



**HAL**  
open science

## Adding constraints to BSDEs with Jumps: an alternative to multidimensional reflections

Romuald Elie, Idris Kharroubi

► **To cite this version:**

Romuald Elie, Idris Kharroubi. Adding constraints to BSDEs with Jumps: an alternative to multidimensional reflections. 2011. hal-00369404v3

**HAL Id: hal-00369404**

**<https://hal.science/hal-00369404v3>**

Preprint submitted on 8 Mar 2011

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Adding constraints to BSDEs with Jumps: an alternative to multidimensional reflections

Romuald ELIE & Idris KHARROUBI  
CEREMADE, CNRS, UMR 7534,  
Université Paris-Dauphine,  
and CREST  
{elie,kharroubi}@ceremade.dauphine.fr

January 2011

## Abstract

This paper is dedicated to the analysis of backward stochastic differential equations (BSDEs) with jumps, subject to an additional global constraint involving all the components of the solution. We study the existence and uniqueness of a minimal solution for these so-called constrained BSDEs with jumps via a penalization procedure. This new type of BSDE offers a nice and practical unifying framework to the notions of constrained BSDEs presented in [19] and BSDEs with constrained jumps introduced in [14]. More remarkably, the solution of a multidimensional Brownian reflected BSDE studied in [11] and [13] can also be represented via a well chosen one-dimensional constrained BSDE with jumps. This last result is very promising from a numerical point of view for the resolution of high dimensional optimal switching problems and more generally for systems of coupled variational inequalities.

**Keywords:** Stochastic control, Switching problems, BSDE with jumps, Reflected BSDE.

**MSC Classification (2000):** 93E20, 60H30, 60J75.

## 1 Introduction

Since their introduction by Pardoux and Peng in [15], Backward Stochastic Differential Equations (BSDEs in short) have been widely studied. In particular, they appear as a very powerful tool to solve partial differential equations (PDEs) and corresponding stochastic optimization problems. Several generalizations of this notion are based on the addition of new constraints on the solution. First, El Karoui et al. [9] study the case where the component  $Y$  is forced to stay above a given process, leading to the notion of reflected BSDEs related to optimal stopping and obstacle problems. Motivated by super replication

issues under portfolio constraints, Cvitanic et al. [7] consider the case where the component  $Z$  is constrained to stay in a fixed convex set. More recently, Kharroubi et al. [14] introduce a constraint on the jump component  $U$  of the BSDE, providing a representation of solutions for a class of PDE, called quasi-variational inequalities, arising from optimal impulse control problems. The generalization of the results of El Karoui et al. [9] to oblique reflections in a multi-dimensional framework was first given in a very special case (e.g. the generator does not depend on  $z$ ) by Ramasubramanian [20], who studied a BSDE reflected in an orthant. Then, Hu and Tang [13] followed by Hamadène and Zhang [11] consider general BSDEs with oblique reflections and connect them with systems of variational inequalities and optimal switching problems. Our paper introduces the notion of constrained BSDEs with jumps, which offers in particular a nice and natural probabilistic representation for these types of switching problems. This new notion essentially unifies and extends the notions of constrained BSDE without jumps, BSDE with constrained jumps as well as multidimensional BSDE with oblique reflections.

Let us illustrate our presentation with the example of the following switching problem

$$\sup_{\alpha} \mathbf{E} \left[ g_{\alpha T}(X_T) + \int_0^T \psi_{\alpha_s}(s, X_s) ds + \sum_{0 < \tau_k \leq T} c_{\alpha_{\tau_k^-}, \alpha_{\tau_k}} \right], \quad (1.1)$$

where  $X$  is an underlying Itô diffusion process,  $\alpha$  is a switching control process valued in  $\mathcal{I} := \{1, \dots, m\}$ ,  $m > 0$ , and  $(\tau_k)_k$  denotes the jump times of the control  $\alpha$ . This type of stochastic control problem is typically encountered by an agent maximizing the production rentability of a given good by switching between  $m$  possible modes of production based on different commodities. A switch is penalized by a given cost function  $c$  and the production rentability functions  $\psi$  and  $g$  depend on the chosen mode of production. As observed in [8], the solution of problem (1.1) starting in mode  $i_0 \in \mathcal{I}$  at time  $t$  rewrites  $Y_t^{i_0}$  where  $(Y^i, Z^i, K^i)_{i \in \mathcal{I}}$  solves the following multidimensional reflected BSDE

$$\begin{cases} Y_t^i = g_i(X_T) + \int_t^T \psi_i(s, X_s) ds - \int_t^T \langle Z_s^i, dW_s \rangle + K_T^i - K_t^i, & 0 \leq t \leq T, \quad i \in \mathcal{I} \\ Y_t^i \geq Y_t^j + c_{i,j}, & 0 \leq t \leq T, \quad i, j \in \mathcal{I}, \\ \int_0^T [Y_t^i - \max_{j \in \mathcal{I}} \{Y_t^j + c_{i,j}\}] dK_t^i = 0, & i \in \mathcal{I}. \end{cases} \quad (1.2)$$

The main difficulty in the derivation of a one-dimensional BSDE representation for this type of problem relies on the dependence of the solution in mode  $i \in \mathcal{I}$  with respect to the global solution in all possible modes. Nevertheless, Tang and Yong [23] interpret the value function associated to this problem as the unique viscosity solution of a given coupled system of variational inequalities. A clever observation of Bouchard [3] concludes that this unique viscosity solution represents also the value function of a well suited stochastic target problem associated to a diffusion with jumps. Using entirely probabilistic arguments, the BSDE representation provided in this paper heavily relies on this type of correspondence. In our approach, we let artificially the strategy jump randomly between the different modes of production. Similarly to the approach of Pardoux et al. [16], this allows to retrieve in the jump component of a one-dimensional backward process, some information

regarding the solution in the other modes of production. Indeed, let us introduce a pure jump process  $(I_t)_{0 \leq t \leq T}$  based on an independent random measure  $\mu$  and consider the following constrained BSDE associated to the two dimensional forward process  $(I, X)$  (called transmutation-diffusion process in [16]) and defined on  $[0, T]$  by:

$$\begin{cases} \tilde{Y}_t &= g_{I_T}(X_T) + \int_t^T \psi_{I_s}(s, X_s) ds + \tilde{K}_T - \tilde{K}_t - \int_t^T \langle \tilde{Z}_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} \tilde{U}_s(i) \mu(ds, di), \\ \tilde{U}_t(i) &\geq c_{i, I_{t-}}, \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \text{ a.e.} \end{cases} \quad (1.3)$$

This BSDE enters into the class of constrained BSDEs studied in the paper and its unique minimal solution relates directly to the solution of (1.2) via the relation  $(\tilde{Y}_t, \tilde{Z}_t, \tilde{U}_t) = (Y_t^{I_t}, Z_t^{I_t}, \{Y_t^i - Y_{t-}^{I_{t-}}\}_{i \in \mathcal{I}})$  for  $t \in [0, T]$ . In particular, the solution of the switching problem (1.1) starting in mode  $I_0$  at time 0 rewrites  $\tilde{Y}_0^{I_0}$ .

In order to unify our results with the one based on multidimensional reflected BSDE considered in [13] or [11], we extend this approach and introduce the notion of constrained BSDE with jumps whose solution  $(Y, Z, U, K)$  satisfies the general dynamics

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds + K_T - K_t - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di), \quad (1.4)$$

a.s., for  $0 \leq t \leq T$ , as well as the constraint

$$h_i(t, Y_{t-}, Z_t, U_t(i)) \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \text{ a.e.}, \quad (1.5)$$

where  $f$  and  $h$  are given random Lipschitz functions, and  $h$  is non-increasing in its last variable. Through a penalization argument, we provide in Section 2 the existence of a unique minimal solution to the constrained BSDE with jumps (1.4)-(1.5). This new type of BSDE mainly extends and unifies the existing literature on BSDEs in three interconnected directions:

- We generalize the notion of BSDE with constrained jumps considered in [14], letting the driver function  $f$  depend on  $U$  and considering a general constraint function  $h$  depending on all the components of the solution.
- We add some jumps in the dynamics of constrained BSDE studied in [19] and let the coefficients depend on the jump component  $U$ .
- Via the addition of artificial jumps, a well chosen one-dimensional constrained BSDE with jumps allows to represent the solution of a multidimensional reflected BSDE, in the framework of [11] or [13].

We believe that the representation of a multidimensional obliquely reflected BSDE by a one-dimensional constrained BSDE with jumps is also numerically very promising. As developed in [2], it offers the possibility to solve high-dimensional optimal switching problems via a natural extension of the entirely probabilistic numerical scheme studied in [4]. Such type of algorithm could also solve high dimensional systems of variational inequalities, which relate directly to multidimensional BSDEs with oblique reflections, see [13] for more

details. The algorithm as well as the Feynman Kac representation of general constrained BSDEs with jumps are presented in [10].

The paper is organized as follows. The next section provides the existence of a unique minimal solution for the new class of constrained BSDEs with jumps (1.4)-(1.5). The connection with multidimensional reflected BSDEs is detailed in Section 3. We regroup in the last section of the paper some technical results on BSDEs, mainly extensions of existing results, which are not the main focus of the paper but present some interest in themselves: we provide a comparison and a monotonic limit theorem for reflected BSDEs with jumps, as well as viability and comparison properties for multidimensional constrained BSDEs. We isolate these results in order to present them in a general framework and to simplify their possible future invocation. All the proofs of the paper only rely on probabilistic arguments and can be applied in a non-Markovian setting.

**Notations.** Throughout this paper we are given a finite terminal time  $T$  and a probability space  $(\Omega, \mathcal{G}, \mathbf{P})$  endowed with a  $d$ -dimensional standard Brownian motion  $W = (W_t)_{t \geq 0}$ , and a Poisson random measure  $\mu$  on  $\mathbb{R}_+ \times \mathcal{I}$ , where  $\mathcal{I} = \{1, \dots, m\}$ , with intensity measure  $\lambda(di)dt$  for some finite measure  $\lambda$  on  $\mathcal{I}$  with  $\lambda(i) > 0$  for all  $i \in \mathcal{I}$ . We set  $\tilde{\mu}(dt, di) = \mu(dt, di) - \lambda(di)dt$  the compensated measure associated to  $\mu$ .  $\sigma(\mathcal{I})$  denotes the  $\sigma$ -algebra of subsets of  $\mathcal{I}$ . For  $x = (x_1, \dots, x_\ell) \in \mathbb{R}^\ell$  with  $\ell \in \mathbb{N}$ , we set  $|x| = \sqrt{|x_1|^2 + \dots + |x_\ell|^2}$  the Euclidean norm. We denote by  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  (resp.  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ ) the augmentation of the natural filtration generated by  $W$  and  $\mu$  (resp. by  $W$ ), and by  $\mathcal{P}_{\mathbb{G}}$  (resp.  $\mathcal{P}_{\mathbb{F}}$ ,  $\mathfrak{P}_{\mathbb{G}}$ ,  $\mathfrak{P}_{\mathbb{F}}$ ) the  $\sigma$ -algebra of  $\mathbb{G}$ -predictable (resp.  $\mathbb{F}$ -predictable  $\mathbb{G}$ -progressive,  $\mathbb{F}$ -progressive) subsets of  $\Omega \times [0, T]$ . We denote by  $\mathcal{S}_{\mathbb{G}}^2$  (resp.  $\mathcal{S}_{\mathbb{F}}^2$ ) the set of real-valued càd-làg  $\mathbb{G}$ -adapted (resp. continuous  $\mathbb{F}$ -adapted) processes  $Y = (Y_t)_{0 \leq t \leq T}$  such that

$$\|Y\|_{\mathcal{S}^2} := \left( \mathbf{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] \right)^{\frac{1}{2}} < \infty.$$

$\mathbf{L}^p(\mathbf{0}, \mathbf{T})$ ,  $p \geq 1$ , is the set of real-valued processes  $\phi = (\phi_t)_{0 \leq t \leq T}$  such that

$$\|\phi\|_{\mathbf{L}^p(\mathbf{0}, \mathbf{T})} := \left( \mathbf{E} \left[ \int_0^T |\phi_t|^p dt \right] \right)^{\frac{1}{p}} < \infty,$$

and  $\mathbf{L}_{\mathbb{F}}^p(\mathbf{0}, \mathbf{T})$  (resp.  $\mathbf{L}_{\mathbb{G}}^p(\mathbf{0}, \mathbf{T})$ ) is the subset of  $\mathbf{L}^p(\mathbf{0}, \mathbf{T})$  consisting of  $\mathfrak{P}_{\mathbb{F}}$ -measurable (resp.  $\mathfrak{P}_{\mathbb{G}}$ -measurable) processes.  $\mathbf{L}_{\mathbb{F}}^p(\mathbf{W})$  (resp.  $\mathbf{L}_{\mathbb{G}}^p(\mathbf{W})$ ),  $p \geq 1$ , is the set of  $\mathbb{R}^d$ -valued  $\mathcal{P}_{\mathbb{F}}$ -measurable (resp.  $\mathcal{P}_{\mathbb{G}}$ -measurable) processes  $Z = (Z_t)_{0 \leq t \leq T} \in \mathbf{L}_{\mathbb{F}}^p(\mathbf{0}, \mathbf{T})$  (resp.  $\mathbf{L}_{\mathbb{G}}^p(\mathbf{0}, \mathbf{T})$ ).  $\mathbf{L}^p(\tilde{\mu})$ ,  $p \geq 1$ , is the set of  $\mathcal{P} \otimes \sigma(\mathcal{I})$ -measurable maps  $U : \Omega \times [0, T] \times \mathcal{I} \rightarrow \mathbb{R}$  such that

$$\|U\|_{\mathbf{L}^p(\tilde{\mu})} := \left( \mathbf{E} \left[ \int_0^T \int_{\mathcal{I}} |U_t(i)|^p \lambda(di) dt \right] \right)^{\frac{1}{p}} < \infty.$$

$\mathbf{A}_{\mathbb{F}}^2$  (resp.  $\mathbf{A}_{\mathbb{G}}^2$ ) is the closed subset of  $\mathcal{S}_{\mathbb{F}}^2$  (resp.  $\mathcal{S}_{\mathbb{G}}^2$ ) consisting of nondecreasing processes  $K = (K_t)_{0 \leq t \leq T}$  with  $K_0 = 0$ . Finally, for  $t \in [0, T]$ ,  $\mathcal{T}_t$  denotes the set of  $\mathbb{F}$ -stopping times  $\tau$  such that  $\tau \in [t, T]$ ,  $\mathbf{P}$ -a.s.. For ease of notation, we omit in all the paper the dependence in  $\omega \in \Omega$ , whenever it is not relevant.

## 2 Constrained Backward SDEs with jumps

This section is devoted to the presentation of constrained Backward SDEs with jumps, generalizing the framework considered in [14] or [19]. Namely:

- We allow the driver function to depend on the jump component of the backward process,
- We extend the class of possible constraint functions by letting them depend on all the components of the solution to the BSDE.

We adapt the arguments developed in [14] in order to derive existence and uniqueness of a minimal solution for this new type of BSDE. No major difficulty appears for the obtention of these results and, from our point of view, the nice feature of such constrained BSDE relies on their relation with multidimensional reflected BSDE, developed in the next section. In order to simplify, the readability of the paper, the required technical extensions of comparison and monotonic limit theorems are reported in Sections 4.1 and 4.2. They are presented in a more abstract framework and can therefore be quoted more conveniently in the future.

### 2.1 Formulation

A constrained BSDE with jumps is characterized by three objects:

- a terminal condition, i.e. a  $\mathcal{G}_T$ -measurable random variable  $\xi$ ,
- a driver function, i.e. a map  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}$ , which is  $\mathfrak{P}_{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m)$ -measurable,
- a constraint function, i.e. a  $\sigma(\mathcal{I}) \otimes \mathfrak{P}_{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ -measurable map  $h : \mathcal{I} \times \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $h_i(\omega, t, y, z, \cdot)$  is non-increasing for all  $(i, \omega, t, y, z) \in \mathcal{I} \times \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$ .

**Definition 2.1.** (i) A solution to the corresponding constrained BSDE with jumps is a quadruple  $(Y, Z, U, K) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds + K_T - K_t - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di), \quad (2.1)$$

for  $0 \leq t \leq T$  a.s., as well as the constraint

$$h_i(t, Y_{t-}, Z_t, U_t(i)) \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \text{ a.e.} \quad (2.2)$$

(ii)  $(Y, Z, U, K)$  is a minimal solution to (2.1)-(2.2) whenever it is solution to (2.1)-(2.2) and for any other solution  $(\check{Y}, \check{Z}, \check{U}, \check{K})$  of (2.1)-(2.2), we have  $Y \leq \check{Y}$  a.s.

We notice that for a minimal solution  $(Y, Z, U, K)$  to (2.1)-(2.2), the component  $Y$  naturally interprets in the terminology of Peng [17] as the smallest supersolution to (2.1)-(2.2).

**Remark 2.1.** In the case where the driver function  $f$  does not depend on  $U$  and the constraint function  $h$  is of the form  $h_i(u + c(t, y, z))$ , observe that this BSDE exactly fits in the framework considered in [14]. Similarly, in the Brownian case (i.e. no jump component), this type of BSDEs was studied in [19]. Therefore, our framework generalizes and unifies those considered in [14] and [19].

In order to work on this class of BSDE, we require the classical Lipschitz and linear growth conditions on the coefficients, as well as a control on the way the driver function depends on the jump component  $U$  of the BSDE. We regroup these conditions in the following assumption.

**(H0)**

- (i) There exists a constant  $k > 0$  such that the functions  $f$  and  $h$  satisfy  $\mathbf{P}$ -a.s. the uniform Lipschitz property:

$$\begin{aligned} |f(t, y, z, u) - f(t, y', z', u')| &\leq k|(y, z, u) - (y', z', u')|, \\ |h_i(t, y, z, u_i) - h_i(t, y', z', u'_i)| &\leq k|(y, z, u_i) - (y', z', u'_i)|, \end{aligned}$$

for all  $\{i, t, (y, z, u), (y', z', u')\} \in \mathcal{I} \times [0, T] \times [\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m]^2$ .

- (ii) The coefficients  $\xi$ ,  $f$  and  $h$  satisfy the following integrability condition

$$\mathbf{E}|\xi|^2 + \int_0^T \mathbf{E}|f(t, 0, 0, 0)|^2 dt + \sum_{i \in \mathcal{I}} \int_0^T \mathbf{E}|h_i(t, 0, 0, 0)|^2 dt < \infty. \quad (2.3)$$

- (iii) There exist two constants  $C_1 \geq C_2 > -1$  such that we can find a  $\mathcal{P}_{\mathbb{G}} \otimes \sigma(\mathcal{I}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^m)$ -measurable map  $\gamma : \Omega \times [0, T] \times \mathcal{I} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow [C_2, C_1]$  satisfying

$$f(t, y, z, u) - f(t, y, z, u') \leq \int_{\mathcal{I}} (u_i - u'_i) \gamma_t^{y, z, u, u'}(i) \lambda(di),$$

for all  $(i, t, y, z, u, u') \in \mathcal{I} \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}^m]^2$ ,  $\mathbf{P}$ -a.s..

**Remark 2.2.** Under Assumption **(H0)** (i) and (ii), existence and uniqueness of a solution  $(Y, Z, U, K)$  to the BSDE (2.1) with  $K = 0$  follows from classical results on BSDEs with jumps, see Lemma 2.4 in [22]. In order to add the  $h$ -constraint (2.2), one needs as usual to relax the dynamics of  $Y$  by injecting the non-decreasing process  $K$  in (2.1). In mathematical finance, the purpose of this new process  $K$  is to increase the super replication price  $Y$  of a contingent claim, under additional portfolio constraints. In order to find a minimal solution to the constrained BSDE (2.1)-(2.2), the nondecreasing property of  $h$  is crucial for stating comparison principles needed in the penalization approach.

**Remark 2.3.** Part (iii) of Assumption **(H0)** constrains the dependence of the driver  $f$  with respect to the jump component of the BSDE. It is inspired by [21] and will ensure comparison results for BSDEs driven by this type of driver, as detailed in Section 4.1.

## 2.2 Approximation by penalization

This paragraph focuses on the existence of a unique minimal solution for the constrained BSDE with jumps (2.1)-(2.2). Our approach requires the addition of an increasing component to the comparison results for BSDEs with jumps, derived by Royer [21] as well as the extension of Peng's monotonic limit theorem [17] to the consideration of BSDEs with jumps. We could not find these properties in the existing literature and report them respectively in Proposition 4.1 and Proposition 4.2 of Section 4.

The proof relies on a classical penalization argument and we introduce the following sequence of BSDEs with jumps

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f(s, Y_s^n, Z_s^n, U_s^n) ds + n \int_t^T \int_{\mathcal{I}} h_i^-(s, Y_s^n, Z_s^n, U_s^n(i)) \lambda(di) ds \\ &\quad - \int_t^T \langle Z_s^n, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s^n(i) \mu(ds, di), \quad 0 \leq t \leq T, \quad n \in \mathbb{N}, \end{aligned} \quad (2.4)$$

where  $h_i^-(\cdot) := \max(-h_i(\cdot), 0)$  is the negative part of the function  $h_i$ ,  $i \in \mathcal{I}$ . Under Assumption **(H0)**, the Lipschitz property of the coefficients  $f$  and  $h$  ensures existence and uniqueness of a solution  $(Y^n, Z^n, U^n) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$  to (2.4), see Theorem 2.1 in [1].

In order to obtain the convergence of the sequence  $(Y^n)_{n \in \mathbb{N}}$ , we require:

**(H1)** There exists  $(\check{Y}, \check{Z}, \check{K}, \check{U}) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  solution of (2.1)-(2.2).

This assumption, which may appear restrictive, is rather classical and we present in Section 3 a large class of cases where **(H1)** is satisfied. Furthermore, as detailed in Remark 2.4 below, **(H1)** can also be replaced by the weaker assumption:

**(H1')** There exists a constant  $M$  such that  $\sup_{n \in \mathbb{N}} \|Y^n\|_{\mathcal{S}^2} \leq M$ .

Under these assumptions, we are now ready to study the convergence of the quadruple  $(Y^n, Z^n, U^n, K^n)_{n \in \mathbb{N}}$ , where the nondecreasing process  $K^n \in \mathbf{A}_{\mathbb{G}}^2$  is defined by

$$K_t^n := n \int_0^t \int_{\mathcal{I}} h_i^-(s, Y_s^n, Z_s^n, U_s^n(i)) \lambda(di) ds, \quad 0 \leq t \leq T, \quad n \in \mathbb{N}.$$

The next theorem states that the sequence  $(Y^n, Z^n, U^n, K^n)_{n \in \mathbb{N}}$  converges indeed to the minimal solution of the constrained BSDE (2.1)-(2.2).

**Theorem 2.1.** *Under **(H0)** and **(H1)**, the following holds.*

- (i) *There exists a unique minimal solution  $(Y, Z, U, K) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  to (2.1)-(2.2), with  $K$  predictable.*



(ii) The sequence  $(Y^n)_{n \in \mathbb{N}}$  converges increasingly to the process  $Y$  and we have

$$\|Y^n - Y\|_{\mathbf{L}^2(\mathbf{0}, \mathbf{T})} + \|Z^n - Z\|_{\mathbf{L}^1(\mathbf{0}, \mathbf{T})} + \|U^n - U\|_{\mathbf{L}^1(\tilde{\mu})} \xrightarrow{n \rightarrow \infty} 0, \quad 1 \leq p < 2.$$

Moreover,  $(Z, U, K)$  is the weak limit of  $(Z^n, U^n, K^n)_{n \in \mathbb{N}}$  in  $\mathbf{L}^2_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{L}^2_{\mathbb{G}}(\mathbf{0}, \mathbf{T})$  and  $K_t$  is the weak limit of  $(K_t^n)_{n \in \mathbb{N}}$  in  $\mathbf{L}^2(\Omega, \mathcal{G}_t, \mathbf{P})$ , for all  $t \in [0, T]$ .

**Proof.** We prove the statements of the theorem in a reverse order. First, we show the convergence of the sequence  $(Y^n, Z^n, U^n, K^n)_{n \in \mathbb{N}}$ . Second, we verify that the limit is a minimal solution to (2.1)-(2.2). Third, we tackle the uniqueness property.

**Step 1:** *Convergence of  $(Y^n, Z^n, U^n, K^n)_{n \in \mathbb{N}}$ .*

For  $n \in \mathbb{N}$ , we introduce the Lipschitz map  $f^n := f + n \int_{\mathcal{I}} h^- d\lambda$ . Since  $f$  satisfies **(H0)**(iii) and  $h$  is Lipschitz and non-increasing, we deduce:

$$\begin{aligned} f^n(t, y, z, u) - f^n(t, y, z, u') &\leq \int_{\mathcal{I}} \{(u_i - u'_i) \gamma_t^{y, z, u, u'}(i) + n(h_i^-(t, y, z, u) - h_i^-(t, y, z, u'))\} \lambda(di), \\ &\leq \int_{\mathcal{I}} (u_i - u'_i) (\gamma_t^{y, z, u, u'}(i) + kn \mathbf{1}_{u_i \geq u'_i}) \lambda(di), \quad \mathbf{P}\text{- a.s.}, \quad n \in \mathbb{N}, \end{aligned}$$

for any  $(t, y, z, u, u') \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m$ . Thus, for any  $n \in \mathbb{N}$ , the coefficients  $f^n$  and  $f^{n+1}$  satisfy **(H0)** as well as  $f^n \leq f^{n+1}$ . We deduce from a simplified version of Proposition 4.1 without the additional increasing process  $K$ , that the sequence  $(Y^n)_{n \in \mathbb{N}}$  is non-decreasing.

Furthermore, for any quadruple  $(\check{Y}, \check{Z}, \check{U}, \check{K}) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  satisfying (2.1)-(2.2), we obtain  $Y^n \leq \check{Y}$  a.s.,  $n \in \mathbb{N}$ , applying once again Proposition 4.1 but with coefficients  $f_1 = f_2 = f^n$  and  $K^2 = \check{K}$ . Therefore, under **(H1)**, the sequence  $(Y^n)_{n \in \mathbb{N}}$  is nondecreasing and upper bounded, ensuring its monotonic convergence to a process  $Y$  with  $\|Y\|_{\mathcal{S}^2} < \infty$ .

Finally, we observe that under **(H0)** and **(H1)**, **(H3)** is satisfied by the generator  $f$  and the sequence  $(Y^n, Z^n, U^n, K^n)$  (see Section 4.2). We are now in position to appeal to Proposition 4.2, which is an extended version of Peng's monotonic limit theorem. Hence, the sequence  $(Y^n, Z^n, U^n, K^n)_{n \in \mathbb{N}}$  converges in the sense specified above. Furthermore, the limit  $(Y, Z, U, K) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  satisfies (2.1) and  $K$  is predictable.

**Step 2:**  *$(Y, Z, U, K)$  is a minimal solution to (2.1)-(2.2).*

Since  $(Y, Z, U, K)$  solves (2.1), we now focus on the constraint property (2.2). From the previous convergence result, we derive in particular that  $(Y^n, Z^n, U^n)_{n \in \mathbb{N}}$  converges in  $\mathbf{L}_{\mathbb{G}}^1(\mathbf{0}, \mathbf{T}) \times \mathbf{L}_{\mathbb{G}}^1(\mathbf{0}, \mathbf{T}) \times \mathbf{L}^1(\tilde{\mu})$  to  $(Y, Z, U)$ . Since  $h$  is Lipschitz, we get

$$\frac{\mathbf{E}[K_T^n]}{n} = \mathbf{E} \left[ \int_0^T \int_{\mathcal{I}} h_i^-(s, Y_s^n, Z_s^n, U_s^n(i)) \lambda(di) ds \right] \rightarrow \mathbf{E} \left[ \int_0^T \int_{\mathcal{I}} h_i^-(s, Y_s, Z_s, U_s(i)) \lambda(di) ds \right],$$

as  $n$  goes to infinity. Since Part (i) of Proposition 4.2 ensures that the sequence  $(K_T^n)_{n \in \mathbb{N}}$  is uniformly bounded in  $\mathbf{L}^1(\Omega, \mathcal{G}_T, \mathbf{P})$ , we deduce that the right hand side of the previous

expression equals zero. Hence the constraint (2.2) is satisfied.

As observed in the previous step, for any quadruple  $(\check{Y}, \check{Z}, \check{U}, \check{K}) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  satisfying (2.1)-(2.2), the sequence  $(Y^n)_{n \in \mathbb{N}}$  is upper bounded by  $\check{Y}$ . Passing to the limit, we deduce that  $(Y, Z, U, K)$  is a minimal solution to (2.1)-(2.2).

**Step 3:** *Uniqueness of the minimal solution.*

From the minimality condition, the uniqueness for the component  $Y$  of the solution is obvious. Suppose now that we have two solutions  $(Y, Z, U, K)$  and  $(Y, Z', U', K')$  in  $\mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  with  $K$  and  $K'$  predictable. Then we have

$$\begin{aligned} \int_0^t [f(s, Y_s, Z_s, U_s) - f(s, Y_s, Z'_s, U'_s)] ds + \int_0^t [Z'_s - Z_s] dW_s \\ + \int_0^t \int_{\mathcal{I}} [U'_s(i) - U_s(i)] \mu(di, ds) + K'_t - K_t = 0, \quad 0 \leq t \leq T. \end{aligned} \quad (2.5)$$

Since  $\mu$  is a Poisson measure, it has unaccessible jumps. Recalling that  $K$  and  $K'$  are predictable and taking the predictable projection in expression (2.5), we get

$$\int_0^t [f(s, Y_s, Z_s, U_s) - f(s, Y_s, Z'_s, U'_s)] ds + \int_0^t [Z'_s - Z_s] dW_s + K'_t - K_t = 0, \quad (2.6)$$

for  $0 \leq t \leq T$ , and

$$\int_0^T \int_{\mathcal{I}} [U'_s(i) - U_s(i)] \mu(di, ds) = 0,$$

which gives  $U' = U$ . Identifying the finite variation and the Brownian parts in (2.6) we get

$$\int_0^T [Z'_s - Z_s] dW_s = 0,$$

which leads to  $Z = Z'$ . The uniqueness of  $K$  finally follows from (2.5).  $\square$

**Remark 2.4.** Observe that the purpose of Assumption **(H1)** is simply to ensure an upper bound in  $\mathcal{S}_{\mathbb{G}}^2$  on the sequence of solutions  $(Y^n)_{n \in \mathbb{N}}$  to the penalized BSDEs. If such an upper bound already exists, there exists a minimal solution to (2.1)-(2.2) and **(H1)** is automatically satisfied. Hence, Theorem 2.1 also holds under **(H0)**-**(H1')**. Particular cases where Assumption **(H1)** is satisfied are for instance presented in Theorem 3.1 below. In a Markovian setting, sufficient conditions for this assumption are also provided in Remark 3.2 of [10].

### 3 Connection with multidimensional reflected BSDEs

In this section, we prove that one-dimensional constrained BSDEs with jumps offer a nice alternative for the representation of solutions to multidimensional reflected BSDEs studied in [13] and [11]. This representation has practical implications, since, for example, it opens

the door to the numerical resolution of multi-dimensional reflected BSDEs via the approximation of a single one-dimensional constrained BSDE with additional artificial jumps. The arguments presented here are purely probabilistic and therefore apply in the non Markovian framework considered in [13]. Furthermore, the proofs require precise comparison results for reflected BSDEs based on viability properties that are reported in Section 4.3 for the convenience of the reader.

### 3.1 Multidimensional reflected BSDEs

Recall that solving a general multidimensional reflected BSDE consists in finding  $m$  triplets  $(Y^i, Z^i, K^i)_{i \in \mathcal{I}} \in (\mathcal{S}_{\mathbb{F}}^2 \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}) \times \mathbf{A}_{\mathbb{F}}^2)^m$  satisfying, for all  $i \in \mathcal{I}$ ,

$$\begin{cases} Y_t^i = \xi^i + \int_t^T \psi_i(s, Y_s^1, \dots, Y_s^m, Z_s^i) ds - \int_t^T \langle Z_s^i, dW_s \rangle + K_T^i - K_t^i, & 0 \leq t \leq T, \\ Y_t^i \geq \max_{j \in A_i} h_{i,j}(t, Y_t^j), & 0 \leq t \leq T, \\ \int_0^T [Y_t^i - \max_{j \in A_i} \{h_{i,j}(t, Y_t^j)\}] dK_t^i = 0, \end{cases} \quad (3.1)$$

where  $\psi_i : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^d \rightarrow \mathbb{R}$  is an  $\mathbb{F}$ -progressively measurable map,  $\xi^i \in \mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbf{P})$ ,  $A_i$  is a nonempty subset of  $\mathcal{I}$  and, for any  $j \in A_i \cup \{i\}$ ,  $h_{i,j} : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a given  $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R})$ -measurable function satisfying  $h_{i,i}(t, y) = y$  for all  $(t, y) \in [0, T] \times \mathbb{R}$ . As detailed in Theorem 3.1 and Theorem 4.2 of [11], existence and uniqueness of a solution to (3.1) is ensured by the following assumption:

#### (H2)

- (i) For any  $i \in \mathcal{I}$  and  $j \in A_i$ , we have  $\xi^i \geq h_{i,j}(T, \xi^j)$ .
- (ii) For any  $i \in \mathcal{I}$ ,  $\mathbf{E}|\xi^i|^2 + \mathbf{E} \int_0^T \sup_{y \in \mathbb{R}^m} |\psi_i(t, y, 0)|^2 \mathbf{1}_{\{y_i=0\}} dt < +\infty$ , and  $\psi_i$  is Lipschitz continuous: there exists a constant  $k_\psi \geq 0$  such that

$$|\psi_i(t, y, z) - \psi_i(t, y', z')| \leq k_\psi (|y - y'| + |z - z'|), \quad (t, y, z, y', z') \in [0, T] \times [\mathbb{R}^m \times \mathbb{R}^d]^2.$$

- (iii) For any  $i \in \mathcal{I}$ , and  $j \neq i$ ,  $\psi_i$  is nondecreasing in its  $(j+1)$ -th variable i.e. for any  $(t, y, y', z) \in \mathcal{I} \times [\mathbb{R}^m]^2 \times \mathbb{R}^d$  such that  $y_k = y'_k$  for  $k \neq j$  and  $y_j \leq y'_j$ , we have

$$\psi_i(t, y, z) \leq \psi_i(t, y', z) \quad \mathbf{P} - a.s.$$

- (iv) For any  $(i, t, y) \in \mathcal{I} \times [0, T] \times \mathbb{R}$  and  $j \in A_i$ ,  $h_{i,j}$  is continuous,  $h_{i,j}(t, \cdot)$  is a 1-Lipschitz increasing function satisfying  $h_{i,j}(t, y) \leq y$ ,  $\mathbf{P}$ -a.s. and we have  $h_{i,j}(\cdot, 0) \in \mathbf{L}^2(\mathbf{0}, \mathbf{T})$ .

- (v) For any  $i \in \mathcal{I}$ ,  $j \in A_i$  and  $l \in A_j$ , we have  $l \in A_i \cup \{i\}$  and

$$h_{i,j}(t, h_{j,l}(t, y)) < h_{i,l}(t, y), \quad (t, y) \in [0, T] \times \mathbb{R}.$$

**Remark 3.1.** Part (ii) and (iii) of Assumption (H2) are classical Lipschitz and monotonicity properties of the driver. Part (iv) ensures a tractable form for the domain of  $\mathbb{R}^m$  where  $(Y^i)_{i \in \mathcal{I}}$  lies, and (i) implies that the terminal condition is indeed in the domain. Recent results in [5] allow to relax the monotonicity condition (iii) for the case of constraint function  $h$  associated to switching problems.

### 3.2 Corresponding constrained BSDE with jumps

We consider now the following one-dimensional constrained BSDE with jump : find a minimal quadruple  $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{0}, \mathbf{T}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  satisfying

$$\begin{aligned} \tilde{Y}_t = & \xi^{I_T} + \int_t^T \psi_{I_{s^-}}(s, \tilde{Y}_s + \tilde{U}_s(1)\mathbf{1}_{I_{s^-} \neq 1}, \dots, \tilde{Y}_s + \tilde{U}_s(m)\mathbf{1}_{I_{s^-} \neq m}, \tilde{Z}_s) ds + \tilde{K}_T - \tilde{K}_t \\ & - \int_t^T \langle \tilde{Z}_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} \tilde{U}_s(i) \mu(ds, di), \quad 0 \leq t \leq T, \text{ a.s.} \end{aligned} \quad (3.2)$$

together with the constraint

$$\mathbf{1}_{A_{I_{t^-}}}(i) \left[ \tilde{Y}_{t^-} - h_{I_{t^-}, i}(t, \tilde{Y}_{t^-} + \tilde{U}_t(i)) \right] \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \text{ a.e.}, \quad (3.3)$$

where the process  $I$  is a pure jump process defined by

$$I_t = I_0 + \int_0^t \int_{\mathcal{I}} (i - I_{s^-}) \mu(ds, di).$$

**Remark 3.2.** If the Poisson measure rewrites  $\sum_{n \geq 0} \delta_{(\kappa_n, L_n)}$ , where  $(\kappa_n)_n$  are the jump times and  $(L_n)_n$  the jump sizes, the pure jump process  $I$  simply coincides with  $L_n$  on each  $[\kappa_n, \kappa_{n+1})$ .

Considering  $I$  as an extra source of randomness, the BSDE (3.2)-(3.4) enters into the class of constrained BSDEs with jumps of the form (2.1)-(2.2) studied above, with the following correspondence

$$\begin{aligned} \xi &= \xi^{I_T}; \\ f(t, y, z, u) &= \psi_{I_{t^-}}(t, (y + u_i \mathbf{1}_{I_{t^-} \neq i})_{i \in \mathcal{I}}, z), \quad (t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m; \\ h_i(t, y, z, v) &= \{y - h_{I_{t^-}, i}(t, y + v)\} \mathbf{1}_{i \in A_{I_{t^-}}} \quad (i, t, y, z, v) \in \mathcal{I} \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}. \end{aligned}$$

As detailed below, Assumption **(H2)** is sufficient to ensure the existence of a one-dimensional minimal solution to the BSDE (3.2)-(3.3). Remarkably, we prove hereafter that this one-dimensional solution directly relates with the multidimensional solution of the reflected BSDE (3.1). Since the new constrained BSDE is one-dimensional, this alternative BSDE representation is promising for the numerical resolution of optimal switching problems. An entirely probabilistic numerical scheme for these equations is given in [10].

We are now ready to state the main result of the paper.

**Theorem 3.1.** *Suppose that Assumption **(H2)** is in force and denote by  $(Y^i, Z^i, K^i)_{i \in \mathcal{I}}$  the unique solution of (3.1). Then, the constrained BSDE (3.2)-(3.3) satisfies **(H0)**-**(H1)** and its unique corresponding minimal solution  $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  verifies*

$$\tilde{Y}_t = Y_t^{I_t}, \quad \tilde{Z}_t = Z_t^{I_t^-}, \quad \tilde{U}_t = (Y_t^i - Y_t^{I_t^-})_{i \in \mathcal{I}}, \quad 0 \leq t \leq T. \quad (3.4)$$

**Proof.** The proof divides in 3 steps. First we prove the existence of a unique minimal solution to (3.2)-(3.3). Then, we introduce a sequence of penalized BSDEs converging to the solution of the multidimensional reflected BSDE (3.1). Finally, we prove that a corresponding sequence of penalized BSDEs with jumps, built via a relation of the form of (3.4), converges indeed to the solution of (3.2)-(3.3).

**Step 1:** *Existence and uniqueness of a minimal solution to (3.2)-(3.3).*

In order to use Theorem 2.1, we need to verify that Assumptions **(H0)** and **(H1)** are satisfied in this context.

First, parts (i) and (ii) of Assumption **(H0)** are direct consequences of **(H2)**(ii) and (iv). Fix any  $(t, y, z, u, u') \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m$ , and define  $v^{(k)} \in \mathbb{R}^m$  by

$$v^{(k)} = (u'_1, \dots, u'_{k-1}, u_k, \dots, u_m), \quad 1 \leq k \leq m+1.$$

From the monotonicity assumption **(H2)**(iii) on the Lipschitz function  $\psi$  we get

$$\begin{aligned} f(t, y, z, u) - f(t, y, z, u') &= \sum_{k=1}^m \psi_{I_{t-}}(t, (y + v_i^{(k)} \mathbf{1}_{I_{t-} \neq i})_{i \in \mathcal{I}}, z) - \psi_{I_{t-}}(t, (y + v_i^{(k+1)} \mathbf{1}_{I_{t-} \neq i})_{i \in \mathcal{I}}, z) \\ &\leq k_\psi \sum_{k=1}^{m-1} (u_k - u'_k) \mathbf{1}_{u_k \geq u'_k} \mathbf{1}_{k \neq I_{t-}}. \end{aligned}$$

Taking  $\gamma_t^{y, z, u, u'}(i) = \frac{k_\psi}{\lambda(i)} \mathbf{1}_{u_k \geq u'_k} \mathbf{1}_{i \neq I_{t-}}$  (which is well defined, since  $\lambda(i) > 0$  for any  $i \in \mathcal{I}$ ), we get **(H0)**-(iii).

In order to prove that **(H1)** holds, one needs to verify the existence of a solution to (3.2)-(3.3). We indeed check hereafter that the candidate  $(\tilde{Y}, \tilde{Z}, \tilde{U})$  defined in (3.4) satisfies (3.2) as well as (3.3). Let define  $N_t := \mu(\mathcal{I} \times [0, t])$  for  $t \in [0, T]$ , the (random) number of stopping times  $\kappa_n$ , associated to the random measure  $\mu$ , which satisfy  $\kappa_n \in [0, t]$ . Then, since  $Y$  is a solution of the reflected BSDE (3.1), we have

$$\begin{aligned} Y_{\kappa_{N_T}}^{L_{N_T}} &= \xi^{L_{N_T}} + \int_{\kappa_{N_T}}^T \psi_{L_{N_T}}(s, (Y_s^{L_{N_T}} + U_s(i) \mathbf{1}_{i \neq L_{N_T}})_{i \in \mathcal{I}}, Z_s^{L_{N_T}}) ds \\ &\quad - \int_{\kappa_{N_T}}^T Z_s^{L_{N_T}} dW_s + K_T^{L_{N_T}} - K_{\kappa_{N_T}}^{L_{N_T}}. \end{aligned}$$

Then, still using the equation (3.1) and identifying the jumps at time  $\kappa_{N_T}$ , we compute:

$$\begin{aligned} Y_{\kappa_{N_T-1}}^{L_{N_T-1}} &= Y_{\kappa_{N_T}}^{L_{N_T}} + \int_{\kappa_{N_T-1}}^{\kappa_{N_T}} \psi_{L_{N_T-1}}(s, (Y_s^{L_{N_T-1}} + U_s(i) \mathbf{1}_{i \neq L_{N_T-1}})_{i \in \mathcal{I}}, Z_s^{L_{N_T-1}}) ds \\ &\quad - \int_{\kappa_{N_T-1}}^{\kappa_{N_T}} Z_s^{L_{N_T-1}} dW_s + K_{\kappa_{N_T}}^{L_{N_T-1}} - K_{\kappa_{N_T-1}}^{L_{N_T-1}} + (Y_{\kappa_{N_T}}^{L_{N_T-1}} - Y_{\kappa_{N_T}}^{L_{N_T}}) \\ &= \xi^{L_{N_T}} + \int_{\kappa_{N_T-1}}^T \psi_{I_{s-}}(s, (Y_s^{I_{s-}} + U_s(i) \mathbf{1}_{i \neq I_{s-}})_{i \in \mathcal{I}}, Z_s^{I_{s-}}) ds - \int_{\kappa_{N_T-1}}^T Z_s^{I_{s-}} dW_s \\ &\quad - \int_{\kappa_{N_T-1}}^T \int_{\mathcal{I}} U_s(i) \mu(di, ds) + K_T^{L_{N_T}} - K_{\kappa_{N_T}}^{L_{N_T}} + K_{\kappa_{N_T}}^{L_{N_T-1}} - K_{\kappa_{N_T-1}}^{L_{N_T-1}}. \end{aligned}$$

Repeating this procedure until time  $\kappa_{N_t+1}$  for  $t \in [0, T]$ , we get

$$\begin{aligned} Y_{\kappa_{N_t+1}}^{L_{N_t+1}} &= \xi^{I_T} + \int_{\kappa_{N_t+1}}^T \psi_{I_{s^-}}(s, (Y_s^{I_s} + U_s(i)\mathbf{1}_{i \neq I_{s^-}})_{i \in \mathcal{I}}, Z_s^{I_{s^-}}) ds - \int_{\kappa_{N_t+1}}^T Z_s^{I_{s^-}} dW_s \\ &\quad - \int_{\kappa_{N_t+1}}^T \int_{\mathcal{I}} U_s(i) \mu(di, ds) + K_T^{L_{N_T}} - K_{\kappa_{N_T}}^{L_{N_T}} + K_{\kappa_{N_T}}^{L_{N_T-1}} - K_{\kappa_{N_T-1}}^{L_{N_T-1}} \\ &\quad + \dots + K_{\kappa_{N_t+2}}^{L_{N_t+1}} - K_{\kappa_{N_t+1}}^{L_{N_t+1}}. \end{aligned}$$

Combining this last expression with the equation satisfied by  $Y^{L_{N_t}}$  between  $t$  and  $\kappa_{N_t+1}$ , we deduce the existence of a square integrable increasing process  $\tilde{K}$  such that  $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$  satisfies equation (3.2). The reflection constraint in (3.1) together with the identification (3.4) imply directly that  $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$  satisfies the constraint (3.3).

Therefore **(H0)** and **(H1)** hold for (3.2)-(3.3) and the existence of a unique minimal solution follows from Theorem 2.1.

**Step 2:** *Penalization of the multidimensional BSDE (3.1).*

We now introduce the following sequence of multidimensional penalized BSDEs: for  $n \in \mathbb{N}$ , find  $m$  couples  $(Y^{i,n}, Z^{i,n})_{i \in \mathcal{I}} \in (\mathcal{S}_{\mathbb{F}}^2 \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}))^m$  satisfying

$$Y_t^{i,n} = \xi^i + \int_t^T \psi_i^n(s, Y_s^{1,n}, \dots, Y_s^{m,n}, Z_s^n) ds - \int_t^T \langle Z_s^{i,n}, dW_s \rangle, \quad 0 \leq t \leq T, \quad i \in \mathcal{I}, \quad (3.5)$$

where the random map  $\psi^n$  is defined on  $[0, T] \times \mathbb{R}^m \times [\mathbb{R}^d]^m$  by

$$\psi_i^n(t, y, z) = \psi_i(t, y, z_i) + n \sum_{j \in A_i} [y_j - h_{i,j}(t, y_j)]^{-\lambda(j)}, \quad (i, t, y) \in \mathcal{I} \times [0, T] \times \mathbb{R}^d.$$

For any  $n \in \mathbb{N}$ , the existence of a unique solution to (3.5) is given in the seminal paper [15] and we prove now that the sequence of solutions to these BSDEs converges to the solution of the multidimensional reflected BSDE (3.1).

In order to prove that the sequence  $(Y^{i,n})_{n \in \mathbb{N}}$  is nondecreasing and convergent for any  $i \in \mathcal{I}$ , we shall appeal to the multidimensional comparison theorem for reflected BSDEs presented in Section 4.3 of the paper. First, since  $\psi_i^n \leq \psi_i^{n+1}$  for any  $i \in \mathcal{I}$  and  $n \in \mathbb{N}$ , Theorem 2.1 in [12] implies that the sequence  $(Y^{\cdot, n})_{n \in \mathbb{N}}$  is nondecreasing componentwise. Second, we compute from the Lipschitz property of  $\psi$  that

$$-2\langle y, \psi^n(t, y', z) - \psi^n(t, y', z') \rangle = -2\langle y, \psi(t, y', z) - \psi(t, y', z') \rangle \leq k_\psi^2 |y|^2 + \sum_{i=1}^m |z_i - z'_i|^2,$$

**P**-a.s., for any  $\{t, y, y', (z, z')\} \in [0, T] \times [\mathbb{R}^+]^m \times \mathbb{R}^m \times [\mathbb{R}^{d \times m}]^2$  and  $n \in \mathbb{N}$ . Therefore, since  $\psi^n(t, Y_t, Z_t) = \psi(t, Y_t, Z_t)$  for  $t \in [0, T]$ , we deduce from Proposition 4.4 below that

$$Y_t^{i,n} \leq Y_t^i, \quad \text{for all } (i, t, n) \in \mathcal{I} \times [0, T] \times \mathbb{N}. \quad (3.6)$$

Introducing the sequence of processes  $K^{i,n} := n \int_0^{\cdot} \int_{A_i} [Y_s^{i,n} - h_{i,j}(s, Y_s^{j,n})]^{-\lambda(dj)} ds$ , for  $i \in \mathcal{I}$  and  $n \in \mathbb{N}$ , we deduce from Peng's monotonic limit theorem [17] the existence of:



Notice that the minimality condition in (3.9) differs from the expected one in (3.1). Nevertheless, those two coincide whenever  $\hat{Y}$  is continuous, property that we verify now.

Suppose on the contrary that  $\hat{Y}_t^{i_1} \neq \hat{Y}_{t^-}^{i_1}$  for some fixed  $(i_1, t) \in \mathcal{I} \times [0, T]$ . Then, we deduce from (3.9) that  $\hat{Y}_t^{i_1} - \hat{Y}_{t^-}^{i_1} = \hat{K}_t^{i_1} - \hat{K}_{t^-}^{i_1} < 0$ , which further implies

$$\hat{Y}_{t^-}^{i_1} = \max_{j \in A_i} h_{i_1, j}(t, \hat{Y}_{t^-}^j) = h_{i_1, i_2}(t, \hat{Y}_{t^-}^{i_2}),$$

for some  $i_2 \neq i_1$ . Using the constraint satisfied by  $\hat{Y}$ , we get

$$h_{i_1, i_2}(t, \hat{Y}_{t^-}^{i_2}) = \hat{Y}_{t^-}^{i_1} > \hat{Y}_t^{i_1} \geq \max_{i \in A_{i_1}} h_{i_1, i}(t, \hat{Y}_t^i) \geq h_{i_1, i_2}(t, \hat{Y}_t^{i_2}).$$

Thus  $\hat{Y}_t^{i_2} < \hat{Y}_{t^-}^{i_2}$ . Repeating this argument we get a finite cyclic sequence  $(i_k)_{1 \leq k \leq N}$  such that  $i_N = i_1$  and

$$\hat{Y}_{t^-}^{i_{k-1}} = h_{i_{k-1}, i_k}(t, \hat{Y}_{t^-}^{i_k}), \quad 2 \leq k \leq N,$$

which contradicts **(H2)** (v).

**Step 3:** *Link between solutions of BSDE (3.1) and BSDE (3.2)-(3.3).*

For  $n \in \mathbb{N}$ , define the process  $(Y^{I, n}, Z^{I, n}, U^{I, n}) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$  by

$$Y_t^{I, n} := Y_t^{I_t, n}, \quad Z_t^{I, n} := Z_t^{I_{t^-}, n} \quad \text{and} \quad U_t^{I, n} := (Y_t^{i, n} - Y_{t^-}^{i, n})_{i \in \mathcal{I}}, \quad 0 \leq t \leq T. \quad (3.10)$$

In order to obtain the correspondence (3.4), it only remains to prove that  $(Y^{I, n}, Z^{I, n}, U^{I, n})_n$  converges to  $(\tilde{Y}, \tilde{Z}, \tilde{U})$ .

As in Step 1, writing the dynamics of (3.5) between each successive stopping times associated to the random measure  $\mu$ , we easily check that  $(Y^{I, n}, Z^{I, n}, U^{I, n})$  is the unique solution of the following penalized BSDE

$$\begin{aligned} Y_t^{I, n} &= \xi^{I_T} + \int_t^T \psi_{I_{s^-}}(s, Y_s^{I, n} + U_s^{I, n}(1)\mathbf{1}_{I_{s^-} \neq 1}, \dots, Y_s^{I, n} + U_s^{I, n}(m)\mathbf{1}_{I_{s^-} \neq m}, Z_s^{I, n}) ds \\ &\quad - \int_t^T \langle Z_s^{I, n}, dW_s \rangle + n \int_t^T \int_{\mathcal{I}} h_i^-(s, Y_s^{I, n}, Z_s^{I, n}, U_s^{I, n}(i)) \lambda(di) ds + \int_t^T \int_{\mathcal{I}} U_s^{I, n}(i) \mu(ds, di), \end{aligned}$$

for  $0 \leq t \leq T$ . Therefore, Step 1 ensures that we can apply Theorem 2.1 and we get

$$\|Y^{I, n} - \tilde{Y}\|_{\mathbf{L}^2(0, T)} + \|Z^{I, n} - \tilde{Z}\|_{\mathbf{L}^p(0, T)} + \|U^{I, n} - \tilde{U}\|_{\mathbf{L}^p(\tilde{\mu})} \longrightarrow 0, \quad p < 2, \quad (3.11)$$

where we recall that  $(\tilde{Y}, \tilde{Z}, \tilde{U})$  is the minimal solution to (3.2)-(3.3). Combining this result with (3.10) and Step 2 concludes the proof.  $\square$



## 4 Subsidiary technical points

This section regroups technical properties which are mainly extensions of existing results but that we could not find as such in the literature. They are not the main focus of the paper but still present some interest in themselves. This dissociation allows to present them in a more abstract setting and simplifies their possible future quotation. We provide a comparison and a monotonic limit theorem for BSDEs with jumps, as well as viability and comparison properties for multidimensional reflected BSDEs.

### 4.1 A comparison theorem for reflected BSDEs with jumps

We derive here a general comparison theorem for reflected BSDEs with jumps. This extends the results of Theorem 2.5 in [21] obtained in the non-reflected case.

**Proposition 4.1.** *Let  $f_1, f_2 : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}$  two generators satisfying Assumption (H0) and  $\xi_1, \xi_2 \in \mathbf{L}^2(\Omega, \mathcal{G}_T, \mathbf{P})$ . Let  $(Y^1, Z^1, U^1) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$  satisfying on  $[0, T]$*

$$Y_t^1 = \xi^1 + \int_t^T f_1(s, Y_s^1, Z_s^1, U_s^1) ds - \int_t^T \langle Z_s^1, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s^1(i) \mu(ds, di), \quad (4.1)$$

and  $(Y^2, Z^2, U^2, K^2) \in \mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  satisfying on  $[0, T]$

$$Y_t^2 = \xi^2 + \int_t^T f_2(s, Y_s^2, Z_s^2, U_s^2) ds - \int_t^T \langle Z_s^2, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s^2(i) \mu(ds, di) + K_T^2 - K_t^2. \quad (4.2)$$

If  $\xi^1 \leq \xi^2$  and  $f_1(t, Y_t^1, Z_t^1, U_t^1) \leq f_2(t, Y_t^1, Z_t^1, U_t^1)$  for all  $t \in [0, T]$ , then we have

$$Y_t^1 \leq Y_t^2, \quad 0 \leq t \leq T.$$

**Proof.** Let us denote  $\bar{Y} := Y^2 - Y^1$ ,  $\bar{Z} := Z^2 - Z^1$ ,  $\bar{U} := U^2 - U^1$ ,  $\bar{f} = f_2(\cdot, Y^2, Z^2, U^2) - f_1(\cdot, Y^1, Z^1, U^1)$  and  $\bar{\xi} = \xi^2 - \xi^1$  so that

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}_s ds - \int_t^T \langle \bar{Z}_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} \bar{U}_s(i) \mu(ds, di) + K_T^2 - K_t^2, \quad 0 \leq t \leq T, \quad (4.3)$$

Let now define the process  $a$  by

$$a_t := \frac{f_2(t, Y_t^2, Z_t^2, U_t^2) - f_2(t, Y_t^1, Z_t^2, U_t^2)}{Y_t} \mathbf{1}_{\{\bar{Y}_t \neq 0\}}, \quad 0 \leq t \leq T,$$

and  $b$  the  $\mathbb{R}^d$ -valued process defined component by component by

$$b_t^k := \frac{f_2(t, Y_t^1, Z_t^{(k-1)}, U_t^2) - f_2(t, Y_t^1, Z_t^{(k)}, U_t^2)}{V_t^k} \mathbf{1}_{\{V_t^k \neq 0\}}, \quad k = 1, \dots, d, \quad 0 \leq t \leq T,$$

where  $Z_t^{(k)}$  is the  $\mathbb{R}^d$ -valued random vector whose  $k$  first components are those of  $Z^1$  and whose  $(d - k)$  lasts are those of  $Z^2$ , and  $V_t^k$  is the  $k$ -th component of  $Z_t^{(k-1)} - Z_t^{(k)}$ .

Notice that the processes  $a$  and  $b$  are  $\mathbf{P}$ -a.s. bounded since  $f_2$  is Lipschitz continuous. Observe also that the process  $\bar{K}$  defined on  $[0, T]$  by

$$\bar{K}_t := K_t^2 - \int_0^t \int_{\mathcal{I}} 2\gamma_s^{Y_s^1, Z_s^1, U_s^1, U_s^2} \bar{U}_s(i) \lambda(di) ds + \int_0^t (f_2(s, Y_s^1, Z_s^1, U_s^2) - f_1(s, Y_s^1, Z_s^1, U_s^1)) ds$$

is a non-decreasing process since  $f_2$  satisfies **(H0)** (iii) with associated bounded process  $2\gamma$ , and  $f_1(t, Y_t^1, Z_t^1, U_t^1) \leq f_2(t, Y_t^1, Z_t^1, U_t^1)$ , for all  $t \in [0, T]$ . With these notations, we rewrite (4.3) as:

$$\begin{aligned} \bar{Y}_t &= \bar{\xi} + \int_t^T \left( a_s \bar{Y}_s + \langle b_s, \bar{Z}_s \rangle + \int_{\mathcal{I}} 2\gamma_s^{Y_s^1, Z_s^1, U_s^1, U_s^2} (i) \bar{U}_s(i) \lambda(di) \right) ds \\ &\quad - \int_t^T \langle \bar{Z}_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} \bar{U}_s(i) \mu(ds, di) + \bar{K}_T - \bar{K}_t. \end{aligned}$$

Consider now the positive process  $\Gamma$  solution of the s.d.e.:

$$d\Gamma_t = \Gamma_{t-} \left( a_t dt + \langle b_t, dW_t \rangle + \int_{\mathcal{I}} 2\gamma_t^{Y_t^1, Z_t^1, U_t^1, U_t^2} (i) \mu(dt, di) \right), \quad \Gamma_0 = 1.$$

Notice that  $\Gamma$  lies in  $\mathcal{S}_{\mathbb{G}}^2$  since  $a, b$  and  $\gamma$  are bounded, and  $\Gamma$  is positive since  $2\gamma > -1$ . A direct application of Itô's formula leads to

$$d(\Gamma \bar{Y})_t = \langle \Gamma_{t-} \bar{Z}_t + \bar{Y}_{t-} \Gamma_{t-} b_t, dW_t \rangle + \Gamma_{t-} \int_{\mathcal{I}} 2\gamma_t^{Y_t^1, Z_t^1, U_t^1, U_t^2} (i) \bar{U}_t(i) \tilde{\mu}(ds, di) - \Gamma_{t-} d\bar{K}_t,$$

recall that  $\tilde{\mu}$  is the compensated measure associated to  $\mu$ . Hence, the process  $\Gamma \bar{Y}$  is a supermartingale since  $\Gamma > 0$ . Therefore

$$\Gamma_t \bar{Y}_t \geq \mathbb{E} [\Gamma_T \bar{Y}_T | \mathcal{G}_t] = \mathbb{E} [\Gamma_T \bar{\xi} | \mathcal{G}_t] \geq 0, \quad 0 \leq t \leq T,$$

leading to  $\bar{Y} \geq 0$ . □

## 4.2 Monotonic limit theorem for BSDE with jumps

This paragraph is devoted to the extension of Peng's monotonic limit theorem to the framework of BSDEs driven by a Brownian motion and a Poisson random measure. In the particular case where the driver  $f$  does not depend on the jump component  $U$ , this extension can be obtained combining several results derived in Section 3 of [14]. For sake of completeness, we provide here a proof of the result.

We consider a sequence  $(Y^n, Z^n, U^n, K^n)_n$  in  $\mathcal{S}_{\mathbb{G}}^2 \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  such that

$$Y_t^n = Y_T^n + \int_t^T g(s, Y_s^n, Z_s^n, U_s^n) ds - \int_t^T \langle Z_s^n, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s^n(i) \mu(ds, di) + K_T^n - K_t^n,$$

for all  $t \in [0, T]$  and all  $n \in \mathbb{N}$ . Here  $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}$ , is  $\mathfrak{P}_{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m)$ -measurable, We also introduce the following assumption :

**(H3)**

(i)  $g(\cdot, 0, 0, 0)$  is square integrable:

$$\mathbb{E} \int_0^T |g(t, 0, 0, 0)|^2 dt < \infty .$$

(ii) There exists a constant  $k > 0$  such that the function  $g$  satisfies **P**-a.s. the uniform Lipschitz property:

$$|g(t, y, z, u) - g(t, y', z', u')| \leq k|(y, z, u) - (y', z', u')| ,$$

for all  $\{t, (y, z, u), (y', z', u')\} \in [0, T] \times [\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m]^2$ .

(iii) For any  $t \in [0, T]$ ,  $(Y_t^n)_n$  converges increasingly to  $Y_t$ , and we have  $\|Y\|_{\mathcal{S}^2} < \infty$ .

(iv)  $K^n$  is a continuous process, for any  $n \in \mathbb{N}$ .

**Proposition 4.2.** *Suppose that (H3) holds true. Then, we have:*

(i) *Up to a modification,  $Y \in \mathcal{S}_{\mathbb{G}}^2$  and there exists  $(Z, U, K) \in \mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^2$  with  $K$   $\mathbb{G}$ -predictable, such that*

$$\|Y^n - Y\|_{\mathbf{L}^2(\mathbf{0}, \mathbf{T})} + \|Z^n - Z\|_{\mathbf{L}^p(\mathbf{0}, \mathbf{T})} + \|U^n - U\|_{\mathbf{L}^p(\tilde{\mu})} \xrightarrow{n \rightarrow \infty} 0, \quad 1 \leq p < 2 ,$$

*and  $K_t$  is the weak limit of  $(K_t^n)_{n \in \mathbb{N}}$  in  $\mathbf{L}^2(\Omega, \mathcal{G}_t, \mathbf{P})$ , for any  $t \in [0, T]$ . Moreover,  $(Z, U, K)$  is the weak limit of  $(Z^n, U^n, K^n)_{n \in \mathbb{N}}$  in  $\mathbf{L}_{\mathbb{G}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{L}_{\mathbb{G}}^2(\mathbf{0}, \mathbf{T})$ .*

(ii) *The quadruple  $(Y, Z, U, K)$  satisfies*

$$\begin{aligned} Y_t &= Y_T + \int_t^T g(s, Y_s, Z_s, U_s) ds - \int_t^T \langle Z_s, dW_s \rangle \\ &\quad - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di) + K_T - K_t, \quad 0 \leq t \leq T. \end{aligned} \quad (4.4)$$

**Proof.** The proof of Proposition 4.2 is an adaptation of the proof of Theorem 2.4 in [17]. The main assumption which allows to extend the arguments of [17] is the continuity of each process  $K^n$ ,  $n \in \mathbb{N}$ . We recall the main steps of the proof and explain how the continuity assumption provides the result.

**1. Uniform estimate.** Since the sequence  $(Y^n)_n$  is monotonic, there exists a constant  $C$  such that

$$\sup_{n \in \mathbb{N}} \|Y^n\|_{\mathcal{S}^2} \leq \|Y^0\|_{\mathcal{S}^2} + \|Y\|_{\mathcal{S}^2} \leq C . \quad (4.5)$$

Applying Itô's formula to  $|Y^n|^2$  and using **(H3)** (ii), we have

$$\begin{aligned}
\mathbb{E}|Y_t^n|^2 &= \mathbb{E}|Y_T^n|^2 + 2\mathbb{E} \int_t^T Y_s^n g(s, Y_s^n, Z_s^n, U_s^n) ds - \mathbb{E} \int_t^T |Z_s^n|^2 ds \\
&\quad - \mathbb{E} \int_t^T \int_{\mathcal{I}} (|Y_{s-}^n + U_s^n(i)|^2 - |Y_{s-}^n|^2) \mu(di, ds) + 2\mathbb{E} \int_t^T Y_s^n dK_s^n \\
&\leq \mathbb{E}|Y_T^n|^2 + 2\mathbb{E} \int_t^T |Y_s^n| (g(s, 0, 0, 0) + k|Y_s^n| + k|Z_s^n| + k|U_s^n|) ds - \mathbb{E} \int_t^T |Z_s^n|^2 ds \\
&\quad - \mathbb{E} \int_t^T \int_{\mathcal{I}} (2Y_s^n U_s^n(i) - |U_s^n(i)|^2) \lambda(di) ds + 2\mathbb{E} \sup_{s \in [0, T]} |Y_s^n| \int_t^T dK_s^n.
\end{aligned}$$

Using the inequality  $2ab \leq \eta|a|^2 + \frac{|b|^2}{\eta}$  for  $a, b \in \mathbb{R}$  and  $\eta > 0$  and **(H3)** (i), we get the existence of a constant  $C$  s.t.

$$\mathbb{E} \int_0^T |Z_s^n|^2 ds + \mathbb{E} \int_0^T \int_{\mathcal{I}} |U_s^n(i)|^2 \lambda(di) ds \leq C \left( \mathbb{E} \sup_{t \in [0, T]} |Y_t^n|^2 + 1 \right) + 2\mathbb{E} K_T^n \sup_{t \in [0, T]} |Y_t^n|. \quad (4.6)$$

Then since

$$K_T^n = Y_0^n - Y_T^n - \int_0^T g(s, Y_s^n, Z_s^n, U_s^n) ds + \int_0^T \langle Z_s^n, dW_s \rangle + \int_0^T \int_{\mathcal{I}} U_s^n(i) \mu(di, ds),$$

we have from **(H3)** (ii), the existence of a positive constant  $C'$  s.t.

$$\mathbb{E}|K_T^n|^2 \leq C' \left( 1 + \mathbb{E} \sup_{t \in [0, T]} |Y_t^n|^2 + \mathbb{E} \int_0^T |Z_t^n|^2 dt + \mathbb{E} \int_0^T \int_{\mathcal{I}} |U_s^n(i)|^2 \lambda(di) ds \right). \quad (4.7)$$

Applying the inequality  $2ab \leq 2C'|a|^2 + \frac{|b|^2}{2C'}$  for  $a, b \in \mathbb{R}$ , we obtain

$$2\mathbb{E} K_T^n \sup_{t \in [0, T]} |Y_t^n| \leq \frac{1}{2} \mathbb{E} \int_0^T |Z_s^n|^2 ds + \frac{1}{2} \mathbb{E} \int_0^T \int_{\mathcal{I}} |U_s^n(i)|^2 \lambda(di) ds + C'' \left( 1 + \mathbb{E} \sup_{t \in [0, T]} |Y_t^n|^2 \right).$$

Combining this last estimate with (4.5) and (4.6), we obtain a constant  $C$  such that

$$\|Y^n\|_{\mathcal{S}^2} + \|Z^n\|_{\mathbf{L}^2(\mathbf{0}, \mathbf{T})} + \|U^n\|_{\mathbf{L}^2(\tilde{\mu})} \leq C, \quad n \in \mathbb{N}.$$

Then combining the previous inequality with (4.7) we get

$$\|Y^n\|_{\mathcal{S}_{\mathbb{C}}^2} + \|Z^n\|_{\mathbf{L}^2(\mathbf{0}, \mathbf{T})} + \|U^n\|_{\mathbf{L}^2(\tilde{\mu})} + \|K^n\|_{\mathcal{S}_{\mathbb{C}}^2} \leq C, \quad n \in \mathbb{N}. \quad (4.8)$$

**2. Weak convergence.** Using the previous uniform estimate and the Hilbert structure of  $\mathbf{L}_{\mathbb{C}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{L}_{\mathbb{C}}^2(\mathbf{0}, \mathbf{T}) \times \mathbf{L}_{\mathbb{C}}^2(\mathbf{0}, \mathbf{T})$ , we deduce the existence of a subsequence of  $(Z^n, U^n, K^n, g(\cdot, Y^n, Z^n, U^n))_n$ , which converges weakly to some process  $(Z, U, K, G)$  in  $\mathbf{L}_{\mathbb{C}}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{L}_{\mathbb{C}}^2(\mathbf{0}, \mathbf{T}) \times \mathbf{L}_{\mathbb{C}}^2(\mathbf{0}, \mathbf{T})$ .

Identifying the limits of  $(Y^n)_n$  and  $(Z^n, U^n, K^n, g(\cdot, Y^n, Z^n, U^n))_n$ , we get

$$Y_t = Y_T + \int_t^T G_s ds - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di) + K_T - K_t, \quad 0 \leq t \leq T. \quad (4.9)$$

The predictability of the process  $K$  comes from the predictability of each  $K^n$  and the completeness of  $\mathbf{L}_{\mathbb{C}}^2(\mathbf{W})$  for the weak topology.

**3. Properties of the process  $K$ .** We first observe from Lemma 2.2 in [17] that the process  $K$  admits a càdlàg modification. We then establish that the contribution of the jumps of  $K$  is mainly concentrated within a finite number of intervals with sufficiently small total length.

As in Lemma 2.3 in [17], for any  $\delta, \epsilon > 0$ , there exists a finite number of pairs of stopping times  $(\sigma_k, \tau_k)_{0 \leq k \leq N}$  with  $0 < \sigma_k \leq \tau_k \leq T$  such that

- (i)  $(\sigma_j, \tau_j] \cap (\sigma_k, \tau_k] = \emptyset$  for  $j \neq k$ ;
- (ii)  $\mathbf{E} \sum_{k=0}^N (\tau_k - \sigma_k) \geq T - \epsilon$ ;
- (iii)  $\mathbf{E} \sum_{k=0}^N \sum_{\sigma_k < t \leq \tau_k} |\Delta K_t|^2 \leq \delta$ .

This result is derived with similar arguments as in [17], relying only on the right continuity of the filtration and the predictability of the process  $K$ . More precisely, its proof is based on Lemma A.1 in [17] and the fact that, since  $K$  is predictable, its jump times are predictable stopping times and hence could be announced. In other words, if  $\tau$  is a jump time of  $K$  then there exist a sequence of stopping times  $(\tau_k)_k$  with  $\tau_k < \tau$  for each  $k$  and  $\tau_k \uparrow \tau$  as  $k$  goes to infinity. Combining these two results, we end this step as in [17].

**4. Strong convergence.** From the previous step, for any  $\delta, \epsilon > 0$ , there exists a finite number of disjoint stochastic intervals  $(\sigma_k, \tau_k]$ ,  $k = 0, \dots, N$ , satisfying

- (i)  $\mathbf{E} \sum_{k=0}^N (\tau_k - \sigma_k) \geq T - \frac{\epsilon}{2}$ ;
- (ii)  $\mathbf{E} \sum_{k=0}^N \sum_{\sigma_k < t \leq \tau_k} |\Delta K_t|^2 \leq \frac{\delta \epsilon}{3}$ .

Then applying Itô's formula to  $|Y^n - Y|$  on  $(\sigma_k, \tau_k]$  and summing over  $k$  we have

$$\begin{aligned} \mathbb{E} \int_{\sigma_k}^{\tau_k} |Z_s^n - Z_s|^2 ds + \mathbb{E} \int_{\sigma_k}^{\tau_k} \int_{\mathcal{I}} |U_s(i) - U_s^n(i)|^2 \lambda(di) ds \leq \\ |Y_{\tau_k}^n - Y_{\tau_k}|^2 + \sum_{t \in (\sigma_k, \tau_k]} |\Delta K_t|^2 + 2\mathbb{E} \int_{\sigma_k}^{\tau_k} |Y_s^n - Y_s| |g(s, Y_s^n, Z_s^n, U_s^n) - G_s| ds \\ + 2\mathbb{E} \int_{\sigma_k}^{\tau_k} \int_{\mathcal{I}} (|U_s(i) - U_s^n(i)| |Y_s - Y_s^n|) \lambda(di) ds + 2\mathbb{E} \int_{\sigma_k}^{\tau_k} (Y_s - Y_s^n) dK_s. \end{aligned}$$

Hence, summing over  $k$ , we obtain the existence of a constant  $C$  such that

$$\begin{aligned} \sum_{k=0}^N \mathbb{E} \int_{\sigma_k}^{\tau_k} |Z_s^n - Z_s|^2 ds + \mathbb{E} \int_{\sigma_k}^{\tau_k} \int_{\mathcal{I}} |U_s(i) - U_s^n(i)|^2 \lambda(di) ds \leq \\ C \left( \mathbb{E} \int_0^T |Y_s - Y_s^n| dK_s + \mathbb{E} \int_0^T |Y_s - Y_s^n| (|g(s, Y_s^n, Z_s^n, U_s^n) - G_s| + 1) ds \right) \\ + C \left( \sum_{k=0}^N \mathbb{E} |Y_{\tau_k}^n - Y_{\tau_k}|^2 + \mathbb{E} \sum_{k=0}^N \sum_{t \in (\sigma_k, \tau_k]} |\Delta K_t|^2 \right). \end{aligned} \quad (4.10)$$

Using Cauchy Schwartz inequality, we have

$$\mathbb{E} \int_0^T |Y_s^n - Y_s| |g(s, Y_s^n, Z_s^n, U_s^n) - G_s| ds \leq C \left( \mathbb{E} \int_0^T |Y_s^n - Y_s|^2 ds \right)^{\frac{1}{2}} \longrightarrow 0, \quad (4.11)$$

as  $n \rightarrow \infty$ . Moreover, since  $|Y_s^0 - Y_s| \geq |Y_s^n - Y_s| \rightarrow 0$  and

$$\mathbb{E} \int_0^T |Y_s^0 - Y_s| dK_s \leq \left( \mathbb{E} \sup_{[0, T]} |Y^0 - Y|^2 \right)^{\frac{1}{2}} (\mathbb{E} |K_T|^2)^{\frac{1}{2}} < \infty,$$

we get from the dominated convergence theorem that

$$\mathbb{E} \int_0^T |Y_s^n - Y_s| dK_s \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.12)$$

Finally, since

$$\sum_{k=0}^N \mathbb{E} |Y_{\tau_k}^n - Y_{\tau_k}|^2 \leq N \mathbb{E} \sup_{[0, T]} |Y^0 - Y|^2 < \infty,$$

we get from the dominated convergence theorem that

$$\sum_{k=0}^N \mathbb{E} |Y_{\tau_k}^n - Y_{\tau_k}|^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.13)$$

Combining (4.10) with (4.11), (4.12) and (4.13), we get

$$\overline{\lim}_{n \rightarrow \infty} \sum_{k=0}^N \mathbb{E} \int_{\sigma_k}^{\tau_k} |Z_s^n - Z_s|^2 ds + \mathbb{E} \int_{\sigma_k}^{\tau_k} \int_{\mathcal{I}} |U_s(i) - U_s^n(i)|^2 \lambda(di) ds \leq \sum_{k=0}^N \sum_{t \in (\sigma_k, \tau_k]} |\Delta K_t|^2 \leq \frac{\varepsilon \delta}{3}.$$

Thus, there exists an integer  $l_{\varepsilon, \delta}$  such that

$$\sum_{k=0}^N \mathbb{E} \int_{\sigma_k}^{\tau_k} |Z_s^n - Z_s|^2 ds + \mathbb{E} \int_{\sigma_k}^{\tau_k} \int_{\mathcal{I}} |U_s(i) - U_s^n(i)|^2 \lambda(di) ds \leq \frac{\varepsilon \delta}{2}, \quad \text{for any } n \geq l_{\varepsilon, \delta}.$$

Therefore, in the product space  $([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{G})$ , we have

$$m \otimes \mathbf{P} \left( (s, \omega) \in \cup_{k=0}^N (\sigma_k, \tau_k] \times \Omega, |Z_s^n - Z_s|^2 \geq \delta \right) \leq \frac{\varepsilon}{2},$$

and, in the product space  $([0, T] \times \Omega \times \mathcal{I}, \mathcal{B}([0, T]) \otimes \mathcal{G} \otimes \sigma(\mathcal{I}))$  we have

$$m \otimes \lambda \otimes \mathbf{P} \left( (s, i, \omega) \in \cup_{k=0}^N (\sigma_k, \tau_k] \times \mathcal{I} \times \Omega, |U_s^n(i) - U_s(i)|^2 \geq \delta \right) \leq \frac{\varepsilon}{2},$$

where  $m$  denotes the lebesgue measure on  $\mathbb{R}^+$ . This implies that

$$\lim_{n \rightarrow \infty} m \otimes \mathbf{P} \left( (s, \omega) \in \cup_{k=0}^N (\sigma_k, \tau_k] \times \Omega, |Z_s^n - Z_s|^2 \geq \delta \right) = 0.$$

and

$$\lim_{n \rightarrow \infty} m \otimes \lambda \otimes \mathbf{P} \left( (s, i, \omega) \in \cup_{k=0}^N (\sigma_k, \tau_k] \times \mathcal{I} \times \Omega, |U_s^n(i) - U_s(i)|^2 \geq \delta \right) = 0.$$

Hence  $(Z^n)_n$  (resp.  $(U^n)_n$ ) converges in measure to  $Z$  (resp.  $U$ ) and since it is bounded in  $\mathbf{L}^2(\mathbf{0}, \mathbf{T})$  (resp.  $\mathbf{L}^2(\tilde{\mu})$ ), it converges in  $\mathbf{L}^p(\mathbf{0}, \mathbf{T})$  (resp.  $\mathbf{L}^p(\tilde{\mu})$ ) for all  $p < 2$ . Then, combining **(H3)** (i) with the previous strong convergence of  $(Z^n, U^n)_n$  to  $(Z, U)$  we get

$$G_s = g(s, Y_s, Z_s, U_s), \quad 0 \leq s \leq T,$$

and from (4.9), we deduce that  $(Y, Z, U, K)$  satisfies (4.4).  $\square$

### 4.3 Viability and comparison property for multi-dimensional BSDEs

We generalize in this paragraph some viability and comparison properties for multidimensional BSDEs in a closed convex cone  $\mathcal{C}$  of  $\mathbb{R}^{2m}$ , whenever we add some reflections on the  $Y$ -component of the BSDE. The two following propositions are respectively extensions of Theorem 2.5 in [6] and a simplifying version of Theorem 2.1 in [12]. Their derivations do not present major difficulty and we choose to detail them for sake of completeness.

Let  $(Y, Z) \in (\mathcal{S}_{\mathbb{F}}^2 \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}))^{2m}$  satisfying

$$Y_t = Y_T + \int_t^T F(s, Y_s, Z_s) ds - \int_t^T \langle Z_s, dW_s \rangle + K_T - K_t, \quad 0 \leq t \leq T, \quad (4.14)$$

where  $F : \Omega \times [0, T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m \times d} \rightarrow \mathbb{R}^{2m}$  is a progressively measurable function satisfying **(H2)** (ii) and  $K$  is an  $\mathbb{R}^{2m}$ -valued finite variation process such that

$$K_t = \int_0^t k_s d|K|_s,$$

with  $k_t \in \mathcal{C}$  and  $|K|_s$  the variation of  $K$  on  $[0, s]$ . We denote by  $d_{\mathcal{C}}$  the distance to  $\mathcal{C}$ , i.e.  $d_{\mathcal{C}} : x \mapsto \min_{y \in \mathcal{C}} |x - y|$ , and introduce  $\Pi_{\mathcal{C}}$  the projection operator onto  $\mathcal{C}$ .

**Proposition 4.3.** *Suppose  $Y_T \in \mathcal{C}$  and there exists a constant  $C^0$  such that  $F$  satisfies*

$$4\langle y - \Pi_{\mathcal{C}}(y), F(t, y, z) \rangle \leq \langle D^2 |d_{\mathcal{C}}|^2(y) z, z \rangle + 2C^0 |d_{\mathcal{C}}|^2(y) \quad \mathbf{P} - a.s., \quad (4.15)$$

for any  $(t, y, z) \in [0, T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m \times d}$  such that  $|d_{\mathcal{C}}|^2$  is twice differentiable at the point  $y$ . Then, we have

$$Y_t \in \mathcal{C}, \quad 0 \leq t \leq T, \quad \mathbf{P} - a.s.$$

**Proof.** The proof presented here is an adaptation of the one of Theorem 2.5 in [6], allowing to tackle the additional difficulty due to the  $dK$  term in the dynamics of  $Y$ .

Let  $\eta \in C^\infty(\mathbb{R}^{2m})$  be a non-negative function, with support in the unit ball, such that  $\int_{\mathbb{R}^{2m}} \eta(x) dx = 1$ . For  $\delta > 0$  and  $x \in \mathbb{R}^{2m}$ , we define

$$\eta_\delta(x) := \frac{1}{\delta^{2m}} \eta\left(\frac{x}{\delta}\right) \quad \text{and} \quad \phi_\delta(x) := \int_{\mathbb{R}^{2m}} |d_{\mathcal{C}}(x - y)|^2 \eta_\delta(y) dy.$$

Via direct computation, one can verify that  $\phi_\delta \in C^\infty(\mathbb{R}^{2m})$  and

$$\begin{cases} 0 \leq \phi_\delta(x) \leq (d_{\mathcal{C}}(x) + \delta)^2, \\ D\phi_\delta(x) = \int_{\mathbb{R}^{2m}} D|d_{\mathcal{C}}(y)|^2 \eta_\delta(x - y) dy \quad \text{and} \quad |D\phi_\delta(x)| \leq 2(d_{\mathcal{C}}(x) + \delta), \\ D^2\phi_\delta(x) = \int_{\mathbb{R}^{2m}} D^2|d_{\mathcal{C}}(y)|^2 \eta_\delta(x - y) dy \quad \text{and} \quad 0 \leq |D^2\phi_\delta(x)| \leq 2I_{2m}, \end{cases} \quad (4.16)$$

for any  $x \in \mathbb{R}^{2m}$ . An application of Itô's formula to  $\phi_\delta(Y)$ , combined with these estimates

and  $d_{\mathcal{C}}(Y_T) = 0$ , leads to

$$\begin{aligned}
\mathbf{E}\phi_{\delta}(Y_t) &= \mathbf{E}\phi_{\delta}(Y_T) + \mathbf{E} \int_t^T \langle D\phi_{\delta}(Y_s), F(s, Y_s, Z_s) \rangle ds - \frac{1}{2} \mathbf{E} \int_t^T \langle D^2\phi_{\delta}(Y_s) Z_s, Z_s \rangle ds \\
&\quad + \mathbf{E} \int_t^T \langle D\phi_{\delta}(Y_s), k_s \rangle d|K|_s \\
&\leq \delta^2 + \mathbf{E} \int_t^T \int_{\mathbb{R}^{2m}} \left[ \langle D|d_{\mathcal{C}}(y)|^2, F(s, y, Z_s) \rangle - \frac{1}{2} \langle D^2|d_{\mathcal{C}}(y)|^2 Z_s, Z_s \rangle \right] \eta_{\delta}(Y_s - y) dy ds \\
&\quad - \mathbf{E} \int_t^T \int_{\mathbb{R}^{2m}} \langle D|d_{\mathcal{C}}(y)|^2, F(s, y, Z_s) - F(s, Y_s, Z_s) \rangle \eta_{\delta}(Y_s - y) dy ds \\
&\quad + \mathbf{E} \int_t^T \int_{\mathbb{R}^{2m}} \langle D|d_{\mathcal{C}}(y)|^2, k_s \rangle \eta_{\delta}(Y_s - y) dy d|K|_s, \quad 0 \leq t \leq T. \tag{4.17}
\end{aligned}$$

Since  $k$  is valued in the closed convex cone  $\mathcal{C}$ , we observe that

$$\langle D|d_{\mathcal{C}}(y)|^2, k_s \rangle \leq 0, \quad 0 \leq s \leq T, \quad y \in \mathbb{R}^{2m}.$$

Then, plugging this expression, (4.15) and inequality  $2d_{\mathcal{C}}(\cdot) \leq 1 + d_{\mathcal{C}}(\cdot)^2$  in (4.17), we get

$$\begin{aligned}
\mathbf{E}\phi_{\delta}(Y_t) &\leq \delta^2 + C^0 \mathbf{E} \int_t^T \int_{\mathbb{R}^{2m}} |d_{\mathcal{C}}(y)|^2 \eta_{\delta}(y - Y_s) dy ds \\
&\quad + 2\mathbf{E} \int_t^T \int_{\mathbb{R}^{2m}} d_{\mathcal{C}}(y) \eta_{\delta}(Y_s - y) \max_{y': |y' - Y_s| \leq \delta} |F(s, y', Z_s) - F(s, Y_s, Z_s)| dy ds \\
&\leq \delta^2 + C^0 \int_t^T \mathbf{E}\phi_{\delta}(Y_s) ds + \mathbf{E} \int_t^T (1 + \phi_{\delta}(Y_s)) \max_{y': |y' - Y_s| \leq \delta} |F(s, y', Z_s) - F(s, Y_s, Z_s)| ds,
\end{aligned}$$

for any  $t \in [0, T]$ . Using the uniform Lipschitz property of  $F$ , we deduce

$$\mathbf{E}\phi_{\delta}(Y_t) \leq C \left\{ \delta^2 + \delta + \int_t^T \mathbf{E}\phi_{\delta}(Y_s) ds \right\}, \quad 0 \leq t \leq T, \quad \delta > 0,$$

and Gronwall's lemma leads to

$$\mathbf{E}\phi_{\delta}(Y_t) \leq C(\delta^2 + \delta), \quad 0 \leq t \leq T, \quad \delta > 0.$$

Finally, from Fatou's Lemma, we have

$$\mathbf{E}|d_{\mathcal{C}}(Y_t)|^2 \leq \liminf_{\delta \rightarrow 0} \mathbf{E}\phi_{\delta}(Y_t) = 0, \quad 0 \leq t \leq T,$$

which concludes the proof.  $\square$

We now turn to the obtention of a multidimensional comparison result for BSDEs, whenever the dominating BSDE suffers additional reflections. This proposition also simplifies the results of Theorem 2.1 in [12] in the case where the  $i^{\text{th}}$  component of each driver only depends on the  $i^{\text{th}}$  component of  $Z$ , for any  $i \leq d$ .

Consider  $(Y^1, Z^1, K^1) \in (\mathcal{S}_{\mathbb{F}}^2 \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}) \times \mathbf{A}_{\mathbb{F}}^2)^m$  satisfying

$$Y_t^1 = Y_T^1 + \int_t^T F_1(s, Y_s^1, Z_s^1) ds - \int_t^T \langle Z_s^1, dW_s \rangle + K_T^1 - K_t^1, \quad 0 \leq t \leq T,$$



and  $(Y^2, Z^2) \in (\mathcal{S}_{\mathbb{F}}^2 \times \mathbf{L}_{\mathbb{F}}^2(\mathbf{W}))^m$  satisfying

$$Y_t^2 = Y_T^2 + \int_t^T F_2(s, Y_s^2, Z_s^2) ds - \int_t^T \langle Z_s^2, dW_s \rangle, \quad 0 \leq t \leq T,$$

where  $F_1$  and  $F_2$  are two driver functions satisfying **(H2)** (ii) and such that the  $i^{\text{th}}$  component of each driver only depends on the  $i^{\text{th}}$  component of the corresponding  $Z$ , for any  $i \leq d$ .

**Proposition 4.4.** *Suppose  $Y_T^1 \geq Y_T^2$  and the existence of a constant  $C^1$  such that*

$$-2\langle y, F_1(t, y', z) - F_2(t, y', z') \rangle \leq C^1 |y|^2 + \sum_{i=1}^m |z_i - z'_i|^2 \quad \mathbf{P} - a.s., \quad (4.18)$$

for any  $(t, y, y', z, z') \in [0, T] \times (\mathbb{R}^+)^m \times \mathbb{R}^m \times [\mathbb{R}^{m \times d}]^2$ . Then  $Y_t^1 \geq Y_t^2$ , for all  $t \in [0, T]$ .

**Proof.** The process  $(Y^1 - Y^2, Y^2)$  is valued in  $\mathbb{R}^{2m}$  and solution of a BSDE of the form (4.14) associated to the driver

$$F : (t, (y, y'), (z, z')) \mapsto (F_1(t, y + y', z + z') - F_2(t, y', z'), F_2(t, y', z')),$$

for any  $\{t, (y, y'), (z, z')\} \in [0, T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m \times d}$ . Introducing the closed convex cone  $\mathcal{C} := (\mathbb{R}^+)^m \times \mathbb{R}^m$  of  $\mathbb{R}^{2m}$ , we see that  $d_{\mathcal{C}}(y, y') = |y^-|$  for  $(y, y') \in \mathbb{R}^{2m}$ . Therefore, we deduce from the Lipschitz property of  $F_1$  and (4.18) that

$$\begin{aligned} & 4\langle (y, y') - \Pi_{\mathcal{C}}(y, y'), F(t, (y, y'), (z, z')) \rangle \\ &= 4\langle -y^-, F_1(t, y + y', z + z') - F_1(t, y', z + z') \rangle + 4\langle -y^-, F_1(t, y', z + z') - F_2(t, y', z') \rangle \\ &\leq 4k|y^-|^2 + 2 \sum_{i=1}^m \mathbf{1}_{y_i < 0} |z_i|^2 + 2C^1 |y^-|^2 \\ &= \langle D^2 |d_{\mathcal{C}}|^2(y, y')(z, z'), (z, z') \rangle + (2C^1 + 4k) |d_{\mathcal{C}}|^2(y, y') \quad \mathbf{P} - a.s., \end{aligned}$$

for any  $\{t, (y, y'), (z, z')\} \in [0, T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m \times d}$ . Applying Proposition 4.3 with  $C^0 = C^1 + 2k$ , we deduce that the process  $(Y^1 - Y^2, Y^2)$  is valued in  $\mathcal{C}$  and complete the proof.  $\square$

## References

- [1] BARLES G., BUCKDAHN R. AND E. PARDOUX (1997), Backward stochastic differential equations and integral-partial differential equations, *Stoch. & Stoch. Reports*, **60**, 57-83.
- [2] BERNHART M., PHAM H., TANKOV P. AND X. WARIN (2011), Swing Options Valuation: a BSDE with Constrained Jumps Approach, *Preprint*.
- [3] BOUCHARD B. (2009), A stochastic target formulation for optimal switching problems in finite horizon, *Stochastics*, **81** (2), 171-197.

- [4] BOUCHARD B. AND R. ELIE (2008), Discrete-time approximation of decoupled forward-backward SDE with jumps, *Stochastic Processes and Applications*, **118**, 53-75.
- [5] CHASSAGNEUX J.F., R. ELIE AND I. KHARROUBI (2010), A note on existence and uniqueness of multidimensional reflected BSDEs, To appear in *Electronic Communications in Probability*.
- [6] BUCKDAHN R., M. QUINCAMPOIX AND A. RASCANU (2000), Viability property for a backward stochastic differential equation and applications to partial differential equations, *Probab. Theory. Relat. Fields*, **116**, 485-504.
- [7] CVITANIC J., KARATZAS I. AND M. SONER (1998), Backward stochastic differential equations with constraints on the gain-process, *Annals of Probability*, **26**, 1522-1551.
- [8] DJEHICHE B., HAMADÈNE S. AND A. POPIER (2009), The finite horizon optimal multiple switching problem. *SIAM Journal fo Control and Optimization*, **48**(4), 2751-2770.
- [9] EL KAROUI N., KAPOUDJIAN C., PARDOUX E., PENG S. AND M.C. QUENEZ (1997), Reflected solutions of Backward SDE's, and related obstacle problems for PDEs, *Annals of Probability*, **25**, 702-737.
- [10] ELIE R. AND I. KHARROUBI (2009), Probabilistic representation and approximation for coupled systems of variational inequalities, *Stat. and Proba. Letters*. **80**, 1388-1396.
- [11] HAMADÈNE S. AND J. ZHANG (2010), Switching problem and related system of reflected BSDEs. *Stochastic Processes and Applications*, **120**(4), 403-426.
- [12] HU Y. AND S. PENG (2006), On comparison theorem for multi-dimensional BSDEs, *C. R. Acad. Sci. Paris*, **343**, 135-140.
- [13] HU Y. AND S. TANG (2010), Multi-dimensional BSDE with oblique Reflection and optimal switching, *Prob. Theory and Related Fields*, **147**(1-2) 89-121.
- [14] KHARROUBI I., MA J., PHAM H. AND J. ZHANG (2008), Backward SDEs with constrained jumps and Quasi-Variational Inequalities, *Annals of Probability*, **38**(2) 794-840.
- [15] PARDOUX E. AND S. PENG (1990), Adapted solution of a backward stochastic differential equation, *Systems and Control Letters*, **14**, 55-61.
- [16] PARDOUX E., F. PRADEILLES AND Z. RAO (1997), Probabilistic interpretation of a system of semi-linear parabolic partial differential equations, *Ann. Inst. Henri Poincaré*, section B, **33** (4), 467-490.
- [17] PENG S. (1999), Monotonic limit theory of BSDE and nonlinear decomposition theorem of Doob-Meyer's type, *Prob. Theory and Related Fields*, **113**, 473-499.
- [18] PENG S. AND M. XU (2005), The smallest g-supermartingale and reflected BSDE with single and double  $L^2$  obstacles, *Ann. I.H.P.* **41**, 605-630.

- [19] PENG S. AND M. XU (2007), Constrained BSDE and viscosity solutions of variation inequalities, *Preprint*.
- [20] RAMASUBRAMANIAN S. (2002), Reflected backward stochastic differential equations in an orthant, *Proc. indian Acad. Sci.*, **112**(2), 347-360.
- [21] ROYER M. (2006), Backward stochastic differential equations with jumps and related nonlinear expectations, *Stochastic Processes and their Applications*, **116**, 1358-1376.
- [22] TANG S. AND X. LI (1994), Necessary conditions for optimal control of stochastic systems with jumps, *SIAM J. Control and Optimization*, **32**, 1447-1475.
- [23] TANG S. AND J. YONG (1993), Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach, *Stoch. & Stoch. Reports*, **45**, 145-176.