|_0\ \} + \varepsilon\|f\|_r.\end{aligned}$$

Taking ε small enough, we get

$$\|f\|_r \le C_K\{\ \|\mathcal{L}f\|_{r-2/3} + \|f\|_0 + \|[\mathcal{L},\ \psi\Lambda^{r-2/3}]f\|_0\ \}.$$

Direct verification gives

$$\begin{aligned}[\mathcal{L},\ \psi\Lambda^{r-2/3}] \quad = \quad & [\partial_t + v\cdot\partial_x,\ \psi\Lambda^{r-2/3}] - \sum_{j=1}^n\{\ \|[a,\ \psi\Lambda^{r-2/3}]\partial_{v_j}^2\\ & + a[\partial_{v_j},\ [\partial_{v_j},\psi\Lambda^{r-2/3}]\ ] + 2a[\partial_{v_j},\ \psi\Lambda^{r-2/3}]\partial_{v_j}\ \},\end{aligned}$$

This along with Lemma 2.2 yields

$$\begin{aligned}\|[\mathcal{L},\ \psi\Lambda^{r-2/3}]f\|_0 &\le C_K\{\ \|f\|_{r-2/3} + \sum_{j=1}^n\|\partial_{v_j}f\|_{r-2/3}\ \}\\ &\le C_K\{\ \|\mathcal{L}f\|_{r-2/3} + \|f\|_{r-2/3}\ \}.\end{aligned}$$

These three estimates gives immediately

$$\|f\|_r \le C_K\{\ \|\mathcal{L}f\|_{r-2/3} + \|f\|_0 + \|f\|_{r-2/3}\ \}.$$

Applying interpolation inequality (2.2) again and taking ε small enough, we prove Proposition 2.1.

We consider now the commutators of the operators $\mathcal{L}$ with derivation and cut-off function.

Proposition 2.5. *Let K be a compact subset of $\mathbb{R}^{2n+1}$. Then for any $r \ge 0$, there exist constants $C_{K,r}, C_{K,r,\varphi}$ such that for any $f \in C_0^\infty(K)$,*

$$\|[\mathcal{L},\ D]f\|_r \quad \le \quad C_{K,r}\{\ \|\mathcal{L}f\|_{r+1-2/3} + \|f\|_0\ \},$$

and

$$\|[\mathcal{L},\ \varphi]f\|_r \le C_{K,r,\varphi}\{\ \|\mathcal{L}f\|_{r-1/3} + \|f\|_0\ \},$$

where $\varphi \in C_b^\infty(\mathbb{R}^{2n+1})$ and we denote by D the differential operator ∂_t, ∂_x or ∂_v.

Proof. By using the positivity of coefficient a, we have

$$\|\triangle_v f\|_r \quad \leq \quad C_K\{\ \|\mathcal{L}f\|_r + \|f\|_{r+1}\ \}.$$

And $[\mathcal{L},\ D] = [\partial_t + v\cdot\partial_x,\ D] - [a,\ D]\triangle_v$ deduce

$$\|[\mathcal{L},\ D]f\|_r \leq C_K\{\ \|f\|_{r+1} + \|\triangle_v f\|_r\ \}.$$

The above two inequalities along with the subelliptic estimate (2.3) yield the first desired inequality in Proposition 2.5.

To treat $\|[\mathcal{L},\ \varphi]f\|_r$, the subelliptic estimate (2.3) give

$$\|\nabla_v f\|_r \leq C_K(\|\mathcal{L}f\|_{r-1/3} + \|f\|_0).$$

Now simple verification gives

$$\begin{aligned}\|[\mathcal{L},\ \varphi]f\|_r \quad &\leq \quad C_K\Big\{\ \|f\|_r + \sum_{j=1}^{n}\ \|\partial_{v_j} f\|_r\ \Big\}\\ &\leq \quad C_{K,r}\big\{\ \|\mathcal{L}f\|_{r-1/3} + \|f\|_0\ \big\}.\end{aligned}$$

This completes the proof of Proposition 2.5.

We prove now the Gevrey hypoellipticity of $\mathcal{L}$. Our starting point is the following result due to M.Durand [9]:

Proposition 2.6. *Let P be a linear differential operator with smooth coefficients in $\mathbb{R}^m_y$ and ϱ, ς two fixed positive numbers. If for any $r \geq 0$, any compact $K \subseteq \mathbb{R}^m$ and any $\varphi \in C^\infty(\mathbb{R}^m)$, there exist constants $C_{K,r}$ and $C_{K,r}(\varphi)$ such that for all $f \in C_0^\infty(K)$,the following conditions are fulfilled:*

$$(H_1) \qquad \|f\|_r \leq C_{K,r}(\|Pf\|_{r-\varrho} + \|f\|_0),$$

$$(H_2) \qquad \|[P,\ D_j]f\|_r \leq C_{K,r}(\|Pf\|_{r+1-\varsigma} + \|f\|_0),$$

$$(H_3) \qquad \|[P,\ \varphi]f\|_r \leq C_{K,r}(\varphi)(\|Pf\|_{r-\varsigma} + \|f\|_0),$$

where

$$D_j = \frac{1}{i}\frac{\partial}{\partial y_j}, j = 1, 2, \cdots, m.$$

Then for $s \geq \max(1/\varsigma, 2/\varrho)$, P is $G^s(\mathbb{R}^m)$ hypoelliptic, provided the coefficients of P are in the class of $G^s(\mathbb{R}^m)$.

Proposition 2.1 shows that the operator $\mathcal{L}$ satisfies the conditions (H_1) with $\varrho = 2/3$, Proposition 2.5 assures the conditions (H_2) and (H_3) with $\varsigma = 1/3$. Then $\mathcal{L}$ is $G^s(\mathbb{R}^{2n+1})$ hypoelliptic, $s \geq 3$, and we have proved Theorem 1.1.

3. Gevrey regularity of nonlinear equations

Let $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ be a weak solution of (1.3). Firstly, we will prove $u \in C^\infty(\mathbb{R}^{2n+1})$. And we need the following nonlinear composition results (see for example [23]).

Lemma 3.1. *Let $F(t, x, v, w, p) \in C^\infty(\mathbb{R}^{2n+2+n})$ and $r \geq 0$. If $u, \nabla_v u \in L^\infty_{loc}(\mathbb{R}^{2n+1}) \cap H^r_{loc}(\mathbb{R}^{2n+1})$, then $F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big) \in H^r_{loc}(\mathbb{R}^{2n+1})$, and*

$$\big\|\phi_1 F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big)\big\|_r \leq \bar{C}\ \{\ \|\phi_2 u\|_r + \|\phi_2 \nabla_v u\|_r\ \}, \tag{3.1}$$

where $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^{2n+1})$ and $\phi_2 = 1$ on the support of ϕ_1, and $\bar{C}$ is a constant depending only on r, ϕ_1, ϕ_2.

Remark. If the nonlinear term F is independent of p or in the form of

$$\nabla_v(F(t,x,v,u)),$$

Then that $u \in L^\infty_{loc}(\mathbb{R}^{2n+1}) \cap H^r_{loc}(\mathbb{R}^{2n+1})$ yields $F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big) \in H^r_{loc}(\mathbb{R}^{2n+1})$.

Lemma 3.2. *Let $u, \nabla_v u \in H^r_{loc}(\mathbb{R}^{2n+1}), r \geq 0$. Then we have*

$$\|\varphi_1 \nabla_v u\|_r \leq C \|\varphi_2 u\|_r , \tag{3.2}$$

where $\varphi_1, \varphi_2 \in C^\infty_0(\mathbb{R}^{2n+1})$ and $\varphi_2 = 1$ on the support of φ_1, and C is a constant depending only on r, φ_1, φ_2.

In fact, we have

$$\|\varphi_1 \nabla_v u\|_r \leq \|[\nabla_v,\ \varphi_1]u\|_r + \|\nabla_v \varphi_1 u\|_r .$$

Clearly, the first term on the right is bounded by $C\|\varphi_2 u\|_r$. For the second term , combining the second inequality in Lemma 2.2 and (3.1), we get the desired estimate (3.2) at once. This completes the proof of Lemma 3.2.

Now we are ready to prove

Proposition 3.3. *Let u be a weak solution of (1.3) such that $u, \nabla_v u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$. Then u is in $C^\infty(\mathbb{R}^{2n+1})$.*

In fact, from the subelliptic estimate (2.3) and the fact $\mathcal{L}u(\cdot) = F(\cdot, u(\cdot), \nabla_v u(\cdot))$, it then follows

$$\|\psi_1 u\|_{r+2/3} \leq \bar{C}\{\ \|\psi_2 F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big)\|_r + \|\psi_2 u\|_0\ \}, \tag{3.3}$$

where $\psi_1, \psi_2 \in C^\infty_0(\mathbb{R}^{2n+1})$ and $\psi_2 = 1$ on the support of ψ_1. Combining (3.1), (3.2) and (3.3), we have $u \in H^\infty_{loc}(\mathbb{R}^{2n+1})$ by standard iteration. This completes the proof of Proposition 3.3.

Now starting from the smooth solution, we prove the Gevrey regularity. It suffices to show the regularity in the open unit ball

$$\Omega = \{(t,x,v) \in \mathbb{R}^{2n+1} : t^2 + |x|^2 + |v|^2 < 1\}.$$

Set

$$\Omega_\rho = \big\{(t,x,v) \in \Omega : \big(t^2 + |x|^2 + |v|^2\big)^{1/2} < 1 - \rho\big\}, \qquad 0 < \rho < 1.$$

Let U be an open subset of $\mathbb{R}^{2n+1}$. Denote by $H^r(U)$ the space consisting of the functions which are defined in U and can be extended to $H^r(\mathbb{R}^{2n+1})$. Define

$$\|u\|_{H^r(U)} = \inf\big\{\|\tilde{u}\|_{H^s(\mathbb{R}^{n+1})} : \tilde{u} \in H^s(\mathbb{R}^{2n+1}), \tilde{u}|_U = u\big\}.$$

We denote $\|u\|_{r,U} = \|u\|_{H^r(U)}$, and

$$\|D^j u\|_r = \sum_{|\beta|=j} \|D^\beta u\|_r.$$

In order to treat the nonlinear term F on the right hand of (1.3), we need the following two lemmas. The first one (see [23] for example) concerns weak solution in some algebra, and the second is an analogue of Lemma 1 in [10]. In the sequel $C_j > 1$ will be used to denote suitable constants depending only on n or the function F.

Lemma 3.4. *Let $r > (2n+1)/2$ and $u_1, u_2 \in H^r(\mathbb{R}^{2n+1})$, Then $u_1 u_2 \in H^r(\mathbb{R}^{2n+1})$, moreover*

$$\|u_1 u_2\|_r \leq \tilde{C} \|u_1\|_r \|u_2\|_r, \tag{3.4}$$

where $\tilde{C}$ is a constant depending only on n, r.

Lemma 3.5. *Let M_j be a sequence of positive numbers and for some $B_0 > 0$, the M_j satisfy the monotonicity conditions*

$$\frac{j!}{i!(j-i)!} M_i M_{j-i} \le B_0 M_j, \ (i = 1, 2, \cdots, j; \ j = 1, 2, \cdots). \tag{3.5}$$

Suppose $F(t, x, v, u, p)$ satisfy

$$\left\| \left(D^j_{t,x,v} D^l_u D^m_p F \right) (\cdot, u(\cdot), \nabla_v u(\cdot)) \right\|_{r+n+1,\Omega} \le C_1^{j+l+m} M_{j-2} M_{m+l-2}, \quad j, m+l \ge 2, \tag{3.6}$$

where r is a real number satisfying $r + n + 1 > (2n+1)/2$. Then there exist two constants C_2, C_3 such that for any H_0, H_1 satisfying $H_0, H_1 \ge 1$ and $H_1 \ge C_2 H_0$, if $u(t, x, v)$ satisfy the following conditions

$$\|D^j u\|_{r+n+1,\Omega_{\tilde\rho}} \le H_0, \quad 0 \le j \le 1, \tag{3.7}$$

$$\|D^j u\|_{r+n+1,\Omega_{\tilde\rho}} \le H_0 H_1^{j-2} M_{j-2}, \quad 2 \le j \le N, \tag{3.8}$$

$$\|D_v D^j u\|_{r+n+1,\Omega_{\tilde\rho}} \le H_0 H_1^{j-2} M_{j-2}, \quad 2 \le j \le N. \tag{3.9}$$

Then for all α with $|\alpha| = N$,

$$\left\| \psi_N D^\alpha \left[F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big) \right] \right\|_{r+n+1} \le C_3 H_0 H_1^{N-2} M_{N-2}, \tag{3.10}$$

where $\psi_N \in C_0^\infty(\Omega_{\tilde\rho})$ is an arbitrary function.

Proof. Denote $p = (p_1, p_2, \cdots, p_n) = \nabla_v u$ and $k = (k_1, k_2 \cdots, k_n)$. From Faa di Bruno' formula, $\psi_N D^\alpha [F(\cdot, u(\cdot), \nabla_v u(\cdot))]$ is the linear combination of terms of the form

$$\frac{\psi_N \partial^{|\tilde\alpha|+l+|k|} F}{\partial^{\tilde\alpha}_{t,x,v} \partial u^l \partial p_1^{k_1} \cdots \partial p_n^{k_n}} \prod_{j=1}^{l} D^{\gamma_j} u \cdot \prod_{i=1}^{n} \prod_{j_i=1}^{k_i} D^{\beta_{j_i}} (\partial_{v_i} u), \tag{3.11}$$

where $|\tilde\alpha| + l + |k| \le |\alpha|$ and

$$\sum_{j=1}^{l} \gamma_i + \sum_{i=1}^{n} \sum_{j_i}^{k_i} \beta_{j_i} = \alpha - \tilde\alpha,$$

and if γ_i or β_{j_i} equals to 0, we just mean $D^{\gamma_i} u$ or $D^{\beta_{j_i}} u$ doesn't appear in (3.11). Choose a function $\tilde\psi \in C_0^\infty(\Omega_{\tilde\rho})$ such that $\tilde\psi = 1$ on Supp ψ_N. Note that $n + 1 + r > (2n+1)/2$, and hence applying Lemma 3.4, we have

(3.12)

$$\begin{aligned}
&\left\| \tfrac{\psi_N \partial^{|\tilde\alpha|+l+|k|} F}{\partial^{\tilde\alpha}_{t,x,v} \partial u^l \partial p_1^{k_1} \cdots \partial p_n^{k_n}} \prod_{j=1}^{l} D^{\gamma_j} u \cdot \prod_{i=1}^{n} \prod_{j_i=1}^{k_i} D^{\beta_{j_i}} (\partial_{v_i} u) \right\|_{r+n+1} \\
= &\left\| \tfrac{\psi_N \partial^{|\tilde\alpha|+l+|k|} F}{\partial^{\tilde\alpha}_{t,x,v} \partial u^l \partial p_1^{k_1} \cdots \partial p_n^{k_n}} \prod_{j=1}^{l} \tilde\psi D^{\gamma_j} u \cdot \prod_{i=1}^{n} \prod_{j_i=1}^{k_i} \tilde\psi \partial_{v_i} D^{\beta_{j_i}} u \right\|_{r+n+1} \\
\le &\tilde C \left\| \psi_N (\partial^{|\tilde\alpha|+l+|k|} F) \right\|_{r+n+1} \cdot \prod_{j=1}^{l} \left\| \tilde\psi D^{\gamma_j} u \right\|_{r+n+1} \times \prod_{i=1}^{n} \prod_{j_i=1}^{k_i} \left\| \tilde\psi \partial_{v_i} D^{\beta_{j_i}} u \right\|_{r+n+1} \\
\le &C_0 \left\| (\partial^{|\tilde\alpha|+l+|k|} F) \right\|_{r+n+1,\Omega} \cdot \prod_{j=1}^{l} \|D^{\gamma_j} u\|_{r+n+1,\Omega_{\tilde\rho}} \times \prod_{i=1}^{n} \prod_{j_i=1}^{k_i} \left\| \partial_{v_i} D^{\beta_{j_i}} u \right\|_{r+n+1,\Omega_{\tilde\rho}}.
\end{aligned}$$

In virtue of (3.7)-(3.9) and (3.12), the situation is entirely similar to [10]. The only difference is that we replace the Hölder norm $|u|_j$ by $\|D^j u\|_{r+n+1,\Omega_{\tilde\rho}}$ and $\left\|D_v D^j u\right\|_{r+n+1,\Omega_{\tilde\rho}}$. Then the same argument as the proof of Lemma 1 in [10] yields (3.10). This completes the proof of Lemma 3.5.

Proposition 3.6. *Let $s \geq 3$. Suppose $u \in C^\infty(\bar{\Omega})$ is a solution of (1.3), and $a(t,x,v) \in G^s(\mathbb{R}^{2n+1})$, $F(t,x,v,w,p) \in G^s(\mathbb{R}^{2n+2+n})$ and $a \geq c_0 > 0$. Then there exits a constant A such that for any $r \in [0,1]$ and any $N \in \mathbb{N}$, $N \geq 3$,*

$$\begin{aligned}(E)_{r,N} \qquad & \|D^\alpha u\|_{r+n+1,\Omega_\rho} + \|D_v D^\alpha u\|_{r-1/3+n+1,\Omega_\rho} \\ \leq\ & \frac{A^{|\alpha|-1}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{sr}, \quad \forall\ |\alpha| = N,\ \forall\ 0<\rho<1.\end{aligned}$$

From $(E)_{r,N}$, we have immediately

Proposition 3.7. *Under the same assumption as Proposition 3.6, we have $u \in G^s(\Omega)$.*

In fact, for any compact sunset K of Ω, we have $K \subset \Omega_{\rho_0}$ for some ρ_0, $0 < \rho_0 < 1$. For any α, $|\alpha| \geq 3$, letting $r = 0$ in $(E)_{r,N}$, we have

$$\|D^\alpha u\|_{L^2(K)} \leq \|D^\alpha u\|_{n+1,\Omega_{\rho_0}} \leq \tfrac{A^{|\alpha|-1}}{{\rho_0}^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s \leq \big(\tfrac{A}{{\rho_0}^s}\big)^{|\alpha|+1}(|\alpha|!)^s.$$

This completes the proof of Proposition 3.7.

Proof of Proposition 3.6. We use induction on N. Assuming $(E)_{r,N-1}$ holds for any r with $0 \leq r \leq 1$, and we will show $(E)_{r,N}$ still holds for any $r \in [0,\ 1]$. For any $\alpha, |\alpha| = N$, and for any ρ, $0 < \rho < 1$, choose a function $\varphi_{\rho,N} \in C_0^\infty(\Omega_{\frac{(N-1)\rho}{N}})$ such that $\varphi_{\rho,N} = 1$ in Ω_ρ. it is easy to see

$$\sup|D^\gamma \varphi_{\rho,N}| \leq C_\gamma(\rho/N)^{-|\gamma|} \leq C_\gamma(N/\rho)^{|\gamma|}, \quad \forall\ \gamma.$$

And we will proceed to prove the truth of $(E)_{r,N}$ by the following lemmas.

Lemma 3.8. *For $r = 0$, we have*

$$\|D^\alpha u\|_{n+1,\Omega_\rho} + \|D_v D^\alpha u\|_{-1/3+n+1,\Omega_\rho} \leq \frac{C_7 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s, \quad \forall\ 0<\rho<1.$$

Proof . Write $|\alpha| = |\beta| + 1$, then $|\beta| = N-1$. Denote $\frac{N-1}{N}\rho$ by $\tilde{\rho}$. In the sequel we will use the following fact frequently

$$\frac{1}{\rho^{sk}} \leq \frac{1}{\tilde{\rho}^{sk}} = \frac{1}{\rho^{sk}} \times \big(\frac{N}{N-1}\big)^{sk} \leq \frac{C_4}{\rho^{sk}}, \quad k = 1,2,\cdots,N-3.$$

Note that $\varphi_{\rho,N} = 1$ in Ω_ρ and hence

$$\begin{aligned}\|D^\alpha u\|_{n+1,\Omega_\rho} &\leq \|\varphi_{\rho,N} D^\alpha u\|_{n+1} \leq \|\varphi_{\rho,N} D^\beta u\|_{1+n+1} + \|(D\varphi_{\rho,N})D^\beta u\|_{n+1} \\ &\leq C_5\{\ \|D^\beta u\|_{1+n+1,\Omega_{\tilde{\rho}}} + (N/\rho)\|D^\beta u\|_{n+1,\Omega_{\tilde{\rho}}}\ \}.\end{aligned}$$

Since $(E)_{r,N-1}$ holds by assumption for any r with $0 \leq r \leq 1$, we have immediately

$$\begin{aligned}& \|D^\beta u\|_{1+n+1,\Omega_{\tilde{\rho}}} + (N/\rho)\|D^\beta u\|_{n+1,\Omega_{\tilde{\rho}}} \\ \leq\ & \frac{A^{|\beta|-1}}{\tilde{\rho}^{s(|\beta|-3)}}\big((|\beta|-3)!\big)^s (N/\tilde{\rho})^s + (N/\rho)\frac{A^{|\beta|-1}}{\tilde{\rho}^{s(|\beta|-3)}}\big((|\beta|-3)!\big)^s \\ \leq\ & \frac{2A^{|\alpha|-2}}{\tilde{\rho}^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s \big(N/(N-3)\big)^s \\ \leq\ & \frac{C_6 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.\end{aligned}$$

Thus

$$\|D^\alpha u\|_{n+1,\Omega_\rho} \leq \frac{C_5 C_6 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s. \tag{3.13}$$

The same arguments as above shows that

$$\|D_vD^\alpha u\|_{-1/3+n+1,\Omega_\rho}\le\frac{C_5C_6A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.$$

This along with (3.13) yields the conclusion.

Lemma 3.9. *For any $0\le r\le 1/3$, we have*

$$\|D^\alpha u\|_{r+n+1,\Omega_\rho}+\|D_vD^\alpha u\|_{r-1/3+n+1,\Omega_\rho}\le\frac{C_{35}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{rs},\quad\forall\ 0<\rho<1.$$

Proof . We firstly prove the conclusion is true for $r=1/3$, i.e., to show

$$\|D^\alpha u\|_{1/3+n+1,\Omega_\rho}+\|D_vD^\alpha u\|_{n+1,\Omega_\rho}\le\tfrac{C_{35}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3},\quad\forall\ 0<\rho<1.$$

And we will proceed in the following four steps.

Step 1. Claim

$$\|[\mathcal{L},\ \varphi_{\rho,N}D^\alpha]u\|_{-1/3+n+1}\le\tfrac{C_{19}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.\tag{3.14}$$

In fact, write $\mathcal{L}=X_0-a\triangle_v$ with $X_0=\partial_t+v\cdot\partial_x$. Then direct verification deduces

$$\begin{aligned}\|[\mathcal{L},\ \varphi_{\rho,N}D^\alpha]u\|_{-1/3+n+1}&\le\|[X_0,\ \varphi_{\rho,N}D^\alpha]u\|_{-1/3+n+1}+\|a[\triangle_v,\ \varphi_{\rho,N}D^\alpha]u\|_{-1/3+n+1}\\&\quad+\|\varphi_{\rho,N}[a,\ D^\alpha]\triangle_vu\|_{-1/3+n+1}\\&=:(I)+(II)+(III).\end{aligned}$$

Denote $[X_0,\ D^\alpha]$ by D^{α_0}. Then $|\alpha_0|\le|\alpha|$ and

$$\begin{aligned}(I)&\le\|[X_0,\ \varphi_{\rho,N}]D^\alpha u\|_{n+1}+\|\varphi_{\rho,N}D^{\alpha_0}u\|_{n+1}\\&\le C_8\big\{\ (N/\rho)\|D^\alpha u\|_{n+1,\Omega_{\tilde\rho}}+\|D^{\alpha_0}u\|_{n+1,\Omega_{\tilde\rho}}\ \big\}.\end{aligned}$$

Note that $s\ge3$. Using Lemma 3.8, we have

$$(I)\le C_8\big(N/\rho+1\big)\tfrac{C_7A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\le\tfrac{C_9A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.\tag{3.15}$$

Next we will estimate (II). It is easy to see that

$$\begin{aligned}\|[\triangle_v,\ \varphi_{\rho,N}]D^\alpha u\|_{-1/3+n+1}&\le2\|[D_v,\ \varphi_{\rho,N}]D_vD^\alpha u\|_{-1/3+n+1}\\&\quad+\|[D_v,\ [D_v,\ \varphi_{\rho,N}]\ ]D^\alpha u\|_{-1/3+n+1}.\end{aligned}\tag{3.16}$$

We firstly treat the first term on the right hand. Using Lemma 3.8 again, we have

$$\begin{aligned}\|[D_v,\ \varphi_{\rho,N}]D_vD^\alpha u\|_{-1/3+n+1}&\le(N/\rho)\|D_vD^\alpha u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\\&\le(N/\rho)\frac{C_7A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\\&\le\frac{C_{10}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.\end{aligned}\tag{3.17}$$

Next we treat $\|[D_v,\ [D_v,\ \varphi_{\rho,N}]\ ]D^\alpha u\|_{-1/3+n+1}$, and we compute

$$\begin{aligned}
&\|[D_v,\ [D_v,\ \varphi_{\rho,N}]\ ]D^\alpha u\|_{-1/3+n+1}\\
\le\ &\|(D^2\varphi_{\rho,N})D^\beta u\|_{2/3+n+1}+\|(D^3\varphi_{\rho,N})D^\beta u\|_{-1/3+n+1}\\
\le\ &C_{11}\big\{\ (N/\rho)^2\|D^\beta u\|_{2/3+n+1,\Omega_{\tilde\rho}}+(N/\rho)^3\|D^\beta u\|_{n+1,\Omega_{\tilde\rho}}\ \big\}\\
\le\ &C_{11}\big\{\ (N/\rho)^2\frac{A^{|\beta|-1}}{\tilde\rho^{s(|\beta|-3)}}\big((|\beta|-3)!\big)^s(N/\tilde\rho)^{2s/3}\\
&+(N/\rho)^3\frac{A^{|\beta|-1}}{\tilde\rho^{s(|\beta|-3)}}\big((|\beta|-3)!\big)^s\ \big\}\\
\le\ &C_{11}\big\{\ (N/\rho)^2(N/\tilde\rho)^{-s/3}\frac{A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\\
&+(N/\rho)^3(N/\tilde\rho)^{-s}\frac{A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\ \big\}\\
\le\ &\frac{C_{12}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.
\end{aligned}$$

This along with (3.16) and (3.17) shows at once

$$(II)\le\frac{C_{13}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \tag{3.18}$$

It remains to treat (III), and using Leibniz' formula,

$$\begin{aligned}
(III)\ &\le\ \sum_{0<|\gamma|\le|\alpha|}\binom{\alpha}{\gamma}\big\|\varphi_{\rho,N}(D^\gamma a)\triangle_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1}\\
&\le\ \sum_{0<|\gamma|\le|\alpha|}\binom{\alpha}{\gamma}\big\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\triangle_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1}.
\end{aligned}$$

Since $a\in G^s(\mathbb{R}^{2n+1})$, then

$$\|D^\gamma a\|_{n+1,\Omega}\le C_{14}^{|\gamma|-2}\big((|\gamma|-3)!\big)^s,\quad |\gamma|\ge 3,$$

and

$$\|D^\gamma a\|_{n+1,\Omega}\le C_{14},\quad |\gamma|=1,\ 2.$$

Moreover, note $|\alpha|-|\gamma|+1\le N$, and hence applying Lemma 3.8, we have for any $\gamma,\ |\gamma|\le|\alpha|-2$,

$$\begin{aligned}
\|\varphi_{\rho,N}\triangle_v D^{\alpha-\gamma}u\|_{-1/3+n+1}\ &\le\ \|D_vD^{\alpha-\gamma+1}u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\\
&\le\ \frac{C_7A^{|\alpha|-|\gamma|+1-2}}{\tilde\rho^{s(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s\\
&\le\ \frac{C_{15}A^{|\alpha|-|\gamma|+1-2}}{\rho^{s(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s.
\end{aligned}$$

Consequently, we compute

$$\begin{aligned}
&\sum_{2\le|\gamma|\le|\alpha|-2}\binom{\alpha}{\gamma}\big\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\triangle_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1}\\
\le\ &\sum_{2\le|\gamma|\le|\alpha|-2}\binom{\alpha}{\gamma}C_{14}^{|\gamma|-2}\big((|\gamma|-2)!\big)^s\frac{C_{15}A^{|\alpha|-|\gamma|+1-2}}{\rho^{s(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s\\
\le\ &\frac{C_{15}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\sum_{2\le|\gamma|\le|\alpha|-2}\big(\frac{C_{14}}{A}\big)^{|\gamma|-1}|\alpha|!\big((|\gamma|-2)!\big)^{s-1}\big((|\alpha|-|\gamma|-2)!\big)^{s-1}\\
\le\ &\frac{C_{15}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\sum_{2\le|\gamma|\le|\alpha|-2}\big(\frac{C_{14}}{A}\big)^{|\gamma|-1}|\alpha|\frac{(|\alpha|-1)(|\alpha|-2)}{(|\alpha|-3)^{s-1}}\\
\le\ &\frac{C_{16}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}\sum_{2\le|\gamma|\le|\alpha|-2}\big(\frac{C_{14}}{A}\big)^{|\gamma|-1}.
\end{aligned}$$

Taking A large enough such that $\sum_{2\le|\gamma|\le|\alpha|-2}\big(\frac{C_{14}}{A}\big)^{|\gamma|-1}\le 1$, then we get

$$\sum_{2\le|\gamma|\le|\alpha|-2}\binom{\alpha}{\gamma}\big\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\triangle_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1}\le\frac{C_{16}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

For $|\gamma|=1,\ |\alpha|-1$ or $|\alpha|$, we can compute directly

$$\binom{\alpha}{\gamma}\big\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\triangle_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1}\le\frac{C_{17}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

Combination of the above two inequalities give that

$$(III)\le\frac{C_{18}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

This along with (3.15) and (3.18) yields the conclusion (3.14).

Step 2. Claim

$$\|\varphi_{\rho,N}D^\alpha[F\big(\cdot,u(\cdot),\nabla_v u(\cdot)\big)]\|_{-1/3+n+1}\le\tfrac{C_{21}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.\tag{3.19}$$

Firstly, we will prove F and u satisfy the conditions (3.7)-(3.9) for some M_j. By Lemma 3.8, we have

$$\|D^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\le\|D^j u\|_{n+1,\Omega_{\tilde\rho}}\le\frac{C_7A^{j-2}}{\tilde\rho^{s(j-3)}}\big((j-3)!\big)^s,\quad 3\le j\le N,\tag{3.20}$$

$$\|D_vD^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\le\frac{C_7A^{j-2}}{\tilde\rho^{s(j-3)}}\big((j-3)!\big)^s,\quad 3\le j\le N,\tag{3.21}$$

and

$$\|D^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\le C_7,\quad 0\le j\le 2.\tag{3.22}$$

Since $F\in G^s(\mathbb{R}^{2n+1}\times\mathbb{R})$, then

$$\|(D^k_{t,x,v}\partial^l_u D^m_p F)\big(\cdot,u(\cdot),\nabla_v u(\cdot)\big)\|_{-1/3+n+1,\Omega}\le C_{20}^{k+l}\big((k-3)!\big)^s\big((l-3)!\big)^s,\quad k,m+l\ge 3.\tag{3.23}$$

Define M_j, H_0, H_1 by setting

$$H_0 = C_7, \quad H_1 = A, \quad M_0 = C_7, \quad M_j = \frac{\big((j-1)!\big)^s}{\tilde{\rho}^{s(j-1)}}, \quad j \geq 1.$$

We can choose A large enough such that $H_1 = A \geq C_2 H_0$. Then (3.20)-(3.23) can be rewritten

$$\|D^j u\|_{-1/3+n+1,\Omega_{\tilde{\rho}}} \leq H_0, \quad 0 \leq j \leq 1, \tag{3.24}$$

$$\|D^j u\|_{-1/3+n+1,\Omega_{\tilde{\rho}}} \leq H_0 H_1^{j-2} M_{j-2}, \quad 2 \leq j \leq |\alpha| = N, \tag{3.25}$$

$$\|D_v D^j u\|_{-1/3+n+1,\Omega_{\tilde{\rho}}} \leq H_0 H_1^{j-2} M_{j-2}, \quad 2 \leq j \leq |\alpha| = N, \tag{3.26}$$

$$\|(D^k_{t,x,v}\partial^l_u D^m_p F)\|_{-1/3+n+1,\Omega} \leq C_{20}^{k+l} M_{k-2} M_{m+l-2}, \qquad k, m+l \geq 2. \tag{3.27}$$

For each j, note that $s \geq 3$ and hence

$$\begin{aligned} \tfrac{j!}{i!(j-i)!} M_i M_{j-i} &= \tfrac{j!}{i(j-i)} \big((i-1)!\big)^{s-1} \big((j-i-1)!\big)^{s-1} \tilde{\rho}^{-s(i-1)} \tilde{\rho}^{-s(j-i-1)} \\ &\leq (j!)\big((j-2)!\big)^{s-1} \tilde{\rho}^{-s(j-1)} \\ &\leq \tfrac{j}{(j-1)^{s-1}} (j-1)! \big((j-1)!\big)^{s-1} \tilde{\rho}^{-s(j-1)} \\ &\leq M_j. \end{aligned} \tag{3.28}$$

Thus M_j satisfy the monotonicity condition (3.5). In virtue of (3.24)-(3.28), using Lemma 3.5, we have

$$\begin{aligned} \|\varphi_{\rho,N} D^\alpha [F(\cdot, u(\cdot))]\|_{-1/3+n+1} &\leq C_3 H_0 H_1^{|\alpha|-2} M_{|\alpha|-2} \\ &\leq \frac{C_3 C_7 A^{|\alpha|-2}}{\tilde{\rho}^{s(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s \\ &\leq \frac{C_{21} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s (N/\rho)^{s/3}. \end{aligned}$$

This completes the proof of conclusion (3.19).

Step 3. Claim

$$\|\mathcal{L}\varphi_{\rho,N} D^\alpha u\|_{-1/3+n+1} \leq \frac{C_{23} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s (N/\rho)^{s/3}. \tag{3.29}$$

In fact,

$$\begin{aligned} \|\mathcal{L}\varphi_{\rho,N} D^\alpha u\|_{-1/3+n+1} &\leq C_{22}\{ \|[\mathcal{L}, \varphi_{\rho,N} D^\alpha] u\|_{-1/3+n+1} + \|\varphi_{\rho,N} D^\alpha \mathcal{L} u\|_{-1/3+n+1} \} \\ &= C_{22}\big\{ \|[\mathcal{L}, \varphi_{\rho,N} D^\alpha] u\|_{-1/3+n+1} \\ &\quad + \big\|\varphi_{\rho,N} D^\alpha [F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big)]\big\|_{-1/3+n+1} \big\}. \end{aligned}$$

This along with (3.14), (3.19) in step 1 and step 2 yields immediately the conclusion (3.29).

Step 4. Claim

$$\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}+\|\varphi_{\rho,N}D_vD^\alpha u\|_{1/3-1/3+n+1}\le \tfrac{C_{31}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \tag{3.30}$$

In fact, applying the subelliptic estimate (2.3), we obtain

$$\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}\le C_{24}\{\ \|\mathcal{L}\varphi_{\rho,N}D^\alpha u\|_{-1/3+n+1}+\|\varphi_{\rho,N}D^\alpha u\|_{n+1}\ \}.$$

Combining Lemma 3.8 and (3.29) in Step 3, we have

$$\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}\le \frac{C_{25}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \tag{3.31}$$

Now it remains to treat $\|\varphi_{\rho,N}D_vD^\alpha u\|_{1/3-1/3+n+1}$, and

$$\|\varphi_{\rho,N}D_vD^\alpha u\|_{1/3-1/3+n+1}\le \|D_v\varphi_{\rho,N}D^\alpha u\|_{n+1}+\|[D_v,\ \varphi_{\rho,N}]D^\alpha u\|_{n+1}.$$

Firstly, we treat the first term on the right. By direct calculation, it follows that

$$\begin{aligned}
&\|D_v\varphi_{\rho,N}D^\alpha u\|_{n+1}^2\\
=\ &\mathrm{Re}\big(\mathcal{L}\varphi_{\rho,N}D^\alpha u,\ a^{-1}\Lambda^{2n+2}\varphi_{\rho,N}D^\alpha u\big)-\mathrm{Re}\big(X_0\varphi_{\rho,N}D^\alpha u,\ a^{-1}\Lambda^{2n+2}\varphi_{\varepsilon,k\varepsilon}D^\alpha u\big)\\
=\ &\mathrm{Re}\big(\mathcal{L}\varphi_{\rho,N}D^\alpha u,\ a^{-1}\Lambda^{2n+2}\varphi_{\rho,N}D^\alpha u\big)-\frac{1}{2}\big(\varphi_{\rho,N}D^\alpha u,\ [a^{-1}\Lambda^{2n+2},\ X_0]\varphi_{\rho,N}D^\alpha u\big)\\
&-\frac{1}{2}\big(\varphi_{\rho,N}D^\alpha u,\ [\Lambda^{2n+2},\ a^{-1}]X_0\varphi_{\rho,N}D^\alpha u\big)\\
\le\ &C_{26}\{\ \|\mathcal{L}\varphi_{\rho,N}D^\alpha u\|_{-1/3+n+1}^2+\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}^2\ \}.
\end{aligned}$$

This along with (3.29) and (3.31) shows at once

$$\|D_v\varphi_{\rho,N}D^\alpha u\|_{r-1/3+n+1}\le \frac{C_{27}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

Moreover Lemma 3.8 yields

$$\begin{aligned}
\|[D_v,\ \varphi_{\rho,N}]D^\alpha u\|_{n+1}&\le C_{28}(N/\rho)\|D^\alpha u\|_{n+1,\Omega_{\tilde\rho}}\\
&\le \frac{C_{28}C_7A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}\\
&\le \frac{C_{29}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.
\end{aligned}$$

From the above two inequalities, we have

$$\|\varphi_{\rho,N}D_vD^\alpha u\|_{1/3+n+1}\le \frac{C_{30}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

This completes the proof of Step 4.

It's clear for any $\rho,\ 0<\rho<1$,

$$\|D^\alpha u\|_{1/3+n+1,\Omega_\rho}+\|D_vD^\alpha u\|_{1/3-1/3+n+1,\Omega_\rho}\le \|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}+\|\varphi_{\rho,N}D_vD^\alpha u\|_{1/3-1/3+n+1}.$$

Thus from Step 4, it follows that the conclusion in Lemma 3.9 is true for $r=1/3$.

Moreover for any $0<r<1/3$, using the interpolation inequality (2.2), we have

$$\begin{aligned}
\|D^\alpha u\|_{r+n+1,\Omega_\rho}&\le \|\varphi_{\rho,N}D^\alpha u\|_{r+n+1}\\
&\le \varepsilon\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}+\varepsilon^{-r/(1/3-r)}\|\varphi_{\rho,N}D^\alpha u\|_{n+1}\\
&\le \varepsilon\frac{C_{31}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}+\varepsilon^{-r/(1/3-r)}\frac{C_{32}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s,
\end{aligned}$$

Taking $\varepsilon = (N/\rho)^{s(r-1/3)}$, then

$$\|D^\alpha u\|_{r+n+1,\Omega_\rho} \le \frac{C_{33}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{rs}.$$

Similarly,

$$\|D_v D^\alpha u\|_{r-1/3+n+1,\Omega_\rho} \le \frac{C_{34}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{rs}.$$

This completes the proof of Lemma 3.9.

Inductively, we have the following

Lemma 3.10. *For any r with $1/3 \le r \le 2/3$,*

$$\|D^\alpha u\|_{r+n+1,\Omega_\rho} + \|D_v D^\alpha u\|_{r-1/3+n+1,\Omega_\rho} \le \frac{C_{38}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{sr}, \quad \forall\ 0<\rho<1. \tag{3.32}$$

Moreover, the above inequality still holds for any r with $2/3 \le r \le 1$.

Proof. Repeating the proof of Lemma 3.9, we have the truth of (3.32) for $1/3 \le r \le 2/3$. The case $2/3 \le r \le 1$ is a little different. The conclusion in Step 1 in the above proof still holds for $r = 1$, and corresponding to Step 2, we have to make some modification to prove

$$\big\|\varphi_{\rho,N} D^\alpha [F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big)]\big\|_{1/3+n+1} \le \tfrac{C_{36}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^s.$$

From the truth of (3.32) for $1/3 \le r \le 2/3$, it follows

$$\|D^j u\|_{1/3+n+1,\Omega_{\tilde\rho}} \le \frac{C_{37}A^{j-2}}{\tilde\rho^{s(j-3)}}\big((j-3)!\big)^s (j/\tilde\rho)^{s/3}, \quad 3 \le j \le N,$$

$$\|D_v D^j u\|_{1/3+n+1,\Omega_{\tilde\rho}} \le \|D_v D^j u\|_{2/3-1/3+n+1,\Omega_{\tilde\rho}} \le \frac{C_{37}A^{j-2}}{\tilde\rho^{s(j-3)}}\big((j-3)!\big)^s (j/\tilde\rho)^{2s/3}, \quad 3 \le j \le N,$$

and

$$\|D^j u\|_{1/3+n+1,\Omega_{\tilde\rho}} \le C_{37}, \qquad 0 \le j \le 2,$$

Hence we need define a new sequence $\bar M_j$ by setting

$$\bar M_0 = C_{37}, \quad \bar M_j = \frac{\big((j-1)!\big)^s}{\tilde\rho^{s(j-1)}}\big((j+2)/\tilde\rho\big)^{2s/3}, \quad j \ge 1.$$

For each j, note that $s \ge 3$ and hence direct computation deduces that for $0 < i < j$,

$$\begin{aligned}
\tfrac{j!}{i!(j-i)!}\bar M_i \bar M_{j-i} &= \tfrac{j!}{i(j-i)}\big((i-1)!\big)^{s-1}\big((j-i-1)!\big)^{s-1} \\
&\quad \times (i+2)^{2s/3}(j-i+2)^{2s/3}\tilde\rho^{-s(j-2)}\tilde\rho^{-4s/3} \\
&\le 4(j!)\big((j-2)!\big)^{s-1}(j+2)^{2s/3-1}(j+1)^{2s/3-1}\tilde\rho^{-s(j-1)}\tilde\rho^{-2s/3}\tilde\rho^{s-2s/3} \\
&\le \tfrac{4j(j+1)^{2s/3-1}}{(j-1)^{s-1}}(j-1)!\big((j-1)!\big)^{s-1}\tilde\rho^{-s(j-1)}\big((j+2)/\tilde\rho\big)^{2s/3} \\
&\le C_{39}\bar M_j.
\end{aligned}$$

In the last inequality we used the fact $s-1 \ge 2s/3$. Thus $\bar M_j$ satisfy the monotonicity condition (3.5). Now the left is entirely similar to the proof of Lemma 3.9. And thus (3.32) holds for $r = 1$ and hence for $2/3 \le r \le 1$ by interpolation inequality (2.2). This completes the proof of Lemma 3.10.

Recall C_7, C_{35} and C_{35} are the constants appearing in Lemma 3.8, Lemma 3.9 and Lemma 3.10. Now taking A large enough such that $A \geq \max\{C_7, C_{35}, C_{38}\}$, and then by the above three Lemmas we have the truth of $(E)_{r,N}$ for any $r \in [0,\ 1]$. This complete the proof of Proposition 3.6.

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