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ASYMPTOTIC PROPERTIES OF ENTROPY SOLUTIONS TO FRACTAL BURGERS EQUATION

NATHAEL ALIBAUD, CYRIL IMBERT, AND GRZEGORZ KARCH

ABSTRACT. We study properties of solutions of the initial value problem for the nonlinear and nonlocal equation $u_t + (-\partial_x^2)^{\alpha/2}u + uu_x = 0$ with $\alpha \in (0, 1]$, supplemented with an initial datum approaching the constant states u_{\pm} ($u_- < u_+$) as $x \rightarrow \pm\infty$, respectively. It was shown by Karch, Miao & Xu (SIAM J. Math. Anal. **39** (2008), 1536–1549) that, for $\alpha \in (1, 2)$, the large time asymptotics of solutions is described by rarefaction waves. The goal of this paper is to show that the asymptotic profile of solutions changes for $\alpha \leq 1$. If $\alpha = 1$, there exists a self-similar solution to the equation which describes the large time asymptotics of other solutions. In the case $\alpha \in (0, 1)$, we show that the nonlinearity of the equation is negligible in the large time asymptotic expansion of solutions.

1. INTRODUCTION

In this work, we continue the study of asymptotic properties of solutions of the Cauchy problem for the nonlocal conservation law

$$(1.1) \quad u_t + \Lambda^\alpha u + uu_x = 0, \quad x \in \mathbb{R}, t > 0,$$

$$(1.2) \quad u(0, x) = u_0(x),$$

where $\Lambda^\alpha = (-\partial^2/\partial x^2)^{\alpha/2}$ is the pseudodifferential operator defined via the Fourier transform $\widehat{(\Lambda^\alpha v)}(\xi) = |\xi|^\alpha \widehat{v}(\xi)$.

Motivated by the recent probabilistic approach to problem (1.1)–(1.2) by Jourdain, Méléard, and Woyczyński [10, 11], we assume that the initial datum u_0 is a function with bounded variation on \mathbb{R} :

$$(1.3) \quad u_0(x) = c + \int_{-\infty}^x m(dy)$$

with $c \in \mathbb{R}$ and $m(dy)$ being a finite signed measure on \mathbb{R} . Moreover, we require that

$$(1.4) \quad u_0 - u_- \in L^1((-\infty, 0)) \quad \text{and} \quad u_0 - u_+ \in L^1((0, +\infty)),$$

where

$$(1.5) \quad u_- = c \quad \text{and} \quad u_+ - u_- = \int_{\mathbb{R}} m(dx) \quad \text{satisfy} \quad u_- < u_+.$$

If $c = 0$ and if $m(dy)$ is a probability measure, the function $u_0(x)$ defined in (1.3) is the *cumulative distribution function* and this property is shared by the solution

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$u(t)$ for every $t > 0$ (see [10, 11]). As a consequence of our results, we describe the asymptotic behavior of the family $\{u(t)\}_{t \geq 0}$ of probability distribution functions as $t \rightarrow \infty$ (see the summary at the end of this section).

It was shown in [12] that, under assumptions (1.3)–(1.4) and for $1 < \alpha \leq 2$, the large time asymptotics of solution to (1.1)–(1.2) is described by the so-called *rarefaction waves*. The goal of this paper is to complete these results and to obtain a universal asymptotic profile of solutions for $0 < \alpha \leq 1$.

1.1. Known results. Let us first recall the results obtained in [12]. For $\alpha \in (1, 2]$, the initial value problem for the fractal Burgers equation (1.1)–(1.2) with $u_0 \in L^\infty(\mathbb{R})$ has the unique, smooth, global-in-time solution (cf. [8, Thm. 1.1], [9, Thm. 7]). If, moreover, the initial datum is of the form (1.3) and satisfies (1.4)–(1.5), the corresponding solution $u = u(x, t)$ converges as $t \rightarrow \infty$ toward the rarefaction wave (cf. [12, Thm. 1.1]). More precisely, for every $p \in (\frac{3-\alpha}{\alpha-1}, \infty]$ there exists $C = C(p)$ such that for all $t > 0$,

$$(1.6) \quad \|u(t) - w^R(t)\|_p \leq Ct^{-\frac{1}{2}[\alpha-1-\frac{3-\alpha}{p}]} \log(2+t).$$

Here, the rarefaction wave $w^R = w^R(x, t)$ is the explicit function

$$(1.7) \quad w^R(x, t) = W^R(x/t) = \begin{cases} u_-, & \frac{x}{t} \leq u_-, \\ \frac{x}{t}, & u_- \leq \frac{x}{t} \leq u_+, \\ u_+, & \frac{x}{t} \geq u_+. \end{cases}$$

It is well-known that w^R is the unique entropy solution of the Riemann problem for the nonviscous Burgers equation $w_t^R + w^R w_x^R = 0$.

The goal of the work is to show that, for $\alpha \in (0, 1]$, one should expect completely different asymptotic profiles of solutions. Let us notice that the initial value problem (1.1)–(1.2) has the unique global-in-time entropy solution for every $u_0 \in L^\infty(\mathbb{R})$ and $\alpha \in (0, 1]$ due to the recent work [1]. We recall that result in Section 2.

1.2. Main results for $\alpha < 1$. In the case where $\alpha < 1$ the Duhamel principle (see equation (3.3) below) shows that the nonlinearity in equation (1.1) is negligible in the asymptotic expansion of solutions.

Theorem 1.1. (Convergence toward the linear part)

Let $0 < \alpha < 1$ and $u = u(x, t)$ be the entropy solution to (1.1)–(1.2) corresponding to the initial condition of the form (1.3) satisfying (1.4)–(1.5). Denote by $S_\alpha(t)u_0$ the solution of the linear initial value problem $u_t + \Lambda^\alpha u = 0$, $u(x, 0) = u_0(x)$. For every $p \in (\frac{1}{1-\alpha}, \infty]$ there exists $C = C(p) > 0$ such that

$$(1.8) \quad \|u(t) - S_\alpha(t)u_0\|_p \leq C \|u_0\|_\infty \|m\| t^{1-\frac{1}{\alpha}(1-\frac{1}{p})}$$

for all $t > 0$.

Remark 1.2. It follows from the proof of Theorem 1.1 that inequality (1.8) is valid for every $p \in [1, \infty]$. However, its right-hand-side decays only for $p \in (\frac{1}{1-\alpha}, \infty]$.

The asymptotic term in (1.8) can be written in a self-similar way.

Corollary 1.3. (Self-similar behavior of the linear part)

Under the assumptions of Theorem 1.1, we have for each $p \in (\frac{1}{1-\alpha}, \infty]$

$$\left\| u(t) - \left(c + H_\alpha(x, t) \int_{\mathbb{R}} m(dx) \right) \right\|_p \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where $H_\alpha(x, t) \equiv \int_{-\infty}^x p_\alpha(y, t) dy$ and $p_\alpha(x, t)$ is the fundamental solution of the linear equation $u_t + \Lambda^\alpha u = 0$.

The function $c + H_\alpha(x, t) \int_{\mathbb{R}} m(dx)$ is nothing else but the solution to $u_t + \Lambda^\alpha u = 0$ with the initial datum U_0 given by (1.9) below. It is well-known that this solution is self-similar with the scaling $H_\alpha(x, t) = H_\alpha(xt^{-1/\alpha}, 1)$, see also the homogeneity property (3.5).

Remark 1.4. An explicit estimate of the rate of the convergence from Corollary 1.3 can be derived under the additional assumption $\int_{\mathbb{R}} |z| |m|(dz) < \infty$, see inequality (5.5) below.

1.3. Convergence results in the case $\alpha = 1$. In this case, we use the uniqueness result from [1] combined with a standard scaling technique to show that equation (1.1) has self-similar solutions.

Theorem 1.5. (Existence of self-similar solutions)

Assume $\alpha = 1$. The unique entropy solution $U = U(x, t)$ of the initial value problem (1.1)–(1.2) with the initial condition

$$(1.9) \quad U_0(x) \equiv \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases}$$

is self-similar, i.e. it has the form $U(x, t) = U(x/t, 1)$ for all $x \in \mathbb{R}$ and all $t > 0$.

Remark 1.6. Note that the function U_0 from (1.9) is of the form (1.3) for the measure $m = (u_+ - u_-)\delta_0$ where δ_0 denotes the Dirac mass at 0.

Our second main convergence result states that the self-similar solution $U = U(x, t)$ describes the large time asymptotics of other solutions to (1.1)–(1.2).

Theorem 1.7. (Convergence toward the self-similar solution)

Let $\alpha = 1$. Let $u = u(x, t)$ be the entropy solution to problem (1.1)–(1.2) corresponding to the initial condition of the form (1.3) satisfying (1.4)–(1.5). Denote by $U = U(x, t)$ the self-similar solution from Theorem 1.5. For every $p \in [1, \infty]$ there exists a constant $C = C(p) > 0$ such that

$$\|u(t) - U(t)\|_p \leq Ct^{-\frac{p-1}{p}} \|u_0 - U_0\|_1$$

for all $t > 0$.

1.4. Qualitative results in the case $\alpha = 1$. Let us complete the result stated in Theorem 1.7 by listing main qualitative properties of the profile $U(1)$.

Theorem 1.8. (Qualitative properties of the self-similar profile)

The self-similar solution $U(x, t) = U(\frac{x}{t}, 1)$ from Theorem 1.5 enjoys the following properties:

- p1. (Regularity) The function $U(1) = U(x, 1)$ is Lipschitz-continuous.
- p2. (Monotonicity and limits) $U(1)$ is non-decreasing and satisfies

$$\lim_{x \rightarrow \pm\infty} U(x, 1) = u_{\pm}.$$

- p3. (Symmetry) For all $y \in \mathbb{R}$, we have

$$U(\bar{c} + y, 1) = \bar{c} - U(\bar{c} - y, 1) \quad \text{where} \quad \bar{c} \equiv \frac{u_- + u_+}{2}.$$

- p4. (Convex/concave) $U(1)$ is convex (resp. concave) on $(-\infty, \bar{c}]$ (resp. on $[\bar{c}, +\infty)$).
- p5. (Decay at infinity) We have

$$U_x(x, 1) \sim \frac{u_+ - u_-}{2\pi^2} |x|^{-2} \quad \text{as} \quad |x| \rightarrow \infty.$$

Actually, the profile $U(1) = U(x, 1)$ is expected to be C_b^∞ or analytic, due to recent regularity results [14, 7, 16] for the critical fractal Burgers equation with $\alpha = 1$. It was shown that the solution is smooth whenever u_0 is either periodic or from $L^2(\mathbb{R})$ or from a critical Besov space. Unfortunately, we do not know if those results can be adapted to any initial condition from $L^\infty(\mathbb{R})$.

Property p3 implies that $U(x(t), t)$ is a constant equal to \bar{c} along the characteristic $x(t) = \bar{c}t$, with the symmetry

$$U(\bar{c}t + y, t) = \bar{c} - U(\bar{c}t - y, t)$$

for all $t > 0$ and $y \in \mathbb{R}$. Thus, the real number \bar{c} can be interpreted as a mean celerity of the profile $U(t)$, which is the same mean celerity as for the rarefaction wave in (1.7).

In property p5, we obtain the decay at infinity which is the same as for the fundamental solutions $p_1(x, t) = t^{-1}p_1(\frac{x}{t}, 1)$ of the linear equation $u_t + \Lambda^1 u = 0$, given by the explicit formula

$$(1.10) \quad p_1(x, 1) = \frac{2}{1 + 4\pi^2 x^2}.$$

Following the terminology introduced in [6], one may say that property p5 expresses a far field asymptotics and is somewhere in relation with the results in [6] for fractal conservation laws with $\alpha \in (1, 2)$, where the Duhamel principle plays a crucial role. This principle is less convenient in the critical case $\alpha = 1$, and our proof of p5 does not use it.

Finally, if $u_- = 0$ and $u_+ - u_- = 1$, property p2 means that $U(1)$ is the cumulative distribution function of some probability law \mathcal{L} with density $U_x(1)$. Property p3 ensures that \mathcal{L} is symmetrically distributed around its median \bar{c} ; notice that any random variable with law \mathcal{L} has no expectation, because of property p5. Properties p4-p5 make precise that the density of \mathcal{L} decays around \bar{c} with the same rate at infinity as for the Cauchy law with density $p_1(x, 1)$.

The probability distributions of both laws around their respective medians can be compared as follows.

Theorem 1.9. (Comparison with the Cauchy law)

Let \mathcal{L} be the probability law with density $U_x(1)$, where $U = U(x, t)$ is the self-similar solution defined in Theorem 1.5, with $u_- = 0$ and $u_+ = 1$. Let X (resp. Y) be a real random variable on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with law \mathcal{L} (resp. the Cauchy law (1.10) (with zero median)). Then, we have for all $r > 0$

$$\mathbb{P}(|X - \bar{c}| < r) < \mathbb{P}(|Y - 0| < r)$$

where \bar{c} denotes the median of X .

Remark 1.10. More can be said in order to compare random variables $X - \bar{c}$ and Y . Indeed, their cumulative distribution functions satisfy $F_{X-\bar{c}}(x) = F_Y(x) - g(x)$ where g is a positive and explicit function depending the self-similar solution of (1.1) (see Eq. (6.26)).

1.5. Probabilistic interpretation of results for $\alpha \in (0, 2)$. To summarize, let us emphasize the probabilistic meaning of the complete asymptotic study of the fractal Burgers equation we have now in hands. We have already mentioned that the solution u of (1.1)–(1.2) supplemented with the initial datum of the form (1.3) with $c = 0$ and with a probability measure m on \mathbb{R} is the cumulative distribution function for every $t \geq 0$. This family of probabilities defined by problem (1.1)–(1.2) converges, as $t \rightarrow \infty$, toward

- the uniform distribution on the interval $[0, t]$ if $1 < \alpha \leq 2$ (see the result from [12] recalled in inequality (1.6) above);

- the one parameter family of laws constructed in Theorem 1.5 if $\alpha = 1$ (see Theorem 1.7);
- the symmetric α -stable laws $p_\alpha(t)$ if $0 < \alpha < 1$ (*cf.* Theorem 1.1 and Corollary 1.3).

1.6. Organization of the article. The remainder of this paper is organized as follows. In the next section, we recall the notion of entropy solutions to (1.1)-(1.2) with $\alpha \in (0, 1]$. Results on the regularized equation (*i.e.* equation (1.1) with an additional term $-\varepsilon u_{xx}$ on the left-hand-side) are gathered in Section 3. The convergence of solutions as $\varepsilon \rightarrow 0$ to the regularized problem is studied in Section 4. The main asymptotic results for (1.1)-(1.2) are proved in Section 5 by passage to the limit as ε goes to zero. Section 6 is devoted to the qualitative study of the self-similar profile for $\alpha = 1$. Finally, technical lemmata used in proofs are gathered in Appendix A.

2. ENTROPY SOLUTIONS FOR $0 < \alpha \leq 1$

2.1. Lévy-Khintchine's representation of Λ^α . It is well-known (see *e.g.* [9]) that the operator $\Lambda^\alpha = (-\partial^2/\partial x^2)^{\alpha/2}$ for $\alpha \in (0, 2)$ has an integral representation: for every Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$ and each $r > 0$, we have

$$(2.1) \quad \Lambda^\alpha \varphi = \Lambda_r^{(\alpha)} \varphi + \Lambda_r^{(0)} \varphi,$$

where the integro-differential operators $\Lambda_r^{(\alpha)}$ and $\Lambda_r^{(0)}$ are defined by

$$(2.2) \quad \Lambda_r^{(\alpha)} \varphi(x) \equiv -G_\alpha \int_{|z| \leq r} \frac{\varphi(x+z) - \varphi(x) - \varphi_x(x)z}{|z|^{1+\alpha}} dz,$$

$$(2.3) \quad \Lambda_r^{(0)} \varphi(x) \equiv -G_\alpha \int_{|z| > r} \frac{\varphi(x+z) - \varphi(x)}{|z|^{1+\alpha}} dz,$$

where $G_\alpha = \frac{\alpha \Gamma(\frac{1+\alpha}{2})}{2\pi^{\frac{1}{2}+\alpha} \Gamma(1-\frac{\alpha}{2})} > 0$ and Γ is Euler's function. On the basis of this formula, we can extend the domain of definition of Λ^α and consider $\Lambda_r^{(0)}$ and $\Lambda_r^{(\alpha)}$ as the operators

$$\Lambda_r^{(0)} : C_b(\mathbb{R}) \rightarrow C_b(\mathbb{R}) \text{ and } \Lambda_r^{(\alpha)} : C_b^2(\mathbb{R}) \rightarrow C_b(\mathbb{R});$$

hence, $\Lambda^\alpha : C_b^2(\mathbb{R}) \rightarrow C_b(\mathbb{R})$.

Let us recall some properties on these operators. First, the so-called Kato inequality can be generalized to Λ^α for each $\alpha \in (0, 2]$: let $\eta \in C^2(\mathbb{R})$ be convex and $\varphi \in C_b^2(\mathbb{R})$, then

$$(2.4) \quad \Lambda^\alpha \eta(u) \leq \eta'(u) \Lambda^\alpha u.$$

Note that for $\alpha = 2$ we have

$$-(\eta(u))_{xx} = -\eta''(u)u_x^2 - \eta'(u)u_{xx} \leq -\eta'(u)u_{xx} \quad \text{since } \eta'' \geq 0.$$

If $\alpha \in (0, 2)$, inequality (2.4) is the direct consequence of the integral representation (2.1)–(2.3) and of the inequalities

$$(2.5) \quad \Lambda_r^{(0)} \eta(u) \leq \eta'(u) \Lambda_r^{(0)} u \quad \text{and} \quad \Lambda_r^{(\alpha)} \eta(u) \leq \eta'(u) \Lambda_r^{(\alpha)} u,$$

resulting from the convexity of the function η .

Finally, these operators satisfy the integration by parts formula: for all $u \in C_b^2(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$(2.6) \quad \int_{\mathbb{R}} \varphi \Lambda u \, dx = \int_{\mathbb{R}} u \Lambda \varphi \, dx,$$

where $\Lambda \in \{\Lambda_r^{(0)}, \Lambda_r^{(\alpha)}, \Lambda^\alpha\}$ for every $\alpha \in (0, 2]$ and all $r > 0$. Notice that $\Lambda\varphi \in L^1(\mathbb{R})$, since it is obvious from (2.2)–(2.3) that $\Lambda_r^{(\alpha)} : W^{2,1}(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ and $\Lambda_r^{(0)} : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$.

Detailed proofs of all these properties are based on the representation (2.1)–(2.3) and are written *e.g.* in [1].

2.2. Existence of the entropy solution. It was shown in [2] (see also [14]) that solutions of the initial value problem for the fractal conservation law

$$(2.7) \quad u_t + \Lambda^\alpha u + (f(u))_x = 0, \quad x \in \mathbb{R}, t > 0,$$

$$(2.8) \quad u(0, x) = u_0(x),$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz-continuous, can become discontinuous in finite time if $0 < \alpha < 1$. Hence, in order to deal with discontinuous solutions, the notion of entropy solutions in the sense of Kruzhkov was extended in [1] to fractal conservation laws (2.7)–(2.8) (see also [13] for the recent generalization to Lévy mixed hyperbolic/parabolic equations). Here, the crucial role is played by the Lévy-Khintchine's representation (2.1)–(2.3) of the operator Λ^α .

Definition 2.1. *Let $0 < \alpha \leq 1$ and $u_0 \in L^\infty(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an entropy solution to (2.7)–(2.8) if for all $\varphi \in \mathcal{D}(\mathbb{R} \times [0, \infty))$, $\varphi \geq 0$, $\eta \in C^2(\mathbb{R})$ convex, $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi' = \eta' f'$, and $r > 0$, we have*

$$\begin{aligned} \int_{\mathbb{R}} \int_0^\infty \left(\eta(u) \varphi_t + \phi(u) \varphi_x - \eta(u) \Lambda_r^{(\alpha)} \varphi - \varphi \eta'(u) \Lambda_r^{(0)} u \right) dx dt \\ + \int_{\mathbb{R}} \eta(u_0(x)) \varphi(x, 0) dx \geq 0. \end{aligned}$$

Note that, due to formula (2.3), the quantity $\Lambda_r^{(0)} u$ in the above inequality is well-defined for any bounded function u .

The notion of entropy solutions allows us to solve the fractal Burgers equation for the range of exponent $\alpha \in (0, 1]$.

Theorem 2.2 ([1]). *Assume that $0 < \alpha \leq 1$ and $u_0 \in L^\infty(\mathbb{R})$. There exists a unique entropy solution $u = u(x, t)$ to problem (2.7)–(2.8). This solution u belongs to $C([0, \infty); L_{loc}^1(\mathbb{R}))$ and satisfies $u(0) = u_0$. Moreover, we have the following maximum principle: $\text{ess inf } u_0 \leq u \leq \text{ess sup } u_0$.*

If $\alpha \in (1, 2]$, all solutions to (2.7)–(2.8) with bounded initial conditions are smooth and global-in-time (see [8, 14, 15]). On the other hand, the occurrence of discontinuities in finite time of entropy solutions to (2.7)–(2.8) with $\alpha = 1$ seems to be unclear. As mentioned in the introduction, regularity results have recently been obtained [14, 7, 16] for a large class of initial conditions which, unfortunately, does not include general L^∞ initial data. Nevertheless, Theorem 2.2 provides the existence and the uniqueness of a global-in-time entropy solution even for the critical case $\alpha = 1$.

3. REGULARIZED PROBLEM

In this section, we gather properties of solutions to the Cauchy problem for the regularized fractal Burgers equation with $\varepsilon > 0$

$$(3.1) \quad u_t^\varepsilon + \Lambda^\alpha u^\varepsilon - \varepsilon u_{xx}^\varepsilon + u^\varepsilon u_x^\varepsilon = 0, \quad x \in \mathbb{R}, t > 0,$$

$$(3.2) \quad u^\varepsilon(x, 0) = u_0(x).$$

Our purpose is to derive asymptotic stability estimates of a solution $u^\varepsilon = u^\varepsilon(x, t)$ (uniform in ε) that will be valid for (1.1)–(1.2) after passing to the limit $\varepsilon \rightarrow 0$.

Most of the results of this section are inspired from [12] and, when it is the case, the reader is referred to the corresponding proofs of that work.

Below, we will use the following integral formulation of the initial value problem (3.1)-(3.2)

$$(3.3) \quad u^\varepsilon(t) = S_\alpha^\varepsilon(t)u_0 - \int_0^t S_\alpha^\varepsilon(t-\tau)u^\varepsilon(\tau)u_x^\varepsilon(\tau) d\tau,$$

where $\{S_\alpha^\varepsilon(t)\}_{t>0}$ denotes the semi-group of linear operators which infinitesimal generator is $-\Lambda^\alpha u^\varepsilon + \varepsilon u_{xx}^\varepsilon$.

If, for each $\alpha \in (0, 2]$, the function $p_\alpha(x, t)$ denotes the fundamental solution of the linear equation $u_t + \Lambda^\alpha u = 0$, then

$$(3.4) \quad S_\alpha^\varepsilon(t)u_0 = p_\alpha(t) * p_2(\varepsilon t) * u_0.$$

It is well-known that $p_\alpha(x, t)$ can be represented via the Fourier transform $\widehat{p}_\alpha(\xi, t) = e^{-t|\xi|^\alpha}$. In particular,

$$(3.5) \quad p_\alpha(x, t) = t^{-\frac{1}{\alpha}} P_\alpha(xt^{-\frac{1}{\alpha}}),$$

where P_α is the inverse Fourier transform of $e^{-|\xi|^\alpha}$. For every $\alpha \in (0, 2]$ the function P_α is smooth, non-negative, $\int_{\mathbb{R}} P_\alpha(y) dy = 1$, and satisfies the estimates (optimal for $\alpha \neq 2$)

$$(3.6) \quad 0 < P_\alpha(x) \leq C(1 + |x|)^{-(\alpha+1)} \quad \text{and} \quad |(P_\alpha)_x(x)| \leq C(1 + |x|)^{-(\alpha+2)}$$

for a constant C and all $x \in \mathbb{R}$.

One can see that problem (3.1)–(3.2) admits a unique global-in-time smooth solution that satisfies the maximum principle.

Theorem 3.1. *Let $\alpha \in (0, 2]$, $\varepsilon > 0$, and $u_0 \in L^\infty(\mathbb{R})$. There exists the unique solution $u^\varepsilon = u^\varepsilon(x, t)$ to problem (3.1)–(3.2) in the following sense: for all $T > 0$,*

- $u^\varepsilon \in C_b((0, T) \times \mathbb{R})$ and $u^\varepsilon \in C_b^\infty((a, T) \times \mathbb{R})$ for all $a \in (0, T)$,
- u^ε satisfies equation (3.1) on $(0, T) \times \mathbb{R}$,
- $\lim_{t \rightarrow 0} u^\varepsilon(t) = u_0$ in $L^\infty(\mathbb{R})$ weak-* and in $L_{loc}^p(\mathbb{R})$ for all $p \in [1, \infty)$.

Moreover, the following inequalities hold true:

$$(3.7) \quad \text{ess inf } u_0 \leq u(t) \leq \text{ess sup } u_0 \quad \text{for all } t > 0.$$

Proof. Here, the results from [8] can be easily modified in order to get the existence and the regularity of solutions to (3.1)–(3.2) with $\varepsilon > 0$. \square

The next proposition provides an estimate on the gradient of u^ε .

Proposition 3.2. *Assume that $0 < \alpha \leq 2$, $\varepsilon > 0$, and u_0 is of the form (1.3) with $c \in \mathbb{R}$ and a finite non-negative measure $m(dx)$ on \mathbb{R} . Denote by $u^\varepsilon = u^\varepsilon(x, t)$ the unique solution of problem (3.1)–(3.2). Then*

- $u_x^\varepsilon(x, t) \geq 0$ for all $x \in \mathbb{R}$ and $t > 0$,
- for every $p \in [1, \infty]$ there exists a constant $C = C(p)$ such that and for all $t > 0$, we have

$$(3.8) \quad \|u_x^\varepsilon(t)\|_p \leq C \min\{t^{-(1-\frac{1}{p})} \|m\|_p^{\frac{1}{p}}, t^{-\frac{1}{\alpha}(1-\frac{1}{p})} \|m\|\}.$$

Proof. The proof that $u_x^\varepsilon(x, t) \geq 0$ for all $x \in \mathbb{R}$ and $t > 0$ is almost identical as [12, Proof of Thm. 2.3.i].

To show the decay estimate (3.8), it suffices to modify slightly the argument from [12] as follows. We multiply the equation for $v = u_x^\varepsilon$

$$v_t + \Lambda^\alpha v - \varepsilon v_{xx} + (u^\varepsilon u_x^\varepsilon)_x = 0$$

by v^{p-1} to obtain (cf. [12, eq. (2.19)])

$$(3.9) \quad \frac{1}{p} \frac{d}{dt} \|v(t)\|_p^p + \int_{\mathbb{R}} v^{p-1} \Lambda^\alpha v \, dx - \varepsilon \int_{\mathbb{R}} v^{p-1} v_{xx} \, dx + \frac{p-1}{p} \int_{\mathbb{R}} v^{p+1} \, dx = 0.$$

Note that, due to the Kato inequality (2.4) (see also [12, inequality (2.7)]), the second term on the left-hand side is nonnegative. The same property holds true for the third term because integrating by parts we have

$$(3.10) \quad -\varepsilon \int_{\mathbb{R}} v_{xx}(x, t) \varphi(v(x, t)) \, dx \geq 0$$

for any nondecreasing function φ .

Now, for the proof of the first decay rate on the right-hand side of (3.8), one should use the following inequality resulting immediately from (3.9)

$$\frac{1}{p} \frac{d}{dt} \|v(t)\|_p^p + \frac{p-1}{p} \int_{\mathbb{R}} v^{p+1} \, dx \leq 0.$$

and follow [12, Proof of Thm 2.3.ii]. On the other hand, for the second decay estimate in (3.8), one should deal with the inequality

$$\frac{1}{p} \frac{d}{dt} \|v(t)\|_p^p + \int_{\mathbb{R}} v^{p-1} \Lambda^\alpha v \, dx \leq 0$$

and follow estimates from [12, Lemma 3.1]. \square

We can now give asymptotic stability estimates uniform in ε .

Theorem 3.3. *Let $\alpha \in (0, 2]$ and $\varepsilon > 0$. Assume that u^ε and $\widetilde{u}^\varepsilon$ are two solutions of the regularized problem (3.1)–(3.2) with initial conditions u_0 and \widetilde{u}_0 associated with finite signed measures m and \widetilde{m} respectively. Suppose, moreover, that the measure \widetilde{m} of \widetilde{u}_0 is non-negative and $u_0 - \widetilde{u}_0 \in L^1(\mathbb{R})$. Then, for every $p \in [1, \infty]$ there exists a constant $C = C(p) > 0$ independent of ε such that for all $t > 0$*

$$(3.11) \quad \|u^\varepsilon(t) - \widetilde{u}^\varepsilon(t)\|_p \leq Ct^{-\frac{1}{\alpha}(1-\frac{1}{p})} \|u_0 - \widetilde{u}_0\|_1.$$

Proof. First, to show inequality (3.11) for $p = 1$, it suffices to copy computations from [12, Proof of Thm. 2.2]. Next, the proof of (3.11) for $p > 1$ follows the arguments from [12, Proof of Lemma 3.1]. In this reasoning, in the case of solutions of the regularized problem (3.1)–(3.2), we have to deal with the additional term $-\varepsilon v_{xx} = -\varepsilon(u^\varepsilon - \widetilde{u}^\varepsilon)_{xx}$ which can be always skipped in calculations thanks to inequality (3.10). \square

Theorem 3.4. *Let $0 < \alpha < 1$ and $\varepsilon > 0$. Assume that $u_0(x) = c + \int_{-\infty}^x m(dy)$ with $c \in \mathbb{R}$ and m being a finite non-negative measure on \mathbb{R} . Denote $S_\alpha^\varepsilon(t)$ the semigroup of linear operators generated by $-t\Lambda^\alpha + \varepsilon t \partial_x^2$. Then, for every $p \in [1, \infty]$ there exists $C = C(p, \alpha) > 0$ independent of ε such that the solution u^ε to (1.1)–(1.2) satisfies*

$$\|u^\varepsilon(t) - S_\alpha^\varepsilon(t)u_0\|_p \leq C \|u_0\|_\infty \|m\| t^{1-\frac{1}{\alpha}(1-\frac{1}{p})}$$

for all $t > 0$.

Proof. Using the integral equation (3.3) we immediately obtain

$$(3.12) \quad \|u^\varepsilon(t) - S_\alpha^\varepsilon(t)u_0\|_p \leq \int_0^t \|S_\alpha^\varepsilon(t-\tau)u^\varepsilon(\tau)u_x^\varepsilon(\tau)\|_p \, d\tau.$$

Now, we estimate the integral in the right-hand side of (3.12) using the L^p -decay of the semigroup $S_\alpha^\varepsilon(t)$ as well as inequalities (3.7) and (3.8). Indeed, it follows from (3.5)–(3.6) that

$$\|p_2(\varepsilon t)\|_1 = 1 \quad \text{and} \quad \|p_\alpha(t)\|_r = t^{-\frac{1}{\alpha}(1-\frac{1}{r})} \|p_\alpha(1)\|_r$$

for every $r \in [1, \infty]$. Hence, by the Young inequality for the convolution and inequalities (3.7), (3.8), we obtain

$$\begin{aligned}
 (3.13) \quad & \|S_\alpha^\varepsilon(t - \tau)u^\varepsilon(\tau)u_x^\varepsilon(\tau)\|_p \\
 & \leq \|p_\alpha(t - \tau) * (u^\varepsilon(\tau)u_x^\varepsilon(\tau))\|_p \\
 & \leq C(t - \tau)^{-\frac{1}{\alpha}(\frac{1}{q} - \frac{1}{p})} \|u^\varepsilon(\tau)\|_\infty \|u_x^\varepsilon(\tau)\|_q \\
 & \leq C(t - \tau)^{-\frac{1}{\alpha}(\frac{1}{q} - \frac{1}{p})} \|u_0\|_\infty \|m\| \tau^{-\frac{1}{\alpha}(1 - \frac{1}{q})},
 \end{aligned}$$

for all $1 \leq q \leq p \leq \infty$, $t > 0$, $\tau \in (0, t)$, and the constant $C > 0$ independent of t, τ, ε .

Next, we decompose the integral on the right-hand side of (3.12) as follows $\int_0^t \dots d\tau = \int_0^{t/2} \dots d\tau + \int_{t/2}^t \dots d\tau$ and we bound both integrands by using inequality (3.13) either with $q = 1$ or with $q = p$. This leads to inequality

$$\begin{aligned}
 (3.14) \quad & \|u^\varepsilon(t) - S_\alpha^\varepsilon(t)u_0\|_p \\
 & \leq C\|u_0\|_\infty \|m\| \left(\int_0^{t/2} (t - \tau)^{-\frac{1}{\alpha}(1 - \frac{1}{p})} d\tau + \int_{t/2}^t \tau^{-\frac{1}{\alpha}(1 - \frac{1}{p})} \tau \right).
 \end{aligned}$$

Computing both integrals with respect to τ , we complete the proof of Theorem 3.4. \square

4. CONVERGENCE TOWARD THE ENTROPY SOLUTION

It is natural to expect that the solution u^ε of (3.1)–(3.2) converges, as $\varepsilon \rightarrow 0$, toward the entropy solution u to (1.1)–(1.2). Here, we prove this property in the case of fractal conservation laws with general nonlinearities. For a first reading, the reader can omit the proof of Theorem 4.1 and proceed directly to Section 5.

Together with the general fractal conservation law (2.7)–(2.8), we study the associated regularized problem

$$(4.1) \quad u_t^\varepsilon + \Lambda^\alpha u^\varepsilon - \varepsilon u_{xx}^\varepsilon + (f(u^\varepsilon))_x = 0, \quad x \in \mathbb{R}, t > 0,$$

$$(4.2) \quad u^\varepsilon(x, 0) = u_0(x)$$

where $f \in C^\infty(\mathbb{R})$. Hence, by results of [8] (see also Theorem 3.1), problem (4.1)–(4.2) admits the unique, global-in-time, smooth solution u^ε .

The main result of this section reads as follows.

Theorem 4.1. *Assume that $u_0 \in L^\infty(\mathbb{R})$ and $\varepsilon > 0$. Let $u^\varepsilon = u^\varepsilon(x, t)$ be the solution to (4.1)–(4.2) and $u = u(x, t)$ be the entropy solution to (2.7)–(2.8). Then, for every $T > 0$, $u^\varepsilon \rightarrow u$ in $C([0, T]; L_{loc}^1(\mathbb{R}))$ as $\varepsilon \rightarrow 0$.*

Remark 4.2. We assume for simplicity that $f \in C^\infty$ but general Lipschitz-continuous nonlinearities can be considered. More generally, this result holds true in the case of multidimensional fractal conservation laws with source terms $h = h(u, x, t)$ and fluxes $f = f(u, x, t)$ (see [9, 8]). However, the study of such equations would lead to several technical difficulties which we prefer to avoid for the sake of clarity.

4.1. Finite-infinite propagation speed property. Inequality from the following proposition is the starting point to prove Theorem 4.1.

Proposition 4.3. *Let $u_0, \tilde{u}_0 \in L^\infty(\mathbb{R})$ and $\varepsilon > 0$. Let $u^\varepsilon = u^\varepsilon(x, t)$ and $\tilde{u}^\varepsilon = \tilde{u}^\varepsilon(x, t)$ be the solutions to (4.1)–(4.2) with the initial data u_0 and \tilde{u}_0 , resp. Then*

$$(4.3) \quad \int_{-R}^R |u^\varepsilon(x, t) - \tilde{u}^\varepsilon(x, t)| dx \leq \int_{-R-Lt}^{R+Lt} S_\alpha^\varepsilon(t) |u_0 - \tilde{u}_0|(x) dx$$

for all $t > 0$ and $R > 0$, where

$$(4.4) \quad L = \max_{z \in [-M, M]} |f'(z)| \quad \text{and} \quad M = \max \{ \|u_0\|_\infty, \|\widetilde{u}_0\|_\infty \}.$$

Even if this result does not appear in [1], its proof is based on ideas introduced in [1, Thm 3.2]. This is the reason why we only sketch the proof of Proposition 4.3; the reader is referred to [1] for more details.

Sketch of proof of Proposition 4.3. The solution u^ε of (4.1)–(4.2) satisfies

$$(4.5) \quad \int_{\mathbb{R}} \int_a^\infty \left(\eta(u^\varepsilon) \varphi_t + \phi(u^\varepsilon) \varphi_x \right) dx dt \\ + \int_{\mathbb{R}} \int_a^\infty \left(-\eta(u^\varepsilon) \Lambda_r^{(\alpha)} \varphi - \varphi \eta'(u^\varepsilon) \Lambda_r^{(0)} u^\varepsilon \right) dx dt \\ - \varepsilon \int_{\mathbb{R}} \int_a^\infty (\eta(u^\varepsilon))_x \varphi_x dx dt + \int_{\mathbb{R}} \eta(u^\varepsilon(x, a)) \varphi(x, a) dx \geq 0,$$

for all $\varphi \in \mathcal{D}(\mathbb{R} \times [0, \infty))$ non-negative, $\eta \in C^2(\mathbb{R})$ convex, $\phi' = \eta' f'$ and $a, r > 0$. To show this inequality, it suffices to multiply (4.1) by $\eta'(u^\varepsilon) \varphi$, use the Kato inequalities (2.4) and integrate by parts over the domain $\mathbb{R} \times [a, \infty)$. Now, let us introduce the so-called Kruzhkov entropy-flux pairs (η_k, ϕ_k) defined for fixed $k \in \mathbb{R}$ and all $u \in \mathbb{R}$ by

$$\eta_k(u) \equiv |u - k| \quad \text{and} \quad \phi_k(u) \equiv \text{sign}(u - k) (f(u) - f(k)),$$

where “sign” denotes the sign function defined by

$$\text{sign}(u) \equiv \begin{cases} 1, & u > 0, \\ -1, & u < 0, \\ 0, & u = 0. \end{cases}$$

Consider a sequence $\{\eta_k^n\}_{n \in \mathbb{N}} \subset C^2(\mathbb{R})$ of convex functions converging toward η_k locally uniformly on \mathbb{R} and such that $(\eta_k^n)' \rightarrow \text{sign}(\cdot - k)$ pointwise on \mathbb{R} by being bounded by 1, as $n \rightarrow \infty$. The associated fluxes $\phi_k^n(u) \equiv \int_k^u \eta_k^n(\tau) f'(\tau) d\tau$ then converge toward ϕ_k pointwise on \mathbb{R} , as $n \rightarrow \infty$, by being pointwise bounded by $|\phi_k^n(u)| \leq \text{sign}(u - k) \int_k^u |f'(\tau)| d\tau$. By the dominated convergence theorem, the passage to the limit in (4.5) with $(\eta, \phi) = (\eta_k^n, \phi_k^n)$ gives

$$(4.6) \quad \int_{\mathbb{R}} \int_a^\infty \left(|u^\varepsilon - k| \varphi_t + \text{sign}(u^\varepsilon - k) (f(u^\varepsilon) - f(k)) \varphi_x \right) dx dt \\ + \int_{\mathbb{R}} \int_a^\infty \left(-|u^\varepsilon - k| \Lambda_r^{(\alpha)} \varphi - \varphi \text{sign}(u^\varepsilon - k) \Lambda_r^{(0)} u^\varepsilon \right) dx dt \\ - \varepsilon \int_{\mathbb{R}} \int_a^\infty \text{sign}(u^\varepsilon - k) u_x^\varepsilon \varphi_x dx dt + \int_{\mathbb{R}} |u^\varepsilon(x, a) - k| \varphi(x, a) dx \geq 0,$$

for all $\varphi \in \mathcal{D}(\mathbb{R} \times [0, \infty))$ non-negative, $a, r > 0$ and $k \in \mathbb{R}$. In the same way, similar inequalities hold true for $\widetilde{u}^\varepsilon$.

On the basis of these inequalities, we claim that the well-known doubling variable technique of Kruzhkov allows us to compare u_ε and $\widetilde{u}_\varepsilon$. To do so, we have to copy almost the same computations from [1], since the beginning of [1, Subsection 4.1] until [1, equation (4.11)] with $u = u^\varepsilon$ and $v = \widetilde{u}^\varepsilon$. The only difference comes from the term $-\varepsilon \int_{\mathbb{R}} \int_a^\infty \text{sign}(u^\varepsilon - k) u_x^\varepsilon \varphi_x dx dt$ in (4.6) and the term $-\varepsilon \int_{\mathbb{R}} \int_a^\infty \text{sign}(\widetilde{u}^\varepsilon - k) \widetilde{u}_x^\varepsilon \varphi_x dx dt$ in the entropy inequalities of $\widetilde{u}^\varepsilon$. But, these new terms do not present any particular difficulty, since u_ε and $\widetilde{u}_\varepsilon$ are smooth. Arguing as in [1], one can

show that for all $\phi \in \mathcal{D}(\mathbb{R} \times [0, \infty))$ non-negative and $a > 0$,

$$\begin{aligned} & \int_{\mathbb{R}} \int_a^{\infty} |u^\varepsilon - \widetilde{u}^\varepsilon| (\phi_t + L|\phi_x| - \Lambda^\alpha \phi) \\ & \quad - \varepsilon \int_{\mathbb{R}} \int_a^{\infty} \text{sign}(u^\varepsilon - \widetilde{u}^\varepsilon) (u^\varepsilon - \widetilde{u}^\varepsilon)_x \phi_x \\ & \quad \quad \quad + \int_{\mathbb{R}} |u^\varepsilon(x, a) - \widetilde{u}^\varepsilon(x, a)| \phi(x, a) dx \geq 0, \end{aligned}$$

where L is defined in (4.4). Since $|u^\varepsilon - \widetilde{u}^\varepsilon|$ is Lipschitz-continuous on $\mathbb{R} \times [a, \infty)$, its a.e. derivative is equal to its distribution derivative with $\text{sign}(u^\varepsilon - \widetilde{u}^\varepsilon) (u^\varepsilon - \widetilde{u}^\varepsilon)_x = (|u^\varepsilon - \widetilde{u}^\varepsilon|)_x$. By integrating by parts, we deduce that

$$\begin{aligned} & \int_{\mathbb{R}} \int_a^{\infty} |u^\varepsilon - \widetilde{u}^\varepsilon| (\phi_t + L|\phi_x| - g[\phi]) dxdt \\ & \quad \quad \quad + \int_{\mathbb{R}} |u^\varepsilon(x, a) - \widetilde{u}^\varepsilon(x, a)| \phi(x, a) dx \geq 0, \end{aligned}$$

where $g[\phi] \equiv (\Lambda^\alpha - \varepsilon \partial_x^2) \phi$. Passing to the limit as $a \rightarrow 0$, thanks to the continuity with values in $L^1_{loc}(\mathbb{R})$ of u^ε and $\widetilde{u}^\varepsilon$ in Theorem 3.1, one can prove that for all non-negative $\phi \in \mathcal{D}(\mathbb{R} \times [0, \infty))$

$$(4.7) \quad \begin{aligned} & \int_{\mathbb{R}} \int_0^{\infty} |u^\varepsilon - \widetilde{u}^\varepsilon| (\phi_t + L|\phi_x| - g[\phi]) dxdt \\ & \quad \quad \quad + \int_{\mathbb{R}} |u_0(x) - \widetilde{u}_0(x)| \phi(x, 0) dx \geq 0. \end{aligned}$$

This is almost the same equation as that in [1, equation (4.11)] with the diffusive operator $g = \Lambda^\alpha - \varepsilon \partial_x^2$ instead of $g = \Lambda^\alpha$. Hence, we can argue exactly as in [1, Subsection 4.2] replacing the kernel of Λ^α by the kernel of the new operator $\Lambda^\alpha - \varepsilon \partial_x^2$. This gives the desired inequality (4.3) in place of the inequality [1, equation (3.1)]. \square

Proof of Theorem 4.1. Now, we are in a position to prove the convergence result in Theorem 4.1. The proof follows two steps: first we show the relative compactness of the family of functions $\mathcal{F} \equiv \{u^\varepsilon : \varepsilon \in (0, 1]\}$ and, next, we pass to the limit in entropy inequalities.

Step 1: compactness. Let us prove that

$$(4.8) \quad \mathcal{F} \text{ is relatively compact in } F \equiv C([0, T]; L^1([-R, R]))$$

for all $T, R > 0$. The space F being a Banach space, the statement (4.8) is equivalent to the precompactness of \mathcal{F} :

$$(4.9) \quad \begin{aligned} & \forall \mu > 0 \quad \exists \mathcal{F}_\mu \subseteq F \quad \text{relatively compact such that} \\ & \quad \quad \quad \lim_{\mu \rightarrow 0} \sup_{u^\varepsilon \in \mathcal{F}} \text{dist}_F(u^\varepsilon, \mathcal{F}_\mu) = 0. \end{aligned}$$

To construct \mathcal{F}_μ , we consider an approximation of the Dirac mass

$$\rho_\mu(x) \equiv \mu^{-1} \rho(\mu^{-1}x)$$

with a smooth, non-negative function $\rho = \rho(x)$, supported in $[-1, 1]$ and such that $\int_{\mathbb{R}} \rho(x) dx = 1$. Then we define

$$\mathcal{F}_\mu \equiv \{u_\mu^\varepsilon : \varepsilon \in (0, 1]\},$$

where $u_\mu^\varepsilon \equiv u^\varepsilon *_x \rho_\mu$ and $*_x$ denotes the convolution product with respect to the space variable.

First, we have to prove that \mathcal{F}_μ is relatively compact in F . By estimate (3.7), it is clear that

$$(4.10) \quad \|u_\mu^\varepsilon\|_\infty \leq \|u_0\|_\infty \quad \text{and} \quad \|(u_\mu^\varepsilon)_x\|_\infty \leq \|u_0\|_\infty \|(\rho_\mu)_x\|_1.$$

Moreover, using equation (4.1) satisfied by u^ε we obtain

$$(4.11) \quad (u_\mu^\varepsilon)_t = -\Lambda^\alpha u_\mu^\varepsilon + \varepsilon(u_\mu^\varepsilon)_{xx} - (f(u^\varepsilon)_x) *_x \rho_\mu = 0.$$

Applying the equalities $\Lambda^\alpha u_\mu^\varepsilon = \Lambda^\alpha(u^\varepsilon *_x \rho_\mu) = u^\varepsilon *_x (\Lambda^\alpha \rho_\mu)$ we see that

$$\|\Lambda^\alpha u_\mu^\varepsilon\|_\infty \leq \|u^\varepsilon\|_\infty \|\Lambda^\alpha \rho_\mu\|_1 \leq \|u_0\|_\infty \|\Lambda^\alpha \rho_\mu\|_1.$$

The same way, one can prove that

$$\|(u_\mu^\varepsilon)_{xx}\|_\infty \leq \|u_0\|_\infty \|(\rho_\mu)_{xx}\|_1 \quad \text{and} \quad \|(f(u^\varepsilon)_x) *_x \rho_\mu\|_\infty \leq C(\|u_0\|_\infty) \|(\rho_\mu)_x\|_1.$$

Consequently, it follows from equation (4.11) that for every fixed $\mu > 0$, the time derivative of u_μ^ε is bounded independently of $\varepsilon \in (0, 1]$. By (4.10) and the Ascoli-Arzelà Theorem, we infer that \mathcal{F}_μ is relatively compact in $C_b([-R, R] \times [0, T])$ and, *a fortiori*, in F .

Next, we have to prove that $\lim_{\mu \rightarrow 0} \sup_{u^\varepsilon \in \mathcal{F}} \text{dist}_F(u^\varepsilon, \mathcal{F}_\mu) = 0$. Applying Theorem 4.3 to the following simple inequality

$$\|u^\varepsilon(t) - u_\mu^\varepsilon(t)\|_{L^1([-R, R])} \leq \int_{-R}^R \int_{-\mu}^\mu |u^\varepsilon(x, t) - u^\varepsilon(x - y, t)| \rho_\mu(y) \, dx dy$$

we get

$$\begin{aligned} \|u^\varepsilon(t) - u_\mu^\varepsilon(t)\|_{L^1([-R, R])} &\leq \sup_{|y| \leq \mu} \int_{-R}^R |u^\varepsilon(x, t) - u^\varepsilon(x - y, t)| \, dx, \\ &\leq \sup_{|y| \leq \mu} \int_{-R-Lt}^{R+Lt} S_\alpha^\varepsilon(t) v_0^y(x) \, dx, \end{aligned}$$

where $v_0^y(x) = |u_0(x) - u_0(x - y)|$. Consequently, by Lemma A.1 in Appendix A, we see that there exists a modulus of continuity ω such that for all $r > 0$ and $\varepsilon \in (0, 1]$

$$\|u^\varepsilon - u_\mu^\varepsilon\|_F \leq \sup_{|y| \leq \mu} \int_{-R-LT-r}^{R+LT+r} v_0^y(x) \, dx + \|v_0^y\|_\infty \omega(1/r).$$

The continuity of the translation in L^1 implies that

$$\lim_{\mu \rightarrow 0} \sup_{|y| \leq \mu} \int_{-R-LT-r}^{R+LT+r} v_0^y(x) \, dx = 0.$$

Hence, it is clear that $\lim_{\mu \rightarrow 0} \sup_{\varepsilon \in (0, 1]} \|u^\varepsilon - u_\mu^\varepsilon\|_F = 0$, which proves (4.9) and thus (4.8).

Conclusion: passage to the limit. It follows from the first step that there exists $v \in C([0, \infty); L^1_{loc}(\mathbb{R}))$ such that $\lim_{\varepsilon \rightarrow 0} u^\varepsilon = v$ (up to a subsequence) in $C([0, T]; L^1_{loc}(\mathbb{R}))$ for all $T > 0$. Passing to another subsequence, if necessary, we can assume that $u^\varepsilon \rightarrow v$ a.e. From inequality (3.7), we deduce that $v \in L^\infty(\mathbb{R} \times (0, \infty))$. What we have to prove is that $v = u$, however, by the uniqueness of entropy solutions (*cf.* Theorem 2.2), it suffices to show that v is an entropy solution to (2.7)–(2.8).

Let $\eta \in C^2(\mathbb{R})$ be convex, $\phi' = \eta' f'$ and $r > 0$. Integrating by parts the term $-\varepsilon \int_{\mathbb{R}} \int_a^\infty (\eta(u^\varepsilon))_x \varphi_x \, dx dt$ in (4.5) and passing to the limit $a \rightarrow 0$ in this inequality, we get

$$\begin{aligned} &\int_{\mathbb{R}} \int_0^\infty \left(\eta(u^\varepsilon) \varphi_t + \phi(u^\varepsilon) \varphi_x - \eta(u^\varepsilon) \Lambda_r^{(\alpha)} \varphi - \varphi \eta'(u^\varepsilon) \Lambda_r^{(0)} u^\varepsilon \right) \, dx dt \\ &\quad + \int_{\mathbb{R}} \eta(u_0(x)) \varphi(x, 0) \, dx \geq -\varepsilon \int_{\mathbb{R}} \int_0^\infty \eta(u^\varepsilon) \varphi_{xx} \, dx dt. \end{aligned}$$

Finally, let us recall that $u^\varepsilon \rightarrow v$ a.e. as $\varepsilon \rightarrow 0$ and that u^ε is bounded in L^∞ -norm by $\|u_0\|_\infty$. Hence, the Lebesgue dominated convergence theorem allows us to pass to the limit, as $\varepsilon \rightarrow 0$, in the inequality above and to deduce that

$$\begin{aligned} \int_{\mathbb{R}} \int_0^\infty \left(\eta(v) \varphi_t + \phi(v) \varphi_x - \eta(v) \Lambda_r^{(\alpha)} \varphi - \varphi \eta'(v) \Lambda_r^{(0)} v \right) dx dt \\ + \int_{\mathbb{R}} \eta(u_0(x)) \varphi(x, 0) dx \geq 0. \end{aligned}$$

Hence, according to Definition 2.1 and Theorem 2.2, the function v is the unique entropy solution to (2.7)–(2.8). The proof of Theorem 4.1 is complete. \square

5. PASSAGE TO THE LIMIT $\varepsilon \rightarrow 0$ AND ASYMPTOTIC STUDY

This section is devoted to the proof of Theorems 1.1–1.7 and Corollary 1.3. Below, we systematically use Theorem 4.1 in order to pass to the limit $\varepsilon \rightarrow 0$ in estimates from Section 3.

Proof of Theorem 1.5. The existence of the solution $U = U(x, t)$ to equation (1.1) with $\alpha = 1$ supplemented with the initial condition (1.9) is provided by Theorem 2.2. To obtain the self-similar form of U , we follow a standard argument based on the uniqueness result from Theorem 2.2. Observe that if U is the solution to (1.1), the rescaled function $U^\lambda(x, t) = U(\lambda x, \lambda t)$ is the solution for every $\lambda > 0$, too. Since, the initial datum (1.9) is invariant under the rescaling $U_0^\lambda(x) = U_0(\lambda x)$, by the uniqueness, we obtain that for all $\lambda > 0$, $U(x, t) = U(\lambda x, \lambda t)$ for a.e. $(x, t) \in \mathbb{R} \times (0, +\infty)$. \square

Theorem 1.7 is a particular case of the following more general result which is an immediate corollary of Theorem 3.3 by passing to the limit as $\varepsilon \rightarrow 0$.

Corollary 5.1. *Let $0 < \alpha \leq 1$. Assume that u and \tilde{u} are two entropy solutions of (1.1)–(1.2) with initial conditions u_0 and \tilde{u}_0 of the form (1.3) with finite signed measures m and \tilde{m} . Suppose, moreover, that the measure \tilde{m} of \tilde{u}_0 is non-negative and $u_0 - \tilde{u}_0 \in L^1(\mathbb{R})$. Then for every $p \in [1, \infty]$ there exists a constant $C = C(p) > 0$ such that for all $t > 0$*

$$(5.1) \quad \|u(t) - \tilde{u}(t)\|_p \leq Ct^{-\frac{1}{\alpha}(1-\frac{1}{p})} \|u_0 - \tilde{u}_0\|_1.$$

Proof. Denote by u^ε and \tilde{u}^ε the solutions to the regularized equation (3.1) with the initial conditions u_0 and \tilde{u}_0 . By Theorem 4.1 and inequality (3.7), we know that $\lim_{\varepsilon \rightarrow 0} u^\varepsilon(t) = u(t)$ and $\lim_{\varepsilon \rightarrow 0} \tilde{u}^\varepsilon(t) = \tilde{u}(t)$ in $L^p_{loc}(\mathbb{R})$ for every $p \in [1, \infty)$ and in $L^\infty(\mathbb{R})$ weak-*. Hence, for each $R > 0$ and $p \in [1, \infty]$, using Theorem 3.3 we have

$$\begin{aligned} \|u(t) - \tilde{u}(t)\|_{L^p((-R, R))} &\leq \liminf_{\varepsilon \rightarrow 0} \|u^\varepsilon(t) - \tilde{u}^\varepsilon(t)\|_{L^p((-R, R))} \\ &\leq Ct^{-\frac{1}{\alpha}(1-\frac{1}{p})} \|u_0 - \tilde{u}_0\|_1. \end{aligned}$$

Since $R > 0$ is arbitrary and the right-hand side of this inequality does not depend on R , we complete the proof of inequality (5.1). \square

Proof of Theorem 1.7. Apply Corollary 5.1 with $\alpha = 1$ and $\tilde{u}_0 = U_0$. \square

Proof of Theorem 1.1. We argue exactly as in the proof of Corollary 5.1, since $\lim_{\varepsilon \rightarrow 0} u^\varepsilon(t) = u(t)$ and $\lim_{\varepsilon \rightarrow 0} \tilde{u}^\varepsilon(t) = \tilde{u}(t)$ in $L^p_{loc}(\mathbb{R})$ for every $p \in [1, \infty)$ and in $L^\infty(\mathbb{R})$ weak-*. Moreover, it is well-known that for fixed $t > 0$

$$\lim_{\varepsilon \rightarrow 0} S_\alpha^\varepsilon(t) u_0 = \lim_{\varepsilon \rightarrow 0} p_2(\varepsilon t) * (p_\alpha(t) * u_0) = S_\alpha(t) u_0 \quad \text{in } L^p(\mathbb{R})$$

for all $p \in [1, \infty]$. Hence, for every $R > 0$ and $p \in [1, \infty]$, by Theorem 3.4, we obtain

$$\begin{aligned} \|u(t) - S_\alpha(t)u_0\|_{L^p((-R,R))} &\leq \liminf_{\varepsilon \rightarrow 0} \|u^\varepsilon(t) - S_\alpha^\varepsilon(t)u_0\|_{L^p((-R,R))} \\ &\leq C\|u_0\|_\infty \|m\| t^{1-\frac{1}{\alpha}(1-\frac{1}{p})}. \end{aligned}$$

The proof is complete by letting $R \rightarrow \infty$. \square

Proof Corollary 1.3. By Theorem 1.1, it suffices to show

$$(5.2) \quad \left\| c + H_\alpha(t) \int_{\mathbb{R}} m(dx) - S_\alpha(t)u_0 \right\|_p \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Moreover, replacing u_0 by $u_0 - c$ (note that $S_\alpha(t)c \equiv c$) we can assume that $c = 0$ and $u_0(x) = \int_{-\infty}^x m(dy)$ in the expression (1.3). Hence, using the Fubini theorem and changing the variables we obtain

$$\begin{aligned} S_\alpha(t)u_0(x) &= \int_{\mathbb{R}} p_\alpha(x-y, t)u_0(y) dy = \int_{-\infty}^{+\infty} \int_{-\infty}^y p_\alpha(x-y, t) m(dz) dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{x-z} p_\alpha(y, t) dy m(dz) = \int_{-\infty}^{+\infty} H_\alpha(x-z, t) m(dz), \end{aligned}$$

and consequently,

$$(5.3) \quad \begin{aligned} S_\alpha(t)u_0(x) - H_\alpha(x, t) \int_{\mathbb{R}} m(dx) \\ = \int_{-\infty}^{+\infty} (H_\alpha(x-z, t) - H_\alpha(x, t)) m(dz). \end{aligned}$$

Observe that $|H_\alpha(x-z, t) - H_\alpha(x, t)| \leq |H_\alpha(x-z, t)| + |H_\alpha(x, t)| \leq 2$. Furthermore, by Taylor's expansion, we have

$$(5.4) \quad \begin{aligned} |H_\alpha(x-z, t) - H_\alpha(x, t)| &= \left| \int_0^1 \frac{d}{ds} H_\alpha(x-sz, t) ds \right|, \\ &= |z| \int_0^1 p_\alpha(x-sz, t) ds. \end{aligned}$$

Now, under the additional assumption $\int_{\mathbb{R}} |z| |m|(dz) < \infty$, using equality (5.3) combined with identity (5.4) we have the following L^p -estimate

$$(5.5) \quad \left\| c + H_\alpha(t) \int_{\mathbb{R}} m(dx) - S_\alpha(t)u_0 \right\|_p \leq C t^{-\frac{1}{\alpha}(1-\frac{1}{p})} \int_{\mathbb{R}} |z| |m|(dz)$$

for each $p \in [1, \infty]$ and all $t > 0$. A standard approximation argument leads to the decay estimate (5.2) (however, without any rate) for every finite measure $m(dx)$. \square

6. QUALITATIVE STUDY OF THE SELF-SIMILAR PROFILE FOR $\alpha = 1$

This section is devoted to the proof of Theorems 1.8 and 1.9.

Proof of properties p1–p4 from Theorem 1.8. The Lipschitz-continuity stated in p1 is an immediate consequence of Proposition 3.2 and Theorem 4.1. Indeed, $U(1)$ is the limit in $L^1_{loc}(\mathbb{R})$ of $u^\varepsilon(1)$ as $\varepsilon \rightarrow 0$, where $u^\varepsilon = u^\varepsilon(x, t)$ is solution to (3.1)–(3.2) with $u_0 = U_0$ defined in (1.9). Moreover, by (3.8), the family $\{u^\varepsilon(1) : \varepsilon > 0\}$ is equi-Lipschitz-continuous, which implies that the limit $U(1)$ is Lipschitz-continuous.

Before proving properties p2–p4, let us reduce the problem to a simpler one. We remark that equation (1.1) is invariant under the transformation

$$(6.1) \quad V(x, t) \equiv U(x + \bar{c}t, t) - \bar{c} \quad \text{where} \quad \bar{c} \equiv \frac{u_- + u_+}{2};$$

that is to say, if U is a solution to (1.1) with $U(x, 0) = U_0(x)$ defined in (1.9), then V is a solution to (1.1) with the initial datum

$$(6.2) \quad V(x, 0) = V_0(x) \equiv \begin{cases} v_+, & x < 0, \\ v_-, & x > 0, \end{cases}$$

where $v_- = -v_+$ and $v_+ \equiv |\bar{c}| \geq 0$. It is clear that U satisfies p2–p4, whenever V enjoys these properties. In the sequel, we thus assume without loss of generality that $u_- = -u_+$ and $u_+ > 0$.

It has been shown in [2, Lemma 3.1] that if $u_0 \in L^\infty(\mathbb{R})$ is non-increasing, odd and convex on $(0, +\infty)$, then the solution $u = u(x, t)$ of (1.1)–(1.2) shares these properties w.r.t. x , for all $t > 0$. The proof is based on a splitting method and on the fact that the “odd, concave/convex” property is conserved by both the hyperbolic equation $u_t + uu_x$ and the fractal equation $u_t + \Lambda^1 u = 0$. The same proof works with minor modifications to show that if u_0 is non-decreasing, odd and convex on $(-\infty, 0)$, then these properties are preserved by problem (1.1)–(1.2). Details are left to the reader since in that case, no shock can be created by the Burgers part and the proof is even easier. By the hypothesis $u_- = -u_+ < 0$ made above, the initial datum in (1.9) is non-decreasing, odd and convex on $(-\infty, 0)$. We conclude that so is the profile $U(1)$. The proof of properties p3–p4 is now complete.

What is left to prove is the limit in property p2. By Theorem 2.2, we have $U(t) \rightarrow U_0$ in $L^1_{loc}(\mathbb{R})$ as $t \rightarrow 0$. In particular, the convergence holds true a.e. along a subsequence $t_n \rightarrow 0$ as $n \rightarrow \infty$ and there exists $\pm x_\pm > 0$ such that $U(x_\pm, t_n) \rightarrow u_\pm$. By the self-similarity of U , we get $U(x_\pm/t_n, 1) \rightarrow u_\pm$ as $n \rightarrow \infty$. Since $x_\pm/t_n \rightarrow \pm\infty$ and $U(1)$ is non-decreasing, we deduce property p2. \square

6.1. Technical lemmata. Property p5 of Theorem 1.5 is the most difficult part to prove. Let us first show technical results that shall be needed in our reasoning.

Lemma 6.1. *Let $v \in L^\infty(\mathbb{R})$ be non-negative, even and non-decreasing on $(-\infty, 0)$. Assume that there exists $\ell > 0$ such that for all $x_0 > 1/2$,*

$$(6.3) \quad \lim_{n \rightarrow \infty} n^{-1} \int_{n(x_0-1/2)}^{n(x_0+1/2)} y^2 v(y) dy = \ell.$$

Then, we have $v(y) \sim_{|y| \rightarrow \infty} \ell |y|^{-2}$ (in the sense that v admits an a.e. representant satisfying this equivalence).

Proof. By assumption, v admits an a.e. representant which is defined everywhere on \mathbb{R} by being non-negative, even and non-decreasing on $(-\infty, 0]$. We still let v denote this representant. For all $x_0 > 1/2$, we have

$$n^{-1} \int_{n(x_0-1/2)}^{n(x_0+1/2)} y^2 v(y) dy \geq n^2 (x_0 - 1/2)^2 v(n(x_0 + 1/2)),$$

thanks to the fact that v is non-increasing on $[0, +\infty)$. Hence, we have

$$n^2 (x_0 + 1/2)^2 v(n(x_0 + 1/2)) \leq \frac{n^2 (x_0 + 1/2)^2}{n^2 (x_0 - 1/2)^2} n^{-1} \int_{n(x_0-1/2)}^{n(x_0+1/2)} y^2 v(y) dy.$$

Taking the upper semi-limit, we get for all $x_0 > 1/2$

$$(6.4) \quad \limsup_{n \rightarrow \infty} n^2 (x_0 + 1/2)^2 v(n(x_0 + 1/2)) \leq \ell \left(\frac{x_0 + 1/2}{x_0 - 1/2} \right)^2,$$

thanks to (6.3). In the same way, one can show that for all $x_0 > 1/2$,

$$(6.5) \quad \ell \left(\frac{x_0 - 1/2}{x_0 + 1/2} \right)^2 \leq \liminf_{n \rightarrow \infty} n^2 (x_0 - 1/2)^2 v(n(x_0 - 1/2)).$$

Moreover, for fixed $x_0 > 1/2$ and all $y \geq x_0 + 1/2$, there exists a unique integer n_y such that

$$n_y(x_0 + 1/2) \leq y < (n_y + 1)(x_0 + 1/2).$$

Using again that v is non-increasing on $[0, +\infty)$, we infer that

$$\begin{aligned} y^2 v(y) &\leq (n_y + 1)^2 (x_0 + 1/2)^2 v(n_y(x_0 + 1/2)), \\ &= \frac{(n_y + 1)^2 (x_0 + 1/2)^2}{n_y^2 (x_0 + 1/2)^2} n_y^2 (x_0 + 1/2)^2 v(n_y(x_0 + 1/2)). \end{aligned}$$

Notice that $n_y \rightarrow \infty$ as $y \rightarrow +\infty$. Therefore, passing to the upper semi-limit as $y \rightarrow +\infty$ in the inequality above, one can show that for all $x_0 > 1/2$

$$\limsup_{y \rightarrow +\infty} y^2 v(y) \leq \ell \left(\frac{x_0 + 1/2}{x_0 - 1/2} \right)^2,$$

thanks to (6.4). In the same way, we deduce from (6.5) that for all $x_0 > 1/2$

$$\ell \left(\frac{x_0 - 1/2}{x_0 + 1/2} \right)^2 \leq \liminf_{y \rightarrow +\infty} y^2 v(y).$$

Letting finally $x_0 \rightarrow +\infty$ in both inequalities above implies that

$$\ell \leq \liminf_{y \rightarrow +\infty} y^2 v(y) \leq \limsup_{y \rightarrow +\infty} y^2 v(y) \leq \ell.$$

Since v is even, we have complete the proof of the lemma. \square

Lemma 6.2. *Let $u \in L^\infty(\mathbb{R})$ and $r > 0$. Then $\Lambda_r^{(0)} u \in L^\infty(\mathbb{R}) \subset L_{loc}^1(\mathbb{R})$ defined in (2.3). Moreover, if $\{u_n\}_{n \in \mathbb{N}}$ is uniformly essentially bounded and $u_n \rightarrow u$ in $L_{loc}^1(\mathbb{R})$, then $\Lambda_r^{(0)} u_n \rightarrow \Lambda_r^{(0)} u$ as $n \rightarrow +\infty$.*

Proof. For $u \in L^\infty(\mathbb{R})$, we have

$$\begin{aligned} &\Lambda_r^{(0)} u(x) \\ &= -G_1 \int_{\mathbb{R}} u(x+z) \mathbf{1}_{\{|z|>r\}} |z|^{-2} dz + u(x) G_1 \int_{|z|>r} |z|^{-2} dz \\ &= u * \rho(x) + Cu(x), \end{aligned}$$

where $\rho(x) \equiv -G_1 \mathbf{1}_{\{|x|>r\}} |x|^{-2} \in L^1(\mathbb{R})$ and $C = \|\rho\|_1 \equiv 2G_1 r^{-1}$. The second part of the lemma is a consequence of classical properties of the convolution product. The proof is now complete. \square

Lemma 6.3. *Let $u \in L^\infty(\mathbb{R})$ be non-decreasing, odd and convex on $(-\infty, 0)$. Then, for the operator defined in (2.2), we have $\Lambda_r^{(1)} u \in L_{loc}^1(\mathbb{R}_*)$ together with the inequality*

$$(6.6) \quad \int_{|x|>R} |\Lambda_r^{(1)} u(x)| dx \leq \frac{4G_1 r}{R-r} \|u\|_\infty$$

for all $R > r > 0$.

Proof. The proof is divided into a sequence of steps.

Step 1: estimates of u_x . The convex function u on $(-\infty, 0)$ is locally Lipschitz-continuous on $(-\infty, 0)$ and *a fortiori* a.e. differentiable, with a distribution derivative equal to its a.e. derivative. By the slopes inequality for convex functions with the points $(x, u(x))$ and $(0, u(0)) = (0, 0)$, we see that for a.e. $x \neq 0$,

$$(6.7) \quad |u_x(x)| \leq \|u\|_\infty |x|^{-1};$$

notice that the slopes inequality gives the inequality for $x < 0$ and the one for $x > 0$ is deduced by oddity of u .

Step 2: estimates of u_{xx} . By convexity of u , u_{xx} is a non-negative Radon measure on $(-\infty, 0)$ in the distribution sense. Hence, $u_x \in BV_{loc}((-\infty, 0))$ satisfies $\int_{[\tilde{x}, x]} u_{yy}(dy) = u_x(x) - u_x(\tilde{x})$, for a.e. $\tilde{x} < x < 0$. Using (6.7) and letting $\tilde{x} \rightarrow -\infty$, we conclude that for a.e. $x < 0$

$$(6.8) \quad \int_{(-\infty, x]} u_{yy}(dy) = u_x(x),$$

thanks to the sup-continuity of non-negative measures. Again by (6.7) and oddity of u_{xx} , this shows for a.e. $x \neq 0$

$$(6.9) \quad \int_{|y| \geq |x|} |u_{yy}|(dy) \leq 2\|u\|_\infty |x|^{-1};$$

notice that by the inf-continuity of non-negative measures, this inequality holds for all $x \neq 0$.

Step 3: estimate of $\Lambda_r^{(1)}u$. Let us prove that $\Lambda_r^{(1)}u$ is well-defined by formula (2.2) for a.e. $x \neq 0$. By the preceding steps, we know that $u \in L^\infty(\mathbb{R}) \cap W_{loc}^{1,\infty}(\mathbb{R}_*)$ and $u_x \in BV_{loc}(\mathbb{R}_*)$. By Taylor's formula (see Lemma A.2 in Appendix A), we infer that for all $R > r > 0$

$$\begin{aligned} I &\equiv \int_{|x| > R} \int_{|z| \leq r} \frac{|u(x+z) - u(x) - u_x(x)z|}{|z|^2} dx dz \\ &\leq \int_{|x| > R} \int_{|z| \leq r} |z|^{-2} \left| \int_{I_{x,z}} |x+z-y| u_{yy}(dy) \right| dx dz, \end{aligned}$$

where $I_{x,z} \equiv (x, x+z)$ if $z > 0$ and $I_{x,z} \equiv (x+z, x)$ in the opposite case. Therefore, we see that

$$\begin{aligned} I &\leq \int_{|x| > R} \int_{|z| \leq r} |z|^{-1} \int_{I_{x,z}} |u_{yy}|(dy) dx dz, \\ &= \int_{\mathbb{R}_*} \int_{\mathbb{R}} |z|^{-1} \mathbf{1}_{\{|z| \leq r\}} \int_{|x| > R} \mathbf{1}_{I_{x,z}}(y) dx |u_{yy}|(dy) dz, \end{aligned}$$

by integrating first w.r.t x ; notice that all the integrands are measurable by Fubini's theorem, since the Radon measure $|u_{yy}|(dy)$ is σ -finite on \mathbb{R}_* . For fixed $(y, z) \in \mathbb{R}_* \times \mathbb{R}$, we have

$$\mathbf{1}_{\{|z| \leq r\}} \int_{|x| > R} \mathbf{1}_{I_{x,z}}(y) dx \leq |z| \mathbf{1}_{\{|z| \leq r\}} \mathbf{1}_{\{|y| \geq R-r\}},$$

because the measure of the set $\{x : y \in I_{x,z}\}$ can be estimated by $|z|$, and if $|z| \leq r$, then $\mathbf{1}_{I_{x,z}}(y) = 0$ for all $|x| > R$ whenever $|y| < R-r$. It follows that

$$I \leq \int_{\mathbb{R}_*} \int_{\mathbb{R}} \mathbf{1}_{\{|z| \leq r\}} \mathbf{1}_{\{|y| \geq R-r\}} |u_{yy}|(dy) dz = 2r \int_{|y| \geq R-r} |u_{yy}|(dy).$$

Recalling the definition of I above and estimate (6.9), we have shown that

$$(6.10) \quad \int_{|x| > R} \int_{|z| \leq r} \frac{|u(x+z) - u(x) - u_x(x)z|}{|z|^2} dx dz \leq 4r\|u\|_\infty (R-r)^{-1}.$$

Fubini's theorem then implies that $\Lambda_r^{(1)}u(x)$ is well-defined by (2.2) for a.e. $|x| > R > r$ by satisfying the desired estimate (6.6).

Step 4: local integrability on \mathbb{R}_ .* Estimate (6.6) implies that $\Lambda_r^{(1)}u \in L_{loc}^1(\mathbb{R} \setminus [-r, r])$. In fact, $\Lambda_r^{(1)}u$ is locally integrable on all \mathbb{R}_* . Indeed, simple computations show that for all $r > \tilde{r} > 0$

$$(6.11) \quad \Lambda_r^{(1)}u + \Lambda_r^{(0)}u = \Lambda_{\tilde{r}}^{(1)}u + \Lambda_{\tilde{r}}^{(0)}u,$$

since their difference evaluated at some x is equal to $\int_{\tilde{r} \leq |z| \leq r} \frac{-u_x(x)z}{|z|^2}$, which is null by oddity of the function $z \rightarrow -u_x(x)z$. By Lemma 6.2 and Step 3, it follows that $\Lambda_r^{(1)}u = \Lambda_{\tilde{r}}^{(1)}u + \Lambda_{\tilde{r}}^{(0)}u - \Lambda_r^{(0)}u \in L_{loc}^1(\mathbb{R} \setminus [-\tilde{r}, \tilde{r}])$, which completes the proof. \square

Remark 6.4. Lemmata 6.2–6.3 imply that $\Lambda_r^{(1)}u + \Lambda_r^{(0)}u \in L_{loc}^1(\mathbb{R}_*)$ whenever $u \in L^\infty(\mathbb{R})$ is non-decreasing, odd and convex on $(-\infty, 0)$. This sum does not depend on $r > 0$ by (6.11). Denoting by Λ^1u this sum, one see from (6.10), Fubini's theorem and (2.1), that for all $\varphi \in \mathcal{D}(\mathbb{R}_*)$, $\int_{\mathbb{R}} \varphi \Lambda^1u \, dx = \int_{\mathbb{R}} u \Lambda^1\varphi \, dx$. This means that this sum corresponds to the distribution fractional Laplacian of u on \mathbb{R}_* .

Here is a corollary of the two previous lemmata.

Corollary 6.5. *Let $\alpha = 1$ and $U = U(x, t)$ be the self-similar solution from Theorem 1.5 with initial datum U_0 in (1.9) for some $u_- = -u_+ < 0$. Then, for all $t \geq 0$, one have $\Lambda^1U(t) \in L_{loc}^1(\mathbb{R}_*)$. Moreover, $\Lambda^1U(t)$ converges toward Λ^1U_0 in $L_{loc}^1(\mathbb{R}_*)$ as $t \rightarrow 0$, where for all $x \neq 0$*

$$\Lambda^1U_0(x) = \frac{u_+ - u_-}{2\pi^2} x^{-1}.$$

Proof. By properties p2–p4 of Theorem 1.8, $U(t) \in L^\infty(\mathbb{R})$ is non-decreasing, odd and convex on $(-\infty, 0)$ for all $t \geq 0$. By Remark 6.4, $\Lambda^1U(t)$ and Λ^1U_0 belong to $L_{loc}^1(\mathbb{R}_*)$. By taking $0 < r < |x|$, simple computations show that

$$(6.12) \quad \Lambda_r^{(1)}U_0(x) = 0 \quad \text{and} \quad \Lambda_r^{(0)}U_0(x) = \frac{u_+ - u_-}{2\pi^2} x^{-1},$$

so that

$$\Lambda^1U_0(x) = \frac{u_+ - u_-}{2\pi^2} x^{-1};$$

here, we have used the equalities $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$ in order to get $G_1 = (2\pi^2)^{-1}$ in (2.2)–(2.3). Moreover, Theorem 2.2 implies that $U(t) \rightarrow U_0$ as $t \rightarrow 0$ in $L_{loc}^1(\mathbb{R})$ with $\|U(t)\|_\infty \leq \|U_0\|_\infty$. We deduce from Lemma 6.2 that for fixed $r > 0$, $\Lambda_r^{(0)}U(t) \rightarrow \Lambda_r^{(0)}U_0$ in $L_{loc}^1(\mathbb{R})$ as $t \rightarrow 0$. It follows that for all $\tilde{R} > R > r$,

$$\begin{aligned} & \limsup_{t \rightarrow 0} \int_{R < |x| < \tilde{R}} |\Lambda^1U(t) - \Lambda^1U_0| \, dx \\ & \leq \limsup_{t \rightarrow 0} \int_{R < |x| < \tilde{R}} |\Lambda_r^{(1)}U(t) - \Lambda_r^{(1)}U_0| \, dx, \\ & = \limsup_{t \rightarrow 0} \int_{R < |x| < \tilde{R}} |\Lambda_r^{(1)}U(t)| \, dx \quad \text{by (6.12),} \\ & \leq \limsup_{t \rightarrow 0} 4G_1r \|U(t)\|_\infty (R-r)^{-1} \quad \text{by (6.6) in Lemma 6.3,} \\ & \leq 4G_1r \|U_0\|_\infty (R-r)^{-1}. \end{aligned}$$

The proof is complete by letting $r \rightarrow 0$. \square

Proof of property p5 from Theorem 1.8. Let us finish the proof of Theorem 1.8. We assume again without loss of generality that $u_- = -u_+ < 0$, thanks to the transformation (6.1); hence, $U_0 \in L^\infty(\mathbb{R})$ is non-decreasing, odd and convex on $(-\infty, 0)$ and so is $U(t)$ for all $t > 0$ by properties p2–p4 of Theorem 1.8. We proceed again in several steps.

Step 1: equation satisfied by $U(1)$. By using $\eta(r) = \pm r$ in Definition 2.1, we obtain (in a classical way) that entropy solutions to (1.1) are distribution solutions, *i.e.*

$$(6.13) \quad U_t + UU_x + \Lambda^1U = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0, \infty)).$$

By property p1 of Theorem 1.8, one has $U(1) \in W^{1,\infty}(\mathbb{R})$. By the self-similarity $U(x,t) = U(\frac{x}{t}, 1)$, one has at least $U_t, U_x \in L_{loc}^\infty(\mathbb{R} \times (0, \infty))$ together with the following equalities for a.e. $t > 0$ and $x \in \mathbb{R}$

$$\begin{aligned} U_t(x,t) &= -xt^{-2}U_x\left(\frac{x}{t}, 1\right), \\ U_x(x,t) &= t^{-1}U_x\left(\frac{x}{t}, 1\right). \end{aligned}$$

By Corollary 6.5, we have also $\Lambda^1 U(1) \in L_{loc}^1(\mathbb{R}_*)$. Using again the self-similarity, it is easy to deduce from (2.1) that $\Lambda^1 U \in L_{loc}^1(\mathbb{R}_* \times (0, \infty))$ with for a.e. $t > 0$ and $x \in \mathbb{R}_*$,

$$\Lambda^1 U(x,t) = t^{-1}\Lambda^1 U\left(\frac{x}{t}, 1\right)$$

(in fact, $\Lambda^1 U \in L_{loc}^\infty(\mathbb{R} \times (0, \infty))$ by (6.13) so that $\Lambda^1 U(1) \in L_{loc}^\infty(\mathbb{R})$). Putting these formulas into (6.13), we get for a.e. $t > 0$ and $x \in \mathbb{R}$,

$$-xt^{-2}U_x\left(\frac{x}{t}, 1\right) + t^{-1}U\left(\frac{x}{t}, 1\right)U_x\left(\frac{x}{t}, 1\right) + t^{-1}\Lambda^1 U\left(\frac{x}{t}, 1\right) = 0.$$

Multiplying by t and changing the variable by $y = t^{-1}x$, one infer that the profile $\mathcal{U}(y) \equiv U(y, 1)$ satisfies for a.e. $y \in \mathbb{R}$

$$(6.14) \quad (\mathcal{U}(y) - y)\mathcal{U}_y(y) + \Lambda^1 \mathcal{U}(y) = 0.$$

Step 2: reduction of the problem. By properties p1–p4, the function $\mathcal{U}_y \in L^\infty(\mathbb{R})$ is non-negative, even and non-decreasing on $(-\infty, 0)$. Then, Lemma 6.1 shows that the proof of p5 can be reduced to the proof of the following property:

$$(6.15) \quad \forall x_0 > 1/2 \quad \lim_{n \rightarrow \infty} n^{-1} \int_{n(x_0-1/2)}^{n(x_0+1/2)} y^2 \mathcal{U}_y(y) dy = \frac{u_+ - u_-}{2\pi^2}.$$

Moreover, equality (6.14) implies that $\mathcal{U}_y(y) = \frac{\Lambda^1 \mathcal{U}(y)}{y - \mathcal{U}(y)}$ (for a.e. $y > \|\mathcal{U}\|_\infty$). One deduce that (6.15) is equivalent to the following property:

$$(6.16) \quad \forall x_0 > 1/2 \quad \lim_{n \rightarrow \infty} n^{-1} \int_{n(x_0-1/2)}^{n(x_0+1/2)} y \Lambda^1 \mathcal{U}(y) dy = \frac{u_+ - u_-}{2\pi^2}.$$

Conclusion: proof of (6.16). Let us change the variable by $y = nx$. Easy computations show that

$$\begin{aligned} n^{-1} \int_{n(x_0-1/2)}^{n(x_0+1/2)} y \Lambda^1 \mathcal{U}(y) dy &= n^{-1} \int_{x_0-1/2}^{x_0+1/2} nx \Lambda^1 U(x/n^{-1}, 1) ndx, \\ &= \int_{x_0-1/2}^{x_0+1/2} x \Lambda^1 U(x, n^{-1}) dx. \end{aligned}$$

Since Corollary 6.5 implies that the function $x \rightarrow x \Lambda^1 U(x, n^{-1})$ converges toward $\frac{u_+ - u_-}{2\pi^2}$ in $L^1((x_0 - 1/2, x_0 + 1/2))$ as $n \rightarrow \infty$, the proofs of (6.16) and thus of property p5 are complete. \square

6.2. Duhamel's representation of the self-similar profile. It remains to prove Theorem 1.9, for which we need the following result.

Proposition 6.6. *Let $\alpha = 1$ and let $U = U(x, t)$ be the self-similar solution of Theorem 1.5 with $u_{\pm} = \pm 1/2$. Then, for all $x \in \mathbb{R}$, we have*

$$(6.17) \quad U(x, 1) = -1/2 + H_1(x, 1) \\ - \int_0^{1/2} \partial_x p_1(1 - \tau) * \frac{U^2(\cdot/\tau, 1)}{2}(x) d\tau \\ - \int_{1/2}^1 \tau^{-1} p_1(1 - \tau) * (U(\cdot/\tau, 1)U_x(\cdot/\tau, 1))(x) d\tau$$

(where $H_1(x, 1) = \int_{-\infty}^x p_1(y, 1) dy$).

Proof. The proof proceeds in several steps.

Step 1: Duhamel's representation of the approximate solution. Notice that formula (6.17) makes sense. Indeed, by the homogeneity property (3.5), we have for all $t > 0$

$$(6.18) \quad \|\partial_x p_1(t)\|_1 = C_0 t^{-1},$$

where $C_0 \equiv \|\partial_x P_1(1)\|_1$ is finite by (3.6). Hence, the integral $\int_0^{1/2} \dots d\tau$ in (6.17) is well-defined since the integration variable τ is far from the singularity at $\tau = 1$. The same way, since $U(1) \in W^{1, \infty}(\mathbb{R})$, the integral $\int_{1/2}^1 \dots d\tau$ is also well-defined.

Let now $u^\varepsilon = u^\varepsilon(x, t)$ be the solution to the regularized equation (3.1), with initial datum U_0 in (1.9). The goal is to pass to the limit in formula (3.3) at time $t = 1$

$$(6.19) \quad u^\varepsilon(x, 1) = S_1^\varepsilon(1)U_0(x) \\ - \int_0^{1/2} p_2(\varepsilon(1 - \tau)) * \partial_x p_1(1 - \tau) * \frac{(u^\varepsilon(\tau))^2}{2}(x) d\tau \\ - \int_{1/2}^1 p_2(\varepsilon(1 - \tau)) * p_1(1 - \tau) * (u^\varepsilon(\tau)u_x^\varepsilon(\tau))(x) d\tau,$$

for all $x \in \mathbb{R}$.

Step 2: pointwise limits and bounds of the integrands. We first remark that

$$\lim_{x \rightarrow \pm\infty} u^\varepsilon(t, x) = u^\pm.$$

Hence, thanks to Dini theorem for cumulative distribution functions, we know that for fixed $t > 0$, $\lim_{\varepsilon \rightarrow 0} u^\varepsilon(t)$ converges toward $U(t)$ uniformly on \mathbb{R} .

Let us next recall that $\partial_x p_1(t) \in L^1(\mathbb{R})$, so that for fixed $\tau \in (0, 1)$

$$\lim_{\varepsilon \rightarrow 0} \partial_x p_1(1 - \tau) * \frac{(u^\varepsilon(\tau))^2}{2} = \partial_x p_1(1 - \tau) * \frac{(U(\tau))^2}{2} \quad \text{uniformly on } \mathbb{R}.$$

It follows from classical approximate unit properties of the heat kernel $p_2(x, t)$ that for all $\tau \in (0, 1)$,

$$(6.20) \quad \lim_{\varepsilon \rightarrow 0} p_2(\varepsilon(1 - \tau)) * \partial_x p_1(1 - \tau) * \frac{(u^\varepsilon(\tau))^2}{2} = \partial_x p_1(1 - \tau) * \frac{U^2(\tau)}{2}$$

uniformly on \mathbb{R} . In particular, for all $\tau \in (0, 1)$, we have also

$$(6.21) \quad \lim_{\varepsilon \rightarrow 0} p_2(\varepsilon(1 - \tau)) * p_1(1 - \tau) * (u^\varepsilon(\tau)u_x^\varepsilon(\tau)) = p_1(1 - \tau) * (U(\tau)U_x(\tau))$$

uniformly on \mathbb{R} , since

$$p_2(\varepsilon(1 - \tau)) * \partial_x p_1(1 - \tau) * \frac{(u^\varepsilon(\tau))^2}{2} \\ = p_2(\varepsilon(1 - \tau)) * p_1(1 - \tau) * (u^\varepsilon(\tau)u_x^\varepsilon(\tau))$$

and $\partial_x p_1(1 - \tau) * \frac{U^2(\tau)}{2} = p_1(1 - \tau) * (U(\tau)U_x(\tau))$.

Moreover, by (3.7), (3.8) with $p = +\infty$ and (6.18), one can see that the integrands of (6.19) are pointwise bounded by

$$(6.22) \quad \left\| p_2(\varepsilon(1 - \tau)) * \partial_x p_1(1 - \tau) * \frac{(u^\varepsilon(\tau))^2}{2} \right\|_\infty \leq C_0(1 - \tau)^{-1} \frac{\|u_0\|_\infty^2}{2},$$

and

$$(6.23) \quad \left\| p_2(\varepsilon(1 - \tau)) * p_1(1 - \tau) * (u^\varepsilon(\tau)u_x^\varepsilon(\tau)) \right\|_\infty \leq \tau^{-1} \|u_0\|_\infty.$$

Step 3: passing to the limit. Recall that

$$\lim_{\varepsilon \rightarrow 0} S_1^\varepsilon(1)U_0 = S_1(1)U_0 = p_1(1) * U_0$$

in $L^p(\mathbb{R})$ for all $p \in [1, \infty]$. Let us recall that $U_0(x) = \pm 1/2$ for $\pm x \geq 0$ and $\int_{\mathbb{R}} p_1(y, 1)dy = 1$, so that for all $x \in \mathbb{R}$

$$1/2 + p_1(1) * U_0(x) = p_1(1) * (U_0 + 1/2)(x) = \int_{-\infty}^x p_1(y, 1)dy = H_1(x, 1).$$

We have proved in particular that $\lim_{\varepsilon \rightarrow 0} S_1^\varepsilon(1)U_0 \rightarrow -1/2 + H_1(1)$ pointwise on \mathbb{R} .

In order to pass to the limit in the integral terms of (6.19), we use the Lebesgue dominated convergence theorem. We deduce from (6.20) and (6.22) that for all $x \in \mathbb{R}$, the first integral term converges toward

$$\int_0^{1/2} \partial_x p_1(1 - \tau) * \frac{(U(\tau))^2}{2}(x) d\tau$$

as $\varepsilon \rightarrow 0$. In the same way, we deduce from (6.21) and (6.23) that the last integral term converges toward

$$\int_{1/2}^1 p_1(1 - \tau) * (U(\tau)U_x(\tau))(x) d\tau.$$

The limit as $\varepsilon \rightarrow 0$ in (6.19) then implies that for all $x \in \mathbb{R}$,

$$\begin{aligned} U(x, 1) = -1/2 + H_1(x, 1) - \int_0^{1/2} \partial_x p_1(1 - \tau) * \frac{U^2(\tau)}{2}(x) d\tau \\ - \int_{1/2}^1 p_1(1 - \tau) * (U(\tau)U_x(\tau))(x) d\tau. \end{aligned}$$

This completes the proof of (6.17), thanks to the self-similarity of U . \square

Proof of Theorem 1.9. We have to prove that for all $r > 0$

$$(6.24) \quad \mathbb{P}(|X - \bar{c}| < r) < \mathbb{P}(|Y - 0| < r).$$

Let us verify that \bar{c} and 0 are the medians of X and Y , respectively. First, a simple computation allows to see that $p_1(x, 1)$, defined by Fourier transform by $\widehat{p}_1(\xi, 1) = e^{-|\xi|}$, also satisfies formula (1.10). This density of probability is even and the median of Y is null. Second, by property p3 of Theorem 1.8, $U_x(1)$ is symmetric w.r.t. to the axis $\{x = \bar{c}\}$ and the median of X is $\bar{c} = \frac{u_- + u_+}{2}$.

In particular, the centered random variable $X - \bar{c}$ admits a density being the even function

$$f_{X - \bar{c}}(x) = U_x(x + \bar{c}, 1).$$

It becomes clear that (6.24) is equivalent to the following property

$$(6.25) \quad \forall x > 0 \quad F_{X - \bar{c}}(x) < F_Y(x),$$

where $F_{X - \bar{c}}$ and F_Y are the cumulative distribution functions of X and Y , respectively.

Let us compute these functions. First, we have seen above that $f_{X-\bar{c}}(x) = V_x(x, 1)$, where V is defined by the transformation (6.1). Let us recall that V is the self-similar solution to (1.1) with initial datum $V(x, 0) = \pm 1/2$ for $\pm x > 0$. Hence, $F_{X-\bar{c}}$ is equal to $V(\cdot, 1)$ up to an additive constant, which has to be $1/2$ by property p2 of Theorem 1.8; that is to say, we have $F_{X-\bar{c}}(x) = 1/2 + V(x, 1)$ for all $x \in \mathbb{R}$. Second, we defined H_1 in Proposition 6.6 such that $F_Y(x) = H_1(x, 1)$. By this proposition, we have for all $x \in \mathbb{R}$,

$$F_{X-\bar{c}}(x) = F_Y(x) - g(x),$$

where $g(x)$ is defined by

$$(6.26) \quad g(x) \equiv \int_0^{1/2} \partial_x p_1(1-\tau) * \frac{V^2(\cdot/\tau, 1)}{2}(x) d\tau \\ + \int_{1/2}^1 \tau^{-1} p_1(1-\tau) * (V(\cdot/\tau, 1)V_x(\cdot/\tau, 1))(x) d\tau.$$

One concludes that the proof of (6.25), and thus of (6.24), is equivalent to the proof of the positivity of $g(x)$ for positive x . But, by definition of g , it suffices to prove that for each $\tau \in (0, 1)$ and $x > 0$,

$$(6.27) \quad p_1(1-\tau) * (V(\cdot/\tau, 1)V_x(\cdot/\tau, 1))(x) > 0.$$

Indeed, the second integral term in (6.26) would be positive, and the first integral term also, since for fixed τ ,

$$\partial_x p_1(1-\tau) * \frac{V^2(\cdot/\tau, 1)}{2}(x) = \tau^{-1} p_1(1-\tau) * (V(\cdot/\tau, 1)V_x(\cdot/\tau, 1))(x).$$

Let us end by proving inequality (6.27), thus concluding Theorem 1.9. It is clear that the function $V(\cdot/\tau, 1)V_x(\cdot/\tau, 1)$ is odd, since $V(1)$ is odd. Moreover, we already know that $V_x(1)$ is non-negative, even and non-increasing on $(0, +\infty)$, since $V(1)$ is non-decreasing, odd and concave on $[0, +\infty)$. By property p5, we conclude that $V_x(1)$ is positive a.e. on $(0, +\infty)$, and thus on \mathbb{R} as even function. In particular, $V(1)$ is increasing and for all $x > 0$, $V(x, 1) > V(0, 1) = 0$.

To summarize, $V(\cdot/\tau, 1)V_x(\cdot/\tau, 1)$ is odd and positive on $(0, +\infty)$. Moreover, it is clear that $p_1(1-\tau)$ is positive, even and decreasing on $(0, +\infty)$, see (1.10). A simple computation then implies that the convolution product in (6.27) is effectively positive for positive x . The proof of Theorem 1.9 is complete. \square

APPENDIX A. ADDITIONAL TECHNICAL LEMMATA

Lemma A.1. *There exists a modulus of continuity ω such that for all $v_0 \in L^\infty(\mathbb{R})$, all $T, R, r > 0$, and all $\varepsilon \in (0, 1]$, we have*

$$\sup_{t \in [0, T]} \int_{-R-Lt}^{R+Lt} S_\alpha^\varepsilon(t) |v_0|(x) dx \leq \int_{-R-LT-r}^{R+LT+r} |v_0(x)| dx + \|v_0\|_\infty \omega(1/r).$$

Proof. First, we write

$$(A.1) \quad \sup_{t \in [0, T]} \int_{-R-Lt}^{R+Lt} S_\alpha^\varepsilon(t) |v_0|(x) dx \\ = \sup_{t \in [0, T]} \int_{-R-Lt}^{R+Lt} p_\alpha(t) * p_2(\varepsilon t) * |v_0|(x) dx \\ \leq \sup_{s \in [0, T]} \sup_{t \in [0, T]} \int_{-R-Lt}^{R+Lt} p_\alpha(t) * p_2(\varepsilon s) * |v_0|(x) dx.$$

Now, for every $s \in [0, T]$, we estimate from above the following function

$$M(s) \equiv \sup_{t \in [0, T]} \int_{-R-Lt}^{R+Lt} p_\alpha(t) * w_0(x) dx,$$

where $w_0 \equiv p_2(\varepsilon s) * |v_0|$. Using properties of the kernel p_α and its self-similarity (see (3.5)) we obtain

$$\begin{aligned} \int_{-R-Lt}^{R+Lt} p_\alpha(t) * w_0(x) dx &= \int_{|x| \leq R+Lt} \int_{|y| \leq r/2} p_\alpha(y, t) w_0(x-y) dx dy \\ &\quad + \int_{|x| \leq R+Lt} \int_{|y| \geq r/2} p_\alpha(y, t) w_0(x-y) dx dy \\ &\leq \|p_\alpha(t)\|_1 \int_{-R-Lt-r/2}^{R+Lt+r/2} |w_0(x)| dx \\ &\quad + \|w_0\|_\infty 2(R+Lt) \int_{|y| \geq r/2} p_\alpha(y, t) dy \\ &= \int_{-R-Lt-r/2}^{R+Lt+r/2} |w_0(x)| dx \\ &\quad + \|w_0\|_\infty 2(R+Lt) \int_{|x| \geq t^{-\frac{1}{\alpha}} r/2} p_\alpha(x, 1) dx. \end{aligned}$$

Computing the supremum with respect to $t \in [0, T]$ we infer that

$$M(s) \leq \int_{-R-LT-r/2}^{R+LT+r/2} |w_0(x)| dx + \|w_0\|_\infty \omega_\alpha(1/r),$$

where $\omega_\alpha : [0, \infty) \rightarrow (0, \infty)$ is defined by

$$\omega_\alpha(1/r) \equiv (2R + 2LT) \int_{|x| \geq T^{-\frac{1}{\alpha}} r/2} p_\alpha(x, 1) dx.$$

It is clear that the modulus of continuity ω_α is non-decreasing and satisfies

$$\lim_{r \rightarrow +\infty} \omega_\alpha(1/r) = 0.$$

Finally, since $\|w_0\|_\infty = \|p_2(\varepsilon s) * |v_0|\|_\infty \leq \|v_0\|_\infty$, we obtain

$$M(s) \leq \int_{-R-LT-r/2}^{R+LT+r/2} |w_0(x)| dx + \|v_0\|_\infty \omega_\alpha(1/r).$$

Analogous computations show now that

$$\begin{aligned} \int_{-R-LT-r/2}^{R+LT+r/2} |w_0(x)| dx &= \int_{-R-LT-r/2}^{R+LT+r/2} p_2(\varepsilon s) * |v_0|(x) dx \\ &\leq \int_{-R-LT-r}^{R+LT+r} |v_0(x)| dx + \|v_0\|_\infty \omega_2(\sqrt{\varepsilon}/r) \\ &\leq \int_{-R-LT-r}^{R+LT+r} |v_0(x)| dx + \|v_0\|_\infty \omega_2(1/r), \end{aligned}$$

because $\varepsilon \leq 1$.

Finally, with the new modulus of continuity $\omega(1/r) \equiv \omega_\alpha(1/r) + \omega_2(1/r)$, we have

$$M(s) \leq \int_{-R-LT-r}^{R+LT+r} |v_0(x)| dx + \|v_0\|_\infty \omega(1/r).$$

Coming back to inequality (A.1), we complete the proof of Lemma A.1. \square

Lemma A.2. *Let I be an open interval of \mathbb{R} and $u \in W^{1,\infty}(I)$ be such that $u' \in BV(I)$. Then, for a.e. $x \in I$ and all $z \in I - x$, we have*

$$u(x+z) = u(x) + u'(x)z + \int_{I_{x,z}} |x+z-y| u''(dy),$$

where $I_{x,z} \equiv (x, x+z)$ if $z > 0$ and $I_{x,z} \equiv (x+z, x)$ if not.

Proof. We can reduce to the case $I = (a, b)$ with $a, b \in \mathbb{R}$. Let us assume without loss of generality that $z > 0$. Since $u_x \in BV(I)$, the function $\tilde{u}_x(x) \equiv c + \int_{(a,x]} u_{yy}(dy)$ is an a.e. representant of u_x , where c is the trace of u_x on the left boundary of I . The trace of $u_x \in BV(I_{x,z})$ onto $\{x\}$ is equal to $\tilde{u}_x(x)$, because $\{x\}$ is the left boundary of I_{x+z} . Simple integration by parts formulas now give

$$\begin{aligned} u(x+z) &= u(x) + \int_{I_{x,z}} u_y(y)dy, \\ &= u(x) - \int_{I_{x,z}} (y-x-z)u_{yy}(y)dy + \tilde{u}_x(x)z. \end{aligned}$$

The proof is complete. \square

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REFERENCES

- [1] N. Alibaud, *Entropy formulation for fractal conservation laws*, J. Evol. Equ. **7** (2007), 145–175.
- [2] N. Alibaud, J. Droniou and J. Vovelle, *Occurrence and non-appearance of shock in fractal Burgers equations*, J. Hyperbolic Differ. Equ **4** (2007), 479–499.
- [3] P. Biler, G. Karch and W. Woyczyński, *Asymptotics for multifractal conservation laws*, Studia Math. **135** (1999), 231–252.
- [4] P. Biler, G. Karch and W. A. Woyczyński, *Asymptotics for conservation laws involving Lévy diffusion generators*, Studia Math. **148** (2001), 171–192.
- [5] P. Biler, G. Karch and W. A. Woyczyński, *Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws*, Ann. Inst. Henri Poincaré, Analyse non-linéaire, **18** (2001), 613–637.
- [6] L. Brandolese and G. Karch, *Far field asymptotics of solutions of convection equations with anomalous diffusion*, J. Evol. Equ., **8** (2008), 307–326.
- [7] C. H. Chan and M. Czubak, *Regularity of solutions for the critical N -dimensional Burgers equation*, (2008), 1–31. arXiv:0810.3055v3 [math.AP].
- [8] J. Droniou, T. Gallouët and J. Vovelle, *Global solution and smoothing effect for a non-local regularization of a hyperbolic equation*, J. Evol. Equ. **3** (2002), 499 – 521.
- [9] J. Droniou and C. Imbert, *Fractal first order partial differential equations*, Arch. Rat. Mech. Anal. **182** (2006), 299–331.
- [10] B. Jourdain, S. Méléard and W. Woyczyński, *A probabilistic approach for nonlinear equations involving the fractional Laplacian and singular operator*, Potential Analysis **23** (2005), 55–81.
- [11] B. Jourdain, S. Méléard and W. Woyczyński, *Probabilistic approximation and inviscid limits for one-dimensional fractional conservation laws*, Bernoulli **11** (2005), 689–714.
- [12] G. Karch, C. Miao and X. Xu, *On convergence of solutions of fractal Burgers equation toward rarefaction waves*, SIAM J. Math. Anal. **39** (2008), 1536–1549.
- [13] K. Karlsen and S. Ulusoy, *Stability of entropy solutions for Lévy mixed hyperbolic/parabolic equations*, Preprint (2009).
- [14] A. Kiselev, F. Nazarov and R. Shterenberg, *Blow up and regularity for fractal Burgers equation*, (2008), 1–35. arXiv:0804.3549v1 [math.AP].
- [15] C. Miao, B. Yuan and B. Zhang, *Well-posedness of the Cauchy problem for fractional power dissipative equations*, Nonlinear Anal. **68** (2008), 461–484.
- [16] C. Miao and G. Wu, *Global well-posedness of the critical Burgers equation in critical Besov spaces*, (2008), 1–21. arXiv:0805.3465v3 [math.AP].
- [17] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series **30**, Princeton Univ. Press, Princeton, N.J., 1970.

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