# The Scott model of Linear Logic is the extensional collapse of its relational model

Thomas Ehrhard

▶ **To cite this version:**

Thomas Ehrhard. The Scott model of Linear Logic is the extensional collapse of its relational model. Theoretical Computer Science, 2012, 424, pp.20-45. 10.1016/j.tcs.2011.11.027 . hal-00369831

**HAL Id: hal-00369831**
**https://hal.science/hal-00369831**

Submitted on 22 Mar 2009

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The Scott model of Linear Logic is the extensional collapse of its relational model

Thomas Ehrhard
Preuves, Programmes et Systèmes, UMR 7126
CNRS and University Paris Diderot - Paris 7 *

March 22, 2009

**Abstract**

We show that the extensional collapse of the relational model of linear logic is the model of prime-algebraic lattices, a natural extension to linear logic of the well known Scott semantics of the lambda-calculus.

## Introduction

Linear Logic arose from denotational investigations of second order intuitionistic logic by Girard (system F [Gir86]). He observed that the qualitative domains[1] used for interpreting system F can be assumed to be generated by a binary relation on a set of vertices (the *web*): such a structure is called a *coherence space*[2]. The category of coherence spaces, with linear maps (stable maps preserving arbitrary existing unions) as morphisms, has remarkable symmetry properties that led him to the sequent calculus of LL, and then to proof-nets [Gir87] and to the Geometry of Interaction.

**Scott semantics of LL.** In spite of Barr's observation [Bar79] that the category of complete lattices and linear maps is $*$-autonomous, it was a common belief in the Linear Logic community that the standard Scott semantics of the lambda-calculus (Scott domains and continuous maps) cannot provide models of classical linear logic. Huth showed however in [Hut94] that prime-algebraic complete lattices and lub-preserving maps provide a model of classical LL whose associated CCC (the Kleisli category of the "!" comonad) is a full-CCC of the category of Scott domains and continuous maps. Huth considered however his model as degenerate, as it identifies the $\otimes$ and $\parr$ connectives of LL[3]. A few years later, Winskel rediscovered the same model in a semantical investigation

*This work as also been partly funded by the ANR project CHOCO: `http://choco.pps.jussieu.fr`.

[1] Qualitative domains can be seen as particular dI-domains [Ber78].

[2] The pure lambda-calculus, or the Turing-complete functional language PCF [Plo77], can also be interpreted in coherence spaces.

[3] The interpretation of proofs in this model is non-trivial and interesting nevertheless. As in the case of the relational model (see below), it is possible to endow this model which additional structures which separate $\otimes$ and $\parr$, *without modifying the interpretation of proofs.*

of concurrency [Win99] (see also the beginning of [Win04] for instance). As a particular case of a more general profunctor construction, he showed indeed that the category whose objects are preordered sets and where the morphisms from a preorder $S$ to a preorder $T$ are the functions from the set $\mathcal{I}(S)$ of downward closed subsets of $S$ to the set $\mathcal{I}(T)$ which preserve arbitrary unions is a model of classical LL. This category is equivalent to Huth's model, but we prefer Winskel's approach, as it insists on considering preorders (and not lattices) as objects: preorders are similar to the webs of coherence spaces, to the sets of the relational model, and represent the prime elements of the corresponding lattices. Moreover, the LL constructions are easier to describe in terms of preorders than in terms of lattices. It is fair to mention also that Krivine [Kri90, Kri93] used the same construction (set $\mathcal{I}(S)$ of "initial segments" of a preorder $S$) for describing models of the pure lambda-calculus and mentioned that these preorders give rise to a model of LL, with linear negation corresponding to taking the opposite preorder.

**Relational semantics.** On the other hand, when one applies the Occam's Razor Principle to the coherence space semantics, one is led to interpreting formulae as sets (the webs, without any structure) and proofs as relations between these sets. Something tricky happens during this process: since coherence vanishes, one cannot restrict the set interpreting an "of course" to contain only finite cliques as Girard did in [Gir86], the best one can do is take all finite subsets. But then, the dereliction relation (from $!X$ to $X$), which is the set of all pairs $(\{a\}, a)$ where $a \in X$, is no more a natural transformation. This problem can easily be solved by replacing finite sets with finite multisets, but the effect of this choice is that the corresponding Kleisli category is no more well-pointed. One defines in that way the *relational semantics of linear logic*, which is certainly its simplest (and, maybe, most canonical) denotational model.

**Coefficients.** One way of turning the CCC associated with the relational model into a well-pointed category is by enriching it with coefficients: instead of taking subset of $X \times Y$ as morphisms from $X$ to $Y$, take elements of $C^{X \times Y}$, where $C$ is a suitable set (or class) of coefficients; a canonical choice consists in taking $C = \mathbf{Set}$, the class of all sets. An element of $\mathbf{Set}^{X \times Y}$ should be considered as a matrix whose rows are indexed by the elements of $Y$, and columns by the elements of $X$: this is basically the idea of Girard's *quantitative semantics* [Gir88], which is presented as a model of intuitionistic logic, but is indeed a model of LL (Girard wrote this paper before he discovered LL), see [Has02]. It is also an instance of the already mentioned profunctor constructions [Win99].

Finite coefficients belonging to more standard algebraic structures (rigs, fields, etc.) can also be considered, but this requires adding some structure to these sets for guaranteeing the convergence of the sums which appear when multiplying the matrices, see [Ehr02, Ehr05, DE08]: the effect of such additional structure is that objects are equipped with a topology for which the (generally infinite) sums involved in multiplying matrices converge.

**Extensional collapse of the relational model.** The other way of making the relational model well-pointed is by performing an *extensional collapse*. This operation is easily understood in the type hierarchy associated with the cartesian closed Kleisli category of the finite multiset comonad on the category of sets and

relations: each type $A$ is interpreted by its relational interpretation $[A]$ (a simple set), together with a partial equivalence relation (PER) $\sim_A$ on $\mathcal{P}([A])$. When $A$ is the type $B \Rightarrow C$, an element of $\mathcal{P}([A])$ is a morphism from $B$ to $C$, and two such morphisms $f$ and $g$ are $\sim_{B\Rightarrow C}$-equivalent if, for any $x, y$ such that $x \sim_A y$, one has $f(x) \sim_B g(y)$. In other words, this PER is a logical relation[4], and the extensional collapse of this type hierarchy is obtained by quotienting each set $\mathcal{P}([A])$ by the PER $\sim_A$ (one considers only the elements $x$ of $\mathcal{P}([A])$ such that $x \sim_A x$, which are often called *invariant* elements).

**Content of the paper.** We prove that this extensional collapse of the relational model coincides precisely with the Scott model of preorders. The first problem we have to face is to give a precise meaning to this statement. We start from the work of Bucciarelli [Buc97], recasting it in a categorical setting: given a CCC $\mathcal{C}$ and a well-pointed CCC $\mathcal{E}$, we want to express what it means for $\mathcal{E}$ to "be" (we shall say to "represent") the extensional collapse of $\mathcal{C}$. For this, we introduce two categorical constructions.

- The *homogeneous collapse category* $\mathsf{e}(\mathcal{C})$, whose objects are pairs $(U, \sim)$ where $U$ is an object of $\mathcal{C}$ and $\sim$ is a partial equivalence relation (PER) on the points of $U$ (that is on $\mathcal{C}(\top, U)$ where $\top$ is the terminal object of $\mathcal{C}$). The morphisms are those of $\mathcal{C}$ which preserve this additional structure, and it is easy to see that this category is a CCC. The important point in this definition is that the object of morphisms from $(U, \sim)$ to $(V, \sim)$ is $(W, \sim_W)$ where $W$ is the object of morphisms from $U$ to $V$ in $\mathcal{C}$ and the relation $\sim_W$ is defined as a logical relation.

- The *heterogeneous collapse category* $\mathsf{e}(\mathcal{C}, \mathcal{E})$, whose objects are triples $(U, E, \Vdash)$ where $U$ is an object of $\mathcal{C}$, $E$ is an object of $\mathcal{E}$ and $\Vdash \subseteq \mathcal{C}(\top, U) \times \mathcal{E}(\top, E)$ should be understood as a realizability predicate: $x \Vdash \zeta$ means intuitively that $\zeta$ represents the "extensional behavior" of $x$. The morphisms are the pairs $(f, \varphi)$ of morphisms which preserve the relation $\Vdash$, and again, it is easy to check that this category is a CCC. The important point is that, when constructing the object of morphisms, $\Vdash$ is defined as a logical relation.

These two constructions are possible for any CCCs $\mathcal{C}$ and $\mathcal{E}$. We say that $\mathcal{E}$ represents the extensional collapse of $\mathcal{E}$ if

- $\mathsf{e}(\mathcal{C}, \mathcal{E})$ contains a "sufficiently large" (in a reasonable sense, to be made precise later) sub-CCC $\mathcal{H}$ whose objects $(U, E, \Vdash)$ are modest, meaning that $\Vdash$ is a *partial surjection* from $\mathcal{C}(\top, U)$ to $\mathcal{E}(\top, E)$, and therefore induces a PER on $\mathcal{C}(\top, U)$ (observe that $\mathcal{E}(\top, E)$ can be considered as the quotient of $\mathcal{C}(\top, U)$ by this PER)

- and the functor $\mathcal{H} \to \mathsf{e}(\mathcal{C})$ which maps $(U, E, \Vdash)$ to $(U, \sim)$, where $\sim$ is the PER induced by $\Vdash$ (and maps a morphism $(f, \varphi)$ to $f$), is a CCC functor (that is, preserves the CCC structure on the nose).

The nice feature of this definition is that it is compatible with the standard one (based on type hierarchies) and that it can easily be extended, for instance,

[4] Logicians would speak of a binary reducibility predicate.

to a simple and general definition of what it means for a model of the pure lambda-calculus to represent the extensional collapse of another one.

It would be nice of course to have a similar definition of the extensional collapse of a categorical model of LL, and not only of CCCs, but since the definition of such a model is already quite complicated, we prefer not to address this issue. Instead, we perform the CCC constructions defined above concretely, in a completely linear setting, obtaining both CCCs $\mathsf{e}(\mathcal{C})$ and $\mathcal{H}$ as Kleisli constructions of suitable exponential comonads: in the present paper, $\mathcal{C}$ is the Kleisli category $\mathbf{Rel}_!$ associated with the LL model of sets and relations, and $\mathcal{E}$ is the Kleisli category $\mathbf{ScottL}_!$ associated with the LL model of preorders and linear maps between the associated complete lattices.

After having introduced the necessary preliminary material, we first build in Section 2.2 a linear version of the category $\mathsf{e}(\mathbf{Rel}_!)$. More precisely, we define a model of LL denoted as **PerL**, whose objects are called PER-objects: they are sets equipped with a PER on their powersets. The Kleisli category $\mathbf{PerL}_!$ is isomorphic to $\mathsf{e}(\mathbf{Rel}_!)$ (or, more precisely, to a full sub-CCC of $\mathsf{e}(\mathbf{Rel}_!)$).

Then, in Section 3, we describe the Scott model **ScottL** of LL. The objects are preordered sets, and a morphism from $S$ to $T$ is a linear map (that is, a map preserving all unions) from $\mathcal{I}(S)$ (the set of all downward-closed subsets of $S$) to $\mathcal{I}(T)$. As far as sets are concerned, the multiplicative and additive constructions in this model coincide with those of the model **Rel** (more things have to be said about the associated preorders: for instance, $S^\perp$ is the set $S$ equipped with the opposite of the preorder of $S$). As to the exponential, the natural choice would be to define $!S$ as the set of finite subsets of $S$ with a suitable preorder: with that choice, the Kleisli category $\mathbf{ScottL}_!$ is a sub-CCC of the CCC of complete lattices and Scott-continuous functions. But we can obtain the same effect by defining $!S$ as the set of all finite multisets of elements of $S$, and this will greatly simplify our constructions, because with this choice, the set interpreting an LL formula in **Rel** coincides with the set interpreting the same formula in **ScottL** (remember that this set is equipped with a preorder).

In Section 4, we introduce the linear version of the "heterogeneous category" $\mathcal{H}$ of the construction described above. An object should be a triple $(X, S, \Vdash)$ where $X$ is a set, $S$ is a preordered set and $\Vdash \subseteq \mathcal{P}(X) \times \mathcal{I}(S)$ (which has to be a partial surjection). By our choice above for the definition of $!S$, we can assume $X = S$, so as a first simplification, we can assume our objects to be pairs $(S, \Vdash)$ where $S$ is a preordered set and $\Vdash \subseteq \mathcal{P}(S) \times \mathcal{I}(S)$ has to be a partial surjection. A careful analysis shows that, when $x \Vdash u$, we must have $u = \downarrow x$ (the downward closure of $x$ in $S$), so that, for defining the partial surjection $\Vdash$, we only need to know its domain $D$. So an object of our category will be a pair $(S, D)$ where $D \subseteq \mathcal{P}(S)$. What condition should satisfy $D$? As usual, it should be equal to its double dual for a suitable notion of duality: here, we say that $x, x' \subseteq S$ are in duality if $x' \cap \downarrow x \neq \emptyset \Rightarrow x' \cap x \neq \emptyset$, that is $x'$ cannot separate $x$ from its downward closure. We show that these objects (called "preorders with projections"), with suitable linear morphisms, form a model of linear logic **PpL**, whose associated Kleisli category $\mathbf{PpL}_!$ can be considered as a full sub-CCC of $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$, of which all objects are modest. And we show that $\mathbf{ScottL}_!$ represents the extensional collapse of $\mathbf{Rel}_!$ in the sense explained above. We actually exhibit a functor from **PpL** to **PerL** which preserves the structure of LL model and which induces the required CCC functor from $\mathbf{PpL}_!$ to $\mathbf{PerL}_!$.

In the course of these constructions, we also build models of the pure lambda-calculus, using notions of inclusions between the various structures we consider, organizing them into complete partially ordered classes, and using the fact that the logical constructions (tensor product, orthogonality etc) are continuous wrt. these inclusions. This provides a simple representation of the extensional collapse of the reflexive object in $\mathbf{Rel}_!$ we introduced in [BEM07], as a reflexive object in the CCC of complete lattices and continuous maps, which is probably isomorphic to Scott's standard $\mathcal{D}_\infty$.

# Contents


**1 Preliminaries** **6**
- 1.1 Notations . . . 6
- 1.2 Cartesian closed categories and models of the pure lambda-calculus 6
- 1.3 Intuitionistic extensional collapse . . . 7
  - 1.3.1 Representing the collapse as an interpretation. . . . 8
  - 1.3.2 Categorical presentation. . . . 8
  - 1.3.3 Connection between the two definitions. . . . 9
  - 1.3.4 Extensional collapse of a reflexive object. . . . 10
- 1.4 New-Seely categories and LL-functors . . . 10

**2 The collapse partial equivalence relation** **11**
- 2.1 The category of sets and relations . . . 11
  - 2.1.1 Linear structure. . . . 11
  - 2.1.2 The associated CCC. . . . 12
  - 2.1.3 Inclusions. . . . 12
- 2.2 The collapse category . . . 12
  - 2.2.1 Pre-PERs, PER objects and morphisms of PER objects. . 12
  - 2.2.2 Orthogonality and strong isomorphisms. . . . 13
  - 2.2.3 Monoidal structure. . . . 13
  - 2.2.4 Additive structure. . . . 14
  - 2.2.5 Exponentials. . . . 15
  - 2.2.6 Fundamental isomorphism and cartesian closeness. . . . 15
- 2.3 The partially ordered class of PER-objects . . . 16
  - 2.3.1 Completeness. . . . 16
  - 2.3.2 Variable PER-objects and fixpoints thereof. . . . 17
  - 2.3.3 An extensional reflexive PER-object. . . . 18

**3 A linear Scott semantics** **18**
- 3.1 Star-autonomous structure . . . 18
  - 3.1.1 Isomorphisms. . . . 19
  - 3.1.2 Monoidal structure. . . . 19
- 3.2 Products and coproducts . . . 20
- 3.3 Exponentials . . . 20
  - 3.3.1 Comonad structure of the exponential. . . . 21
  - 3.3.2 Weakening and contraction. . . . 21
- 3.4 The Kleisli category . . . 21
  - 3.4.1 The Kleisli category of preorders. . . . 22
- 3.5 The partially ordered class of preorders . . . 23

**4 The category of preorders with projections** **23**
4.1 A duality on preorders . . . . . 23
4.2 The linear category . . . . . 25
4.2.1 Identity and composition. . . . . 25
4.2.2 Tensor product. . . . . 25
4.2.3 Strong isomorphisms. . . . . 26
4.2.4 Associativity and symmetry isomorphisms. . . . . 26
4.2.5 Linear function space and monoidal closeness. . . . . 26
4.2.6 The "par" connective. . . . . 27
4.2.7 The morphism mix is not an isomorphism in general. . . . 27
4.3 The additives . . . . . 27
4.4 The exponentials . . . . . 28
4.4.1 Fundamental isomorphism. . . . . 28
4.4.2 Structural maps. . . . . 29
4.4.3 Cartesian closeness. . . . . 29
4.5 The partially ordered class of PPs . . . . . 29
4.5.1 Completeness. . . . . 30
4.5.2 Variable PPs and least fixpoints thereof. . . . . 31
4.5.3 An extensional reflexive PP. . . . . 31
4.6 PPs are heterogeneous logical relations . . . . . 31
4.6.1 Heterogeneous relation associated with a PP. . . . . 31
4.7 A functor from PPs to PER-objects . . . . . 33
4.7.1 Continuity of $\varepsilon$. . . . . 35
4.7.2 Image of the reflexive object of $\mathbf{PpL}_!$. . . . . 36
4.8 A functor from PPs to preorders . . . . . 36

# 1 Preliminaries

## 1.1 Notations

A finite multiset $p$ of elements of $S$ is a map $p : S \to \mathbb{N}$ such that $p(a) = 0$ for almost all $a \in S$. We write $a \in p$ for $p(a) > 0$, and use $\mathsf{supp}(p)$ for the support of $p$ which is the set $\{a \in S \mid a \in p\}$. We use $p + q$ for the pointwise sum of multisets, and 0 for the empty multiset.

Given a category $\mathcal{C}$ and two morphisms $f \in \mathcal{E}(E, F)$ and $x \in \mathcal{C}(\top, E)$ (where $\top$ is the terminal object of $\mathcal{C}$ that we assume to exist), we write $f(x)$ instead of $f \circ x$ because we consider $x$ as a "point" (an "element") of $E$.

## 1.2 Cartesian closed categories and models of the pure lambda-calculus

We briefly recall that a category $\mathcal{C}$ is cartesian closed (is a CCC) if each finite family $(E_i)_{i\in I}$ of objects of $\mathcal{C}$ has a cartesian product $\&_{i\in I} E_i$ (in particular, it has a terminal object $\top$) together with projections $\pi_j \in \mathcal{C}(\&_{i\in I} E_i, E_j)$ such that, for any family $(f_i)_{i\in I}$ with $f_i \in \mathcal{C}(F, E_i)$ there is an unique morphism $\langle f_i\rangle_{i\in I} \in \mathcal{C}(F, \&_{i\in I} E_i)$ such that $\pi_j \circ \langle f_i\rangle_{i\in I} = f_j$ for each $j$ and if, given two objects $E$ and $F$ of $\mathcal{C}$, there is a pair $(E \Rightarrow F, \mathsf{Ev})$, called the *object of morphisms from $E$ to $F$*, together with an evaluation morphism $\mathsf{Ev} \in \mathcal{C}((E \Rightarrow F) \,\&\, E, F)$

such that, for any $f \in \mathcal{C}(G \,\&\, E, F)$, there is an unique $\mathsf{Cur}(f) \in \mathcal{C}(G, E \Rightarrow F)$ such that $\mathsf{Ev} \circ (\mathsf{Cur}(f) \,\&\, \mathsf{Id}_E) = f$.

Given two CCCs $\mathcal{C}$ and $\mathcal{D}$, a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ will be said to be a *cartesian closed functor* if it preserves the cartesian closed structure on the nose. This means that $\mathcal{F}(\&_{i\in I}\, E_i) = \&_{i\in I}\, \mathcal{F}(E_i)$, $\mathcal{F}(\pi_i) = \pi_i$, $\mathcal{F}(E \Rightarrow F) = \mathcal{F}(E) \Rightarrow \mathcal{F}(F)$ and $\mathcal{F}(\mathsf{Ev}) = \mathsf{Ev}$.

A *reflexive object* in a CCC $\mathcal{C}$ is a triple $(H, \mathsf{app}, \mathsf{lam})$ where $H$ is an object of $\mathcal{C}$, $\mathsf{app} \in \mathcal{C}(H, H \Rightarrow H)$ and $\mathsf{lam} \in \mathcal{C}(H \Rightarrow H, H)$ satisfy $\mathsf{app} \circ \mathsf{lam} = \mathsf{Id}_{H\Rightarrow H}$. One says moreover that $(H, \mathsf{app}, \mathsf{lam})$ is *extensional*[5] if $\mathsf{lam} \circ \mathsf{app} = \mathsf{Id}_H$. If $(H, \mathsf{app}, \mathsf{lam})$ is a reflexive object in $\mathcal{C}$ and if $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a CCC functor, then $(\mathcal{F}(H), \mathcal{F}(\mathsf{app}), \mathcal{F}(\mathsf{lam}))$ is a reflexive object in $\mathcal{D}$, which is extensional if $(H, \mathsf{app}, \mathsf{lam})$ is extensional.

Let $(H, \mathsf{app}, \mathsf{lam})$ be a reflexive object in the CCC $\mathcal{C}$. Then, given any lambda-term $M$ and any repetition-free list of variables $\vec{x} = x_1, \ldots, x_n$ which contains all the free variables of $M$ (such a list will be said to be *adapted* to $M$), one defines $[M]^H_{\vec{x}} \in \mathcal{C}(H^n, H)$ by induction on $M$ ($[x_i]^H_{\vec{x}} = \pi_i$, $[\lambda x\, N]^H_{\vec{x}} = \mathsf{lam} \circ \mathsf{Cur}([N]^H_{\vec{x},x})$ and $[(N)\, P]^H_{\vec{x}} = \mathsf{Ev} \circ \langle \mathsf{app} \circ [N]^H_{\vec{x}}, [P]^H_{\vec{x}} \rangle$). If $M$ and $M'$ are $\beta$-equivalent and $\vec{x}$ is adapted to $M$ and $M'$, we have $[M]^H_{\vec{x}} = [M']^H_{\vec{x}}$. If $(H, \mathsf{app}, \mathsf{lam})$ is extensional, we have $[M]^H_{\vec{x}} = [M']^H_{\vec{x}}$ when $M$ and $M'$ are $\beta\eta$-equivalent.

If $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a CCC functor then, for any lambda-term $M$, we have $\mathcal{F}([M]^H_{\vec{x}}) = [M]^{\mathcal{F}(H)}_{\vec{x}}$ where $[M]^{\mathcal{F}(H)}_{\vec{x}}$ is the interpretation of $M$ in the reflexive object $(\mathcal{F}(H), \mathcal{F}(\mathsf{app}), \mathcal{F}(\mathsf{lam}))$.

## 1.3 Intuitionistic extensional collapse

The present analysis of the extensional collapse of a model of the typed lambda-calculus is based on [Buc97].

From the usual intuitionistic viewpoint, the extensional collapse is a *logical relation*. More specifically, consider the hierarchy of simple types based on some type atoms $\alpha$, $\beta\ldots$, and intuitionistic implication $\Rightarrow$. Consider a cartesian closed category $\mathcal{C}$ (with terminal object $\top$, cartesian product $\&$ and function space $\Rightarrow$). Given a valuation $I$ from type atoms to objects of $\mathcal{C}$, we have an interpretation of types $[A]_I \in \mathcal{C}$. The *extensional collapse* of this interpretation is a type-indexed family of partial equivalence relations $(\sim_A)$, where $\sim_A \subseteq \mathcal{C}(\top, [A]_I)^2$. This relation is defined by induction on types.

- At each basic type $\alpha$, the relation $\sim_\alpha$ coincides with equality on $\mathcal{C}(\top, I(\alpha))$.
- Then, given $f, g \in \mathcal{C}(\top, [A \Rightarrow B]_I) = \mathcal{C}(\top, [A]_I \Rightarrow [B]_I) \simeq \mathcal{C}([A]_I, [B]_I)$, one has $f \sim_{A\Rightarrow B} g$ if, for all $x, y \in \mathcal{C}(\top, [A]_I)$ such that $x \sim_A y$, one has $f(x) \sim_B g(y)$ (where we recall that we write $f(x)$ instead of $f \circ x$ when the source of $x$ is the terminal object).

By induction on types, one proves easily that $\sim_A$ is a PER on $\mathcal{C}(\top, [A]_I)$ for each type $A$. Since the family of PERs $(\sim_A)$ is defined as a logical relation, it is compatible with the syntax of the simply typed lambda-calculus, in the sense that, if $M$ is a closed term of type $A$, its semantics $[M]_I \in \mathcal{C}(\top, [A]_I)$

[5] This notion of extensionality, which corresponds to the $\eta$ conversion rule of the lambda-calculus, should not be confused with the notion of extensionality we are dealing with in this paper, which is related to the categorical notion of well-pointedness.

satisfies $[M]_I \sim_A [M]_I$. This property can be extended to *functional* enriched versions of the simply typed lambda-calculus (such as PCF) under some mild assumptions on $\mathcal{C}$ and $I$.

**1.3.1 Representing the collapse as an interpretation.** Let $\mathcal{E}$ be another cartesian closed category, that we assume to be well-pointed (meaning that, if $\varphi, \psi \in \mathcal{E}(E,F)$ satisfy $\varphi(\zeta) = \psi(\zeta)$ for all $\zeta \in \mathcal{E}(\top, E)$, then $\varphi = \psi$). Let $J$ be a valuation of type atoms in $\mathcal{E}$ and, for each type atom $\alpha$, let $\Vdash_\alpha \subseteq \mathcal{C}(\top, I(\alpha)) \times \mathcal{E}(\top, J(\alpha))$ be a *bijection* (to be understood as expressing an equality relation between the elements of the two models at ground types). Then we define $\Vdash_A \subseteq \mathcal{C}(\top, [A]_I) \times \mathcal{E}(\top, [A]_J)$ for all type $A$ as a logical relation (called the *heterogeneous relation*), that is

$$f \Vdash_{A \Rightarrow B} \psi \Leftrightarrow (\forall x, \zeta \quad x \Vdash_A \zeta \Rightarrow f(x) \Vdash_B \varphi(\zeta)) .$$

If $\Vdash_A$ is surjective for all type $A$ (that is $\forall \zeta \in \mathcal{E}(\top, [A]_J)\, \exists x \in \mathcal{C}(\top, [A]_I)\, x \Vdash_A \zeta$), then all the relations $\Vdash_A$ are functional (in the sense that if $x \Vdash_A \zeta$ and $x \Vdash_A \zeta'$, then $\zeta = \zeta'$). This is easy to check by induction on types and is due to the well-pointedness of $\mathcal{E}$.

We say that $(\Vdash_A)$ is a *representation of the collapse* of the interpretation $I$ by the interpretation of $J$ if, for all type $A$, $\Vdash_A$ is surjective (and bijective when $A = \alpha$ is a basic type) and one has

$$\forall x, y \in \mathcal{C}(\top, [A]_I) \quad x \sim_A y \Leftrightarrow (\exists \zeta \in \mathcal{E}(\top, [A]_J)\; x \Vdash_A \zeta \text{ and } y \Vdash_A \zeta) .$$

This means that, at each type $A$, the relation $\Vdash_A$ induces a bijection between $\mathcal{E}(\top, [A]_J)$ and the quotient[6] $\mathcal{C}(\top, [A]_I)/\sim_A$.

Assume that $(\Vdash_A)$ is such a representation. Since it is defined as a logical relation, we have $[M]_I \Vdash_A [M]_J$ for each closed lambda-term of type $A$, we have $[M]_I \sim_A [N]_I$ iff $[M]_J = [N]_J$ for all closed terms $M$ and $N$ of type $A$.

**1.3.2 Categorical presentation.** There is another, more conceptual way of describing the situation above. First one defines the *collapse category* $\mathsf{e}(\mathcal{C})$ of $\mathcal{C}$. Its objects are pairs $U = (\ulcorner U \urcorner, \sim_U)$ where $\ulcorner U \urcorner$ is an object of $\mathcal{C}$ and $\sim_U \subseteq \mathcal{C}(\top, \ulcorner U \urcorner)^2$ is a PER. Given two objects $U$ and $V$ of $\mathsf{e}(\mathcal{C})$, the elements of $\mathsf{e}(\mathcal{C})(U,V)$ are the morphisms $f \in \mathcal{C}(\ulcorner U \urcorner, \ulcorner V \urcorner)$ such that

$$\forall x, x' \in \mathcal{C}(\top, \ulcorner U \urcorner) \quad x \sim_U x' \Rightarrow f(x) \sim_V f(x') .$$

If the category $\mathcal{C}$ is cartesian, then so is $\mathsf{e}(\mathcal{C})$ (with cartesian products defined in the most obvious way). And if $\mathcal{C}$ is cartesian closed, so is $\mathsf{e}(\mathcal{C})$. Given two objects $U$ and $V$ of $\mathcal{C}$, one defines $U \Rightarrow V = (\ulcorner U \urcorner \Rightarrow \ulcorner V \urcorner, \sim_{U \Rightarrow V})$ with $f \sim_{U \Rightarrow V} f'$ iff $f(x) \sim_Y f'(x')$ for all $x, x' \in \mathcal{C}(\top, \ulcorner U \urcorner)$ such that $x \sim_U x'$ (for $f, f' \in \mathcal{C}(\top, \ulcorner U \Rightarrow V \urcorner) \simeq \mathcal{C}(\ulcorner U \urcorner, \ulcorner V \urcorner)$). The evaluation morphism $\mathsf{Ev} \in \mathsf{e}(\mathcal{C})((U \Rightarrow V)\ \&\ U, V)$ is the evaluation morphism of the category $\mathcal{C}$, which is also a morphism in $\mathsf{e}(\mathcal{C})$. We say that an object $U$ of $\mathsf{e}(\mathcal{C})$ is *discrete* if $\sim_U$ coincides with equality.

Similarly, one defines the *heterogeneous category* $\mathsf{e}(\mathcal{C}, \mathcal{E})$ of $\mathcal{C}$ and $\mathcal{E}$. Its objects are triples $X = (\ulcorner X \urcorner, \llcorner X \lrcorner, \Vdash_X)$ where $\ulcorner X \urcorner$ is an object of $\mathcal{C}$, $\llcorner X \lrcorner$

[6] When quotienting a set by a PER, one considers only the elements of the set which are equivalent to themselves.

is an object of $\mathcal{E}$ and $\Vdash_X \subseteq \mathcal{C}(\top, \ulcorner X \urcorner) \times \mathcal{E}(\top, \llcorner X \lrcorner)$. A morphism $\theta$ from $X$ to $Y$ in that category is a pair $(\ulcorner \theta \urcorner, \llcorner \theta \lrcorner)$ where $\ulcorner \theta \urcorner \in \mathcal{C}(\ulcorner X \urcorner, \ulcorner Y \urcorner)$ and $\llcorner \theta \lrcorner \in \mathcal{E}(\llcorner X \lrcorner, \llcorner Y \lrcorner)$ satisfy $\ulcorner \theta \urcorner(x) \Vdash_Y \llcorner \theta \lrcorner(\zeta)$ for all $(x, \zeta)$ such that $x \Vdash_X \zeta$.

Again, if both categories $\mathcal{C}$ and $\mathcal{E}$ are cartesian, so is $\mathsf{e}(\mathcal{C}, \mathcal{E})$, and if they are cartesian closed, so is $\mathsf{e}(\mathcal{C}, \mathcal{E})$, with $X \Rightarrow Y$ defined as follows: $\ulcorner X \Rightarrow Y \urcorner = \ulcorner X \urcorner \Rightarrow \ulcorner Y \urcorner$, $\llcorner X \Rightarrow Y \lrcorner = \llcorner X \lrcorner \Rightarrow \llcorner Y \lrcorner$ and, given $f \in \mathcal{C}(\top, \ulcorner X \Rightarrow Y \urcorner) \simeq \mathcal{C}(\ulcorner X \urcorner, \ulcorner Y \urcorner)$ and $\varphi \in \mathcal{E}(\top, \llcorner X \Rightarrow Y \lrcorner) \simeq \mathcal{C}(\llcorner X \lrcorner, \llcorner Y \lrcorner)$, we have $f \Vdash_{X \Rightarrow Y} \varphi$ if $f(x) \Vdash_Y \varphi(\zeta)$ for all $(x, \zeta)$ such that $x \Vdash_X \zeta$.

Let us say that an object $X$ of $\mathsf{e}(\mathcal{C}, \mathcal{E})$ is *modest*[7] if the relation $\Vdash_X$ is a *partial surjection* from $\mathcal{C}(\top, \ulcorner X \urcorner)$ to $\mathcal{E}(\top, \llcorner X \lrcorner)$. Let $\mathsf{e}_{\mathsf{mod}}(\mathcal{C}, \mathcal{E})$ be the full subcategory of $\mathsf{e}(\mathcal{C}, \mathcal{E})$ whose objects are the modest objects. If $\mathcal{C}$ and $\mathcal{E}$ are cartesian, then $\mathsf{e}_{\mathsf{mod}}(\mathcal{C}, \mathcal{E})$ is a sub-cartesian category of $\mathsf{e}(\mathcal{C}, \mathcal{E})$. But in general, $\mathsf{e}_{\mathsf{mod}}(\mathcal{C}, \mathcal{E})$ is not cartesian closed. It can be noticed that, if $X$ and $Y$ are objects of $\mathsf{e}(\mathcal{C}, \mathcal{E})$ which are modest (so that, again, $X \Rightarrow Y$ is well defined but not necessarily modest) and if $\Vdash_{X \Rightarrow Y}$ is surjective, then $\Vdash_{X \Rightarrow Y}$ is functional, and hence $X \Rightarrow Y$ is modest.

There is a cartesian closed "second projection" functor $\sigma : \mathsf{e}(\mathcal{C}, \mathcal{E}) \to \mathcal{E}$ (it maps an object $X$ to $\llcorner X \lrcorner$ and a morphism $\theta$ to $\llcorner \theta \lrcorner$). There is also a functor $\varepsilon : \mathsf{e}_{\mathsf{mod}}(\mathcal{C}, \mathcal{E}) \to \mathsf{e}(\mathcal{C})$ which maps an object $X$ to $(\ulcorner X \urcorner, \sim_{\varepsilon(X)})$, where $x_1 \sim_{\varepsilon(X)} x_2$ if $x_1 \Vdash_X \zeta$ and $x_2 \Vdash_X \zeta$ for some (necessarily unique) $\zeta$. Given $\theta \in \mathsf{e}(\mathcal{C}, \mathcal{E})(X, Y)$, we set $\varepsilon(\theta) = \ulcorner \theta \urcorner$. Indeed, let $x_1, x_2 \in \mathcal{C}(\top, \ulcorner X \urcorner)$ such that $x_1 \sim_{\varepsilon(X)} x_2$ (with $\zeta \in \mathcal{E}(\top, \llcorner X \lrcorner)$ such that $x_1 \Vdash_X \zeta$ and $x_2 \Vdash_X \zeta$), we have $\ulcorner \theta \urcorner(x_1) \Vdash_Y \llcorner \theta \lrcorner(\zeta)$ and $\ulcorner \theta \urcorner(x_2) \Vdash_Y \llcorner \theta \lrcorner(\zeta)$, and hence $\ulcorner \theta \urcorner(x_1) \sim_Y \ulcorner \theta \urcorner(x_2)$, so that $\ulcorner \theta \urcorner \in \mathsf{e}(\mathcal{C})(\varepsilon(X), \varepsilon(Y))$.

We say that the category $\mathcal{E}$ *represents the extensional collapse of the category* $\mathcal{C}$ if there exists a sub-CCC $\mathcal{H}$ of $\mathsf{e}(\mathcal{C}, \mathcal{E})$ such that

- each object of $\mathcal{H}$ is modest;
- the functor $\varepsilon : \mathcal{H} \to \mathsf{e}(\mathcal{C})$ is cartesian closed
- and, for any[8] *discrete* object $U$ of $\mathsf{e}(\mathcal{C})$, there is an object $X$ of $\mathcal{H}$ such that $\varepsilon(X) = U$ (so that $\ulcorner X \urcorner = U$ and $\Vdash_X$ is a bijection).

**1.3.3 Connection between the two definitions.** The motivation of this definition is that, in that situation, if $I$ is a type valuation in $\mathcal{C}$ then, for each ground type $\alpha$, we can find an object $J(\alpha)$ of $\mathcal{E}$ such that $K(\alpha) = (I(\alpha), J(\alpha), \Vdash_\alpha)$ is an object of $\mathcal{H}$, for some bijection $\Vdash_{K(\alpha)}$. We can extend $(K(\alpha))$ into an interpretation of types $([A]_K)$ in the CCC $\mathcal{H}$ which satisfies $[A]_K = ([A]_I, [A]_J, \Vdash_A)$ where $\Vdash_A$ coincides with the heterogeneous logical relation defined in 1.3.1. Then our assumption that $\mathcal{E}$ represents the extensional collapse of $\mathcal{C}$ implies that $(\Vdash_A)$ is a representation of the extensional collapse of $I$ by $J$, in the sense of 1.3.1.

The benefit of this abstraction is that the concept of a CCC $\mathcal{E}$ representing the extensional collapse of a CCC $\mathcal{C}$ is quite flexible and independent of any type hierarchy given *a priori*. For instance, it provides a natural definition of the extensional collapse of a model of the pure lambda-calculus.

[7] This is compatible with the standard terminology of realizability, see e.g. [AC98].

[8] We actually don't need this property for all discrete $U$s, but only for those which are intended to represent the basic types of the functional language we have in mind. For the sake of simplicity, we adopt this stronger hypothesis.

**1.3.4 Extensional collapse of a reflexive object.** Assume indeed that $\mathcal{E}$ represents the extensional collapse of $\mathcal{C}$ in the sense above, with $\mathcal{H}$ as heterogeneous collapse CCC. Let $(Z, \mathsf{app}, \mathsf{lam})$ be a reflexive object in $\mathcal{H}$. Then $(\varepsilon(Z), \ulcorner \mathsf{app} \urcorner, \ulcorner \mathsf{lam} \urcorner)$ is a reflexive object in $\mathsf{e}(\mathcal{C})$, $(\ulcorner Z \urcorner, \ulcorner \mathsf{app} \urcorner, \ulcorner \mathsf{lam} \urcorner)$ is a reflexive object in $\mathcal{C}$ and $(\llcorner Z \lrcorner, \llcorner \mathsf{app} \lrcorner, \llcorner \mathsf{lam} \lrcorner)$ is a reflexive object in $\mathcal{E}$.

In that case, we say that the reflexive object $(\llcorner Z \lrcorner, \llcorner \mathsf{app} \lrcorner, \llcorner \mathsf{lam} \lrcorner)$ is a representation of the extensional collapse of the reflexive object $(\ulcorner Z \urcorner, \ulcorner \mathsf{app} \urcorner, \ulcorner \mathsf{lam} \urcorner)$.

*Remark*: The precise syntactical meaning of this definition is not completely clear yet. In this paper, we shall give a representation of the extensional collapse of the relational model of the lambda-calculus introduced in [BEM07] (in the sense above), and these two models will clearly be quite different. However, both models induce the same equational theory on lambda-terms (namely, the theory $\mathcal{H}^*$, according to which two terms $M$ and $M'$ are equivalent if, for any context $C$, the term $C[M]$ is solvable iff the term $C[M']$ is solvable). With the notations above, this means that, when restricted to the interpretations of lambda-terms, the relation $\sim_Z$ is just equality. Extending for instance the lambda-calculus with a *parallel composition* construction based on the *mix* rule of Linear Logic as in [DK00, BEM08], the situation becomes more interesting and the theories induced by the two models on the language are distinct.

## 1.4 New-Seely categories and LL-functors

Following [Bie95], we define a model $\mathcal{L}$ of LL as a *New-Seely* category. This consists of

- a symmetric monoidal closed star-autonomous category (also denoted with $\mathcal{L}$) which has all finite products (the unit of the tensor product is denoted with 1, the dualizing object with $\bot$, the terminal object $\top$ and the cartesian product of $X$ and $Y$ is denoted with $X \,\&\, Y$),
- a comonad $! : \mathcal{L} \to \mathcal{L}$ (the structure morphisms $\mathsf{d}^{\mathcal{L}}_X \in \mathcal{L}(!X, X)$ is called *dereliction* and $\mathsf{p}^{\mathcal{L}}_X \in \mathcal{L}(!X, !!X)$ is called *digging*),
- and two natural isomorphisms $!\top \simeq 1$ and $!(X \,\&\, Y) \simeq !X \otimes !Y$

such that the adjunction between $\mathcal{L}$ and its Kleisli category $\mathcal{L}_!$ (which is cartesian closed by the hypotheses above) is a monoidal adjunction.

Given a function $I$ (valuation) from the propositional atoms of LL to objects of $\mathcal{L}$, the interpretation $[A]^{\mathcal{L}}_I$ of an LL-formula $A$ is defined by induction on $A$, using the above mentioned structures of $\mathcal{L}$, e.g. $[A \otimes B]^{\mathcal{L}}_I = [A]^{\mathcal{L}}_I \otimes^{\mathcal{L}} [B]^{\mathcal{L}}_I$. Similarly, given a proof $\pi$ of $A$, one defines $[\pi]^{\mathcal{L}}_I \in \mathcal{L}(1, [A]^{\mathcal{L}}_I)$ by induction on $\pi$ (expressed in the standard sequent calculus of LL [Gir87]).

Given two New-Seely categories $\mathcal{L}$ and $\mathcal{M}$, a functor $F : \mathcal{L} \to \mathcal{M}$ will be called an *LL-functor* if it commutes on the nose with all the constructions required for interpreting LL, e.g. $F(X \otimes_{\mathcal{L}} Y) = F(X) \otimes_{\mathcal{M}} F(Y)$, $F(\mathsf{d}^{\mathcal{L}}_X) = \mathsf{d}^{\mathcal{M}}_X$ etc. Then one has $F([A]^{\mathcal{L}}_I) = [A]^{\mathcal{M}}_{F \circ I}$ and $F([\pi]^{\mathcal{L}}_I) = [\pi]^{\mathcal{M}}_{F \circ I}$ for all formula $A$ and proof $\pi$ of LL.

Such an LL-functor $F$ functor induces a cartesian closed functor (still denoted with $F$) from $\mathcal{L}_!$ to $\mathcal{M}_!$.

# 2 The collapse partial equivalence relation

We define a category whose objects are sets equipped with a partial equivalence relation (PER) on their powersets, the intuition being that two subsets are equivalent if they have the same "extensional" behavior. These PERs are defined as logical relations, in the sense that, when we define function spaces, two morphisms are equivalent iff they map equivalent sets to equivalent sets.

## 2.1 The category of sets and relations

This category underlies the collapse category we want to define. More precisely, the collapse category we define in Section 2.2 is an enrichment of the category of sets and relations where each object is endowed with a partial equivalence relation expressing when two sets are extensionally equivalent, as in 1.3.2.

**2.1.1 Linear structure.** The category of sets and relations **Rel** has sets as objects, and, given two sets $E$ and $F$, the set of morphisms from $E$ to $F$ is $\mathbf{Rel}(E,F) = \mathcal{P}(E \times F)$. Composition is defined in the standard relational way: the composition of $s \in \mathbf{Rel}(E,F)$ and $t \in \mathbf{Rel}(F,G)$ is $t \cdot s \in \mathbf{Rel}(E,G)$. The identity morphism is the diagonal relation $\mathsf{Id} \in \mathbf{Rel}(E,E)$. This category has a quite simple monoidal structure: the tensor product is $E \otimes F = E \times F$ and the unit of the tensor is $1 = \{*\}$. This tensor product is a functor: given $s_i \in \mathbf{Rel}(E_i, F_i)$ for $i = 1,2$, then $s_1 \otimes s_2 = \{((a_1,a_2),(b_1,b_2)) \mid (a_i,b_i) \in s_i \text{ for } i = 1,2\}$. Equipped with this tensor product, **Rel** is symmetric monoidal closed (the associativity, neutrality and symmetry isomorphisms are defined in the usual obvious way), with an object of linear morphisms $E \multimap F = E \times F$ and linear evaluation morphism $\mathsf{ev} \in \mathbf{Rel}((E \multimap F) \otimes E, F)$ given by $\mathsf{ev} = \{(((a,b),a),b) \mid a \in E \text{ and } b \in F\}$.

The symmetric monoidal closed category **Rel** is a star-autonomous category, with dualizing object $\bot = 1$, and the corresponding duality is trivial: $E^\bot = E$. So $E \parr F = E \multimap F = E \otimes F = E \times F$ in this model.

*Remark*: Again, this category is a "degenerate model" of LL in the sense that it identifies $\otimes$ and $\parr$, just as **ScottL** (and even worse, since it equates a formula with its linear negation!). We showed in [BE01] how this model can be enriched with various structures *without modifying the interpretation of proofs*, making $\otimes$ and $\parr$ non-isomorphic operations. This can be considered as one of the most striking features of LL: this logical system is so robust that it survives (in the sense that proofs are not trivialized) in such a degenerate framework.

Given $s \in \mathbf{Rel}(E,F)$ and $x \subseteq E$, one sets $s \cdot x = \{b \mid \exists a \in x \text{ and } (a,b) \in s\}$.

The category **Rel** is cartesian. The cartesian product of a family $(E_i)_{i \in I}$ of sets is $\&_{i \in I} E_i = \bigcup_{i \in I}(\{i\} \times E_i)$, with projections $\pi_j = \{((j,a),a) \mid a \in E_j\} \in \mathbf{Rel}(\&_{i \in I} E_i, E_j)$. Given a family of morphisms $s_i \in \mathbf{Rel}(F, E_i)$, the corresponding morphism $\langle s_i \rangle_{i \in I} \in \mathbf{Rel}(F, \&_{i \in I} E_i)$ is given by $\langle s_i \rangle_{i \in I} = \{(b,(i,a)) \mid i \in I \text{ and } (b,a) \in s_i\}$. The terminal object is $\top = \emptyset$.

The exponential comonad is $!E = \mathcal{M}_{\mathrm{fin}}(E)$, with action on morphisms defined as follows: $!s = \{([a_1,\dots,a_n],[b_1,\dots,b_n]) \mid (a_i,b_i) \in s \text{ for } i = 1,\dots,n\} \in \mathbf{Rel}(!E,!F)$ for $s \in \mathbf{Rel}(E,F)$. Dereliction is given by $\mathsf{d}_E = \{([a],a) \mid a \in S\} \in \mathbf{Rel}(!E,E)$ and digging by $\mathsf{p}_E = \{(m_1 + \cdots + m_n, [m_1,\dots,m_n]) \mid n \in \mathbb{N} \text{ and } m_1,\dots,m_n \in !E\} \in \mathbf{Rel}(!E,!!E)$. Given $x \subseteq E$, one defines $x^! = \mathcal{M}_{\mathrm{fin}}(x)$. Observe that, as usual, $!s \cdot x^! = (s \cdot x)^!$, $\mathsf{d}_E \cdot x^! = x$ and $\mathsf{p}_E \cdot x^! = x^{!!}$.

The isomorphism $!\top \simeq 1$ identifies $[]$ and $*$, and the isomorphism $!(E \,\&\, F) \simeq !E \otimes !F$ maps the element $[(1,a_1),\dots,(1,a_l),(2,b_1),\dots,(2,b_r)]$ of $!(E \,\&\, F)$ to $([a_1,\dots,a_l],[b_1,\dots,b_r]) \in !E \otimes !F$ (this is called the *fundamental isomorphism* in the present paper).

All these data define a new Seely category, see Section 1.4.

**2.1.2 The associated CCC.** The Kleisli category $\mathbf{Rel}_!$ is cartesian closed. Given a set $E$, a point of $E$ in $\mathbf{Rel}_!$ is by definition a morphism in $\mathbf{Rel}(!\top, E)$, that is, a subset of $E$. The terminal object is $\top$, the cartesian product of $(E_i)_{i\in I}$ is $E = \&_{i\in I} E_i$, with projections $\pi_i \circ \mathsf{d}_E$ (still denoted as $\pi_i$). The object of morphisms $E \Rightarrow F$ is $!E \multimap F$, with evaluation map $\mathsf{Ev} = \mathsf{ev} \circ (\mathsf{d}_{E\Rightarrow F} \otimes \mathsf{Id}_{!E})$, that is

$$\mathsf{Ev} = \{(([[(m,b)],m),b) \mid m \in !E \text{ and } b \in F\}\,.$$

Applying a morphism $s \in \mathbf{Rel}_!(E,F) = \mathbf{Rel}(!E,F)$ to a point $x \subseteq E$ consists in composing $s$ with $x$ (considered as a morphism from $\top$ to $E$) in $\mathbf{Rel}_!$; the result is

$$s(x) = s \cdot x^! = \{b \mid \exists m\ (m,b) \in s \text{ and } \mathsf{supp}(m) \subseteq x\}\,.$$

The category $\mathbf{Rel}_!$ is not well pointed, in the sense that two distinct morphisms $s_1, s_2 \in \mathbf{Rel}_!(E,F)$ can satisfy $\forall x \subseteq E\ s_1(x) = s_2(x)$; take for instance $s_1 = \{([a],b)\}$ and $s_2 = \{([a,a],b)\}$.

The purpose of the collapse PER is precisely to make it explicit when two such morphisms should be identified. This depends of course on the PERs $E$ and $F$ are equipped with: the collapse PER is a *logical relation*. We shall present this construction as a new category.

**2.1.3 Inclusions.** Let $E$ and $F$ be two sets such that $E \subseteq F$. Then we denote by $\eta_{E,F}$ and $\rho_{E,F}$ the relations

$$\eta_{E,F} = (E \times F) \cap \mathsf{Id}_E \quad \text{and} \quad \rho_{E,F} = (F \times E) \cap \mathsf{Id}_E\ .$$

Observe that $\rho_{E,F} \circ \eta_{E,F} = \mathsf{Id}_E$.

We denote by **RelC** the class of all sets, ordered by inclusion. This is a partially ordered class, which is complete in the sense that any family $(E_\gamma)_{\gamma\in\Gamma}$ of elements of **RelC** admits a least upper bound. We shall consider actually only directed families (that is, where $\Gamma$ is a directed poset, and $\gamma \leq \delta \Rightarrow E_\gamma \subseteq E_\delta$).

## 2.2 The collapse category

We equip now the objects of **Rel** with a partial equivalence relation whose purpose is to identify morphisms which yield equivalent values when applied to equivalent arguments.

**2.2.1 Pre-PERs, PER objects and morphisms of PER objects.** Let $E$ be a set. Given a binary relation $B$ on $\mathcal{P}(E)$, we define another binary relation $B^\perp$ on $\mathcal{P}(E)$, called the *dual* of $B$, as follows:

$$x'\ B^\perp\ y' \quad \text{if} \quad \forall x,y \in \mathcal{P}(E)\ x\ B\ y \Rightarrow (x \cap x' \neq \emptyset \Leftrightarrow y \cap y' \neq \emptyset)\,.$$

As usual, one has $B \subseteq C \Rightarrow C^\perp \subseteq B^\perp$ and $B \subseteq B^{\perp\perp}$ (as subsets of $\mathcal{P}(E)^2$). We say that the relation $B$ is a *pre-PER* if it is symmetric and satisfies $x\ B$

$y \Rightarrow x\ B\ x$. Clearly, any PER is a pre-PER and if $B$ is a pre-PER, then $B^\perp$ is a PER.

A *PER-object* is a pair $U = (|U|, \sim_U)$, where $|U|$ is a set and $\sim_U$ is a binary relation on $\mathcal{P}(|U|)$ which is a pre-PER such that $\sim_U^{\perp\perp} = \sim_U$. This simply means that, given $x, y \subseteq |U|$, one has $x \sim_U y$ as soon as $x \cap x' \neq \emptyset \Leftrightarrow y \cap y' \neq \emptyset$, for all $x', y' \subseteq |U|$ such that $x' \sim_U^\perp y'$. By this condition, $\sim_U$ is automatically a PER (indeed, $\sim_U$ is pre-PER, hence $\sim_U^\perp$ is a PER, and therefore $\sim_U = \sim_U^{\perp\perp}$ is a PER).

Let **PerL** be the category whose objects are the PER-objects, and where a morphism from $U$ to $V$ is a relation $t \subseteq |U| \times |V|$ such, for all $x, y \in \mathcal{P}(|X|)$, if $x \sim_X y$ then $t \cdot x \sim_Y t \cdot y$.

*Remark*: Let $U$ be a PER-object and $\mathcal{A} \subseteq \mathcal{P}(|U|)$ such that $\forall x_1, x_2 \in \mathcal{A}\ x_1 \sim_U x_2$. Then $\forall x \in \mathcal{A}\ x \sim_C \bigcup \mathcal{A}$. Indeed, let $x_1', x_2' \subseteq |U|$ be such that $x_1' \sim_{U^\perp} x_2'$. If $x \cap x_1' \neq \emptyset$, then $x \cap x_2' \neq \emptyset$ because $x \sim_U x$, and hence $\bigcup \mathcal{A} \cap x_2' \neq \emptyset$. Conversely, if $\bigcup \mathcal{A} \cap x_2' \neq \emptyset$, there is some $y \in \mathcal{A}$ such that $y \cap x_2' \neq \emptyset$ and we conclude since $x \sim_U y$. So each equivalence class of $\sim_U$ has a maximal element, which is the union of all the elements of the class. These particular elements $x$ of $\mathcal{P}(|U|)$ are characterized by the two following properties:

- $x \sim_U x$
- and $\forall y \in \mathcal{P}(|U|) \quad y \sim_U x \Rightarrow y \subseteq x$.

**Lemma 1** *Let $U$ be a PER-object and let $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ be families of elements of $\mathcal{P}(|U|)$ be such that $x_i \sim_U y_i$ for each $i \in I$. Then $\bigcup_{i\in I} x_i \sim_U \bigcup_{i\in I} y_i$.*

The proof is straightforward. In particular $\emptyset \sim_U \emptyset$, for any PER-object $U$.

**2.2.2 Orthogonality and strong isomorphisms.** We define the PER-object $U^\perp$ by $|U^\perp| = |U|$ and $\sim_{U^\perp} = \sim_U^\perp$, so that $U^{\perp\perp} = U$.

**Lemma 2** *Given two PER-objects $U$ and $V$, any bijection $\theta : |U| \to |V|$ such that, for all $x, y \in \mathcal{P}(|X|)$, one has $x \sim_U y$ iff $\theta(x) \sim_V \theta(y)$ is an isomorphism from $U$ to $V$. Such a bijection will be called a* strong isomorphism *from $U$ to $V$.*

Straightforward verification. Of course, $\theta^{-1}$ is a strong isomorphism from $V$ to $U$.

Observe that any strong isomorphism $\theta$ from $U$ to $V$ is also a strong isomorphism from $U^\perp$ to $V^\perp$. Indeed, let $x_1', x_2' \subseteq |U|$. Assume first that $x_1' \sim_{U^\perp} x_2'$ and let us show that $\theta(x_1') \sim_{V^\perp} \theta(x_2')$. So let $y_1, y_2 \subseteq |V|$ be such that $y_1 \sim_V y_2$. We have $\theta(x_1') \cap y_1 \neq \emptyset \Leftrightarrow x_1' \cap \theta^{-1}(y_1) \neq \emptyset$ and we conclude since $\theta^{-1}$ is a strong isomorphism from $V$ to $U$. The converse implication $\theta(x_1') \sim_{V^\perp} \theta(x_2') \Rightarrow x_1' \sim_{U^\perp} x_2'$ is proven similarly.

**2.2.3 Monoidal structure.** We define $U \otimes V$ as follows. We take $|U \otimes V| = |U| \times |V|$, and $\sim_{U\otimes V} = E^{\perp\perp}$ where

$$E = \{(x_1 \times y_1, x_2 \times y_2) \mid x_1 \sim_U x_2 \text{ and } y_1 \sim_U y_2\} \subseteq \mathcal{P}(|U \otimes V|)^2 .$$

Since this relation $E$ is a pre-PER (but not a PER *a priori*, since one cannot recover $x$ and $y$ from $x \times y$ when one of these two sets is empty), the relation $\sim_{U \otimes V}$ is a PER, and $U \otimes V$ so defined is a PER-object. We define $U \multimap V = (U \otimes V^\perp)^\perp$.

**Lemma 3** *One has $|U \multimap V| = |U| \times |V|$. If $t_1, t_2 \in \mathcal{P}(|U \multimap V|)$, one has $t_1 \sim_{U \multimap V} t_2$ iff for all $x_1, x_2 \subseteq |U|$ such that $x_1 \sim_U x_2$, one has $t_1 \cdot x_1 \sim_Y t_2 \cdot x_2$. Moreover, one has $t_1 \sim_{U \multimap V} t_2 \Leftrightarrow {}^\natural t_1 \sim_{V^\perp \multimap U^\perp} {}^\natural t_2$.*

*Proof.* This is due to the fact that, for any $t \subseteq |U \multimap V|$, $x \subseteq |U|$ and $y' \subseteq |V|$, one has $t \cap (x \times y') \neq \emptyset \Leftrightarrow (t \cdot x) \cap y' \neq \emptyset$ □

So the morphisms from $U$ to $V$ are exactly the $t \in \mathcal{P}(|U \multimap V|)$ such that $t \sim_{U \multimap V} t$, and if $t \in \mathbf{PerL}(U, V)$ then ${}^\natural t \in \mathbf{PerL}(V^\perp, U^\perp)$.

**Lemma 4** *The obvious bijection $\lambda$ from $|U \otimes V \multimap W|$ to $|U \multimap (V \multimap W)|$ defines a strong isomorphism between the PER-objects $U \otimes V \multimap W$ and $U \multimap (V \multimap W)$. In particular, for $s_1, s_2 \in \mathcal{P}(|U \otimes V \multimap W|)$, one has $s_1 \sim_{U \otimes V \multimap W} s_2$ iff for any $x_1, x_2 \in \mathcal{P}(|U|)$ and $y_1, y_2 \in \mathcal{P}(|V|)$ such that $x_1 \sim_U x_2$ and $y_1 \sim_U y_2$, one has $s_1 \cdot (x_1 \times y_1) \sim_W s_2 \cdot (x_2 \times y_2)$.*

*Proof.* Let $t_1, t_2 \subseteq \mathcal{P}(U \otimes V \multimap W)$. Assume first that $t_1 \sim_{U \otimes V \multimap W} t_2$, we want to prove that $\lambda(t_1) \sim_{U \multimap (V \multimap W)} \lambda(t_2)$. But this is clear since, if $x_1, x_2 \subseteq |U|$ and $y_1, y_2 \subseteq |V|$ satisfy $x_1 \sim_U x_2$ and $y_1 \sim_V y_2$, then we have $x_1 \times y_2 \sim_{U \otimes V} x_2 \times y_2$, and therefore $(\lambda(t_1) \cdot x_1) \cdot y_1 = t_1 \cdot (x_1 \times y_1) \sim_W t_2 \cdot (x_2 \times y_2) = (\lambda(t_2) \cdot x_2) \cdot y_2$. Assume conversely that $\lambda(t_1) \sim_{U \multimap (V \multimap W)} \lambda(t_2)$, we prove that $t_1 \sim_{U \otimes V \multimap W} t_2$. For this, we proceed as above, showing that ${}^\natural t_1 \sim_{W^\perp \multimap (U \otimes V)^\perp} {}^\natural t_2$ and applying Lemma 3. □

**Lemma 5** *The obvious bijection $\alpha : |(U \otimes V) \otimes W| \to |U \otimes (V \otimes W)|$ is an isomorphism of PER-objects from $(U \otimes V) \otimes W$ to $U \otimes (V \otimes W)$.*

*Proof.* By 2.2.2, it suffices to prove that $\alpha$ is an isomorphism from $((U \otimes V) \otimes W)^\perp$ to $(U \otimes (V \otimes W))^\perp$, and this results from Lemma 4. □

Given $s \in \mathbf{PerL}(U_1, U_2)$ and $t \in \mathbf{PerL}(V_1, V_2)$, one defines $s \otimes t \subseteq |U_1 \otimes V_1| \times |U_2 \otimes V_2|$ as in 4.2.2. Then one shows using Lemma 4 that $s \otimes t \in \mathbf{PerL}(U_1 \otimes V_1, U_2 \otimes V_2)$, and one checks that the category **PerL** equipped with this $\otimes$ binary functor, together with the associativity isomorphism of Lemma 5 (as well as the symmetry isomorphism etc.) is a symmetric monoidal category, which is closed (with $U \multimap V$ as object of linear morphisms from $U$ to $V$) by Lemma 4. The linear evaluation morphism is $\mathsf{ev}$, as defined in Section 2.1.

**PerL** is star-autonomous, with $\perp = (\{*\}, =)$ as dualizing object.

**2.2.4 Additive structure.** Given a family $(U_i)_{i \in I}$ of PER-objects, one defines $U = \&_{i \in I} U_i$ by setting $|U| = \prod_{i \in I} (\{i\} \times |U_i|)$, and by saying that, for any $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in \mathcal{P}(|U|)$ (identifying this latter set with a product), one has $x \sim_U y$ if one has $x_i \sim_{U_i} y_i$ for all $i \in I$. Using the fact that $\emptyset \sim_V \emptyset$ in any PER-object $V$, one shows that $\sim_U^\perp = \sim_{\&_{i \in I} U_i^\perp}$ and it follows that $U$ is a PER-object. It is routine to check that $\&_{i \in I} U_i$ so defined is the cartesian

product of the $U_i$s in the category **PerL**, and that this cartesian product is also a coproduct. In particular, if $U$ is a PER-object and $I$ is a set, we denote with $U^I$ the product $\&_{i\in I} U_i$ where $U_i = U$ for each $U$.

In particular, **PerL** has a terminal object $\top$, given by $|\top| = \emptyset$ and $\emptyset \sim_\top \emptyset$. Observe that this is the only PER-object with an empty web.

**2.2.5 Exponentials.** Given a PER-object $U$, we define $!U$ by $|!U| = \mathcal{M}_{\mathrm{fin}}(|U|)$, and $\sim_{!U} = E^{\perp\perp}$ where

$$E = \{(x_1^!, x_2^!) \mid x_1, x_2 \in \mathcal{P}(|U|)\ x_1 \sim_U x_2\}$$

where we recall that $x^! = \mathcal{M}_{\mathrm{fin}}(x)$. Since $E$ is a pre-PER (and actually a PER, because $x$ can be recovered from $x^!$ using *dereliction*: $x = \{a \mid [a] \in x^!\}$), the relation $\sim_{!U}$ is a PER. We recall that, if $s \subseteq |!U \multimap V|$ and $x \subseteq |U|$, then we denote with $s(x)$ the subset $s \cdot x^!$ of $|Y|$, see Section 2.1.

**Lemma 6** *Let $U$ and $V$ be PER-objects and let $s_1, s_2 \subseteq |!U \multimap V|$. One has $s_1 \sim_{!U\multimap V} s_2$ iff*

$$\forall x_1, x_2 \subseteq |U| \quad x_1 \sim_U x_2 \Rightarrow s_1(x_1) \sim_V s_2(x_2)\,.$$

*Proof.* The $\Rightarrow$ direction is trivial. For the converse, one assumes that the stated condition holds, and one checks that ${}^{\mathrm{t}}s_1 \sim_{V^\perp \multimap (!U)^\perp} {}^{\mathrm{t}}s_2$, and for this purpose, it suffices to apply Lemma 3. □

Given $s \in \mathbf{PerL}(U, V)$, one defines $!s \subseteq |!U| \times |!V|$ as in the standard relational model by setting

$$!s = \{([a_1, \ldots, a_n], [b_1, \ldots, b_n]) \mid n \in \mathbb{N},\ (a_i, b_i) \in s \text{ for } i = 1, \ldots, n\}\,.$$

Then, since $!s \cdot x^! = (s \cdot x)^!$, we have $!s_1 \sim_{!U\multimap !V} !s_2$ as soon as $s_1 \sim_{U\multimap V} s_2$ (by Lemma 6); in particular, if $s \in \mathbf{PerL}(U, V)$, one has $!s \in \mathbf{PerL}(!U, !V)$ and so the operation $s \mapsto !s$ is an endofunctor on **PerL**.

One defines $\mathsf{d}_U \subseteq |!U| \times |U|$ as $\mathsf{d}_U = \{([a], a) \mid a \in |U|\}$, and since $\mathsf{d}_U \cdot x^! = x$ for all $x \subseteq |U|$, we get easily $\mathsf{d}_U \in \mathbf{PerL}(!U, U)$. Similarly, one defines $\mathsf{p}_U \subseteq |!U| \times |!!U|$ as $\mathsf{p}_U = \{(m_1 + \cdots + m_k, [m_1, \ldots, m_k]) \mid m_1, \ldots, m_k \in |!U|\}$. Since $\mathsf{p}_U \cdot x^! = x^{!!}$, we get $\mathsf{p}_U \in \mathbf{PerL}(!U, !!U)$. The naturality in $U$ of these morphisms is clear (it holds in the relational model), and $!\_$ equipped with these two natural transformations is a comonad. Moreover, the fundamental isomorphism also holds in this setting.

**2.2.6 Fundamental isomorphism and cartesian closeness.** Let $U$ and $V$ be PER-objects. Let $\theta : |!(U \,\&\, V)| \to |!U \otimes !V|$ be the usual bijection defined by

$$\theta([(1, a_1), \ldots, (1, a_l), (2, b_1), \ldots, (2, b_r)]) = ([a_1, \ldots, a_l], [b_1, \ldots, b_r])$$

Using Lemma 6, one shows easily that $\theta \in \mathbf{PerL}(!(U \,\&\, V), !U \otimes !V)$ (as a relation). For showing that $\theta^{-1} \in \mathbf{PerL}(!U \otimes !V, !(U \,\&\, V))$, one applies Lemma 4 and then Lemma 6, twice. This shows that $\theta$ is a strong isomorphism of PER-objects.

So the category of PER-objects (together with the monoidal and exponential structure explained above) is a new-Seely category, in the sense of [Bie95].

The associated Kleisli category $\mathbf{PerL}_!$ is cartesian closed. The object of morphisms from $U$ to $V$ is $U \Rightarrow V = !U \multimap V$ and we have seen that the associated PER $\sim_{U\Rightarrow V}$ is such that, given two elements $s_1$ and $s_2$ of $\mathbf{PerL}_!(U,V)$, one has $s_1 \sim_{U\Rightarrow V} s_2$ iff $s_1(x_1) \sim_V s_2(x_2)$ for all $x_1, x_2 \subseteq |U|$ such that $x_1 \sim_U x_2$. The evaluation morphism is $\mathsf{Ev}$, as defined in 2.1.2.

## 2.3 The partially ordered class of PER-objects

Let $U$ and $V$ be PER objects. We say that $U$ is a subobject of $V$ and write $U \sqsubseteq V$ if $|U| \subseteq |V|$, and moreover $\eta_{|U|,|V|} \in \mathbf{PerL}(U,V)$ and $\rho_{|U|,|V|} \in \mathbf{PerL}(V,U)$. This means that the two following conditions are satisfied

$$\forall x_1, x_2 \subseteq |U| \quad x_1 \sim_U x_2 \Rightarrow x_1 \sim_V x_2$$

and

$$\forall y_1, y_2 \subseteq |V| \quad y_1 \sim_V y_2 \Rightarrow y_1 \cap |U| \sim_U y_2 \cap |U| \,.$$

Observe that $\sqsubseteq$ a partial order relation and let $\mathbf{PerC}$ be the partially ordered class of PER-objects ordered by $\sqsubseteq$.

One of the main features of this definition is that linear negation is covariant with respect to the subobject partial order.

**Lemma 7** *If $U \sqsubseteq V$ then $U^\perp \sqsubseteq V^\perp$.*

*Proof.* We have $|U^\perp| = |U| \subseteq |V| = |V^\perp|$. Moreover ${}^t\eta_{|U|,|V|} = \rho_{|U|,|V|}$ and ${}^t\rho_{|U|,|V|} = \eta_{|U|,|V|}$. The result follows. □

### 2.3.1 Completeness.

**Lemma 8** *Let $\Gamma$ be a directed set and let $(U_\gamma)_{\gamma\in\Gamma}$ be a directed family of PERs (meaning that $\gamma \leq \delta \Rightarrow U_\gamma \sqsubseteq U_\delta$). We define $U = \bigsqcup_{\gamma\in\Gamma} U_\gamma$ by $|U| = \bigcup_{\gamma\in\Gamma} |U_\gamma|$ and, for $x_1, x_2 \subseteq |U|$, $x_1 \sim_U x_2$ iff $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$ for all $\gamma \in \Gamma$. Then $U$ is a PER-object. Moreover $U^\perp = \bigsqcup_{\gamma\in\Gamma} U_\gamma^\perp$.*

*Proof.* Let $U' = \bigsqcup_{\gamma\in\Gamma} U_\gamma^\perp$, it will be enough to show that $U = U'^\perp$. Let $x_1, x_2 \subseteq |U|$. Assume first that $x_1 \sim_U x_2$ and let us show that $x_1 \sim_{U'^\perp} x_2$. So let $x'_1, x'_2 \subseteq |U|$ be such that $x'_1 \sim_{U'} x'_2$ and assume that $x_1 \cap x'_1 \neq \emptyset$. Let $\gamma \in \Gamma$ be such that $x_1 \cap x'_1 \cap |U_\gamma| \neq \emptyset$. By definition of $U$ and $U'$, we have $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$ and $x'_1 \cap |U_\gamma| \sim_{U_\gamma^\perp} x'_2 \cap |U_\gamma|$, and therefore $x_2 \cap x'_2 \cap |U_\gamma| \neq \emptyset$, and hence $x_2 \cap x'_2 \neq \emptyset$ as required. Assume next that $x_1 \sim_{U'^\perp} x_2$ and let us show that $x_1 \sim_U x_2$. So let $\gamma \in \Gamma$ and let us prove that $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$. So let $x'_1, x'_2 \subseteq |U_\gamma|$ be such that $x'_1 \sim_{U_\gamma^\perp} x'_2$ and assume that $(x_1 \cap |U_\gamma|) \cap x'_1 \neq \emptyset$, that is $x_1 \cap x'_1 \neq \emptyset$.

We show that $x'_1 \sim_{U'} x'_2$. Let $\delta \in \Gamma$ and let us show that $x'_1 \cap |U_\delta| \sim_{U_\delta^\perp} x'_2 \cap |U_\delta|$. So let $\varepsilon \in \Gamma$ be such that $\gamma, \delta \leq \varepsilon$. Let $y_1, y_2 \subseteq |U_\delta|$ be such that $y_1 \sim_{U_\delta} y_2$ and $x'_1 \cap |U_\delta| \cap y_1 \neq \emptyset$. Since $U_\delta \sqsubseteq U_\varepsilon$ and $U_\delta^\perp \sqsubseteq U_\varepsilon^\perp$ (by Lemma 7), we have $x'_1 \sim_{U_\varepsilon^\perp} x'_2$ and $y_1 \sim_{U_\varepsilon} y_2$. Therefore $x'_2 \cap y_2 \neq \emptyset$, that is $x'_2 \cap |U_\delta| \cap y_2 \neq \emptyset$ (since $y_2 \subseteq |U_\delta|$) as required.

Since $x_1 \sim_{U'^\perp} x_2$ and $x'_1 \sim_{U'} x'_2$, we have $x_2 \cap x'_2 \neq \emptyset$, that is $(x_2 \cap |U_\gamma|) \cap x'_2 \neq \emptyset$ (since $x'_2 \subseteq |U_\gamma|$) as required. □

**Lemma 9** *If $(U_\gamma)_{\gamma\in\Gamma}$ is a directed family of PER-objects, then $\bigsqcup_{\gamma\in\Gamma} U_\gamma$ is its lub in* **PerC**.

*Proof.* For showing that $U_\delta \sqsubseteq \bigsqcup_{\gamma\in\Gamma} U_\gamma$, one must show that, if $x_1 \sim_{U_\delta} x_2$, then $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$ for any given $\gamma \in \Gamma$; one picks some $\varepsilon \in \Gamma$ such that $\gamma, \delta \leq \varepsilon$ and one proceeds as in the proof of Lemma 8. Let $V$ be a PER-object an assume that $U_\gamma \sqsubseteq V$ for all $\gamma \in \Gamma$, we must show that $U = \bigsqcup_{\gamma\in\Gamma} U_\gamma \sqsubseteq V$. Let first $x_1, x_2 \subseteq |U|$ and assume that $x_1 \sim_U x_2$, and let us prove that $x_1 \sim_V x_2$. So let $y'_1, y'_2 \subseteq |V|$ be such that $y'_1 \sim_{V^\perp} y'_2$, and assume that $x_1 \cap y'_1 \neq \emptyset$. Let $\gamma \in \Gamma$ be such that $x_1 \cap y'_1 \cap |U_\gamma| \neq \emptyset$. Since $U_\gamma^\perp \sqsubseteq V^\perp$ by Lemma 7, we have $y'_1 \cap |U_\gamma| \sim_{U_\gamma^\perp} y'_2 \cap |U_\gamma|$ and hence $x_2 \cap y'_2 \cap |U_\gamma| \neq \emptyset$ and so $x_2 \cap y'_2 \neq \emptyset$. Let now $y_1, y_2 \subseteq |V|$ be such that $y_1 \sim_V y_2$ and let us show that $y_1 \cap |U| \sim_U y_2 \cap |U|$, that is $y_1 \cap |U_\gamma| \sim_U y_2 \cap |U_\gamma|$ for all $\gamma \in \Gamma$, which holds since $U_\gamma \sqsubseteq V$ by assumption. □

**2.3.2 Variable PER-objects and fixpoints thereof.** A functor (that is, a "monotone" class function) $\Phi : \mathbf{PerC}^n \to \mathbf{PerC}$ which commutes with the lubs of directed families (of $n$-tuples) of PER-objects will be said to be continuous, or to be a *variable PER-object*. Let $\Psi : \mathbf{PerC} \to \mathbf{PerC}$ be a variable PER-object. Then $\Psi$ has a least fixpoint $\mathsf{fix}(\Psi) = \bigsqcup_{k\in\mathbb{N}} \Psi^k(\top)$ where $\top$ is the empty PER-object (see 2.2.4). Of course, given a PER-object $\Phi : \mathbf{PerC}^{n+1} \to \mathbf{PerC}$, the operation $\mathbf{PerC}^n \to \mathbf{PerC}$ which maps $(U_1, \ldots, U_n)$ to $\mathsf{fix}(\Phi(U_1, \ldots, U_n, \_))$ is a variable PER-object. We have already seen that the map $U \to U^\perp$ is a variable PER-objects.

**Lemma 10** *The operations $(U, V) \mapsto U \otimes V$, $U \mapsto U^I$ and $U \mapsto !U$ are variable PER-objects.*

*Proof.* We observe first that $\otimes$ is monotone, in the sense that if $U \sqsubseteq U'$ and $V \sqsubseteq V'$, then $U \otimes V \sqsubseteq U' \otimes V'$. This results from the fact that $|U \otimes V| \subseteq |U' \otimes V'|$ and from the obvious equations $\eta_{|U\otimes V|,|U'\otimes V'|} = \eta_{|U|,|U'|} \otimes \eta_{|V|,|V'|}$ and $\rho_{|U\otimes V|,|U'\otimes V'|} = \rho_{|U|,|U'|} \otimes \rho_{|V|,|V'|}$. We check similarly that $!\_$ and $(\_)^I$ are monotone.

We show that $(U, V) \mapsto (U \multimap V)$ is a variable PER-object. It is monotone by the considerations above. Let $(U_\gamma)_{\gamma\in\Gamma}$ and $(V_\gamma)_{\gamma\in\Gamma}$ be directed families of PER-objects. We show that $U \multimap V = \bigsqcup_{\gamma\in\Gamma}(U_\gamma \multimap V_\gamma)$ where $U = \bigsqcup_{\gamma\in\Gamma} U_\gamma$ and $V = \bigsqcup_{\gamma\in\Gamma} V_\gamma$. Let $t_1, t_2 \subseteq |U \multimap V|$. Assume first that $t_1 \sim_{U\multimap V} t_2$; one has $t_1 \cap |U_\gamma \multimap V_\gamma| \sim_{U_\gamma \multimap V_\gamma} t_2 \cap |U_\gamma \multimap V_\gamma|$ because, if $x_1 \sim_{U_\gamma} x_2$, one has $(t_i \cap |U_\gamma \multimap V_\gamma|) \cdot x_i = (t_i \cdot x_i) \cap |V_\gamma|$. Conversely, assume that $t_1 \sim_{\bigsqcup_{\gamma\in\Gamma}(U_\gamma \multimap V_\gamma)} t_2$ and let us show that $t_1 \sim_{U\multimap V} t_2$. So let $x_1, x_2 \subseteq |U|$ be such that $x_1 \sim_U x_2$, and let us show that $t_1 \cdot x_1 \sim_V t_2 \cdot x_2$. We have $t_i \cdot x_i = \bigcup_{\gamma\in\Gamma} (t_i \cap |U_\gamma \multimap V_\gamma|) \cdot (x_i \cap |U_\gamma|)$ and $(t_1 \cap |U_\gamma \multimap V_\gamma|) \cdot (x_1 \cap |U_\gamma|) \sim_{V_\gamma} (t_2 \cap |U_\gamma \multimap V_\gamma|) \cdot (x_2 \cap |U_\gamma|)$ for each $\gamma \in \Gamma$. We conclude applying Lemma 1 and using the fact that $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$ for all $\gamma \in \Gamma$. Since $U \otimes V = (U \multimap V^\perp)^\perp$, this shows that $(U, V) \mapsto U \otimes V$ is a variable PER-object.

One proves easily that $U \mapsto U^I$ is a variable PER-object.

To conclude, let us prove that $\Phi : U \mapsto (!U)^\perp$ is a variable PER-object. It is a monotone operation because $!\_$ is monotone as we have seen. So let $(U_\gamma)_{\gamma\in\Gamma}$ be a directed family of PER-objects and let us show that $\Phi(U) = \bigsqcup_{\gamma\in\Gamma}\Phi(U_\gamma)$, where $U = \bigsqcup_{\gamma\in\Gamma} U_\gamma$. Let $A'_1, A'_2 \subseteq \mathcal{M}_{\rm fin}(|!U|)$. Assume first that $A'_1 \sim_{\Phi(U)} A'_2$ and let $\gamma \in \Gamma$, we prove that $A'_1 \cap |\Phi(U_\gamma)| \sim_{\Phi(U_\gamma)} A'_2 \cap |\Phi(U_\gamma)|$. So let $x_1, x_2 \subseteq |U_\gamma|$ with $x_1 \sim_{U_\gamma} x_2$ and assume that $A'_1 \cap |\Phi(U_\gamma)| \cap x_1^! \neq \emptyset$. We have $x_1 \sim_U x_2$ and hence $A'_2 \cap x_2^! \neq \emptyset$, that is $A'_2 \cap |\Phi(U_\gamma)| \cap x_2^! \neq \emptyset$. Conversely, assume that $A'_1 \sim_{\bigsqcup_{\gamma\in\Gamma}\Phi(U_\gamma)} A'_2$ and let us prove that $A'_1 \sim_{\Phi(U)} A'_2$. So let $x_1, x_2 \subseteq |U|$ with $x_1 \sim_U x_2$ and assume that $A'_1 \cap x_1^! \neq \emptyset$; let $m$ be an element of that intersection. Since $\Gamma$ is directed and $m$ is a *finite* multiset, one can find $\gamma \in \Gamma$ such that $m \in |\Phi(U_\gamma)|$. By assumption, we have $A'_1 \cap |\Phi(U_\gamma)| \sim_{\Phi(U_\gamma)} A'_2 \cap |\Phi(U_\gamma)|$ and $x_1 \cap |U_\gamma| \sim_{U_\gamma} x_2 \cap |U_\gamma|$. We conclude using the fact that $(x_1 \cap |U_\gamma|)^! = x_1^! \cap |\Phi(U_\gamma)|$: we have $A'_1 \cap x_1^! \cap |\Phi(U_\gamma)| \neq \emptyset$, that is $(A'_1 \cap |\Phi(U_\gamma)|) \cap (x_1 \cap |U_\gamma|)^! \neq \emptyset$ and hence $(A'_2 \cap |\Phi(U_\gamma)|) \cap (x_2 \cap |U_\gamma|)^! \neq \emptyset$ which implies $A'_2 \cap x_2^! \neq \emptyset$. □

**2.3.3 An extensional reflexive PER-object.** Consider the mapping of PER-object $\Phi_{\mathsf{e}}$ defined by $\Phi_{\mathsf{e}}(U) = (!(U^{\mathbb{N}}))^\perp$. By Lemmas 7 and 10, $\Phi_{\mathsf{e}}$ is a variable PER-object, and has therefore a least fixpoint, namely the PER-object $\mathcal{D}_{\mathsf{e}} = \bigsqcup_{k\in\mathbb{N}} \Phi_{\mathsf{e}}^k(\top)$. One has $\mathcal{D}_{\mathsf{e}} \Rightarrow \mathcal{D}_{\mathsf{e}} = (!\mathcal{D}_{\mathsf{e}})^\perp \parr \mathcal{D}_{\mathsf{e}} = (!\mathcal{D}_{\mathsf{e}})^\perp \parr \Phi_{\mathsf{e}}(\mathcal{D}_{\mathsf{e}}) = (!\mathcal{D}_{\mathsf{e}})^\perp \parr (!(\mathcal{D}_{\mathsf{e}}^{\mathbb{N}}))^\perp \simeq (!(\mathcal{D}_{\mathsf{e}} \mathbin{\&} \mathcal{D}_{\mathsf{e}}^{\mathbb{N}}))^\perp$ by the fundamental isomorphism of 2.2.6. We conclude since $\mathcal{D}_{\mathsf{e}} \mathbin{\&} \mathcal{D}_{\mathsf{e}}^{\mathbb{N}} \simeq \mathcal{D}_{\mathsf{e}}^{\mathbb{N}}$ (by the strong isomorphism which maps $(1, a)$ to $(0, a)$ and $(2, (i, a))$ to $(i + 1, a)$). Therefore $\mathcal{D}_{\mathsf{e}}$ is an extensional model of the pure lambda-calculus in the Kleisli category $\mathbf{PerL}_!$.

The underlying set $|\mathcal{D}_{\mathsf{e}}|$ is the relational model of the pure lambda-calculus described in [BEM07]. We denote it as $\mathcal{D}_{\mathsf{r}}$. It is the least fixpoint (in the partially ordered class of sets) of the monotone and continuous operation $E \mapsto \mathcal{M}_{\rm fin}(\mathbb{N} \times E)$.

# 3 A linear Scott semantics

Given a preordered set $(S, \leq)$, we denote with $S^{\rm op}$ the opposite preorder. Given $x \subseteq S$, we denote with $\downarrow_S x$ (or simply $\downarrow x$ if the ambient preorder is clear from the context) the set $\{a \in S \mid \exists b \in x\ a \leq b\}$. And we set $\uparrow_S x = \downarrow_{S^{\rm op}} x$. We also define

$$\mathcal{I}(S) = \{x \subseteq S \mid \underset{S}{\downarrow} x = x\}$$

which, ordered by inclusion, is a prime-algebraic lattice.

## 3.1 Star-autonomous structure

Let $S$ and $T$ be preorders. A function $f : \mathcal{I}(S) \to \mathcal{I}(T)$ is *linear* if it commutes with arbitrary lubs. In other words, for any family $(x_i)_{i\in I}$ of elements of $\mathcal{I}(S)$, we must have $f\left(\bigcup_{i\in I} x_i\right) = \bigcup_{i\in I} f(x_i)$. This implies in particular that $f$ is monotone, and that $f(\emptyset) = \emptyset$ (of course, we do not necessarily have $f(S) = T$). We denote with **ScottL** the corresponding category.

We equip the hom-set $\mathbf{ScottL}(S, T)$ with the ordinary pointwise order: $f \leq g$ if $\forall x \in \mathcal{I}(X)\ f(x) \subseteq g(x)$.

Given such a linear map $f \in \mathbf{ScottL}(S,T)$, we define its *linear trace* as

$$\mathsf{tr}^{\mathsf{S}}(f) = \{(a,b) \in S \times T \mid b \in f(\underset{S}{\downarrow}\{a\})\}\,.$$

This is similar to the usual definition of the trace of a stable linear map (see [Gir87, AC98]).

Then it is easily checked that $\mathsf{tr}^{\mathsf{S}}(f) \in \mathcal{I}(S^{\mathrm{op}} \times T)$. Conversely, given any $t \in \mathcal{I}(S^{\mathrm{op}} \times T)$, we define a function

$$\begin{array}{rcl} \mathsf{fun}^{\mathsf{S}}(t) : \mathcal{I}(S) & \to & \mathcal{P}(T) \\ x & \mapsto & t \cdot x \end{array}$$

and it is easy to check that $\mathsf{fun}^{\mathsf{S}}(t)$ takes its values in $\mathcal{I}(T)$ and is linear from $\mathcal{I}(S)$ to $\mathcal{I}(T)$.

**Proposition 11** *The maps* $\mathsf{tr}^{\mathsf{S}}$ *and* $\mathsf{fun}^{\mathsf{S}}$ *define an order isomorphism between the posets* $\mathbf{ScottL}(S,T)$ *and* $\mathcal{I}(S^{\mathrm{op}} \times T)$. *Moreover, these isomorphisms commute with composition (of maps and relations respectively).*

Therefore, we set $S \multimap T = S^{\mathrm{op}} \times T$. Thanks to the lemma above, we can consider the morphisms of the category **ScottL** as linear functions or as relations. For instance, as a function, the identity map $S \to S$ is of course the identity function $\mathcal{I}(S) \to \mathcal{I}(S)$, but as a relation, it is $\mathsf{Id}^{\mathsf{S}} = \{(a,b) \in S \times S \mid b \leq a\}$. In this paper, we prefer the relational viewpoint on morphisms.

The following observation is trivial but useful.

**Lemma 12** *Let* $t \subseteq S \times T$ *and let* $x \in \mathcal{I}(S)$. *One has* $\downarrow_T (t \cdot x) = (\downarrow_{S \multimap T} t) \cdot x$.

**3.1.1 Isomorphisms.** An isomorphism (in the usual categorical sense) from $S$ to $T$ is a relation $t \in \mathcal{I}(S \multimap T)$ such that $\mathsf{fun}^{\mathsf{S}}(t) : \mathcal{I}(S) \to \mathcal{I}(T)$ is an order isomorphism. As a relation, an isomorphism from $S$ to $T$ has no reason to be a bijection, not even a function. For instance, if $S = \{0\}$ and $T = \mathbb{N}$ (with the largest preorder, in which $n \leq m$ for all $n, m \in \mathbb{N}$), then the relation $\{(0,n) \mid n \in \mathbb{N}\}$ is an isomorphism from $S$ to $T$ (it is actually the only non-empty morphism from $S$ to $T$).

We shall call *strong isomorphism* from $S$ to $T$ any function $\varphi : S \to T$ which is an isomorphism of preorders (that is, $\varphi$ is bijective and $a \leq_S b$ iff $\varphi(a) \leq_T \varphi(b)$). Such a $\varphi$ is not an isomorphism (in the categorical sense above) in general, but $\downarrow_{S \multimap T} \varphi$ is. And we shall say that $S$ and $T$ are strongly isomorphic if there is a strong isomorphism from $S$ to $T$.

**3.1.2 Monoidal structure.** The *tensor product* of preorders is given by $S \otimes T = S \times T$. It is easily seen to be functorial. Indeed, let $s \in \mathcal{I}(S_1 \multimap S_2)$ and $t \in \mathcal{I}(T_1 \multimap T_2)$. Then, we set

$$s \otimes t = \{((a_1,b_1),(a_2,b_2)) \in (S_1 \otimes T_1) \multimap (S_2 \otimes T_2) \mid (a_1,a_2) \in s \text{ and } (b_1,b_2) \in t\}\,.$$

One can check that $s \otimes t \in \mathcal{I}((S_1 \otimes T_1) \multimap (S_2 \otimes T_2))$ and that $(s' \otimes t') \circ (s \otimes t) = (s' \circ s) \otimes (t' \circ t)$.

The neutral element of the tensor product is $1 = \{\star\}$ (actually, any non-empty preorder such that $a \leq b$ for all $a, b$ is isomorphic to 1, and therefore is

neutral for $\otimes$). The so defined symmetric monoidal category **ScottL** is monoidal closed, with linear evaluation morphism $\mathsf{ev}^{\mathsf{S}} \in \mathbf{ScottL}((S \multimap T) \otimes S, T)$ given by

$$\mathsf{ev}^{\mathsf{S}} = \{(((a,b),a'),b') \mid b' \leq_{|T|} b \text{ and } a \leq_{|S|} a'\}\,.$$

We use the same object 1 as dualizing object, but when used in that way, we denote it with $\bot$.

It is clear that $S \multimap \bot = S^{\mathrm{op}}$ (up to the identification of $a \in S$ with $(a,\star) \in S \multimap \bot$), and that the canonical map $S \to (S \multimap \bot) \multimap \bot$ coincides with the identity, so the monoidal category of preorders and linear maps is a star-autonomous category in the sense of [Bar79].

## 3.2 Products and coproducts

Let $(S_i)_{i\in I}$ be a collection of preorders, the cartesian product of this family is denoted with $\&_{i\in I} S_i$ and is the disjoint union $\bigcup_{i\in I}(\{i\} \times S_i)$, endowed with the disjoint union of the preorder relations. One has $\mathcal{I}(\&_{i\in I}) = \prod_{i\in I} \mathcal{I}(S_i)$ up to a trivial and canonical isomorphism. The $i$-th projection $\pi_i^{\mathsf{S}} : \&_{i\in I} S_i \to S_i$ is given by

$$\pi_i^{\mathsf{S}} = \{((i,a),b) \mid a,b \in S_i\ b \leq a\}\,.$$

And given morphisms $t_i : T \to S_i$, the unique morphism $t = \langle t_i\rangle_{i\in I} : T \to \&_{i\in I} S_i$ characterized by $\forall i\, \pi_i^{\mathsf{S}} \circ t = t_i$ is given by

$$t = \bigcup_{i\in I} \{(b,(i,a)) \mid (b,a) \in t_i)\}\,.$$

The sum $\oplus_{i\in I} S_i = (\&_{i\in I} {S_i}^{\mathrm{op}})^{\mathrm{op}}$ is the operation dual to this product, and coincides with it as easily checked.

If $S$ is a preorder and $I$ is a set, we use $S^I$ for the product $\&_{i\in I} S_i$ where $S_i = S$ for each $I$. We use $\top$ for the product of the empty family of preorders: it is the terminal object, and, as a preorder, it is empty (so $\mathcal{I}(\top) = \{\emptyset\}$). It is obviously isomorphic to its dual, denoted with 0.

## 3.3 Exponentials

Given a preorder $S$, we define the preorder $!S$, whose elements are the finite multisets of elements of $S$, with the following preorder relation: given $p, q \in !S$, one has $p \leq_{!S} q$ if $\forall a \in \mathsf{supp}(p)\ \exists b \in \mathsf{supp}(q)\ a \leq_S b$. Of course we could have taken $!S = \mathcal{P}_{\mathrm{fin}}(S)$, with a similarly defined preorder, and the associated lattices of initial segments would have been trivially isomorphic. We choose multisets because our goal is to compare this preorder model with the relational model, where the exponentials are defined with finite multisets. This choice makes the study of the collapse much simpler.

Given $x \subseteq S$, we set $x^! = \mathcal{M}_{\mathrm{fin}}(x)$.

**Lemma 13** *Let $x \subseteq S$. We have $(\downarrow_{|X|} x)^! = \downarrow_{|!S|} (x^!)$.*

We'll use this remark quite often, tacitly. It implies that, if $x \in \mathcal{I}(S)$, then $x^! \in \mathcal{I}(!S)$. Given $t : S \to T$, we set

$$!t = \{(p,q) \in !S \times !T \mid \forall b \in q \exists a \in p\ (a,b) \in t\}\,.$$

Then one shows easily that $!t : !S \to !T$, and that this operation on morphisms is functorial. Moreover, it is quite useful to observe that

$$\forall x \in \mathcal{I}(S) \quad !t \cdot x^! = (t \cdot x)^! \,.$$

And this property actually characterizes the morphism $!t$.

**3.3.1 Comonad structure of the exponential.** As it is usual in models of linear logic, this functor $!\_$ has a structure of comonad, which is given by the natural morphism

$$\mathsf{d}^{\mathsf{S}}_S = \{(p, b) \in !S \times S \mid \exists a \in p \; b \leq a\} : !S \to S$$

usually called *dereliction* and

$$\mathsf{p}^{\mathsf{S}}_S = \{(p, [p_1, \ldots, p_n]) \in !S \times !!S \mid p_1 + \cdots + p_n \leq_{!S} p\} : !S \to !!S$$

usually called *digging*. Observe that $\mathsf{d}^{\mathsf{S}}_S \cdot x^! = x$ and that $\mathsf{p}^{\mathsf{S}}_S \cdot x^! = (x^!)^!$, and that these equations characterize the morphisms $\mathsf{d}^{\mathsf{S}}_S$ and $\mathsf{p}^{\mathsf{S}}_S$. With these observations, it is trivial to check that these morphisms are natural (as announced) and provide the functor $!\_$ with a comonad structure.

**3.3.2 Weakening and contraction.** Given two preorders $S_1$ and $S_2$, there is a canonical and natural strong isomorphism between the preorders $!(S_1 \,\&\, S_2)$ and $!S_1 \otimes !S_2$, which is actually the preorder isomorphism

$$[(1, a_1), \ldots, (1, a_n), (2, b_1), \ldots, (2, b_m)] \mapsto ([a_1, \ldots, a_n], [b_1, \ldots, b_m]) \,.$$

Similarly, there is a trivial isomorphism between $!\top$ and $1$ (both are the one-point preorder). Using these isomorphisms, and applying the $!\_$ functor to the diagonal map $\delta_S : S \to S \,\&\, S$ (which, as easily checked, is the set $\{(a, (1, b)) \mid b \leq a\} \cup \{(a, (2, b)) \mid b \leq a\}$) and to the unique map $S \to \top$ (the empty map), we get the contraction and weakening maps:

$$\begin{array}{rcl} \mathsf{contr}^{\mathsf{S}}_S & = & \{(p, (q_1, q_2)) \mid q_1 + q_2 \leq_{!S} p)\} : !S \to !S \otimes !S \\ \mathsf{weak}^{\mathsf{S}}_S & = & \{(p, \star) \mid p \in !S\} : !S \to 1 \,. \end{array}$$

With all these structures, **ScottL** is a new-Seely category in the sense of [Bie95], see Section 1.4): this is the model discovered independently by Huth [Hut94] and Winskel [Win99].

## 3.4 The Kleisli category

We denote with $\mathbf{ScottL}_!$ the associated Kleisli category; remember that a morphism from $S$ to $T$ in this category is a linear morphism $t : !S \to T$:

$$\mathbf{ScottL}_!(S, T) = \mathbf{ScottL}(!S, T) \,.$$

Given such a morphism $t : !S \to T$, we can define a map

$$\begin{array}{rcl} \mathsf{Fun}^{\mathsf{S}}(t) : \mathcal{I}(S) & \to & \mathcal{I}(T) \\ x & \mapsto & t \cdot x^! \end{array}$$

In other words, $\mathsf{Fun}^{\mathsf{S}}(t)(x) = \{b \in T \mid \exists p \in !S \ \ \mathsf{supp}(p) \subseteq x \text{ and } (p,b) \in t\}$

Observe that the function $S \to !S$ which maps $x$ to $x^!$ is never linear (since it maps $\emptyset$ to $\{[]\}$; it is actually the "most non-linear" map from $S$ to $S\ldots$), but is Scott continuous. Therefore, the map $\mathsf{Fun}^{\mathsf{S}}(t)$ is Scott-continuous as well.

Conversely, observe that $\mathcal{I}(S)$ is a Scott domain, whose compact elements are the finitely generated elements of $\mathcal{I}(S)$, that is, the elements $x_0$ of $\mathcal{I}(S)$ such that $x_0 = \downarrow_S u$ for some finite $u \subseteq S$. Given a Scott-continuous function $f : \mathcal{I}(S) \to \mathcal{I}(T)$, one defines the set

$$\mathsf{Tr}^{\mathsf{S}}(f) = \{(p,b) \in \mathcal{M}_{\mathrm{fin}}(S) \times T \mid b \in f(\underset{S}{\downarrow}(\mathsf{supp}(p)))\}\,.$$

that we call the *trace* of $f$. This is similar to the definition of the trace of a stable function (see [Gir86, AC98]), with the essential difference that there is no minimality requirement on $p$ (such a requirement would not make sense in general because usually our preorders are not well-founded).

**Lemma 14** *Let $S$ and $T$ be preorders. The maps $\mathsf{Tr}^{\mathsf{S}}$ and $\mathsf{Fun}^{\mathsf{S}}$ define an order isomorphism between $\mathcal{I}(!S \multimap T)$ and the set of Scott-continuous functions from $\mathcal{I}(S)$ to $\mathcal{I}(T)$, endowed with the pointwise order.*

*Proof.* Let $f, g : \mathcal{I}(S) \to \mathcal{I}(T)$ be Scott-continuous functions such that $f \leq g$ for the pointwise order. Let $(p,b) \in \mathsf{Tr}^{\mathsf{S}}(f)$. Then $b \in f(\downarrow_S(\mathsf{supp}(p))) \subseteq g(\downarrow_S(\mathsf{supp}(p)))$, so $(p,b) \in \mathsf{Tr}^{\mathsf{S}}(g)$ and hence the map $\mathsf{Tr}^{\mathsf{S}}$ is monotone. Let $s, t \in \mathcal{I}(!S \multimap T)$ be such that $s \subseteq t$, let $x \in \mathcal{I}(S)$ and let $b \in \mathsf{Fun}^{\mathsf{S}}(s)(x)$. This means that there exists $p \in !S$ such that $(p,b) \in s$ and $\mathsf{supp}(p) \subseteq x$. Then $(p,b) \in t$ and hence we also have $b \in \mathsf{Fun}^{\mathsf{S}}(t)(x)$, and this shows that the map $\mathsf{Fun}^{\mathsf{S}}$ is monotone as well.

Let $f : \mathcal{I}(S) \to \mathcal{I}(T)$ be continuous, $f' = \mathsf{Fun}^{\mathsf{S}}(\mathsf{Tr}^{\mathsf{S}}(f))$ and let $x \in \mathcal{I}(S)$. Let $b \in f(x)$. Since $f$ is continuous, there is a finite subset $u$ of $x$ such that $b \in f(\downarrow_S(u))$. Let $p \in !S$ be such that $\mathsf{supp}(p) = u$. Then we have $(p,b) \in \mathsf{Tr}^{\mathsf{S}}(f)$ and hence $b \in f'(x)$. Conversely, if $b \in f'(x)$, let $p \in !S$ be such that $(p,b) \in \mathsf{Tr}^{\mathsf{S}}(f)$ and $\mathsf{supp}(p) \subseteq x$, then $b \in f(\downarrow_S(\mathsf{supp}(p))) \subseteq f(x)$ and we have shown that $f'(x) = f(x)$ for all $x \in \mathcal{I}(S)$, so $\mathsf{Fun}^{\mathsf{S}} \circ \mathsf{Tr}^{\mathsf{S}}$ is the identity map.

Conversely, let $t \in \mathcal{I}(!S \multimap T)$ and let $t' = \mathsf{Tr}^{\mathsf{S}}(\mathsf{Fun}^{\mathsf{S}}(t))$. Let $(p,b) \in t$, then $b \in \mathsf{Fun}(t)(\downarrow_S(\mathsf{supp}(p)))$, and hence $(p,b) \in t'$. Let $(p,b) \in t'$, then $b \in \mathsf{Fun}^{\mathsf{S}}(t)(\downarrow_S(\mathsf{supp}(p)))$ and hence there exists $q \in !S$ such that $(q,b) \in t$ and $\mathsf{supp}(q) \subseteq \downarrow_S(\mathsf{supp}(p))$, that is, $q \leq_{!S} p$. Since $(p,b) \leq_{!S \multimap T} (q,b) \in t$ and $t \in \mathcal{I}(!S \multimap T)$, we have $(p,b) \in t$, and this shows that $\mathsf{Tr}^{\mathsf{S}} \circ \mathsf{Fun}^{\mathsf{S}}$ is the identity map. □

**3.4.1 The Kleisli category of preorders.** This isomorphism is compatible with composition, as easily checked, so that we can consider $\mathbf{ScottL}_!$ as a full subcategory of the category of Scott domains and continuous functions. Moreover, it is easily checked that the cartesian products and function space constructions in both categories coincide: the cartesian product in $\mathbf{ScottL}_!$ of $S$ and $T$ is $S \,\&\, T$, and we have seen that $\mathcal{I}(S \,\&\, T) \simeq \mathcal{I}(S) \times \mathcal{I}(T)$ (with the product order) and their function space is $S \Rightarrow T = !S \multimap T$, and we have seen that $\mathcal{I}(!S \multimap T)$ is isomorphic (as a poset) to the space of continuous maps from $\mathcal{I}(S)$ to $\mathcal{I}(T)$, endowed with the pointwise order, which is precisely the function space of $\mathcal{I}(S)$

and $\mathcal{I}(T)$ in the category of Scott domains and continuous functions. The evaluation map $\mathsf{Ev}^{\mathsf{S}} \in \mathbf{ScottL}_!((S \Rightarrow T)\ \&\ S, T) \simeq \mathbf{ScottL}(!(S \Rightarrow T) \otimes !S, T)$ satisfies

$$\mathsf{Ev}^{\mathsf{S}} = \{((r,p),b) \mid \exists (p',b') \in r \quad b \leq_T b' \text{ and } p' \leq_{!S} p\}$$

as easily checked using that fact that $\mathsf{Ev}^{\mathsf{S}} =$

So $\mathbf{ScottL}_!$ is a full sub-CCC of the CCC of Scott domains and continuous functions.

### 3.5 The partially ordered class of preorders

We say that the preorder $S$ is a substructure of the preorder $T$, and we write $S \sqsubseteq T$ if, for any $a_1, a_2 \in S$, one has $a_1 \leq_S a_2 \Leftrightarrow a_1 \leq_T a_2$. We denote with $\mathbf{ScottC}$ the corresponding partially ordered class. It is easy to check that $\mathbf{ScottC}$ is complete (any directed family $(S_\gamma)_{\gamma\in\Gamma}$ has a lub $\bigsqcup_{\gamma\in\Gamma} S_\gamma$), and that all the constructions we have introduced on preorders are *variables preorders*, that is, continuous class functions $\mathbf{ScottC}^n \to \mathbf{ScottC}$. Any variable preorder $\Phi : \mathbf{ScottC} \to \mathbf{ScottC}$ admits a least fixpoint. In particular, the operation $\Phi_{\mathsf{s}} : \mathbf{ScottC} \to \mathbf{ScottC}$ defined by $\Phi_{\mathsf{s}}(S) = (!(S^{\mathbb{N}}))^{\perp}$ is a variable preorder and therefore admits a least fixpoint $\mathcal{D}_{\mathsf{s}}$, which is an extensional model of the pure lambda-calculus (same computation as in 2.3.3).

## 4 The category of preorders with projections

### 4.1 A duality on preorders

Now comes the most important definition of the paper. Let $S$ be a preorder. Given $x, x' \subseteq S$, we shall say that $x$ and $x'$ are *in duality* (with respect to $S$) and write $x \perp_S x'$ if

$$x \cap x' = \emptyset \Rightarrow (\underset{S}{\downarrow} x) \cap x' = \emptyset\,.$$

Of course, the converse implication always holds so that, when it holds, the implication above is actually an equivalence. The intuition is clear: $x$ and $x'$ are in duality if $x'$ cannot separate $x$ from its downward closure.

This duality relation is symmetric in the following sense: since clearly $(\downarrow_S x) \cap x' = \emptyset \Leftrightarrow x \cap (\uparrow_S x') = \emptyset \Leftrightarrow (\downarrow_S x) \cap (\uparrow_S x') = \emptyset$, we have

$$\forall x, x' \subseteq S \quad x \perp_S x' \Leftrightarrow x' \perp_{S^{\mathrm{op}}} x\,.$$

If $D \subseteq \mathcal{P}(S)$, we set

$$D^{\perp s} = \{x' \subseteq S \mid \forall x \in D \quad x \perp_S x'\}$$

With this definition, we have $D \subseteq D^{\perp s \perp s^{\mathrm{op}}}$. Indeed, let $x \in D$ and let $x' \in D^{\perp s}$. We have $x \perp_S x'$, that is $x' \perp_{S^{\mathrm{op}}} x$, and since this holds for all $x' \in D^{\perp s}$, we have $x \in D^{\perp s \perp s^{\mathrm{op}}}$. Moreover, if $D, E \subseteq \mathcal{P}(S)$, we have $D \subseteq E \Rightarrow E^{\perp s} \subseteq D^{\perp s}$. Therefore, one always has $D^{\perp s \perp s^{\mathrm{op}} \perp s} = D^{\perp s}$.

Let $D \subseteq \mathcal{P}(S)$ be such that $D = D^{\perp s \perp s^{\mathrm{op}}}$ (equivalently, $D = E^{\perp s^{\mathrm{op}}}$ for some $E \subseteq \mathcal{P}(S)$). Then $\mathcal{I}(S) \subseteq D \subseteq \mathcal{P}(S)$. And one checks easily that $\mathcal{P}(S)^{\perp s} = \mathcal{I}(S^{\mathrm{op}})$ and $\mathcal{I}(S)^{\perp s} = \mathcal{P}(S^{\mathrm{op}})$. Let $(x_i)_{i\in I}$ be a family of elements of $D$. Then $\bigcup_{i\in I} x_i \in D$. Indeed, since $D = D^{\perp s \perp s^{\mathrm{op}}}$, it suffices to show

that $\left(\bigcup_{i\in I} x_i\right) \perp_S x'$ for all $x' \in D^{\perp_S}$. So let $x' \in D^{\perp_S}$, and let us assume that $\left(\bigcup_{i\in I} x_i\right) \cap x' = \emptyset$. Then, for any $i \in I$, we have $x_i \cap x' = \emptyset$ and hence $\downarrow_S x_i \cap x' = \emptyset$ (since $x_i \in \mathsf{D}(X)$) and therefore $\left(\bigcup_{i\in I} \downarrow_S x_i\right) \cap x' = \emptyset$. We conclude because clearly $\left(\bigcup_{i\in I} \downarrow_S x_i\right) = \downarrow_S \left(\bigcup_{i\in I} x_i\right)$. So $D$, endowed with inclusion, is a complete lattice, whose least element is $\emptyset$, and largest element is $S$.

A *preorder with projection* (a PP for short; the reason for this terminology will appear later) is a pair $X = (|X|, \mathsf{D}(X))$ where $|X|$ is a preorder and $\mathsf{D}(X) \subseteq \mathcal{P}(|X|)$ satisfies $\mathsf{D}(X) = \mathsf{D}(X)^{\perp_{|X|}\perp_{|X|^{\mathrm{op}}}}$. We define then

$$X^\perp = (|X|^{\mathrm{op}}, \mathsf{D}(X)^{\perp_{|X|}}) .$$

By definition, we have $X^{\perp\perp} = X$. Remember that $\mathcal{I}(|X|) \subseteq \mathsf{D}(X) \subseteq \mathcal{P}(|X|)$.

Given two PPs $X$ and $Y$, we define $X \otimes Y$ by setting $|X \otimes Y| = |X| \times |Y|$, endowed with the product order. Then $\mathsf{D}(X \otimes Y)$ is given by

$$\mathsf{D}(X \otimes Y) = \{x \times y \mid x \in \mathsf{D}(X) \text{ and } y \in \mathsf{D}(Y)\}^{\perp_{|X|\times|Y|}\perp_{|X|^{\mathrm{op}}\times|Y|^{\mathrm{op}}}}$$

We define accordingly $X \multimap Y = (X \otimes Y^{\perp_{|Y|}})^{\perp_{|X|\times|Y|^{\mathrm{op}}}}$, so that $|X \multimap Y| = |X|^{\mathrm{op}} \times |Y|$ and, for $t \subseteq |X \multimap Y|$, one has $t \in \mathsf{D}(X \multimap Y)$ iff, for all $x \in \mathsf{D}(X)$ and for all $y' \in \mathsf{D}(Y^\perp)$, one has

$$t \cap (x \times y') = \emptyset \Rightarrow t \cap (\underset{|X|}{\downarrow} x \times \underset{|Y|}{\uparrow} y') = \emptyset .$$

Given $t \subseteq |X| \times |Y|$, remember that the transpose of $t$ is ${}^{\mathrm{t}}t = \{(b, a) \mid (a, b) \in t\} \subseteq |Y| \times |X|$. One checks easily that $t \in \mathsf{D}(X \multimap Y)$ iff ${}^{\mathrm{t}}t \in \mathsf{D}(Y^\perp \multimap X^\perp)$.

Fortunately, there is an easy functional characterization of the elements of $\mathsf{D}(X \multimap Y)$.

**Proposition 15** *Let $X$ and $Y$ be PPs. Let $t \subseteq |X| \times |Y|$. One has $t \in \mathsf{D}(X \multimap Y)$ iff the two following conditions are satisfied.*

- *For all $x \in \mathsf{D}(X)$, one has $t \cdot x \in \mathsf{D}(Y)$*
- *and, for all $x \in \mathsf{D}(X)$, one has $\downarrow_{|Y|} (t \cdot x) = \downarrow_{|X\multimap Y|} t \cdot \downarrow_{|X|} x$ .*

*The second condition is equivalent to $\forall x \in \mathsf{D}(X)\ \downarrow_{|X\multimap Y|} t \cdot \downarrow_{|X|} x \subseteq \downarrow_{|Y|} (t \cdot x)$, which in turn is equivalent to $\forall x \in \mathsf{D}(X)\ \downarrow_{|Y|} (t \cdot \downarrow_{|X|} x) \subseteq \downarrow_{|Y|} (t \cdot x)$, that is to $\forall x \in \mathsf{D}(X)\ t \cdot \downarrow_{|X|} x \subseteq \downarrow_{|Y|} (t \cdot x)$.*

*Proof.* The equivalences at the end of the statement result from Lemma 12.

Assume first that $t \in \mathsf{D}(X \multimap Y)$. Let $x \in \mathsf{D}(X)$. We show first that $t \cdot x \in \mathsf{D}(Y) = \mathsf{D}(Y^\perp)^{\perp_{|Y|^{\mathrm{op}}}}$, so let $y' \in \mathsf{D}(Y^\perp)$ and let us assume that $(t \cdot x) \cap y' = \emptyset$. This is equivalent to $t \cap (x \times y') = \emptyset$, and since $t \in \mathsf{D}(X \multimap Y)$, we have $t \cap \uparrow_{X\multimap Y} (x \times y') = \emptyset$, that is $t \cap (\downarrow_{|X|} x \times \uparrow_{|Y|} y') = \emptyset$. But this implies $t \cap (x \times \uparrow_{|Y|} y') = \emptyset$, that is, $(t \cdot x) \cap \uparrow_{|Y|} y' = \emptyset$. Since this holds for all $y' \in \mathsf{D}(Y^\perp)$, we have shown that $t \cdot x \in \mathsf{D}(Y)$.

We must show now that $\downarrow_{|X\multimap Y|} t \cdot \downarrow_{|X|} x \subseteq \downarrow_{|Y|} (t \cdot x)$. So let $b \in \downarrow_{|X\multimap Y|} t \cdot \downarrow_{|X|} x$, we have $\uparrow_{|Y|} b \in \mathsf{D}(Y^\perp)$ and $\downarrow_{|X\multimap Y|} t \cap (\downarrow_{|X|} x \times \uparrow_{|Y|} b) \neq \emptyset$, that is $\downarrow_{X\multimap Y} t \cap \uparrow_{X\multimap Y} (x \times \{b\}) \neq \emptyset$. Since $t \in \mathsf{D}(X \multimap Y)$, this shows that $t \cap (x \times \uparrow_{|Y|} b) \neq \emptyset$, that is $(t \cdot x) \cap \uparrow_{|Y|} b \neq \emptyset$, that is $b \in \downarrow_{|Y|} (t \cdot x)$ as required.

Assume conversely that the two conditions of the statement are satisfied, and let us show that $t \in \mathsf{D}(X \multimap Y)$. So let $x \in \mathsf{D}(X)$ and $y' \in \mathsf{D}(Y^\perp)$, and assume that $t \cap \uparrow_{X \multimap Y} (x \times y') \neq \emptyset$. Equivalently, we have $t \cap (\downarrow_{|X|} x \times \uparrow_{|Y|} y') \neq \emptyset$, that is $(t \cdot \downarrow_{|X|} x) \cap \uparrow_{|Y|} y' \neq \emptyset$. By our second assumption, we have therefore $\downarrow_{|Y|} (t \cdot x) \cap \uparrow_{|Y|} y' \neq \emptyset$, and hence $t \cap (x \times y') \neq \emptyset$ since $t \cdot x \in \mathsf{D}(Y)$ and $y' \in \mathsf{D}(Y^\perp)$. □

## 4.2 The linear category

Let **PpL** be the category whose objects are the PPs, and with $\mathbf{PpL}(X, Y) = \mathsf{D}(X \multimap Y)$, composition defined as the usual relational composition.

**4.2.1 Identity and composition.** Indeed, by Proposition 15, the identity relation $\mathsf{Id} \subseteq |X| \times |X|$ belongs to $\mathsf{D}(X \multimap X)$.

As to composition, let $s \in \mathsf{D}(X \multimap Y)$ and $t \in \mathsf{D}(Y \multimap Z)$, then we show that the relational composition $u = t \cdot s$ of these morphisms belongs to $\mathsf{D}(Y \multimap Z)$, using Proposition 15. So let $x \in \mathsf{D}(X)$. First, we have $u \cdot x = t \cdot (s \cdot x) \in \mathsf{D}(Z)$ since $s \cdot x \in \mathsf{D}(Y)$. Next $\downarrow_{|Z|} (u \cdot x) = \downarrow_{|Z|} (t \cdot (s \cdot x)) = (\downarrow_{Y \multimap Z} t) \cdot \downarrow_Y (s \cdot x)$ (by Proposition 15 and the fact that $s \cdot x \in \mathsf{D}(Y)$). Hence we have $\downarrow_{|Z|} (u \cdot x) = ((\downarrow_{Y \multimap Z} t) \circ (\downarrow_{X \multimap Y} s)) \cdot \downarrow_{|X|} x$. To conclude, it suffices to check that $\downarrow_{X \multimap Z} u = (\downarrow_{Y \multimap Z} t) \circ (\downarrow_{X \multimap Y} s)$. The "$\subseteq$" inclusion is straightforward, we check the converse. Let $(a, c) \in (\downarrow_{Y \multimap Z} t) \circ (\downarrow_{X \multimap Y} s)$. Let $b \in |Y|$ be such that $(b, c) \in \downarrow_{Y \multimap Z} t$ and $(a, b) \in \downarrow_{X \multimap Y} s$. Let $(a', b') \in s$ be such that $a' \leq_{|X|} a$ and $b' \geq_{|Y|} b$, and let $(b'', c') \in t$ be such that $b'' \leq_{|Y|} b$ and $a' \geq_{|Z|} a$. We have $b'' \leq b'$ and hence (e.g.) $(b', c) \leq_{|Y \multimap Z|} (b'', c') \in t$ and $(a, b') \leq_{|X \multimap Y|} (a', b') \in s$ and we conclude.

**4.2.2 Tensor product.**

**Lemma 16** *Let $X_1$, $X_2$ and $Y$ be PPs. Let $t \subseteq |X_1 \otimes X_2 \multimap Y|$. One has $t \in \mathbf{PpL}(X_1 \otimes X_2, Y)$ iff, for all $x_1 \in \mathsf{D}(X_1)$ and $x_2 \in \mathsf{D}(X_2)$, one has*

- $t \cdot (x_1 \otimes x_2) \in \mathsf{D}(Y)$
- *and* $\downarrow_{|Y|} (t \cdot (x_1 \otimes x_2)) = (\downarrow_{|X_1 \otimes X_2 \multimap Y|} t) \cdot (\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2)$.

*The second condition is equivalent to $t \cdot (\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2) \subseteq \downarrow_{|Y|} (t \cdot (x_1 \otimes x_2))$.*

*Proof.* The conditions are necessary by Proposition 15. We prove that they are sufficient, so assume that they hold. We prove that ${}^\mathsf{t}t \in \mathsf{D}(Y^\perp \multimap (X_1 \otimes X_2)^\perp)$, using Proposition 15, so let $y' \in \mathsf{D}(Y^\perp)$.

We show first that ${}^\mathsf{t}t \cdot y' \in \mathsf{D}((X_1 \otimes X_2)^\perp)$. So let $x_1 \in \mathsf{D}(X_1)$ and $x_2 \in \mathsf{D}(X_2)$ and assume that $({}^\mathsf{t}t \cdot y') \cap (x_1 \otimes x_2) = \emptyset$, hence $(t \cdot (x_1 \otimes x_2)) \cap y' = \emptyset$. But we have $t \cdot (x_1 \otimes x_2) \in \mathsf{D}(Y)$, and hence $(t \cdot (x_1 \otimes x_2)) \cap \uparrow_{|Y|} y' = \emptyset$, and hence, by our second hypothesis, $(\downarrow_{|X_1 \otimes X_2 \multimap Y|} t) \cdot (\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2) \cap \uparrow_{|Y|} y' = \emptyset$. Therefore ${}^\mathsf{t}(\downarrow_{|X_1 \otimes X_2 \multimap Y|} t) \cdot \uparrow_{|Y|} y' \cap (\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2) = \emptyset$, which clearly implies that ${}^\mathsf{t}t \cdot y' \cap (\downarrow_{|X_1|} x_1 \otimes \downarrow_{|X_2|} x_2) = \emptyset$, and this shows that ${}^\mathsf{t}t \cdot y' \in \mathsf{D}((X_1 \otimes X_2)^\perp)$.

Next, we must show that $\uparrow_{|X_1 \otimes X_2|} ({}^\mathsf{t}t \cdot y') = \downarrow_{|Y^\perp \multimap (X_1 \otimes X_2)^\perp|} {}^\mathsf{t}t \cdot (\uparrow_Y y')$, and the only non-trivial inclusion is "$\supseteq$", so let $(a_1, a_2) \in \downarrow_{|Y^\perp \multimap (X_1 \otimes X_2)^\perp|} {}^\mathsf{t}t \cdot (\uparrow_Y y')$. This means that $\downarrow_{X_1 \otimes X_2} \{(a_1, a_2)\} \cap \downarrow_{|Y^\perp \multimap (X_1 \otimes X_2)^\perp|} {}^\mathsf{t}t \cdot (\uparrow_Y y') \neq \emptyset$, that is

$\downarrow_{|(X_1 \otimes X_2) \multimap Y|} t \cdot \downarrow_{X_1 \otimes X_2} \{(a_1, a_2)\} \cap \uparrow_{|Y|} y' \neq \emptyset$, that is, by our second assumption, we have $\downarrow_Y (t \cdot \{(a_1, a_2)\}) \cap \uparrow_Y y' \neq \emptyset$. □

Let $t_i \in \mathbf{PpL}(X_i, Y_i)$ for $i = 1, 2$. Let $t_1 \otimes t_2 \subseteq |(X_1 \otimes X_2) \multimap (Y_1 \otimes Y_2)|$ be defined as usual as $t_1 \otimes t_2 = \{((a_1, a_2), (b_1, b_2)) \mid (a_i, b_i) \in t_i \text{ for } i = 1, 2\}$. Then we show that $t_1 \otimes t_2 \in \mathbf{PpL}(X_1 \otimes X_2, Y_1 \otimes Y_2)$ using Lemma 16. So let $x_i \in \mathsf{D}(X_i)$ for $i = 1, 2$. We have $(t_1 \otimes t_2) \cdot (x_1 \otimes x_2) = (t_1 \cdot x_1) \otimes (t_2 \cdot x_2) \in \mathsf{D}(Y_1 \otimes Y_2)$ since we have $t_i \cdot x_i \in \mathsf{D}(Y_i)$ for $i = 1, 2$. Moreover, we have

$$\begin{aligned} t_1 \otimes t_2 \cdot (\underset{|X_1|}{\downarrow} x_1 \otimes \underset{|X_2|}{\downarrow} x_2) &= (t_1 \cdot (\underset{|X_1|}{\downarrow} x_1)) \otimes (t_2 \cdot (\underset{|X_2|}{\downarrow} x_2)) \\ &\subseteq \underset{|Y_1|}{\downarrow} (t_1 \cdot x_1) \otimes \underset{|Y_2|}{\downarrow} (t_2 \cdot x_2) \\ &= \underset{|Y_1 \otimes Y_2|}{\downarrow} ((t_1 \otimes t_2) \cdot (x_1 \otimes x_2)) \end{aligned}$$

applying Proposition 15 to $t_1$ and $t_2$.

**4.2.3 Strong isomorphisms.** Let $X$ and $Y$ be PPs. A *strong isomorphism* from $X$ to $Y$ is a preorder isomorphism $\theta : |X| \to |Y|$ such that, for any $x \subseteq |X|$, one has $x \in \mathsf{D}(X)$ iff $\theta(x) \in \mathsf{D}(Y)$. A strong isomorphism from $X$ to $Y$ is an isomorphism (in the categorical sense), as easily seen using Lemma 16.

**4.2.4 Associativity and symmetry isomorphisms.** The obvious bijection $\alpha : |(X_1 \otimes X_2) \otimes X_3| \to |X_1 \otimes (X_2 \otimes X_3)|$. Then $\alpha$ is a preorder isomorphism which is also a PP strong isomorphism. Similarly, the bijection $\sigma : |X_1 \otimes X_2| \to |X_2 \otimes X_1|$ is a strong isomorphism. This shows that the category **PpL**, equipped with the above defined tensor product, is a monoidal category (of course, the unit of this tensor product is the PP $1 = (\{*\}, \{\emptyset, \{*\}\})$.

**4.2.5 Linear function space and monoidal closeness.** We have already defined $X \multimap Y = (X \otimes Y^\perp)^\perp$. We show that this object is the linear function space from $X$ to $Y$.

**Lemma 17** *The obvious bijection $\lambda : |(Z \otimes X) \multimap Y| \to |X \multimap (Y \multimap Z)|$ is a strong isomorphisms from $(Z \otimes X) \multimap Y$ to $X \multimap (Y \multimap Z)$.*

*Proof.* We already know that $\lambda$ is a preorder isomorphism.

Let $t \in \mathsf{D}((Z \otimes X) \multimap Y)$ and let us prove that $t' = \lambda(t) \in \mathsf{D}(Z \multimap (X \multimap Y))$, using Lemma 15. So let $z \in \mathsf{D}(Z)$, we show first that $t' \cdot z \in \mathsf{D}(X \multimap Y)$. Let $x \in \mathsf{D}(X)$, we have $(t' \cdot z) \cdot x = t \cdot (z \otimes x) \in \mathsf{D}(Y)$. Next, we have $(t' \cdot z) \cdot \downarrow_{|X|} x = t \cdot (z \otimes \downarrow_{|X|} x) \subseteq t \cdot (\downarrow_{|Z|} z \otimes \downarrow_{|X|} x) \subseteq \downarrow_{|Y|} (t \cdot (z \otimes x)) = \downarrow_{|Y|} ((t' \cdot z) \cdot x)$ by Lemma 15 applied to $t$, and hence, by the same lemma applied to $t' \cdot z$, we have $t' \cdot z \in \mathsf{D}(X \multimap Y)$. We must show now that $t' \cdot \downarrow_{|Z|} z \subseteq \downarrow_{|X \multimap Y|} (t' \cdot z)$, so let $(a, b) \in t' \cdot \downarrow_{|Z|} z$. We have $b \in (t' \cdot \downarrow_{|Z|} z) \cdot \downarrow_{|X|} a = t \cdot (\downarrow_{|Z|} z \otimes \downarrow_{|X|} a) \subseteq \downarrow_Y (t \cdot (z \otimes \downarrow_{|X|} a))$ so we can find $b' \in |Y|$ with $b' \geq b$, $c \in z$ and $a' \leq a$ such that $((c, a'), b') \in t$, that is $(c, (a', b')) \in t'$. Hence $(a', b') \in t' \cdot z$, and therefore $(a, b) \in \downarrow_{|X \multimap Y|} (t' \cdot z)$ as required. □

Since we have taken $\mathbf{PpL}(X, Y) = \mathsf{D}(X \multimap Y)$ it results easily from that lemma that the monoidal category **PpL** is monoidal closed, with $X \multimap Y$ as function space.

The category **PpL** is clearly star-autonomous (with $\bot = 1^\bot = 1$ as dualizing object), since $X \multimap \bot = (X \otimes 1)^\bot$ and this latter PP is isomorphic to $X^\bot$ by the strong PP isomorphism which maps $a \in |X|$ to $(a, *)$ (one should check that the induced isomorphism $X \to (X \multimap \bot) \multimap \bot$, which maps $a$ to $((a, *), *)$ is the canonical morphism between these two spaces as explained in [Bar79], but this is quite easy).

**4.2.6 The "par" connective.** The co-tensor product, or *par*, is defined as $X \parr Y = (X^\bot \otimes Y^\bot)^\bot = X^\bot \multimap Y$ and has the same associativity and symmetry properties as the tensor product. Also, there is a *mix* morphism $\mathsf{mix} : X \otimes Y \to X \parr Y$, which is the diagonal set $\mathsf{mix} = \{((a,b),(a,b)) \mid a \in |X| \text{ and } b \in |Y|\}$. As it is well known, the fact that this relation is a morphism results from the fact that $1 = 1^\bot = \bot$. A natural question is whether this morphism is an isomorphism, as in both categories **ScottL** and **RelL** (these categories are *compact closed*), and we shall provide a counter-example showing that this is not the case in general.

**4.2.7 The morphism mix is not an isomorphism in general.** Let $X$ be the PP defined by $|X| = \mathbb{N}$ (the natural numbers, with the usual order) and $\mathsf{D}(X) = \mathcal{P}(\mathbb{N})$, and let $Y = X^\bot$. We check first that the "successor" relation $s = \{(n, n+1) \mid n \in \mathbb{N}\}$ belongs to $\mathsf{D}(Y \parr X) = \mathsf{D}(X \multimap X)$. Let $x \in \mathsf{D}(X) = \mathcal{P}(\mathbb{N})$. Obviously $s \cdot x \in \mathsf{D}(X)$, and, if $b \in s \cdot {\downarrow_X} x$, then we have $b > 0$ and $b - 1 \in {\downarrow_X} x$. Let $c \in x$ such that $c \geq b - 1$. We have $c + 1 \in s \cdot x$ and hence $b \in {\downarrow_X}(s \cdot x)$.

On the other hand, we have $\mathsf{Id} \in \mathsf{D}(Y \multimap Y) = \mathsf{D}((Y \otimes X)^\bot)$ and, since $|Y|$ is $\mathbb{N}$ with the opposite order, we have $s \cap {\downarrow_{|Y \multimap Y|}} \mathsf{Id} \neq \emptyset$ (indeed $s \subseteq {\downarrow_{|Y \multimap Y|}} \mathsf{Id}$). But $s \cap \mathsf{Id} = \emptyset$, therefore $s = \mathsf{mix}^{-1} \cdot s \notin \mathsf{D}(Y \otimes X)$, which shows that $\mathsf{mix}^{-1} \notin \mathbf{PpL}(Y \parr X, Y \otimes X)$.

This strongly suggests that **PpL** is not compact closed.

## 4.3 The additives

Given a family $(X_i)_{i \in I}$ of PPs, we define their cartesian product $X = \&_{i \in I} X_i$ by setting $|X| = \bigcup_{i \in I} \{i\} \times |X_i|$ and saying that a set $x \subseteq |X|$ belongs to $\mathsf{D}(X)$ if, for all $i \in I$, one has $\pi_i \cdot x \in \mathsf{D}(X_i)$ (where $\pi_i \subseteq |X \multimap X_i|$ is $\pi_i = \{((i,a),a) \mid a \in |X_i|\}$, so that $\pi_i \cdot x = \{a \in |X_i| \mid (i,a) \in x\}$; we shall use the notation $x_i$ for $\pi_i \cdot x$ in the sequel).

One must check that $\mathsf{D}(X) = \mathsf{D}(X)^{\bot_{|X|}\bot_{|X|^{\mathrm{op}}}}$. For this it will suffice to show that, for all $x' \subseteq |X|$, one has $x' \in \mathsf{D}(X)^{\bot_{|X|}}$ iff $x'_i \in \mathsf{D}(X_i)^{\bot_{|X_i|}}$ for all $i \in I$; this will show that $X$ defined above is a PP, with $X^\bot = \&_{i \in I} X_i^\bot$. Assume first that $x'_i \in \mathsf{D}(X_i)^{\bot_{|X_i|}}$ for all $i \in I$ and assume that ${\downarrow_{|X|}} x \cap x' \neq \emptyset$ for some $x \in \mathsf{D}(X)$. There exists $i \in I$ such that ${\downarrow_{|X_i|}} x_i \cap x'_i \neq \emptyset$, and therefore $x_i \cap x'_i \neq \emptyset$, and hence $x \cap x' \neq \emptyset$. Conversely, assume that $x' \in \mathsf{D}(X)^{\bot_{|X|}}$ and let $i \in I$, we must show that $x'_i \in \mathsf{D}(X_i)^{\bot_{|X_i|}}$. So let $y \in \mathsf{D}(X_i)$ and assume that ${\downarrow_{|X_i|}} y \cap x'_i \neq \emptyset$. Let $x = \{i\} \times y \subseteq |X|$, we have $x \in \mathsf{D}(X)$ (remember the definition of $\mathsf{D}(X)$ and the fact that $\emptyset \in \mathsf{D}(Y)$ for any PP $Y$) and ${\downarrow_{|X|}} x \cap x' \neq \emptyset$. Therefore we have $x \cap x' \neq \emptyset$, that is $y \cap x'_i \neq \emptyset$.

It is straightforward to check that $\&_{i \in I} X_i$ is the cartesian product of the $X_i$s, with the relations $\pi_i$ as projections.

## 4.4 The exponentials

Let $X$ be a PP. We define $!X$ by setting $|!X| = !|X|$; remember that this means that $|!X|$ is the set of all finite multisets of elements of $|X|$, with the preorder defined as follows: $p \leq q$ iff $\forall a \in p \, \exists b \in q \; a \leq_{|X|} b$. Given $x \subseteq |X|$, we set $x^! = \mathcal{M}_{\mathrm{fin}}(x)$, and remember that we have the following property:

$$\underset{|!X|}{\downarrow}(x^!) = (\underset{|X|}{\downarrow} x)^! \,. \tag{1}$$

We set

$$\mathsf{D}(!X) = \{x^! \mid x \in \mathsf{D}(X)\}^{\perp_{|!X|}\perp_{|!X|^{\mathrm{op}}}} \,.$$

**Lemma 18** *Let $X$ and $Y$ be PPs and let $t \subseteq |!X \multimap Y|$. We have $t \in \mathsf{D}(!X \multimap Y)$ iff, for all $x \in \mathsf{D}(X)$,*

- $t \cdot x^! \in \mathsf{D}(Y)$
- *and* $\downarrow_{|Y|}(t \cdot x^!) = (\downarrow_{|!X \multimap Y|} t) \cdot (\downarrow_{|X|} x)^!$

*and the second condition is equivalent to $t \cdot (\downarrow_{|X|} x)^! \subseteq \downarrow_{|Y|}(t \cdot x^!)$.*

The proof is similar to that of Lemma 16.

Let $t \in \mathbf{PpL}(X, Y)$, we define $!t \subseteq |!X \multimap !Y|$ by

$$!t = \{([a_1, \ldots, a_n], [b_1, \ldots, b_n]) \mid (a_i, b_i) \in t \text{ for all } i = 1, \ldots, n\} \,.$$

Using Lemma 18, we prove that $!t \in \mathbf{PpL}(!X, !Y)$. So let $x \in \mathsf{D}(X)$. We have $!t \cdot x^! = (t \cdot x)^! \in \mathsf{D}(!Y)$ since $t \cdot x \in \mathsf{D}(Y)$. Next we have $!t \cdot (\downarrow_{|X|} x)^! = (t \cdot \downarrow_{|X|} x)^! \subseteq (\downarrow_{|Y|}(t \cdot x))^!$ by Proposition 15 applied to $t$, and we conclude because $(\downarrow_{|Y|}(t \cdot x))^! = \downarrow_{|!Y|}(t \cdot x)^! = \downarrow_{|!Y|}(!t \cdot x^!)$, using Equation (1).

We check that the usual comonad structure of the exponential in the relational model gives rise to a comonad structure for the $!\_$ functor we have just defined.

We define first $\mathsf{d}_X$ as $\mathsf{d}_X = \mathsf{d}_{|X|} = \{([a], a) \mid a \in |X|\} \subseteq |!X \multimap X|$. Given $x \in \mathsf{D}(X)$, we have $\mathsf{d}_X \cdot x^! = x$ and $\mathsf{d}_X \cdot (\downarrow_{|X|} x)^! = \downarrow_{|X|} x = \downarrow_{|X|}(\mathsf{d}_X \cdot x^!)$ and so $\mathsf{d}_X \in \mathbf{PpL}(!X, X)$ by Lemma 18. Similarly, we define $\mathsf{p}_X$ as $\mathsf{p}_X = \mathsf{p}_{|X|} = \{(m_1 + \cdots + m_n, [m_1, \ldots, m_n]) \mid m_1, \ldots, m_n \in |!X|\} \subseteq |!X \multimap !!X|$ and we show that $\mathsf{p}_X \in \mathsf{D}(!X \multimap !!X)$, using Lemma 18 again. So let $x \in \mathsf{D}(X)$, we have $\mathsf{p}_X \cdot x^! = x^{!!} \in \mathsf{D}(!!X)$, since $x^! \in \mathsf{D}(!X)$. Next we have $\mathsf{p}_X \cdot (\downarrow_{|X|} x)^! = (\downarrow_{|X|} x)^{!!} = \downarrow_{|!!X|}(x^{!!}) = \downarrow_{|!!X|}(\mathsf{p}_X \cdot x^!)$ and this completes the proof that $\mathsf{p}_X$ is a morphism.

**4.4.1 Fundamental isomorphism.** We show that the PPs $!(X \,\&\, Y)$ and $!X \otimes !Y$ are isomorphic, by the bijection $\theta : |!(X \,\&\, Y)| \to |!X \otimes !Y|$ which maps the multiset $[(1, a_1), \ldots, (1, a_l), (2, b_1), \ldots, (2, b_r)]$ (with $a_i \in |X|$ and $b_j \in |Y|$) to $([a_1, \ldots, a_l], [b_1, \ldots, b_r])$.

We show that $\theta$ is a morphism from $!(X \,\&\, Y)$ to $!X \otimes !Y$. So let $x \in \mathsf{D}(X)$ and $y \in \mathsf{D}(Y)$. We have $\theta \cdot \langle x, y \rangle^! = x^! \otimes y^! \in \mathsf{D}(!X \otimes !Y)$ which shows by Lemma 18 that $\theta$ is a morphism, since it is a preorder isomorphism (so that the second condition of the lemma is trivially satisfied). Conversely, let $\rho = \theta^{-1}$ and let $\rho' \subseteq |!X| \times |(!Y \multimap !(X \,\&\, Y))|$ be given by

$$\rho' = \{(p, (q, m)) \mid m = \theta(p, q)\} \,.$$

By monoidal closeness, it suffices to prove that $\rho'$ is a morphism from $!X$ to $!Y \multimap !(X \,\&\, Y)$, and for this, we apply twice Lemma 18 as follows. First, let $x \in \mathsf{D}(X)$, we must show that $\rho' \cdot x^! \in \mathsf{D}(!Y \multimap !(X \,\&\, Y))$. For this, let $y \in \mathsf{D}(Y)$, we have $(\rho' \cdot x^!) \cdot y^! = \langle x, y\rangle^! \in \mathsf{D}(!(X \,\&\, Y))$. Next, we have $(\rho' \cdot x^!) \cdot (\downarrow_{|Y|} y)^! = \langle x, \downarrow_{|Y|} y\rangle^!$ on the one hand and $\downarrow_{|!(X\&Y)|} ((\rho' \cdot x^!) \cdot y^!) = \downarrow_{|!(X\&Y)|} \langle x, y\rangle^! = (\downarrow_{|X\&Y|} \langle x, y\rangle)^!$ on the other hand, from which it clearly results that $(\rho' \cdot x^!) \cdot (\downarrow_{|Y|} y)^! \subseteq \downarrow_{|!(X\&Y)|} ((\rho' \cdot x^!) \cdot y^!)$ and therefore $\rho' \cdot x^! \in \mathsf{D}(!Y \multimap !(X \,\&\, Y))$ by Lemma 18. To conclude, we must show that $\rho' \cdot (\downarrow_{|X|} x)^! \subseteq \downarrow_{|!Y\multimap!(X\&Y)|} (\rho' \cdot x^!)$, so let $q \in |!Y|$ and $m \in |!(X \,\&\, Y)|$ and assume that $(q, m) \in \rho' \cdot (\downarrow_{|X|} x)^!$. There exists $p \in |!X|$ such $p \in (\downarrow_{|X|} x)^!$ and $m = \theta(p, q)$. Since $p \in (\downarrow_{|X|} x)^!$, we can find $p' \in x^!$ such that $p \leq_{|!X|} p'$. Let $m' = \theta(p', q)$, we have $(q, m') \in \rho' \cdot x^!$ and hence $(q, m) \in \downarrow_{|!Y\multimap!(X\&Y)|} (\rho' \cdot x^!)$ since $m \leq_{|(X\&Y)^!|} m'$.

Observe that there is also an obvious isomorphism from $!\top$ to 1 (the "0-ary version" of the isomorphism above).

**4.4.2 Structural maps.** Using these fundamental isomorphisms, it is easy to define the weakening and contraction maps, which endow $!X$ with a coalgebra structure: it suffices to apply the functor $!\_$ to the "terminal map" in $\mathbf{PpL}(X, \top)$ and to the diagonal map in $\mathbf{PpL}(X, X \,\&\, X)$ and then to compose the resulting map with the suitable fundamental isomorphism. In that way, we get $\mathsf{weak}_X \in \mathbf{PpL}(!X, 1)$, given by $\mathsf{weak}_X = \{([], *)\}$ and $\mathsf{contr}_X \in \mathbf{PpL}(!X \otimes !X, !X)$ given by $\mathsf{contr}_X = \{(p, q, p + q) \mid p, q \in |!X|\}$, which satisfy all the diagrams required (see [Bie95]).

**4.4.3 Cartesian closeness.** Equipped with this structure (the comonad $(!\_, \mathsf{d}, \mathsf{p})$, the fundamental isomorphisms), the cartesian star-autonomous category **PpL** is a model of linear logic in the sense of Section 1.4. It gives rise therefore to a cartesian closed category, which is the Kleisli category $\mathbf{PpL}_!$ of that comonad. The cartesian product of $(X_i)_{i\in I}$ in $\mathbf{PpL}_!$ is $X = \&_{i\in I} X_i$ with projections $\pi_i \circ \mathsf{d}_X$ (simply denoted as $\pi_i$). The object of morphisms from $X$ to $Y$ is $X \Rightarrow Y = !X \multimap Y$ with evaluation morphism $\mathsf{Ev}$ (defined in Section 2.1).

## 4.5 The partially ordered class of PPs

Let $X$ and $Y$ be two PPs. We say that $X$ is a subobject of $Y$ and we write $X \sqsubseteq Y$ if $|X| \sqsubseteq |Y|$ (in the sense of Section 3.5) and if $\eta_{|X|,|Y|} \in \mathbf{PpL}(X, Y)$ and $\rho_{|X|,|Y|} \in \mathbf{PpL}(Y, X)$. This means that the two following conditions must hold:

$$\forall x \subseteq |X| \quad x \in \mathsf{D}(X) \Rightarrow x \in \mathsf{D}(Y)$$
$$\forall y \subseteq |Y| \quad y \in \mathsf{D}(Y) \Rightarrow (y \cap |X| \in \mathsf{D}(X) \text{ and } (\underset{|Y|}{\downarrow} y) \cap |X| \subseteq \underset{|X|}{\downarrow} (y \cap |X|)) .$$

Observe that, in the second condition, the converse inclusion always holds because $|X| \sqsubseteq |Y|$.

It is clear that $\sqsubseteq$ is an order relation on the class of PPs; let us denote with **PpC** the corresponding partially ordered class.

As usual, the first thing to observe is that linear negation is covariant with respect to this notion.

**Lemma 19** *If $X \sqsubseteq Y$ then $X^\perp \sqsubseteq Y^\perp$.*

*Proof.* Same proof as for Lemma 7. □

### 4.5.1 Completeness.

**Lemma 20** *Let $(X_\gamma)_{\gamma\in\Gamma}$ a directed family of PPs. Let $X = \bigsqcup_{\gamma\in\Gamma} X_\gamma$ be defined as follows: $|X| = \bigsqcup_{\gamma\in\Gamma} |X_\gamma|$ (in the partially ordered class* **ScottC***) and $\mathsf{D}(X) = \{x \subseteq |X| \mid \forall \gamma \in \Gamma\ x \cap |X_\gamma| \in \mathsf{D}(X_\gamma)\}$. Then $X$ is a PP.*

*Proof.* Observe first that, if $x \in \mathsf{D}(X_\gamma)$, then $x \in \mathsf{D}(X)$. Indeed, let $\delta \in \Gamma$, we must check that $x \cap |X_\delta| \in \mathsf{D}(X_\delta)$. So let $\varepsilon \in \Gamma$ be such that $\gamma, \delta \leq \varepsilon$. Since $X_\gamma \sqsubseteq X_\varepsilon$, we have $x \in \mathsf{D}(X_\varepsilon)$, and since $X_\delta \sqsubseteq X_\varepsilon$, we have $x \cap |X_\delta| \in \mathsf{D}(X_\delta)$.

For proving the lemma, we build $X' = \bigsqcup_{\gamma\in\Gamma} X_\gamma^\perp$ (this makes sense since the family $(X_\gamma^\perp)_{\gamma\in\Gamma}$ is directed by Lemma 19), and we show that $X = X'^\perp$. Since obviously $|X| = |X'^\perp|$ (as preorders), it remains to show that $\mathsf{D}(X) = \mathsf{D}(X')^\perp$.

First, let $x \in \mathsf{D}(X)$ and let us show that $x \in \mathsf{D}(X')^\perp$. So let $x' \in \mathsf{D}(X')$ and assume that $\downarrow_{|X|} x \cap x' \neq \emptyset$. Let $a \in x$ and let $a' \in x'$ be such that $a' \leq_{|X|} a$. Let $\gamma \in \Gamma$ be such that $a, a' \in |X_\gamma|$ (so that $a' \leq_{|X_\gamma|} a$). We have $x \cap |X_\gamma| \in \mathsf{D}(X_\gamma)$, $x' \cap |X_\gamma| \in \mathsf{D}(X_\gamma^\perp)$ and $a' \in \downarrow_{|X_\gamma|} (x \cap |X_\gamma|) \cap (x' \cap |X_\gamma|)$, and hence $x \cap x' \neq \emptyset$.

Conversely, let $x \in \mathsf{D}(X')^\perp$, and let us show that $x \in \mathsf{D}(X)$. So let $\gamma \in \Gamma$ and let us show that $x \cap |X_\gamma| \in \mathsf{D}(X_\gamma)$. Let $x' \in \mathsf{D}(X_\gamma^\perp)$ and assume that $\downarrow_{|X_\gamma|} x \cap x' \neq \emptyset$. By our initial observation, we have $x' \in \mathsf{D}(X')$. Since $\downarrow_{|X_\gamma|} x \cap x' \neq \emptyset$, we have $\downarrow_{|X|} x \cap x' \neq \emptyset$ and hence $x \cap x' \neq \emptyset$. □

**Lemma 21** *$\bigsqcup_{\gamma\in\Gamma} X_\gamma$ is the least upper bound of the family $(X_\gamma)_{\gamma\in\Gamma}$ in the partially ordered class* **PpC***.*

*Proof.* Let $\delta \in \Gamma$, we check that $X_\delta \sqsubseteq \bigsqcup_{\gamma\in\Gamma} X_\gamma = X$. We have already seen that, if $x \in \mathsf{D}(X_\delta)$, then $x \in \mathsf{D}(X)$. So let $x \in \mathsf{D}(X)$. By definition, we have $x \cap |X_\delta| \in \mathsf{D}(X_\delta)$. We have to check that $\downarrow_{|X|} x \cap |X_\delta| \subseteq \downarrow_{|X_\delta|} (x \cap |X_\delta|)$, so let $a' \in \downarrow_{|X|} x \cap |X_\delta|$ and let $a \in x$ such that $a' \leq_{|X|} a$. We can find $\varepsilon \geq \delta$ such that $a, a' \in |X_\varepsilon|$. Then $a' \in \downarrow_{|X_\varepsilon|} x \cap |X_\delta|$ and since $X_\delta \sqsubseteq X_\varepsilon$, we have $\downarrow_{|X_\varepsilon|} x \cap |X_\delta| \subseteq \downarrow_{|X_\delta|} (x \cap |X_\delta|)$ and hence $a' \in \downarrow_{|X_\delta|} (x \cap |X_\delta|)$ as required.

Let $Y$ be a PP such that $X_\gamma \sqsubseteq Y$ for each $\gamma \in \Gamma$ and let us show that $X = \bigsqcup_{\gamma\in\Gamma} X_\gamma \sqsubseteq Y$. We already know that $\bigsqcup_{\gamma\in\Gamma} |X_\gamma| \sqsubseteq |Y|$. First, let $x \in \mathsf{D}(X)$ and let us show that $x \in \mathsf{D}(Y)$. So let $y' \in \mathsf{D}(Y^\perp)$ and assume that $\downarrow_{|X|} x \cap y' \neq \emptyset$. Let $a' \in \downarrow_{|X|} x \cap y'$ and let $a \in x$ be such that $a' \leq_{|X|} a$. Let $\delta \in \Gamma$ be such that $a, a' \in |X_\delta|$, so that $a' \leq_{|X_\delta|} a$. We have $a' \in \downarrow_{|X_\delta|} (x \cap |X_\delta|) \cap (y' \cap |X_\delta|)$, $x \cap |X_\delta| \in \mathsf{D}(X_\delta)$ (by definition of $X$) and $y' \cap |X_\delta| \in \mathsf{D}(X_\delta^\perp)$ (since $X_\delta \sqsubseteq Y$, and by Lemma 19). Hence $x \cap y' \neq \emptyset$, and this shows that $x \in \mathsf{D}(X)$.

Next, let $y \in \mathsf{D}(Y)$. We must show first that $y \cap |X| \in \mathsf{D}(X)$, but this results immediately from the definition of $X$ and from the fact that $X_\delta \sqsubseteq Y$ for each $\delta \in \Gamma$. Last, we must show that $\downarrow_{|Y|} y \cap |X| \subseteq \downarrow_{|X|} (y \cap |X|)$. Let $a' \in \downarrow_{|Y|} y \cap |X|$. Let $\delta \in \Gamma$ be such that $a' \in |X_\delta|$. Since $X_\delta \sqsubseteq Y$, we have $\downarrow_{|Y|} y \cap |X_\delta| \subseteq \downarrow_{|X_\delta|} (y \cap |X_\delta|)$ and we conclude because $a' \in \downarrow_{|Y|} y \cap |X_\delta|$ and, obviously, $\downarrow_{|X_\delta|} (y \cap |X_\delta|) \subseteq \downarrow_{|X|} (y \cap |X|)$. □

**4.5.2 Variable PPs and least fixpoints thereof.** A *variable PP* is a functor $\Phi : \mathbf{PpC}^n \to \mathbf{PpC}$ which commutes with the lubs of directed families of PPs (as usual we say then that $\Phi$ is *continuous*).

**Lemma 22** *The operations* $(X,Y) \mapsto X \otimes Y$, $X \to X^I$ *and* $X \mapsto !X$ *are variable PPs.*

*Proof.* We observe first that these operations are monotone, as in the proof of Lemma 10.

So the operation $(X,Y) \mapsto (X \multimap Y)$ is monotone, we prove that it is continuous. Let $(X_\gamma)_{\gamma\in\Gamma}$ and $(Y_\gamma)_{\gamma\in\Gamma}$ be directed families of PPs, and let $X$ and $Y$ be their lubs. Then $(X_\gamma \multimap Y_\gamma)_{\gamma\in\Gamma}$ is a directed family of PPs (we have just seen that $\_ \multimap \_$ is monotonous wrt. $\sqsubseteq$), let $Z$ be its lub. We must show that $Z = X \multimap Y$. We already know that $|Z| = |X \multimap Y|$ and that $Z \sqsubseteq X \multimap Y$, so it will be enough to show that $\mathsf{D}(X \multimap Y) \subseteq \mathsf{D}(Z)$. So let $t \in \mathsf{D}(X \multimap Y)$ and let $\gamma \in \Gamma$, we must prove that $t_\gamma = t \cap |X_\gamma \multimap Y_\gamma| \in \mathsf{D}(X_\gamma \multimap Y_\gamma)$. Let $x \in \mathsf{D}(X_\gamma)$, we have $x \in \mathsf{D}(X)$ and $t_\gamma \cdot x = (t \cdot x) \cap |Y_\gamma| \in \mathsf{D}(Y_\gamma)$. Moreover, $t_\gamma \cdot \downarrow_{|X_\gamma|} x = (t \cdot \downarrow_{|X_\gamma|} x) \cap |Y_\gamma| \subseteq (t \cdot \downarrow_{|X|} x) \cap |Y_\gamma| \subseteq \downarrow_{|Y|} (t \cdot x) \cap |Y_\gamma|$ since $t \in \mathsf{D}(X \multimap Y)$. Therefore, since $Y_\gamma \sqsubseteq Y$, we have $t_\gamma \cdot \downarrow_{|X_\gamma|} x \subseteq \downarrow_{|Y_\gamma|} ((t \cdot x) \cap |Y_\gamma|) = \downarrow_{|Y_\gamma|} (t_\gamma \cdot x)$ (remember that $x \in \mathsf{D}(X_\gamma)$) and this concludes the proof that $t_\gamma \in \mathsf{D}(X_\gamma \multimap Y_\gamma)$, and therefore also the proof that $\_ \multimap \_$ is a variable PP.

The operation $\Phi : X \mapsto (!X)^\perp$ is monotone, and we conclude by proving that it is continuous. Let $(X_\gamma)_{\gamma\in\Gamma}$ be a directed family, let $X$ be its lub, and let $Y$ be the lub of the directed family $(\Phi(X_\gamma))_{\gamma\in\Gamma}$. We have $Y \sqsubseteq \Phi(X)$ and $|Y| = |\Phi(X)|$, so it will be sufficient to prove that $\mathsf{D}(\Phi(X)) \subseteq \mathsf{D}(Y)$. Let $A' \in \mathsf{D}(\Phi(X))$ and let $\gamma \in \Gamma$, we must prove that $A' \cap |\Phi(X_\gamma)| \in \mathsf{D}(\Phi(X_\gamma))$. Let $x \in \mathsf{D}(X_\gamma)$ and assume that $A' \cap \downarrow_{|!X_\gamma|} (x^!) \neq \emptyset$. Then we have $A' \cap \downarrow_{|!X|} (x^!) \neq \emptyset$ and hence $A' \cap x^! \neq \emptyset$, since $x \in \mathsf{D}(X)$, that is $(A' \cap |\Phi(X_\gamma)|) \cap x^! \neq \emptyset$. □

Of course, any variable PP $\Phi : \mathbf{PpC} \to \mathbf{PpC}$ admits a least fixpoint, namely $\bigsqcup_{k\in\mathbb{N}} \Phi^k(\top)$ (remember that $\top = (\emptyset, \{\emptyset\})$, so that $\top$ is the least element of $\mathbf{PpC}$ for the preorder $\sqsubseteq$).

**4.5.3 An extensional reflexive PP.** The operation $\Phi_\mathsf{h} : \mathbf{PpC} \to \mathbf{PpC}$ defined by $\Phi_\mathsf{h}(X) = (!(X^\mathbb{N}))^\perp$ is a variable PP and has therefore a least fixpoint that we denote with $\mathcal{D}_\mathsf{h}$. One checks easily (as in 2.3.3) that $\mathcal{D}_\mathsf{h}$ is an extensional reflexive object in the CCC $\mathbf{PpL}_!$.

## 4.6 PPs are heterogeneous logical relations

We know that $\mathbf{Rel}_!$ and $\mathbf{ScottL}_!$ are CCCs and that $\mathbf{ScottL}_!$ is well-pointed, so we can apply to these categories the constructions of 1.3.2. We shall see that, up to canonical isomorphisms, $\mathbf{PpL}_!$ is a sub-cartesian closed category of $\mathsf{e}_\mathsf{mod}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$.

If $E$ is a set considered as an object of $\mathbf{Rel}_!$, a point of $E$ (that is an element of $\mathbf{Rel}_!(\top, E)$) is just a subset of $E$. And if $S$ is a preordered set considered as an object of $\mathbf{ScottL}_!$, a point of $S$ is an element of $\mathcal{I}(S)$.

**4.6.1 Heterogeneous relation associated with a PP.** Given a PP $X$, we can define an object $\mathsf{h}(X)$ of the category $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$ by setting $\ulcorner \mathsf{h}(X) \urcorner =$

$|X|$ (considered as a simple set), $\llcorner \mathsf{h}(X) \lrcorner = |X|$ (considered as a preordered set) and

$$x \Vdash_{\mathsf{h}(X)} u \quad \text{if} \quad x \in \mathsf{D}(X) \text{ and } u = \underset{|X|}{\downarrow} x\,.$$

Given a morphism $t \in \mathbf{PpL}_!(X,Y)$, we define a pair of morphisms $\mathsf{h}(t) = (\ulcorner \mathsf{h}(t) \urcorner, \llcorner \mathsf{h}(t) \lrcorner)$ with $\ulcorner \mathsf{h}(t) \urcorner = t \in \mathbf{Rel}_!(\ulcorner \mathsf{h}(X) \urcorner, \ulcorner \mathsf{h}(Y) \urcorner)$ and $\llcorner \mathsf{h}(t) \lrcorner = \downarrow_{|!X \multimap Y|} t$, which belongs to $\mathbf{ScottL}_!(\llcorner \mathsf{h}(X) \lrcorner, \llcorner \mathsf{h}(Y) \lrcorner)$.

**Theorem 23** *The operation* $\mathsf{h}$ *defined above is a full and faithful cartesian closed functor from* $\mathbf{PpL}_!$ *to* $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$.

*Proof.* Observe first that $\mathsf{h}(t) \in \mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)(\mathsf{h}(X), \mathsf{h}(Y))$ (with the notations above). Indeed, due to the definition of $\Vdash_{\mathsf{h}(X)}$ and of $\Vdash_{\mathsf{h}(Y)}$, this amounts to checking that, for any $x \in \mathsf{D}(X)$, one has $t \cdot x^! \in \mathsf{D}(Y)$ and $\downarrow_{|Y|} (t \cdot x^!) = \downarrow_{|!X \multimap Y|} t \cdot (\downarrow_{|X|} x)^!$. This holds by Lemma 18.

Let us check the functoriality of $\mathsf{h}$, so let $s \in \mathbf{PpL}_!(X,Y)$ and $t \in \mathbf{PpL}_!(Y,Z)$. One has first $\ulcorner \mathsf{h}(t \circ s) \urcorner = t \circ s = \ulcorner \mathsf{h}(t) \urcorner \circ \ulcorner \mathsf{h}(s) \urcorner$. Next, we have $\llcorner \mathsf{h}(t \circ s) \lrcorner = \downarrow_{|!X \multimap Z|} (t \circ s)$. Let $x \in \mathsf{D}(X)$. We have, applying again Lemma 18,

$$\begin{aligned}
\llcorner \mathsf{h}(t \circ s) \lrcorner \cdot (\underset{|X|}{\downarrow} x)^! &= \underset{|!X \multimap Z|}{\downarrow} (t \circ s) \cdot (\underset{|X|}{\downarrow} x)^! \\
&= \underset{|Z|}{\downarrow} ((t \circ s) \cdot x^!) \\
&= \underset{|Z|}{\downarrow} (t \cdot ((s \cdot x^!)^!)) \\
&= \underset{|!Y \multimap Z|}{\downarrow} t \cdot (\underset{|Y|}{\downarrow} (s \cdot x^!))^! \\
&= \underset{|!Y \multimap Z|}{\downarrow} t \cdot (\underset{|!X \multimap Y|}{\downarrow} s \cdot (\underset{|X|}{\downarrow} x)^!)^! \\
&= (\underset{|!Y \multimap Z|}{\downarrow} t \circ \underset{|!X \multimap Y|}{\downarrow} s) \cdot (\underset{|X|}{\downarrow} x)^!
\end{aligned}$$

and hence $\llcorner \mathsf{h}(t \circ s) \lrcorner = \llcorner \mathsf{h}(t) \lrcorner \circ \llcorner \mathsf{h}(s) \lrcorner$ because the category $\mathbf{ScottL}_!$ is well-pointed, and because any element of $\mathcal{I}(|X|)$ can be written $\downarrow_{|X|} x$ for some $x \in \mathsf{D}(X)$ (remember that $\mathcal{I}(|X|) \subseteq \mathsf{D}(X)$). One proves similarly that identities are preserved.

Fullness of $\mathsf{h}$ results again from Lemma 18 (used in the converse direction). It remains to prove that this functor is cartesian closed.

Let $(X_i)_{i \in I}$ be a finite family of PPs and let $X = \&_{i \in I} X_i$, so that $\ulcorner \mathsf{h}(X) \urcorner = \&_{i \in I} \ulcorner \mathsf{h}(X_i) \urcorner$ and $\llcorner \mathsf{h}(X) \lrcorner = \&_{i \in I} \llcorner \mathsf{h}(X_i) \lrcorner$. Moreover, $\ulcorner \mathsf{h}(\pi_i) \urcorner = \pi_i$ and $\llcorner \mathsf{h}(\pi_i) \lrcorner = \downarrow_{|!X_i \multimap X_i|} \pi_i = \pi_i^{\mathsf{S}}$. Last, given $x = \langle x_i \rangle_{i \in I} \in \mathcal{P}(|X|)$ and $u = \langle u_i \rangle_{i \in I} \in \mathcal{I}(|X|)$, we have $x \Vdash_{\mathsf{h}(X)} u$ iff $x \in \mathsf{D}(X)$ and $\downarrow_{|X|} x = u$. The first of these two conditions is equivalent to $\forall i \in I\, x_i \in \mathsf{D}(X_i)$ and the second one is equivalent to $\forall i \in I\, \downarrow_{|X_i|} x_i = u_i$ and therefore $x \Vdash_{\mathsf{h}(X)} u \Leftrightarrow \forall i \in I\, x_i \Vdash_{X_i} u_i$ and this shows that $\mathsf{h}$ commutes with cartesian products.

It remains to show that $\mathsf{h}$ commutes with the function space construction, so let $X$ and $Y$ be PPs and let $Z = (X \Rightarrow Y) = (!X \multimap Y)$. We clearly have $\ulcorner \mathsf{h}(Z) \urcorner = \ulcorner \mathsf{h}(X) \urcorner \Rightarrow \ulcorner \mathsf{h}(Y) \urcorner$ and $\llcorner \mathsf{h}(Z) \lrcorner = \llcorner \mathsf{h}(X) \lrcorner \Rightarrow \llcorner \mathsf{h}(Y) \lrcorner$. Next we have $\ulcorner \mathsf{h}(\mathsf{Ev}) \urcorner = \mathsf{Ev}$ and $\llcorner \mathsf{h}(\mathsf{Ev}) \lrcorner = \downarrow_{|Z|} \mathsf{Ev} = \mathsf{Ev}^{\mathsf{S}}$ (see 3.4.1). Finally, let $t \in \mathcal{P}(|Z|)$ and let $w \in \mathcal{I}(|Z|)$. Assume first that $t \Vdash_{\mathsf{h}(Z)} w$, that is $t \in \mathsf{D}(Z)$ and $\downarrow_{|Z|} t = w$.

We must prove that $t \Vdash_{\mathsf{h}(X)\Rightarrow\mathsf{h}(Y)} w$. So let $x \in \mathcal{P}(|X|)$ and $u \in \mathcal{I}(|X|)$ be such that $x \Vdash_X u$, that is $x \in \mathsf{D}(X)$ and $\downarrow_{|X|} x = u$. By definition of $t(x)$ and $w(u)$ (see Section 1.1), we have $t(x) = t \cdot x^!$ and $w(u) = w \cdot u^! = (\downarrow_{|Z|} t) \cdot (\downarrow_{|X|} x)^! = \downarrow_{|Y|} (t(x))$ by Lemma 18. By the same lemma, we have $t(x) \in \mathsf{D}(Y)$, and hence $t(x) \Vdash_{\mathsf{h}(Y)} w(u)$ as required. Conversely, assume that $t \Vdash_{\mathsf{h}(X)\Rightarrow\mathsf{h}(Y)} w$; we must prove that $t \Vdash_{\mathsf{h}(Z)} w$. We apply again Lemma 18, so let $x \in \mathsf{D}(X)$. We have $x \Vdash_X \downarrow_{|X|} x$ and hence $t(x) \in \mathsf{D}(Y)$ (that is $t \cdot x^! \in \mathsf{D}(Y)$) and $\downarrow_{|Y|} (t \cdot x^!) = w \cdot (\downarrow_{|X|} x)^!$ (by definition of $\Vdash_{\mathsf{h}(X)\Rightarrow\mathsf{h}(Y)}$). We prove that $\downarrow_{|Z|} t = w$. Let $(m, b) \in |Z|$. We have $\downarrow_{|Y|} (t \cdot (\downarrow_{|X|} \mathsf{supp}(m))^!) = w \cdot (\downarrow_{|X|} \mathsf{supp}(m))^!$. Assume first that $(m, b) \in \downarrow_{|Z|} t$ and let $(m', b') \in t$ be such that $(m, b) \leq_{|Z|} (m', b')$. Then $m' \in (\downarrow_{|X|} \mathsf{supp}(m))^!$ and hence $b \in \downarrow_{|Y|} (t \cdot (\downarrow_{|X|} \mathsf{supp}(m))^!)$. So let $m'' \in (\downarrow_{|X|} \mathsf{supp}(m))^!$ be such that $(m'', b) \in w$. Since $w \in \mathcal{I}(|Z|)$, we have $(m, b) \in w$. Conversely, assume that $(m, b) \in w$. Since $m \in (\downarrow_{|X|} \mathsf{supp}(m))^!$, we have $b \in \downarrow_{|Y|} (t \cdot (\downarrow_{|X|} \mathsf{supp}(m))^!)$ so we can find $(m', b') \in t$ such that $m' \in (\downarrow_{|X|} \mathsf{supp}(m))^!$ and $b \leq b'$, that is $(m, b) \leq_{|Z|} (m', b')$, which show that $(m, b) \in \downarrow_{|Z|} t$. Therefore, $x$ being an element of $\mathsf{D}(X)$, we have $\downarrow_{|Y|} (t \cdot x^!) = \downarrow_{|Z|} t \cdot (\downarrow_{|X|} x)^!$ and so $t \in \mathsf{D}(Z)$ by Lemma 18. This concludes the proof that $t \Vdash_Z w$, and therefore we have $\mathsf{h}(Z) = \mathsf{h}(X) \Rightarrow \mathsf{h}(Y)$. Therefore $\mathsf{h}$ is a CCC functor. □

So we can consider $\mathbf{PpL}_!$ as a sub-CCC of $\mathsf{e}(\mathbf{Rel}_!, \mathbf{ScottL}_!)$.

## 4.7 A functor from PPs to PER-objects

Given a PP $X$, we obviously define a PER (denoted with $B_X$ for the time being) on $\mathcal{P}(|X|)$ by saying that $x \; B_X \; y$ if $x, y \in \mathsf{D}(X)$ and $\downarrow_X x = \downarrow_X y$. Observe that $x \; B_X \; \downarrow_X x$ for any $x \in \mathsf{D}(X)$.

**Lemma 24** *For any PP $X$, one has $B_X^\perp = B_{X^\perp}$ and therefore $B_X^{\perp\perp} = B_X$.*

*Proof.* Let $x', y' \subseteq |X|$. Assume first that $x' \; B_X^\perp \; y'$ and let us show that $x' \; B_{X^\perp} \; y'$. We prove first that $x' \in \mathsf{D}(X)^\perp$, so let $x \in \mathsf{D}(X)$, and assume that $x' \cap \downarrow_{|X|} x \neq \emptyset$, we must show that $x' \cap x \neq \emptyset$. This results from the fact that $x \; B_X \downarrow_{|X|} x$. Similarly we get $y' \in \mathsf{D}(X)^\perp$. We must show now that $\uparrow_{|X|} x' = \uparrow_{|X|} y'$, so let $a \in \uparrow_{|X|} x'$. This means that $\downarrow_{|X|} a \cap x' \neq \emptyset$. Since $\downarrow_{|X|} a \; B_X \downarrow_{|X|} a$, we get $\downarrow_{|X|} a \cap y' \neq \emptyset$, that is $a \in \uparrow_{|X|} y'$.

Conversely, assume that $x' \; B_{X^\perp} \; y'$ and let us show that $x' \; B_X^\perp \; y'$. So let $x, y \subseteq |X|$ be such that $x \; B_X \; y$, and assume that $x \cap x' \neq \emptyset$; we must show that $y \cap y' \neq \emptyset$. We have *a fortiori* $\downarrow_{|X|} x \cap \uparrow_{|X|} x' \neq \emptyset$, that is $\downarrow_{|X|} y \cap \uparrow_{|X|} y' \neq \emptyset$. But then, since $y \in \mathsf{D}(X)$ and $y' \in \mathsf{D}(X)^\perp$, we get $y \cap y' \neq \emptyset$. □

We can rephrase this result as follows.

**Lemma 25** *For any PP $X$, $\varepsilon(X) = (|X|, B_X)$ is a PER-object and we have $\varepsilon(X^\perp) = \varepsilon(X)^\perp$.*

The relation $B_X$ can therefore also be denoted with $\sim_{\varepsilon(X)}$.

**Lemma 26** *Let $X$ and $Y$ be PPs and let $s_1, s_2 \in \mathcal{P}(|X \multimap Y|)$. One has $s_1 \sim_{\varepsilon(X \multimap Y)} s_2$ iff for all $x_1, x_2 \in \mathcal{P}(|X|)$, if $x_1 \sim_{\varepsilon(X)} x_2$ then $s_1 \cdot x_1 \sim_{\varepsilon(Y)} s_2 \cdot x_2$. This means that $\varepsilon(X \multimap Y) = \varepsilon(X) \multimap \varepsilon(Y)$.*

*Proof.* Assume first that $s_1 \sim_{\varepsilon(X \multimap Y)} s_2$. Let $x_1, x_2 \subseteq |X|$ be such that $x_1 \sim_{\varepsilon(X)} x_2$, we want to show that $s_1 \cdot x_1 \sim_{\varepsilon(Y)} s_2 \cdot x_2$. Let $y_1', y_2' \subseteq |Y|$ be such that $y_1' \sim_{\varepsilon(Y^\perp)} y_2'$. One has $(s_1 \cdot x_1) \cap y_1' \neq \emptyset$ iff $s_1 \cap (x_1 \times y_1') \neq \emptyset$ and, since $x_1 \in \mathsf{D}(X)$ and $y_1' \in \mathsf{D}(Y)^\perp$, this latter condition holds iff $s_1 \cap \downarrow_{|X \otimes Y^\perp|} (x_1 \times y_1') \neq \emptyset$, which in turn is equivalent to $\downarrow_{|X \multimap Y|} s_1 \cap \downarrow_{|X \otimes Y^\perp|} (x_1 \times y_1') \neq \emptyset$ since $s_1 \in \mathsf{D}(X \multimap Y)$. Since $\downarrow_{|X \multimap Y|} s_1 = \downarrow_{|X \multimap Y|} s_2$ (because $s_1 \sim_{\varepsilon(X \multimap Y)} s_2$) and $\downarrow_{|X \otimes Y^\perp|} (x_1 \times y_1') = \downarrow_{|X \otimes Y^\perp|} (x_2 \times y_2')$ (because $x_1 \sim_{\varepsilon(X)} x_2$ and $y_1' \sim_{\varepsilon(Y^\perp)} y_2'$), we conclude that $(s_1 \cdot x_1) \cap y_1' \neq \emptyset \Leftrightarrow (s_1 \cdot x_2) \cap y_2' \neq \emptyset$, and this shows that $s_1 \cdot x_1 \sim_{\varepsilon(Y)} s_2 \cdot x_2$ by Lemma 24.

Conversely, assume that $s_1 \cdot x_1 \sim_{\varepsilon(Y)} s_2 \cdot x_2$ whenever $x_1 \sim_{\varepsilon(X)} x_2$, and let us show that $s_1 \sim_{\varepsilon(X \multimap Y)} s_2$. Observe that our assumption implies that $s_1 \cdot x_1 \sim_{\varepsilon(Y)} s_1 \cdot x_2$ (indeed, $x_2 \sim_{\varepsilon(X)} x_2$, hence $s_1 \cdot x_2 \sim_{\varepsilon(Y)} s_2 \cdot x_2$ and we can apply transitivity of the relation $\sim_{\varepsilon(Y)}$). We show first that $s_1 \in \mathsf{D}(X \multimap Y)$. So let $x \in \mathsf{D}(X)$. We have $x \sim_{\varepsilon(X)} x$ and hence $s_1 \cdot x \sim_{\varepsilon(Y)} s_2 \cdot x$, which implies $s_1 \cdot x \in \mathsf{D}(X)$. Let $b \in s_1 \cdot \downarrow_{|X|} x$, we show that $b \in \downarrow_{|Y|} (s_1 \cdot x)$. We have $x \sim_{\varepsilon(X)} \downarrow_{|X|} x$ and hence $s_1 \cdot x \sim_{\varepsilon(Y)} s_1 \cdot \downarrow_{|X|} x$ which implies $\downarrow_{|Y|} (s_1 \cdot x) = \downarrow_{|Y|} (s_1 \cdot \downarrow_{|X|} x)$ and we conclude since $b \in \downarrow_{|Y|} (s_1 \cdot \downarrow_{|X|} x)$. By Proposition 15, we have $s_1 \in \mathsf{D}(X \multimap Y)$, and of course the same holds for $s_2$ by symmetry. It remains to show that $\downarrow_{|X \multimap Y|} s_1 = \downarrow_{|X \multimap Y|} s_2$.

Let $(a, b) \in \downarrow_{|X \multimap Y|} s_1$. This means that $\downarrow_{|X \otimes Y^\perp|} (a, b) \cap s_1 \neq \emptyset$, that is $(s_1 \cdot \downarrow_{|X|} a) \cap \uparrow_{|Y|} b \neq \emptyset$. But $\downarrow_X a \sim_{\varepsilon(X)} \downarrow_X a$ and hence $s_1 \cdot \downarrow_{|X|} a \sim_{\varepsilon(Y)} s_2 \cdot \downarrow_{|X|} a$ and since $\uparrow_{|Y|} b \sim^\perp_{\varepsilon(Y)} \uparrow_{|Y|} b$, we have $(s_2 \cdot \downarrow_{|X|} a) \cap \uparrow_{|Y|} b \neq \emptyset$, that is $(a, b) \in \downarrow_{|X \multimap Y|} s_2$. □

In particular, for any PPs $X$ and $Y$, one has $\mathbf{PpL}(X, Y) = \mathbf{PerL}(\varepsilon(X), \varepsilon(Y))$ and so the operation $\varepsilon$ is a full and faithful functor, which is the identity on morphisms. Indeed, composition of morphisms is defined in the same way in both categories, as the standard composition of relations.

**Lemma 27** *Let $X$ and $Y$ be PPs. We have $\varepsilon(X \otimes Y) = \varepsilon(X) \otimes \varepsilon(Y)$, that is, the functor $\varepsilon$ is strict monoidal.*

*Proof.* Apply the fact that $X \otimes Y = (X \multimap Y^\perp)^\perp$, Lemma 25 and Lemma 26. □

**Lemma 28** *The functor $\varepsilon$ commutes with all cartesian products.*

**Lemma 29** *Let $X$ be a PP, one has $\varepsilon(!X) = !\varepsilon(X)$.*

*Proof.* By Lemma 25, it suffices to show that $\varepsilon(!X)^\perp = (!\varepsilon(X))^\perp$. Let $A_1', A_2' \subseteq |!X|$.

On the one hand, $A_1' \sim_{\varepsilon(!X)^\perp} A_2'$ means that $A_1' \sim^\perp_{\varepsilon(!X)} A_2'$, that is

$$\forall A_1, A_2 \subseteq |!X| \; A_1 \sim_{\varepsilon(!X)} A_2 \Rightarrow (A_1 \cap A_1' \neq \emptyset \Leftrightarrow A_2 \cap A_2' \neq \emptyset)\,,$$

and remember that $A_1 \sim_{\varepsilon(!X)} A_2$ means that $A_1, A_2 \in \mathsf{D}(!X)$ and $\downarrow_{|!X|} A_1 = \downarrow_{|!X|} A_2$. By Lemma 25, $A'_1 \sim_{\varepsilon(!X)^\perp} A'_2$ is also equivalent to $A'_1 \sim_{\varepsilon((!X)^\perp)} A'_2$, that is

$$A'_1, A'_2 \in \mathsf{D}(!X)^\perp \quad \text{and} \quad \uparrow_{|!X|} A'_1 = \uparrow_{|!X|} A'_2 . \tag{2}$$

On the other hand, $A'_1 \sim_{(!\varepsilon(X))^\perp} A'_2$ means that $A'_1 \sim^\perp_{!\varepsilon(X)} A'_2$, that is

$$\forall x_1, x_2 \subseteq |X|\ x_1 \sim_{\varepsilon(X)} x_2 \Rightarrow (x_1^! \cap A'_1 \neq \emptyset \Leftrightarrow x_2^! \cap A'_2 \neq \emptyset)$$

and remember that $x_1 \sim_{\varepsilon(X)} x_2$ means that $x_1, x_2 \in \mathsf{D}(X)$ and $\downarrow_{|X|} x_1 = \downarrow_{|X|} x_2$.

Hence $x_1 \sim_{\varepsilon(X)} x_2$ implies $x_1^!, x_2^! \in \mathsf{D}(!X)$ and $\downarrow_{|!X|} x_1^! = (\downarrow_{|X|} x_1)^! = (\downarrow_{|X|} x_2)^! = \downarrow_{|!X|} x_2^!$, that is $x_1^! \sim_{\varepsilon(!X)} x_2^!$ and hence $A'_1 \sim^\perp_{\varepsilon(!X)} A'_2 \Rightarrow A'_1 \sim^\perp_{!\varepsilon(X)} A'_2$.

Let us prove the converse implication, so assume that $A'_1 \sim^\perp_{!\varepsilon(X)} A'_2$ and let us prove that property (2) holds. We prove first that $A'_1 \in \mathsf{D}(!X)^\perp$. So let $x \in \mathsf{D}(X)$ and assume that $A'_1 \cap x^! = \emptyset$. Since $x \sim_{\varepsilon(X)} \downarrow_{|X|} x$, we have $x^! \sim_{!\varepsilon(X)} (\downarrow_{|X|} x)^! = \downarrow_{|!X|} (x^!)$, and hence $A'_1 \cap \downarrow_{|!X|} (x^!) = \emptyset$ since we have $A'_1 \sim^\perp_{!\varepsilon(X)} A'_1$. It remains to show that $\uparrow_{|!X|} A'_1 = \uparrow_{|!X|} A'_2$, we only prove the "$\subseteq$" inclusion. So let $m \in |!X|$ and assume that $m \in \uparrow_{|!X|} A'_1$. This means that $A'_1 \cap \downarrow_{|!X|} m \neq \emptyset$, and since $\downarrow_{|!X|} m \sim_{!\varepsilon(X)} \downarrow_{|!X|} m$, we have $m \in \uparrow_{|!X|} A'_2$. □

**Theorem 30** *The functor $\varepsilon$ is an LL-functor.*

*Proof.* This results from Lemmas 26, 27, 28 and 29, from the fact that $\varepsilon$ acts trivially on morphisms and from the fact that the operations on morphisms are defined in the same way in both categories. □

It follows that $\varepsilon$ is a cartesian closed functor from $\mathbf{PpL}_!$ to $\mathbf{PerL}_!$.

**4.7.1 Continuity of $\varepsilon$.** Let $X$ and $Y$ be PPs such that $X \sqsubseteq Y$. Since $\eta_{|X|,|Y|} \in \mathbf{PpL}(X, Y)$ and since $\varepsilon$ acts trivially on morphisms, we have $\eta_{|X|,|Y|} \in \mathbf{PerL}(\varepsilon(X), \varepsilon(Y))$. Similarly, we have $\rho_{|X|,|Y|} \in \mathbf{PerL}(\varepsilon(Y), \varepsilon(X))$. Therefore $\varepsilon(X) \sqsubseteq \varepsilon(Y)$, that is $\varepsilon$ is a monotone class function from $\mathbf{PpC}$ to $\mathbf{PerC}$.

**Theorem 31** *The monotone class function $\varepsilon : \mathbf{PpC} \to \mathbf{PerC}$ is continuous.*

*Proof.* Let $(X_\gamma)_{\gamma\in\Gamma}$ be a directed family of PPs and let $X = \bigsqcup_{\gamma\in\Gamma} X_\gamma \in \mathbf{PpC}$. We already know that $|X| = \bigcup_{\gamma\in\Gamma} |X_\gamma|$ and so we have to prove that, given $x, y \subseteq |X|$, the two following conditions are equivalent:

1. $x, y \in \mathsf{D}(X)$ and $\downarrow_{|X|} x = \downarrow_{|X|} y$
2. for all $\gamma \in \Gamma$, $x \cap |X_\gamma|$, $y \cap |X_\gamma| \in \mathsf{D}(X_\gamma)$ and $\downarrow_{|X_\gamma|} (x \cap |X_\gamma|) = \downarrow_{|X_\gamma|} (y \cap |X_\gamma|)$.

That (1) implies (2) results from the monotonicity of $\varepsilon$ (for each $\gamma \in \Gamma$, we have $X_\gamma \sqsubseteq X$ and hence $\varepsilon(X_\gamma) \sqsubseteq \varepsilon(X)$), so let us prove the converse and assume that (2) holds. That $x, y \in \mathsf{D}(X)$ results directly from the definition of $X$ (see 4.5.1). We conclude by checking that $\downarrow_{|X|} x \subseteq \downarrow_{|X|} y$. For this, it is sufficient to have $x \subseteq \downarrow_{|X|} y$, so let $a \in x$. Let $\gamma \in \Gamma$ be such that $a \in x \cap |X_\gamma|$. By assumption, $a \in \downarrow_{|X_\gamma|} (y \cap |X_\gamma|)$, so let $b \in y \cap |X_\gamma|$ be such that $a \leq_{|X_\gamma|} b$. Since $|X|$ is the

lub of the $|X_\gamma|$s in the partially ordered class **ScottC**, we have $a \leq_{|X|} b$ and this concludes the proof. □

**4.7.2 Image of the reflexive object of $\mathbf{PpL}_!$.** Remember from 4.5.3 that we have defined a reflexive object $\mathcal{D}_\mathsf{h}$ in $\mathbf{PpL}_!$ as the least fixpoint of a continuous class function $\Phi_\mathsf{h} : \mathbf{PpC} \to \mathbf{PpC}$, in other words $\mathcal{D}_\mathsf{h} = \bigsqcup_{n\in\mathbb{N}} \Phi_\mathsf{h}^n(\top)$. By continuity of $\varepsilon$, we have $\varepsilon(\mathcal{D}_\mathsf{h}) = \bigsqcup_{n\in\mathbb{N}} \varepsilon(\Phi_\mathsf{h}^n(\top)) = \bigsqcup_{n\in\mathbb{N}} \Phi_\mathsf{e}^n(\top) = \mathcal{D}_\mathsf{e}$ (see 2.3.3) since $\varepsilon$ is an LL-functor from **PpL** to **PerL**

## 4.8 A functor from PPs to preorders

We define a functor $\sigma$ from **PpL** to **ScottL**. Given a PP $X$, we set $\sigma(X) = |X|$, which is a preorder. Given two PPs $X$ and $Y$ and $t \in \mathbf{PpL}(X,Y) = \mathsf{D}(X \multimap Y)$, we set

$$\sigma(t) = \underset{|X\multimap Y|}{\downarrow} t \in \mathcal{I}(|X \multimap Y|) \simeq \mathbf{ScottL}(|X|,|Y|)\,.$$

In other words, the linear map $\sigma(t) : \mathcal{I}(|X|) \to \mathcal{I}(|Y|)$ is given by $\sigma(t)(x) = \downarrow_{|Y|}(t \cdot x)$ (see Lemma 12).

**Lemma 32** *The operation $\sigma$ on morphisms is a functor, that is $\sigma(\mathsf{Id}_X) = \mathsf{Id}^\mathsf{S}_X$ and, given $s \in \mathbf{PpL}(X,Y)$ and $t \in \mathbf{PpL}(Y,Z)$, one has $\sigma(t\cdot s) = \sigma(t)\cdot\sigma(s)$.*

*Proof.* See 4.2.1, where the proof is given. □

**Theorem 33** *The functor $\sigma$ is an LL-functor.*

*Proof.* This is a routine verification.

As an example, let $X$ and $Y$ be PPs. We have $\sigma(!X) = |!X| = !|X| = !\sigma(X)$. Let $t \in \mathbf{PpL}(X,Y)$, we prove that $\sigma(!t) = !\sigma(t)$. Let $(p,q) \in |!X| \times |!Y|$. If $(p,q) \in \sigma(!t)$, we can find $(p',q') \in !t$ such that $p' \leq_{|!X|} p$ and $q \leq_{|!Y|} q'$; we show that $(p,q) \in !\sigma(t) = !(\downarrow_{|X\multimap Y|} t)$. Let $b \in q$, let $b' \in q'$ such that $b \leq_{|Y|} b'$. Let $a' \in p'$ be such that $(a',b') \in t$ (since $(p',q') \in !t$). Let $a \in p$ be such that $a' \leq_{|X|} a$ (since $p' \leq_{|!X|} p$). We have $(a',b') \in t$ and $(a,b) \leq_{|X\multimap Y|} (a',b')$, hence $(a,b) \in \sigma(t)$ and this shows that $(p,q) \in !\sigma(t)$. Assume conversely that $(p,q) \in !\sigma(t)$ and let us show that $(p,q) \in \sigma(!t)$. For each $b \in q$, let us choose $\mathsf{l}(b) \in p$ such that $(\mathsf{l}(b),b) \in \sigma(t) = \downarrow_{|X\multimap Y|} t$. Let $(\mathsf{ul}(b),\mathsf{ur}(b)) \in t$ be such that $\mathsf{ul}(b) \leq_{|X|} \mathsf{l}(b)$ and $b \leq_{|Y|} \mathsf{ur}(b)$. We pick $p' \in !|X|$ and $q' \in !|Y|$ such that $\mathsf{supp}(p') = \{\mathsf{ul}(b) \mid b \in \mathsf{supp}(q)\}$ and $\mathsf{supp}(q') = \{\mathsf{ur}(b) \mid b \in \mathsf{supp}(q)\}$. We have $p' \leq_{|!X|} p$ (if $a' \in p'$, we can choose $b \in q$ such that $a' = \mathsf{ul}(b)$, and hence $a' \leq_{|X|} \mathsf{l}(b) \in p$), $q \leq_{|!Y|} q'$ (if $b \in q$, we have $b \leq_{|Y|} \mathsf{ur}(b) \in q'$) and $(p',q') \in !t$ (let $b' \in q'$; we can choose $b \in q$ such that $b' = \mathsf{ur}(b)$, we have $\mathsf{ul}(b) \in p'$ and $(\mathsf{ul}(b),\mathsf{ur}(b)) \in t$). This shows that $(p,q) \in \sigma(!t)$.

Last, let us check that $\sigma(\mathsf{p}_X) = \mathsf{p}^\mathsf{S}_{\sigma(X)}$. Let $(p,P) \in !|X| \times !!|X|$, so that $P$ can be written $P = [p_1,\ldots,p_n]$ with $p_1,\ldots,p_n \in |!X|$. Assume first that $(p,P) \in \sigma(\mathsf{p}_X) = \downarrow_{|!X\multimap !!X|} \mathsf{p}_X$ and let us show that $(p,P) \in \mathsf{p}^\mathsf{S}_{\sigma(X)}$, that is $p_1 + \cdots + p_n \leq_{!|X|} p$. So let $a \in p_1 + \cdots + p_n$, and let $i \in \{1,\ldots,n\}$ be such that $a \in p_i$. Let $(p',P') \in \mathsf{p}_X$ be such that $p' \leq_{!|X|} p$ and $P \leq_{!!|X|} P'$, so that $P' = [p'_1,\ldots,p'_k]$ with $p' = p'_1 + \cdots + p'_k$. Let $j \in \{1,\ldots,k\}$

be such that $p_i \leq_{!|X|} p'_j$. Let $a' \in p'_j$ be such that $a \leq_{|X|} a'$ (remember that $a \in p_i$). Then we have $a' \in p'$ and hence we can find $a'' \in p$ such that $a' \leq_{|X|} a''$. This shows that $p_1 + \cdots + p_n \leq_{!|X|} p$ as required. Conversely, assume that $(p, P) \in \mathsf{p}^{\mathsf{S}}_{\sigma(X)}$ (that is $p_1 + \cdots + p_n \leq_{!|X|} p$) and let us show that $(p, P) \in \sigma(\mathsf{p}_X)$. We have $(p_1 + \cdots + p_n, P) \in \mathsf{p}_X$ by definition of $\mathsf{p}_X$ and we have $(p, P) \leq_{|!X \multimap !!X|} (p_1 + \cdots + p_n, P)$ since $p_1 + \cdots + p_n \leq_{!|X|} p$. Therefore $(p, P) \in \sigma(\mathsf{p}_X)$ as announced. □

It follows that $\sigma$ is a cartesian closed functor from $\mathbf{PpL}_!$ to $\mathbf{ScottL}_!$.

It is straightforward from the definition of **PpC** that $\sigma$ is a continuous class function from **PpC** to **ScottC**. Since $\sigma$ is also an LL-functor from **PpL** to **ScottL**, it follows that $\sigma(\mathcal{D}_\mathsf{h}) = \mathcal{D}_\mathsf{s}$ (as in 4.7.2). According to the definitions of Section 1.3.2, we can summarize the situation as follows (at least as far as CCCs are concerned).

**Theorem 34** *The CCC* $\mathbf{ScottL}_!$ *represents the extensional collapse of the CCC* $\mathbf{Rel}_!$ *in the sense of 1.3.2. The reflexive object* $\mathcal{D}_\mathsf{s}$ *of* $\mathbf{ScottL}_!$ *represents the extensional collapse of the reflexive object* $\mathcal{D}_\mathsf{r}$ *of* $\mathbf{Rel}_!$ *in the sense of 1.3.4.*

Of course, the results presented in this paper are more general than what is stated in that theorem, since they concern the *linear* structures of the models, not only their cartesian closed structures.

# References


[AC98] Roberto Amadio and Pierre-Louis Curien. *Domains and lambda-calculi*, volume 46 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1998.

[Bar79] Michael Barr. $*$*-autonomous categories*. Number 752 in Lecture Notes in Mathematics. Springer-Verlag, 1979.

[BE01] Antonio Bucciarelli and Thomas Ehrhard. On phase semantics and denotational semantics: the exponentials. *Annals of Pure and Applied Logic*, 109(3):205–241, 2001.

[BEM07] Antonio Bucciarelli, Thomas Ehrhard, and Giulio Manzonetto. Not enough points is enough. In *Proceedings of the 21st Annual Conference of the European Association for Computer Science Logic (CSL'07)*, Lecture Notes in Computer Science. Springer-Verlag, September 2007.

[BEM08] Antonio Bucciarelli, Thomas Ehrhard, and Giulio Manzonetto. A relational model of a parallel and non-deterministic lambda-calculus. Technical report, Preuves, Programmes et Systèmes, 2008. Submitted for publication.

[Ber78] Gérard Berry. Stable models of typed lambda-calculi. In *Proceedings of the 5th International Colloquium on Automata, Languages and Programming*, number 62 in Lecture Notes in Computer Science. Springer-Verlag, 1978.

[Bie95] Gavin Bierman. What is a categorical model of intuitionistic linear logic? In Mariangiola Dezani-Ciancaglini and Gordon D. Plotkin, editors, *Proceedings of the second Typed Lambda-Calculi and Applications conference*, volume 902 of *Lecture Notes in Computer Science*, pages 73–93. Springer-Verlag, 1995.

[Buc97] Antonio Bucciarelli. Logical relations and *lambda*-theories. In *Advances in Theory and Formal Methods of Computing, proceedings of the 3rd Imperial College Workshop*, pages 37–48. Imperial College Press, 1997.

[DE08] Vincent Danos and Thomas Ehrhard. On probabilistic coherence spaces. Technical report, Preuves, Programmes et Systèmes, 2008. Submitted for publication.

[DK00] Vincent Danos and Jean-Louis Krivine. Disjunctive tautologies and synchronisation schemes. In Peter Clote and Helmut Schwichtenberg, editors, *CSL*, volume 1862 of *Lecture Notes in Computer Science*, pages 292–301. Springer, 2000.

[Ehr02] Thomas Ehrhard. On Köthe sequence spaces and linear logic. *Mathematical Structures in Computer Science*, 12:579–623, 2002.

[Ehr05] Thomas Ehrhard. Finiteness spaces. *Mathematical Structures in Computer Science*, 15(4):615–646, 2005.

[Gir86] Jean-Yves Girard. The system F of variable types, fifteen years later. *Theoretical Computer Science*, 45:159–192, 1986.

[Gir87] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[Gir88] Jean-Yves Girard. Normal functors, power series and the $\lambda$-calculus. *Annals of Pure and Applied Logic*, 37:129–177, 1988.

[Has02] Ryu Hasegawa. Two applications of analytic functors. *Theoretical Computer Science*, 272(1-2):113–175, 2002.

[Hut94] Michael Huth. Linear domains and linear maps. In *Mathematical Foundations of Programming Semantics*, number 802 in Lecture Notes in Computer Science, pages 438–453. Springer-Verlag, 1994.

[Kri90] Jean-Louis Krivine. *Lambda-Calcul : Types et Modèles*. Études et Recherches en Informatique. Masson, 1990.

[Kri93] Jean-Louis Krivine. *Lambda-Calculus, Types and Models*. Ellis Horwood Series in Computers and Their Applications. Ellis Horwood, 1993. Translation by René Cori from French 1990 edition (Masson).

[Plo77] Gordon Plotkin. LCF considered as a programming language. *Theoretical Computer Science*, 5:223–256, 1977.

[Win99] Glynn Winskel. A linear metalanguage for concurrency. In Armando Martin Haeberer, editor, *AMAST*, volume 1548 of *Lecture Notes in Computer Science*, pages 42–58. Springer-Verlag, 1999.

[Win04] Glynn Winskel. Linearity and non linearity in distributed computation. In Thomas Ehrhard, Jean-Yves Girard, Paul Ruet, and Philip Scott, editors, *Linear Logic in Computer Science*, volume 316 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2004.