



HAL
open science

Cartesian effect categories are Freyd-categories

Jean-Guillaume Dumas, Dominique Duval, Jean-Claude Reynaud

► **To cite this version:**

Jean-Guillaume Dumas, Dominique Duval, Jean-Claude Reynaud. Cartesian effect categories are Freyd-categories. 2009. hal-00369328v1

HAL Id: hal-00369328

<https://hal.science/hal-00369328v1>

Preprint submitted on 19 Mar 2009 (v1), last revised 12 Jun 2009 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Cartesian effect categories are Freyd-categories

Jean-Guillaume Dumas* Dominique Duval* Jean-Claude Reynaud†
{Jean-Guillaume.Dumas,Dominique.Duval,Jean-Claude.Reynaud}@imag.fr

March 19., 2009

Abstract. Although categorical composition and finite products can be used for dealing with the substitution of terms, they do not deal with the order of evaluation of arguments, which may have major consequences when there are side-effects. In this paper cartesian effect categories are introduced for solving this issue, and they are compared with strong monads, Freyd-categories and Haskell's Arrows. It is proved that a cartesian effect category is a Freyd-category where the premonoidal structure is provided by a kind of binary product, called the sequential product.

Keywords. Categorical logic, computational effects, monads, Freyd-categories, premonoidal categories, Arrows, sequential product, cartesian effect categories.

1 Introduction

Roughly speaking, a categorical semantics for a programming language associates an object to each type, a morphism to each term, and uses composition and finite products for dealing with the substitution of terms. This framework does behave very well in a simple equational setting, but it has to be adapted as soon as there is some kind of side-effects, for instance non-termination or state updating in imperative languages. In this paper we focus on the following *sequentiality* issue: the categorical products do not deal with the order of evaluation of arguments, although this order may have major consequences when there are side-effects. We introduce *cartesian effect categories* for this purpose.

Other approaches include strong monads [Mog89], Freyd-categories [PR97] and Arrows [Hug00], as well as [HPP06] and [HLPP07] for the combination of effects. These frameworks are quite similar from several points of view [HJ06, Atk08], while our framework is more precise, and thus less generic. A first draft for cartesian effect categories can be found in [DDR07], and a similar approach in [DR05].

A Freyd-category roughly consists in a category K with a wide subcategory C (wide means with the same objects), such that C is cartesian (i.e., it has finite products) and K is symmetric premonoidal. Quoting [Pow06]: “a Freyd-category is a subtle generalisation of the notion of a category with finite products”. The aim of this paper is similar, but the resulting structure, called a *cartesian effect category*, is more precise and more homogeneous than a Freyd-category: like the symmetric monoidal structure on C derives from its product, here the symmetric premonoidal structure on K derives from some kind of product, called a *sequential product*, which extends the product of C and which generalizes the usual categorical product. Moreover, there are two steps in our definition. First an *effect category* is defined, without mentioning any kind of product, as a category K with a wide subcategory C and with a *consistency relation* \preceq , which can

*Laboratoire Jean Kuntzmann, Université de Grenoble and CNRS, France. [ljk.imag.fr/membres/{Jean-Guillaume.Dumas, Dominique.Duval}](http://ljk.imag.fr/membres/{Jean-Guillaume.Dumas,Dominique.Duval})

†Malhivert, Claix, France.

be seen as an *up-to-effects* relation. Then a *sequential product* on an effect category is defined by a property that generalizes the categorical product property.

A binary product on a category C provides a bifunctor \times on C such that for all $v_1 : X_1 \rightarrow Y_1$ and $v_2 : X_2 \rightarrow Y_2$, the morphism $v_1 \times v_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is characterized by the following diagram, where the p_i 's and q_i 's are the projections. This property is symmetric in v_1 and v_2 . When C is the category of sets, this means that $v_1 \times v_2(x_1, x_2) = (v_1(x_1), v_2(x_2))$.

$$\begin{array}{ccc}
 X_1 & \xrightarrow{v_1} & Y_1 \\
 p_1 \uparrow & = & \uparrow q_1 \\
 X_1 \times X_2 & \xrightarrow{v_1 \times v_2} & Y_1 \times Y_2 \\
 p_2 \downarrow & = & \downarrow q_2 \\
 X_2 & \xrightarrow{v_2} & Y_2
 \end{array}$$

On the other hand, a left sequential product of $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ in a category K should provide a morphism $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ satisfying some property for formalizing the notion of *sequentiality*: “first f_1 , then f_2 ”, and symmetrically a right sequential product $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ for “first f_2 , then f_1 ”. The left sequential product $f_1 \times f_2$ can be seen as the composition of two simpler operations, which are both denoted \times (it will be proved later that there is no ambiguity in this notation): $f_1 \times f_2 = (\text{id}_1 \times f_2) \circ (f_1 \times \text{id}_2)$ where id_1 and id_2 denote the identities of Y_1 and X_2 , respectively. For instance, when K is the category of sets with partial functions, $f_1 \times \text{id}_2$ is the partial function such that $f_1 \times \text{id}_2(x_1, x_2) = (f_1(x_1), x_2)$ whenever $f_1(x_1)$ is defined, otherwise $f_1 \times \text{id}_2(x_1, x_2)$ is not defined. This means that $f_1 \times \text{id}_2$ is characterized by the following diagram, where \preceq is the usual partial order on partial functions, so that the bottom square means that $q_2 \circ (f_1 \times \text{id}_2) \preceq p_2$.

$$\begin{array}{ccc}
 X_1 & \xrightarrow{f_1} & Y_1 \\
 p_1 \uparrow & = & \uparrow q_1 \\
 X_1 \times X_2 & \xrightarrow{f_1 \times \text{id}_2} & Y_1 \times Y_2 \\
 p_2 \downarrow & \preceq & \downarrow q_2 \\
 X_2 & \xrightarrow{\text{id}_2} & X_2
 \end{array}$$

Let us look more closely at the partial order relation \preceq on partial functions, in order to define a similar relation for other kinds of effects. Let us say that two partial functions f and f' have *the same effect* if they have the same domain of definition. This defines an equivalence relation \approx by: $f \approx f'$ if and only if $\mathcal{D}(f) = \mathcal{D}(f')$. Then we can say that the relation \preceq is an *up to effects* relation, because whenever $f \approx f'$ and $f \preceq f'$ then $f = f'$. In this paper, we introduce a framework where the notions of *having the same effect* and *being consistent up to effects* are well defined, so that we are able to axiomatize the consistency relation \preceq and to define sequential products.

There are some variants in the use of the word “cartesian” for categories, in this paper a *cartesian category* is simply a category with a binary product, it is not required that it has a terminal object nor that it has all finite limits. First effect categories, then cartesian effect categories, are defined in section 2. In section 3 we give a precise meaning to the notions of “having the same effect” and “being consistent up to effects”, which provides the motivation for the name “effect categories” in the previous section. Cartesian effect categories are related to Freyd-categories, Arrows and strong monads in section 4. Three examples are presented: partiality and state throughout the paper, and non-determinism in section 4.

2 Cartesian effect categories

The aim of this paper is to generalize the binary product functor in a way that formalizes the idea of sequentiality, as explained in the introduction. We define *effect categories* in section 2.1. Then *cartesian effect categories* are defined in section 2.3 as effect categories with *semi-pure products*, which themselves are defined by generalizing the universal property of products. Then *sequential products* are defined in section 2.4 from the composition of semi-pure products. Various premonoidal-like properties of cartesian effect categories are proved in section 2.5, while a description of sequential products by a generalization of the universal property of semi-pure products is given in section 2.6. Examples are studied in section 2.7. The link between effect categories and computational effects will be given in section 3. Note: in many proofs, when a result is a pair of symmetric properties, only the first one is proved.

2.1 Effect categories

Definition 2.1. A subcategory C of a category K is *wide* if it has the same objects as K , this is denoted $C \subseteq K$. Given a category K with a wide subcategory $C \subseteq K$, a morphism of K is called *pure* if it is in C ; in the latter case it is denoted with “ \rightsquigarrow ”.

Example 2.2. Let C_0 be a category with a monad (M, μ, η) , let K be the Kleisli category of M , and let C denote the image of C_0 in K by the functor $J : C_0 \rightarrow K$ associated with M , which is the identity on objects. Then C is a wide subcategory of K . A morphism $f : X \rightarrow Y$ in K is, or *stands for*, a morphism $[f] : X \rightarrow M(Y)$ in C_0 , and each pure morphism $v : X \rightsquigarrow Y$ stands for a morphism $[v] = \eta_Y \circ J(v_0) : X \rightarrow M(Y)$ for some morphism $v_0 : X \rightarrow Y$ in C_0 . Let us assume that the *mono requirement* is satisfied by the monad, which means that η_X is a mono for every object X , or equivalently that the functor J is faithful. Then the correspondence between pure morphisms $v : X \rightsquigarrow Y$ in K and morphisms $v_0 : X \rightarrow Y$ in C_0 defined by $[v] = \eta_Y \circ v_0$ is one-to-one.

Definition 2.3. Let K be a category with a wide subcategory C . A *consistency* \preceq on $C \subseteq K$ is a *reflexive* and *transitive* relation between parallel morphisms that satisfies:

- *substitution*: for all $f : X \rightarrow Y$ and $g, g' : Y \rightarrow Z$, if $g \preceq g'$ then $g \circ f \preceq g' \circ f$;
- *pure replacement*: for all $f, f' : X \rightarrow Y$ and $w : Y \rightsquigarrow Z$, if $f \preceq f'$ then $w \circ f \preceq w \circ f'$.
- *equality on pure morphisms*: for all $v, v' : X \rightsquigarrow Y$, if $v \preceq v'$ then $v = v'$.

Then $C \subseteq K$ with \preceq is called an *effect category* (for short, the consistency relation \preceq may be implicit).

The (*full*) *replacement* property, which means that for all $f, f' : X \rightarrow Y$ and $g : Y \rightarrow Z$, if $f \preceq f'$ then $g \circ f \preceq g \circ f'$, is not required. In some examples, it happens that pure morphisms are *maximal* for the preorder \preceq , which means that whenever $v \preceq f$ with v pure then $v = f$.

2.2 Cartesian categories

In this paper, a *cartesian category* is a category with a binary product, it is not required that it has a terminal object. Let us remind some facts about binary products. We introduce the (unusual) notion of *product skeleton* because it will be useful later on.

Definition 2.4. A *binary product skeleton* on a category C associates to each pair of objects (X_1, X_2) in C an object $X_1 \times X_2$ and a pair of morphisms (called *projections*) $X_1 \xleftarrow{p_1} X_1 \times X_2 \xrightarrow{p_2} X_2$ in C . A graph homomorphism $C^2 \rightarrow C$ is *compatible* with the binary product skeleton if they agree on objects. In the latter case, the graph homomorphism may also be denoted \times .

We will use the letters p, q, r, s, t, \dots for the projections.

Definition 2.5. A *binary product* on a category C is made of a binary product skeleton \times on C and a compatible graph homomorphism $\times : C^2 \rightarrow C$ such that:

- for all $v_1 : X_1 \rightarrow Y_1$ and $v_2 : X_2 \rightarrow Y_2$, the morphism $v_1 \times v_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is the unique morphism that satisfies the *binary product property*:

$$\begin{cases} q_1 \circ (v_1 \times v_2) = v_1 \circ p_1 \\ q_2 \circ (v_1 \times v_2) = v_2 \circ p_2 \end{cases}$$

Then the category C with the binary product \times is called a *cartesian category* (for short, the binary product \times may be implicit).

$$\begin{array}{ccc} X_1 & \xrightarrow{v_1} & Y_1 \\ p_1 \uparrow & = & \uparrow q_1 \\ X_1 \times X_2 & \xrightarrow{v_1 \times v_2} & Y_1 \times Y_2 \\ p_2 \downarrow & = & \downarrow q_2 \\ X_2 & \xrightarrow{v_2} & Y_2 \end{array}$$

Proposition 2.6. Let C be a cartesian category, then the graph homomorphism $\times : C^2 \rightarrow C$ is a functor. If in addition C has a terminal object 1 , then C with \times and 1 is a symmetric monoidal category [Mac97].

Remark 2.7. Let us assume that for all $v, v' : X \rightarrow Y_1 \times Y_2$, if $q_1 \circ v = q_1 \circ v'$ and $q_2 \circ v = q_2 \circ v'$ then $v = v'$. Then obviously if a morphism $v_1 \times v_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ satisfies the binary product property, it is unique.

2.3 Semi-pure products

In this section, we define the *semi-pure products* on an effect category $C \subseteq K$ as two graph homomorphisms $\ltimes : C \times K \rightarrow K$ and $\rtimes : K \times C \rightarrow K$ satisfying some generalization of the binary product property.

Definition 2.8. Let $C \subseteq K$ be an effect category with a binary product skeleton \times on C . A *left semi-pure product* on $C \subseteq K$ (with respect to \times) is a graph homomorphism $\ltimes : C \times K \rightarrow K$ compatible with \times , which maps C^2 to C and such that:

- for all $v_1 : X_1 \rightsquigarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$, the morphism $v_1 \ltimes f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is the unique morphism that satisfies the *left semi-pure product property*:

$$\begin{cases} q_1 \circ (v_1 \ltimes f_2) \preceq v_1 \circ p_1 \\ q_2 \circ (v_1 \ltimes f_2) = f_2 \circ p_2 \end{cases}$$

Symmetrically, a *right semi-pure product* on $C \subseteq K$ is a graph homomorphism $\rtimes : K \times C \rightarrow K$ compatible with \times which maps C^2 to C and such that:

- for all $f_1 : X_1 \rightarrow Y_1$ and $v_2 : X_2 \rightsquigarrow Y_2$, the morphism $f_1 \rtimes v_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is the unique morphism that satisfies the *right semi-pure product property*:

$$\begin{cases} q_1 \circ (f_1 \rtimes v_2) = f_1 \circ p_1 \\ q_2 \circ (f_1 \rtimes v_2) \preceq v_2 \circ p_2 \end{cases}$$

Then the effect category $C \subseteq K$ with the semi-pure products \ltimes, \rtimes is called a *cartesian effect category* (for short, the binary product skeleton \times and the semi-pure products \ltimes, \rtimes may be implicit).

$$\begin{array}{ccc}
X_1 & \xrightarrow{v_1} & Y_1 \\
\uparrow p_1 & \cong & \uparrow q_1 \\
X_1 \times X_2 & \xrightarrow{v_1 \times f_2} & Y_1 \times Y_2 \\
\downarrow p_2 & = & \downarrow q_2 \\
X_2 & \xrightarrow{f_2} & Y_2
\end{array}
\qquad
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\uparrow p_1 & = & \uparrow q_1 \\
X_1 \times X_2 & \xrightarrow{f_1 \times v_2} & Y_1 \times Y_2 \\
\downarrow p_2 & \cong & \downarrow q_2 \\
X_2 & \xrightarrow{v_2} & Y_2
\end{array}$$

Proposition 2.9. *Let $C \subseteq K$ be a cartesian effect category. Then the restrictions of \times and \bowtie to C^2 coincide and define a binary product functor \times on C .*

Proof. Let $v_1 : X_1 \rightsquigarrow Y_1$ and $v_2 : X_2 \rightsquigarrow Y_2$. Since \times maps C^2 to C , the morphism $v_1 \times v_2$ is pure, and since consistency is the equality on pure morphisms, the left semi-pure product property for $v_1 \times v_2$ coincides with the binary product property. Therefore, the restriction of \times to C^2 defines a binary product functor on C . Symmetrically, the restriction of \bowtie to C^2 defines a binary product functor on C . Both binary products are compatible with the same product skeleton \times , hence the binary product unicity property implies that they agree. \square

Corollary 2.10. *Let $C \subseteq K$ be a cartesian effect category. Then for all X_1 and X_2 :*

$$\text{id}_{X_1} \times \text{id}_{X_2} = \text{id}_{X_1} \bowtie \text{id}_{X_2} = \text{id}_{X_1} \times \text{id}_{X_2} = \text{id}_{X_1 \times X_2}.$$

2.4 Sequential products

In accordance with the intended meaning of “sequential”, we define sequential products as composed from semi-pure products.

Definition 2.11. Let $C \subseteq K$ be a cartesian effect category. The pair of *sequential products* composed from the semi-products \times , \bowtie is made of the graph homomorphisms $\times_{\text{seq}}, \bowtie_{\text{seq}} : K^2 \rightarrow K$ (the *left* and *right* sequential products, respectively) defined as follows:

- for all $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$:

$$f_1 \times_{\text{seq}} f_2 = (\text{id}_{Y_1} \times f_2) \circ (f_1 \times \text{id}_{X_2})$$

- for all $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$:

$$f_1 \bowtie_{\text{seq}} f_2 = (f_1 \times \text{id}_{Y_2}) \circ (\text{id}_{X_1} \times f_2)$$

$$\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{\text{id}} & Y_1 \\
\uparrow p_1 & & \uparrow r_1 & \cong & \uparrow q_1 \\
X_1 \times X_2 & \xrightarrow{f_1 \times \text{id}} & Y_1 \times X_2 & \xrightarrow{\text{id} \times f_2} & Y_1 \times Y_2 \\
\downarrow p_2 & \cong & \downarrow r_2 & = & \downarrow q_2 \\
X_2 & \xrightarrow{\text{id}} & X_2 & \xrightarrow{f_2} & Y_2
\end{array}
\qquad
\begin{array}{ccc}
X_1 & \xrightarrow{\text{id}} & X_1 & \xrightarrow{f_1} & Y_1 \\
\uparrow p_1 & \cong & \uparrow s_1 & = & \uparrow q_1 \\
X_1 \times X_2 & \xrightarrow{\text{id} \times f_2} & X_1 \times Y_2 & \xrightarrow{f_1 \times \text{id}} & Y_1 \times Y_2 \\
\downarrow p_2 & = & \downarrow s_2 & \cong & \downarrow q_2 \\
X_2 & \xrightarrow{f_2} & Y_2 & \xrightarrow{\text{id}} & Y_2
\end{array}$$

Proposition 2.12. *Let $C \subseteq K$ be a cartesian effect category, with the sequential products $\times_{\text{seq}}, \bowtie_{\text{seq}}$ composed from \times , \bowtie . Then, as graph homomorphisms, \times_{seq} and \bowtie_{seq} extend \times and \bowtie , respectively.*

Proof. Let $v_1 : X_1 \rightsquigarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$. Since $v_1 \times_{\text{seq}} f_2 = (\text{id}_{Y_1} \times f_2) \circ (v_1 \times \text{id}_{X_2})$ and since \times coincides with the binary product \times on C^2 (by proposition 2.9) we have:

$$v_1 \times_{\text{seq}} f_2 = (\text{id}_{Y_1} \times f_2) \circ (v_1 \times \text{id}_{X_2}) .$$

The left semi-pure product property yields:

$$q_1 \circ (\text{id}_{Y_1} \times f_2) \preceq r_1 \quad \text{and} \quad q_2 \circ (\text{id}_{Y_1} \times f_2) = f_2 \circ r_2$$

so that by substitution:

$$q_1 \circ (v_1 \times_{\text{seq}} f_2) \preceq r_1 \circ (v_1 \times \text{id}_{X_2}) \quad \text{and} \quad q_2 \circ (v_1 \times_{\text{seq}} f_2) = f_2 \circ r_2 \circ (v_1 \times \text{id}_{X_2})$$

hence from the binary product property we get:

$$q_1 \circ (v_1 \times_{\text{seq}} f_2) \preceq v_1 \circ p_1 \quad \text{and} \quad q_2 \circ (v_1 \times_{\text{seq}} f_2) = f_2 \circ p_2$$

which is the left semi-pure product property. Thus, \times_{seq} extends \times , as required. \square

Notation 2.13. It follows from proposition 2.12 that we may drop the subscript ‘‘seq’’.

2.5 Some properties of cartesian effect categories

2.5.1 Pure morphisms are central

The next definition is similar to the definition of central morphisms in a binoidal category, see section 4.1.

Definition 2.14. Let $C \subseteq K$ be a cartesian effect category. A morphism f_1 is *central* if for all morphism f_2 , $f_1 \times f_2 = f_1 \times f_2$ and symmetrically $f_2 \times f_1 = f_2 \times f_1$.

Remark 2.15. Definition 2.11 and corollary 2.10 imply that the identities are central:

- for all f_1 and X_2 , $f_1 \times \text{id}_{X_2} = f_1 \times \text{id}_{X_2}$;
- and symmetrically, for all X_1 and f_2 , $\text{id}_{X_1} \times f_2 = \text{id}_{X_1} \times f_2$.

Theorem 2.16. Let $C \subseteq K$ be a cartesian effect category. Then every pure morphism is central.

Remark 2.17. Theorem 2.16 means that the graph homomorphisms \times, \times coincide on $C \times K$ and on $K \times C$:

- for all $f_1 : X_1 \rightarrow Y_1$ and $v_2 : X_2 \rightsquigarrow Y_2$:

$$f_1 \times v_2 = f_1 \times v_2 ;$$

- and symmetrically, for all $v_1 : X_1 \rightsquigarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$:

$$v_1 \times f_2 = v_1 \times f_2 .$$

It may be noted that, according to the definitions, $f_1 \times v_2$ and $v_1 \times f_2$ are semi-pure products, while $f_1 \times v_2$ and $v_1 \times f_2$ are not.

Proof. Let $h = f_1 \times v_2 = (\text{id}_{Y_1} \times v_2) \circ (f_1 \times \text{id}_{X_2})$. Since \times coincide with \times on C^2 by proposition 2.9, we get:

$$h = (\text{id}_{Y_1} \times v_2) \circ (f_1 \times \text{id}_{X_2}) .$$

The binary product property yields:

$$q_1 \circ h = r_1 \circ (f_1 \times \text{id}_{X_2}) \quad \text{and} \quad q_2 \circ h = v_2 \circ r_2 \circ (f_1 \times \text{id}_{X_2})$$

and the right semi-pure product property:

$$r_1 \circ (f_1 \times \text{id}_{X_2}) = f_1 \circ p_1 \quad \text{and} \quad r_2 \circ (f_1 \times \text{id}_{X_2}) \preceq p_2$$

so that, using pure replacement, we get:

$$q_1 \circ h = f_1 \circ p_1 \quad \text{and} \quad q_2 \circ h \preceq v_2 \circ p_2$$

which means that h satisfies the right semi-pure product property, hence $h = f_1 \times v_2$ as required. \square

Using definition 2.11, theorem 2.16 can also be stated as follows.

Corollary 2.18. *Let $C \subseteq K$ be a cartesian effect category. Then:*

- for all $f_1 : X_1 \rightarrow Y_1$ and $v_2 : X_2 \rightsquigarrow Y_2$:

$$(\text{id}_{Y_1} \times v_2) \circ (f_1 \times \text{id}_{X_2}) = (f_1 \times \text{id}_{Y_2}) \circ (\text{id}_{X_1} \times v_2);$$

- and symmetrically, for all $v_1 : X_1 \rightsquigarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$:

$$(v_1 \times \text{id}_{Y_2}) \circ (\text{id}_{X_1} \times f_2) = (\text{id}_{Y_1} \times f_2) \circ (v_1 \times \text{id}_{X_2}).$$

Notation 2.19. According to theorem 2.16 there would not be any ambiguity in denoting \times for the semi-pure products \times and \bowtie , as in the introduction and in [DDR07], however we will not use this opportunity.

2.5.2 Functoriality properties

As reminded in proposition 2.6, the binary product in a cartesian category is a functor. In this section it is proved that similarly the semi-pure products in a cartesian effect category are functors.

Lemma 2.20. *Let $C \subseteq K$ be a cartesian effect category. Then:*

- for all $X_1, f_2 : X_2 \rightarrow Y_2$ and $g_2 : Y_2 \rightarrow Z_2$:

$$(\text{id}_{X_1} \times g_2) \circ (\text{id}_{X_1} \times f_2) = \text{id}_{X_1} \times (g_2 \circ f_2)$$

- and symmetrically for all $f_1 : X_1 \rightarrow Y_1, g_1 : Y_1 \rightarrow Z_1$ and X_2 :

$$(g_1 \times \text{id}_{X_2}) \circ (f_1 \times \text{id}_{X_2}) = (g_1 \circ f_1) \times \text{id}_{X_2}.$$

$$\begin{array}{ccc} X_1 \xrightarrow{\text{id}} X_1 \xrightarrow{\text{id}} X_1 & & X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z_1 \\ p_1 \downarrow \cong & \cong & \downarrow r_1 = \downarrow r'_1 \\ X_1 \times X_2 \xrightarrow{\text{id} \times f_2} X_1 \times Y_2 \xrightarrow{\text{id} \times g_2} X_1 \times Z_2 & & X_1 \times X_2 \xrightarrow{f_1 \times \text{id}} Y_1 \times X_2 \xrightarrow{g_1 \times \text{id}} Z_1 \times X_2 \\ p_2 \downarrow = & \cong & \downarrow r_2 \cong \downarrow r'_2 \\ X_2 \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z_2 & & X_2 \xrightarrow{\text{id}} X_2 \xrightarrow{\text{id}} X_2 \end{array}$$

Proof. Let $h = (\text{id}_{X_1} \times g_2) \circ (\text{id}_{X_1} \times f_2)$. The left semi-pure product property yields:

$$s'_1 \circ (\text{id}_{X_1} \times g_2) \preceq s_1 \quad \text{and} \quad s'_2 \circ (\text{id}_{X_1} \times g_2) = g_2 \circ s_2$$

and also:

$$s_1 \circ (\text{id}_{X_1} \times f_2) \preceq p_1 \quad \text{and} \quad s_2 \circ (\text{id}_{X_1} \times f_2) = f_2 \circ p_2.$$

Hence by substitution and transitivity:

$$s'_1 \circ h \preceq p_1 \quad \text{and} \quad s'_2 \circ h = g_2 \circ f_2 \circ p_2$$

which proves that $h = \text{id}_{X_1} \times (g_2 \circ f_2)$. \square

Theorem 2.21. *Let $C \subseteq K$ be a cartesian effect category. Then:*

- *for all $f_1 : X_1 \rightarrow Y_1$, $w_1 : Y_1 \rightsquigarrow Z_1$, $f_2 : X_2 \rightarrow Y_2$ and $g_2 : Y_2 \rightarrow Z_2$:*

$$(w_1 \times g_2) \circ (f_1 \times f_2) = (w_1 \circ f_1) \times (g_2 \circ f_2)$$

- *and symmetrically for all $f_1 : X_1 \rightarrow Y_1$, $g_1 : Y_1 \rightarrow Z_1$, $f_2 : X_2 \rightarrow Y_2$ and $w_2 : Y_2 \rightsquigarrow Z_2$:*

$$(g_1 \times w_2) \circ (f_1 \times f_2) = (g_1 \circ f_1) \times (w_2 \circ f_2).$$

Proof. According to definition 2.11:

$$(w_1 \times g_2) \circ (f_1 \times f_2) = (\text{id}_{Z_1} \times g_2) \circ (w_1 \times \text{id}_{Y_2}) \circ (\text{id}_{Y_1} \times f_2) \circ (f_1 \times \text{id}_{X_2}).$$

We know from corollary 2.18 that:

$$(w_1 \times \text{id}_{Y_2}) \circ (\text{id}_{Y_1} \times f_2) = (\text{id}_{Z_1} \times f_2) \circ (w_1 \times \text{id}_{X_2}),$$

so that:

$$(w_1 \times g_2) \circ (f_1 \times f_2) = (\text{id}_{Z_1} \times g_2) \circ (\text{id}_{Z_1} \times f_2) \circ (w_1 \times \text{id}_{X_2}) \circ (f_1 \times \text{id}_{X_2}).$$

The result now follows from lemma 2.20 and definition 2.11 again. \square

Corollary 2.22. *Let $C \subseteq K$ be a cartesian effect category. Then the graph homomorphisms $\times : C \times K \rightarrow K$, $\times : K \times C \rightarrow K$ are functors.*

Proof. By corollary 2.10 for identities and theorem 2.21 for composition. \square

Remark 2.23. Although we are here more interested in pure morphisms than in central morphisms, it is worth noting that, according to the proof of theorem 2.21, this theorem is valid as soon as w_1 and w_2 are central morphisms. A consequence is that the objects of K together with the central morphisms form a wide subcategory C_K of K , called the *center*, and that corollary 2.22 can be stated more generally as follows.

Let $C \subseteq K$ be a cartesian effect category. Then the restrictions of the sequential products are functors $\times : C_K \times K \rightarrow K$ and $\times : K \times C_K \rightarrow K$.

2.5.3 Naturality properties

Let $C \subseteq K$ be a cartesian effect category, then according to proposition 2.9 the category C is cartesian. Let us assume that in addition $C \subseteq K$ has a *pure terminal object*, which means, an object 1 that is terminal in C . As reminded in proposition 2.6, then C with $\times : C^2 \rightarrow C$ and 1 is a symmetric monoidal category, which means that the projections can be combined in order to get natural isomorphisms a, r, l, c with components:

- $a_X = a_{X_1, X_2, X_3} : (X_1 \times X_2) \times X_3 \rightarrow X_1 \times (X_2 \times X_3)$,
- $r_X : 1 \times X \rightarrow X$, $l_X : X \times 1 \rightarrow X$,
- $c_X = c_{X_1, X_2} : X_1 \times X_2 \rightarrow X_2 \times X_1$,

which satisfy the symmetric monoidal coherence conditions [Mac97]. In this section we prove that a, r, l, c satisfy more general naturality conditions, involving the sequential products \times, \times .

Lemma 2.24. *Let $C \subseteq K$ be a cartesian effect category,*

- *For all f_1, f_2 and pure v_1, v_2 :*

$$\begin{cases} c_Y \circ (v_1 \times f_2) = (f_2 \times v_1) \circ c_X \\ c_Y \circ (f_1 \times v_2) = (v_2 \times f_1) \circ c_X \end{cases}$$

- For all f_1, f_2, f_3 and pure v_1, v_2, v_3 :

$$\begin{cases} a_Y \circ (f_1 \times (v_2 \times v_3)) = ((f_1 \times v_2) \times v_3) \circ a_X \\ a_Y \circ (v_1 \times (f_2 \times v_3)) = ((v_1 \times f_2) \times v_3) \circ a_X \\ a_Y \circ (v_1 \times (v_2 \times f_3)) = ((v_1 \times v_2) \times f_3) \circ a_X \end{cases}$$

Proof. Straightforward. □

Theorem 2.25. Let $C \subseteq K$ be a cartesian effect category,

- For all $f_1 : X_1 \rightarrow Y_1, f_2 : X_2 \rightarrow Y_2$ and $f_3 : X_3 \rightarrow Y_3$:

$$\begin{cases} a_Y \circ (f_1 \times (f_2 \times f_3)) = ((f_1 \times f_2) \times f_3) \circ a_X \\ a_Y \circ (f_1 \times (f_2 \times f_3)) = ((f_1 \times f_2) \times f_3) \circ a_X \end{cases}$$

- For all $f : X \rightarrow Y$:

$$\begin{cases} r_Y \circ (\text{id}_1 \times f) = f \circ r_X \\ l_Y \circ (f \times \text{id}_1) = f \circ l_X \end{cases}$$

- For all $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$:

$$\begin{cases} c_Y \circ (f_1 \times f_2) = (f_2 \times f_1) \circ c_X \\ c_Y \circ (f_1 \times f_2) = (f_2 \times f_1) \circ c_X \end{cases}$$

Proof. For l and r , this comes directly from the semi-pure product properties. For a , let us use the definition of sequential products:

$$f_1 \times (f_2 \times f_3) = (\text{id} \times (f_2 \times f_3)) \circ (f_1 \times \text{id})$$

and:

$$f_2 \times f_3 = (\text{id} \times f_3) \circ (f_2 \times \text{id})$$

hence by lemma 2.20:

$$\text{id} \times (f_2 \times f_3) = (\text{id} \times (\text{id} \times f_3)) \circ (\text{id} \times (f_2 \times \text{id}))$$

and finally:

$$f_1 \times (f_2 \times f_3) = (\text{id} \times (\text{id} \times f_3)) \circ (\text{id} \times (f_2 \times \text{id})) \circ (f_1 \times \text{id}).$$

In a symmetric way:

$$(f_1 \times f_2) \times f_3 = (\text{id} \times f_3) \circ ((\text{id} \times f_2) \times \text{id}) \circ ((f_1 \times \text{id}) \times \text{id}).$$

Hence the result follows from lemma 2.24, from which the three lines relative to a are used, and from corollary 2.10 for dealing with identities. For c the proof is similar, though shorter. □

2.6 The left and right sequential product properties

We have defined the semi-pure products in a way similar to the binary products (definition 2.8), then the sequential products as compositions of semi-pure products (definition 2.11). In this section we prove that sequential products do also satisfy some *left and right sequential product properties*, which generalize the binary product property and the semi-pure products properties. In section 3, we will see that under some additional assumptions the left and right sequential product properties characterize the sequential products.

Definition 2.26. Let $C \subseteq K$ be an effect category with a pair of graph homomorphisms $\times_{\text{seq}}, \times_{\text{seq}} : K^2 \rightarrow K$ compatible with \times and which maps C^2 to C . Then:

- the *left sequential product property* says that for all $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$, the morphism $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ satisfies:

$$\begin{cases} q_1 \circ (f_1 \times f_2) \preceq f_1 \circ p_1 \\ q_2 \circ (f_1 \times f_2) = f_2 \circ r_2 \circ (f_1 \times \text{id}_{X_2}) \end{cases}$$

- and symmetrically, the *right sequential product property* says that for all $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$, the morphism $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ satisfies:

$$\begin{cases} q_1 \circ (f_1 \times f_2) = f_1 \circ s_1 \circ (\text{id}_{X_1} \times f_2) \\ q_2 \circ (f_1 \times f_2) \preceq f_2 \circ p_2 \end{cases}$$

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \uparrow p_1 & \cong & \uparrow q_1 \\ X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & Y_1 \times Y_2 \\ \searrow f_1 \times \text{id} & = & \downarrow q_2 \\ & Y_1 \times X_2 & \\ & \downarrow r_2 & \\ & X_2 & \xrightarrow{f_2} & Y_2 \end{array} \quad \begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \uparrow s_1 & = & \uparrow q_1 \\ X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & Y_1 \times Y_2 \\ \downarrow p_2 & \cong & \downarrow q_2 \\ X_2 & \xrightarrow{f_2} & Y_2 \end{array}$$

Let us check that the sequential product properties extend the semi-pure product properties.

Proposition 2.27. *For all $v_1 : X_1 \rightsquigarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$, the left sequential product property for $v_1 \times f_2$ is the left semi-pure product property. Symmetrically, for all $f_1 : X_1 \rightarrow Y_1$ and $v_2 : X_2 \rightsquigarrow Y_2$, the right sequential product property for $f_1 \times v_2$ is the right semi-pure product property.*

Proof. The left sequential product property for $v_1 \times f_2$ is:

$$q_1 \circ (v_1 \times f_2) \preceq v_1 \circ p_1 \quad \text{and} \quad q_2 \circ (v_1 \times f_2) = f_2 \circ r_2 \circ (v_1 \times \text{id}_{X_2})$$

We know from proposition 2.9 that $v_1 \times \text{id}_{X_2} = v_1 \times \text{id}_{X_2}$, so that $r_2 \circ (v_1 \times \text{id}_{X_2}) = p_2$, hence:

$$q_1 \circ (v_1 \times f_2) \preceq v_1 \circ p_1 \quad \text{and} \quad q_2 \circ (v_1 \times f_2) = f_2 \circ p_2$$

which is the left semi-pure product property. \square

Remark 2.28. The left and right sequential product properties appear as mutually recursive, however proposition 2.27 proves that this recursivity has only two steps: the left sequential product property uses a right semi-pure product, then the right semi-pure product property does not use any kind of product, and symmetrically.

Theorem 2.29. *Let $C \subseteq K$ be a cartesian effect category. Then the sequential products \times, \times do satisfy the sequential product properties.*

Proof. The left sequential product is defined (in definition 2.11) as $f_1 \times_{\text{seq}} f_2 = (\text{id}_{Y_1} \times f_2) \circ (f_1 \times \text{id}_{X_2})$. The left semi-pure product property yields:

$$q_1 \circ (\text{id}_{Y_1} \times f_2) \preceq r_1 \quad \text{and} \quad q_2 \circ (\text{id}_{Y_1} \times f_2) = f_2 \circ r_2$$

so that by substitution:

$$q_1 \circ (f_1 \times f_2) \preceq r_1 \circ (f_1 \times \text{id}_{X_2}) \quad \text{and} \quad q_2 \circ (f_1 \times f_2) = f_2 \circ r_2 \circ (f_1 \times \text{id}_{X_2}).$$

The right semi-pure product property implies that $r_1 \circ (f_1 \times \text{id}_{X_2}) = f_1 \circ p_1$, so that:

$$q_1 \circ (f_1 \times f_2) \preceq f_1 \circ p_1 \quad \text{and} \quad q_2 \circ (f_1 \times f_2) = f_2 \circ r_2 \circ (f_1 \times \text{id}_{X_2})$$

which is the left sequential product property. \square

2.7 Examples

2.7.1 Partiality

Let $K = \mathcal{P}art$ be the category of sets with partial functions and $C = \mathcal{S}et$ the wide subcategory of sets, so that the pure morphisms are the total functions. Hence, in this paper, we use “ \rightarrow ” and “ \rightsquigarrow ” for denoting partial functions and total functions, respectively, instead of the more usual “ \dashrightarrow ” and “ \rightarrow ”. For every partial function $f : X \rightarrow Y$ let $\mathcal{D}(f)$ denote the *domain of definition* of f (while X is the *domain* of f). The usual partial order relation between partial functions is denoted \preceq , it is defined by: for all $f, f' : X \rightarrow Y$,

$$f \preceq f' \iff \mathcal{D}(f) \subseteq \mathcal{D}(f') \text{ and } f, f' \text{ agree on } \mathcal{D}(f).$$

Then \preceq is a consistency relation, so that $\mathcal{S}et \subseteq \mathcal{P}art$ with \preceq is an effect category. In addition, \preceq satisfies the replacement property and pure morphisms are maximal for \preceq . The left semi-product property can be illustrated as follows, with two cases: either $f_2(x_2)$ is defined or not, in the second case we use the symbol \perp .

$$\begin{array}{ccc} x_1 \rightsquigarrow^{v_1} y_1 & & x_1 \rightsquigarrow^{v_1} y_1 \neq \perp \\ \uparrow \text{ } \downarrow & = & \uparrow \text{ } \downarrow \\ \langle x_1, x_2 \rangle \xrightarrow{v_1 \times f_2} \langle y_1, y_2 \rangle & & \langle x_1, x_2 \rangle \xrightarrow{v_1 \times f_2} \perp \\ \downarrow \text{ } \uparrow & = & \downarrow \text{ } \uparrow \\ x_2 \xrightarrow{f_2} y_2 & & x_2 \xrightarrow{f_2} \perp \end{array} \quad \text{or}$$

It follows that the left semi-pure product $v_1 \times f_2$ is such that $\mathcal{D}(v_1 \times f_2) = X_1 \times \mathcal{D}(f_2)$ and:

$$\forall (x_1, x_2) \in X_1 \times \mathcal{D}(f_2), v_1 \times f_2(x_1, x_2) = (v_1(x_1), f_2(x_2)),$$

then the definition of the left sequential product yields $\mathcal{D}(f_1 \times f_2) = \mathcal{D}(f_1) \times \mathcal{D}(f_2)$ and:

$$\forall (x_1, x_2) \in \mathcal{D}(f_1) \times \mathcal{D}(f_2), f_1 \times f_2(x_1, x_2) = (f_1(x_1), f_2(x_2)).$$

It may be noted that in this example $f_1 \times f_2 = f_1 \times f_2$, which means that all morphisms are central.

Remark 2.30. On the other hand, there is an actual product in the category $\mathcal{P}art$: the product of X_1 and X_2 has vertex $(X_1 \times X_2) + X_1 + X_2$, where \times and $+$ denote the cartesian product and the disjoint union of sets, respectively. But this product does not fit with the semantics of non-termination, in contrast with the sequential product.

2.7.2 State

Let S be some fixed set, called *the set of states* (or *stores*), and for each set X let $\pi_X : S \times X \rightarrow X$ and $\sigma_X : S \times X \rightarrow S$ denote the projections. Let K_S be the category with the sets as objects and with a morphism $f : X \rightarrow Y$ for each map $[f] : S \times X \rightarrow S \times Y$ in $\mathcal{S}et$; we will say that the morphism f in K_S *stands for* the map $[f]$ in $\mathcal{S}et$. Let C_S be the wide subcategory of K_S with the pure morphisms $v : X \rightsquigarrow Y$ standing for the maps of the form $[v] = \text{id}_S \times v_0 : S \times X \rightarrow S \times Y$ for some map $v_0 : X \rightarrow Y$ in $\mathcal{S}et$. For all $f, f' : X \rightarrow Y$ in K_S , let:

$$f \smile_S f' \iff \pi_Y \circ [f] = \pi_Y \circ [f'].$$

Then \smile_S is a consistency relation, so that $C_S \subseteq K_S$ with \smile_S is an effect category where the consistency relation is symmetric (hence its notation). The left semi-product property can be illustrated as follows:

$$\begin{array}{ccc}
(s, x_1) \dashv\!\!\!\dashv [v_1] \Rightarrow (s, y_1) \neq (s_2, y_1) & & \\
\uparrow \text{~~~~~} & \smile & \uparrow \text{~~~~~} \\
(s, x_1, x_2) \xrightarrow{[v_1 \times f_2]} (s_2, y_1, y_2) & & \\
\downarrow \text{~~~~~} & = & \downarrow \text{~~~~~} \\
(s, x_2) \xrightarrow{[f_2]} (s_2, y_2) & &
\end{array}$$

It follows that the left semi-pure product $v_1 \times f_2$ is such that:

$$\forall (x_1, x_2) \in X_1 \times X_2, \forall s \in S, [v_1 \times f_2](s, x_1, x_2) = (s_2, y_1, y_2)$$

where $[v_1](s, x_1) = (s, y_1)$ and $[f_2](s, x_2) = (s_2, y_2)$. Then the definition of the left sequential product yields

$$\forall (x_1, x_2) \in X_1 \times X_2, \forall s \in S, [f_1 \times f_2](s, x_1, x_2) = (s_2, y_1, y_2)$$

where $[f_1](s, x_1) = (s_1, y_1)$ and $[f_2](s_1, x_2) = (s_2, y_2)$. Hence the left sequential product $f_1 \times f_2$ is usually distinct from the right sequential product $f_1 \rtimes f_2$.

3 Computational effects

Semi-pure and sequential products have been defined in section 2 in the framework of an *effect category*. In this section we focus on the relation between effect categories and “computational effects”. As mentioned in the introduction, the key point is that, given some computational effect, for an effect category $C \subseteq K$ with consistency \preceq to be considered as a formalization of this notion, we must be able to interpret the relation \preceq as an “up-to-effects” relation. So, we now provide definitions for the informal notions of “computational effects” and “up-to-effects”. We do not claim that our definitions exhaust the various meanings of these informal notions. However we claim that our definitions are relevant. This claim is supported by a comparison with other approaches in section 4 and by several examples: the examples about partiality and state have been used as guidelines for forging our notions, while the example about non-determinism in section 4.4.3 has been considered afterwards.

First we define the *symmetrization* of a consistency relation in section 3.1. In section 3.2, starting from a definition of *effects* that fits with the notion of computational effects, we define a *same-effect* equivalence relation \approx . Then in section 3.3, given a same-effect equivalence relation \approx , we define what it means for a consistency relation \preceq to be an *up-to-effects* relation. In section 3.4 we come back to products and we prove some additional results about cartesian effect categories. Examples are studied in section 3.5.

3.1 Symmetric consistency

Given a consistency relation \preceq , we define a new relation \smile such that $f \smile f'$ when f and f' have a common upper bound with respect to \preceq .

Definition 3.1. Let $C \subseteq K$ be an effect category. The *symmetrization* of \preceq is the relation \smile between parallel morphisms defined by:

- *symmetrization*: for all $f, f' : X \rightarrow Y$, $f \smile f'$ if and only if there is some $f'' : X \rightarrow Y$ such that $f \preceq f''$ and $f' \preceq f''$.

Clearly, \smile always contains \preceq , and whenever \preceq itself is symmetric then \smile is the same as \preceq , so that the notation in example 2.7.2 is sound. Some properties of \smile are stated in proposition 3.2, they are quite similar to the properties of \preceq , although of course \preceq is transitive while \smile is symmetric. Let us recall that the consistency relation \preceq is reflexive and transitive, satisfies substitution and pure replacement, and is the equality on pure morphisms.

Proposition 3.2. *In an effect category, the symmetrization \smile of \preceq is reflexive and symmetric and it satisfies substitution and pure replacement. If in addition \preceq is symmetric or if pure morphisms are maximal for \preceq , then \smile is the equality on pure morphisms.*

Proof. The first part is straightforward, let us focus on the equality on pure morphisms. When \preceq is symmetric then \smile is the same as \preceq , which is the equality on pure morphisms. When pure morphisms are maximal for \preceq , let $v, v' : X \rightsquigarrow Y$ be such that $v \smile v'$, then there is some $f'' : X \rightarrow Y$ such that $v \preceq f''$ and $v' \preceq f''$, and the maximality of pure morphisms implies that $v = f'' = v'$. \square

3.2 Same-effect equivalence

Let K be a category and C a wide subcategory. This section is independent from sections 2 and 3.1, the relation will be established in section 3.3.

Remark 3.3. We claim that the *effect* of a morphism $f : X \rightarrow Y$ should provide some kind of measure of how far f is from being pure, by wiping out the pure part of f . Let us assume that there is a pure terminal object 1 (which means, as in section 2.5.3, that 1 is terminal in C), and for each object X let the unique pure morphism from X to 1 be denoted $\langle \rangle_X : X \rightsquigarrow 1$. We claim that the *effect* of a morphism $f : X \rightarrow Y$ can be defined as the morphism $\langle \rangle_Y \circ f : X \rightarrow 1$. Indeed, every pure morphism $v : X \rightsquigarrow Y$ is *effect-free*, in the sense that its effect $\langle \rangle_Y \circ v$ is $\langle \rangle_X$, which does not bear any information on v since it is also the effect of the identity id_X . More generally, if a morphism $f : X \rightarrow Y$ is composed with a pure morphism $v : Y \rightsquigarrow Z$ then $v \circ f$ has the same effect as f . The examples will further justify the soundness of this definition. In definition 3.4 we keep the main properties of the relation *having the same effect*, without assuming that there is a terminal object in C , hence without defining explicitly the effect of a morphism (see example 3.10).

Definition 3.4. Let K be a category with a wide subcategory C . A *same-effect* relation \approx on $C \subseteq K$ is an *equivalence* relation \approx on morphisms with the same domain that satisfies:

- *substitution*: for all $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, $g' : Y \rightarrow Z'$, if $g \approx g'$ then $g \circ f \approx g' \circ f$;
- *pure morphisms are effect-free*: for all $v : X \rightsquigarrow Y$, $v \approx \text{id}_X$.

A morphism $f : X \rightarrow Y$ is *effect-free* if $f \approx \text{id}_X$.

Here are some straightforward consequences of definition 3.4.

Proposition 3.5. *Let K be a category with a wide subcategory C and let \approx be a same-effect relation on $C \subseteq K$. Then:*

- for all $v : X \rightsquigarrow Y$ and $v' : X \rightsquigarrow Y'$, $v \approx v'$;
- for all $f : X \rightarrow Y$ and $v : Y \rightsquigarrow Z$, $v \circ f \approx f$;
- for all $f : X \rightarrow Y$, $f' : X \rightarrow Y'$ and $v : Y \rightsquigarrow Z$, $v' : Y' \rightsquigarrow Z'$, if $v \circ f \approx v' \circ f'$ then $f \approx f'$.

Example 3.6. As in example 2.2, let $C \subseteq K$ be associated to a monad (M, μ, η) on a category C_0 , that satisfies the mono requirement. Let us assume that there is a terminal object 1 in C_0 , which means, a pure terminal object 1 in K . For each object X , the pure morphism $\langle \rangle_X : X \rightsquigarrow 1$ stands for $[\langle \rangle_X] = \eta_1 \circ \langle \rangle_X : X \rightarrow M(1)$ in C_0 , and for each morphism $f : X \rightarrow Y$ in K the effect $\langle \rangle_Y \circ f$ of f , in the sense of remark 3.3, stands for:

$$[\langle \rangle_Y \circ f] = M([\langle \rangle_Y]) \circ [f] : X \rightarrow M(1) \text{ in } C_0.$$

For all $f : X \rightarrow Y$ and $f' : X \rightarrow Y'$ let $f \approx f'$ if and only if f and f' have the same effect:

$$f \approx f' \text{ if and only if } M(\langle \rangle_Y) \circ [f] = M(\langle \rangle_{Y'}) \circ [f'] : X \rightarrow M(1) \text{ in } C_0 .$$

We claim that this notion is relevant in many situations where a monad is used for dealing with some kind of effect. Examples are provided in section 4.4.

3.3 Up-to-effects consistency

Given a same-effect relation \approx on $C \subseteq K$, we define what it means for a consistency relation \preceq on $C \subseteq K$ to be an *up-to-effects* relation. We rely on definition 3.4 for same-effect relations and on definition 2.3 for consistency relations.

Definition 3.7. Let $C \subseteq K$ with \preceq be an effect category, and let \smile be the symmetrization of \preceq . Let \approx be a same-effect relation on $C \subseteq K$. Then \preceq is an *up-to-effects* consistency relation with respect to \approx if:

- *complementarity*: for all $f, f' : X \rightarrow Y$, if $f \approx f'$ and $f \smile f'$ then $f = f'$.

When \approx is not explicitly needed, we simply say that \preceq is an *up-to-effects* relation.

Remark 3.8. The complementarity property means that every clique for \smile (a clique is a set of pairwise related elements) has at most one element in each equivalence class for \approx . Clearly this property can be expressed without \smile :

- *complementarity*: for all $f, f', f'' : X \rightarrow Y$, if $f \approx f'$ and $f \preceq f''$ and $f' \preceq f''$ then $f = f'$.

Then, due to the reflexivity of \preceq , a consequence of complementarity is that:

- for all $f, f' : X \rightarrow Y$, if $f \approx f'$ and $f \preceq f'$ then $f = f'$.

Proposition 3.2 can now be completed.

Proposition 3.9. *In an effect category $C \subseteq K$, if \preceq is an up-to-effects relation, then the symmetrization \smile of \preceq is the equality on pure morphisms.*

Proof. Let $v, v' : X \rightsquigarrow Y$, then $v \approx v'$ (proposition 3.5). Thus, if $v \smile v'$ then by complementarity $v = v'$. \square

Example 3.10. Most examples of same-effect relations in this paper are built as in remark 3.3, by mapping everything to 1; here is a simple example of another kind of same-effect relation. The set \mathbb{Z} of integers with the addition is a monoid, which is seen as a category K with one object X , a morphism $f : X \rightarrow X$ for each $f \in \mathbb{Z}$, and with $f \circ f' = f + f'$ and $\text{id}_X = 0$. Let us consider some fixed integer $m \geq 2$, then the subset $m\mathbb{Z}$ of \mathbb{Z} defines a wide subcategory C of K . Let \approx be the congruence modulo m , clearly it is a same-effect equivalence relation on $C \subseteq K$. In addition, this example provides a relation \smile that satisfies the properties of proposition 3.2 without being the symmetrization of a consistency relation. The relation \smile is defined on \mathbb{Z} by $f \smile f'$ if and only if $|f - f'| < m$. Then \smile is symmetric and reflexive, not transitive, it is such that if $g \smile g'$ then $g + f \smile g' + f$ (this is both substitution and replacement), and \smile is the equality on pure morphisms. In addition, \smile is a complement of \approx in the sense that if $f \approx f'$ and $f \smile f'$ then $f = f'$. More precisely, every maximal clique for \smile has exactly one element in each equivalence class for \approx . However, there is no consistency relation \preceq on \mathbb{Z} such that \smile is the symmetrization of \preceq . Indeed, let us assume that such a relation exists, and let f, f' be such that $f \smile f'$ with $f \neq f'$ (for instance $f = 0$ and $f' = m - 1$). Then there is some f'' such that $f \preceq f''$ and $f' \preceq f''$. We can assume that $f'' \neq f$ (otherwise we have $f'' = f'$, so that a similar argument holds). Let $k = f'' - f \neq 0$, then from $f \preceq f + k$ we get (by substitution) $f + k \preceq f + 2k$, and so on until $f + (l-1)k \preceq f + lk$ for some l such that $|lk| \geq m$. It follows that $f \preceq f + lk$ (by transitivity of \preceq) so that $f \smile f + lk$, which contradicts the definition of \smile .

3.4 Unicity up-to-effects

Here is a new result about sequential products which holds as soon as the consistency relation \preceq is an up-to-effects relation.

Definition 3.11. Let $C \subseteq K$ be an effect category where \preceq is an up-to-effects relation, and let \smile be the symmetrization of \preceq . Let us assume that there is a product skeleton \times on C . Then the *unicity up-to-effects* property is:

- *unicity up-to-effects*: for all $f, f' : X \rightarrow Y_1 \times Y_2$, if $q_1 \circ f \smile q_1 \circ f'$ and $q_2 \circ f \smile q_2 \circ f'$ then $f \smile f'$.

Lemma 3.12. Let $C \subseteq K$ be an effect category where \preceq is an up-to-effects relation, with a binary product skeleton \times on C that satisfies the unicity up-to-effects property. Let $f, f' : X \rightarrow Y_1 \times Y_2$, $g_1 : X \rightarrow Y_1$ and $g_2 : X \rightarrow Y_2$ be such that:

$$\left\{ \begin{array}{l} q_1 \circ f \preceq g_1 \\ q_2 \circ f = g_2 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} q_1 \circ f' \preceq g_1 \\ q_2 \circ f' = g_2 \end{array} \right.$$

then $f = f'$.

Proof. Since $q_1 \circ f \preceq g_1$ and $q_1 \circ f' \preceq g_1$ we have $q_1 \circ f \smile q_1 \circ f'$ (definition 3.1). Since $q_2 \circ f = g_2$ and $q_2 \circ f' = g_2$ we have $q_2 \circ f = q_2 \circ f'$, hence from the reflexivity of \smile and \approx we get $q_2 \circ f \smile q_2 \circ f'$ and $q_2 \circ f \approx q_2 \circ f'$. Then the unicity up-to-effects property implies that $f \smile f'$, and on the other hand proposition 3.5 implies that $f \approx f'$. The result follows from the complementarity of \smile and \approx . \square

Proposition 3.13. Let $C \subseteq K$ be an effect category where \preceq is an up-to-effects relation, with a binary product skeleton \times on C that satisfies the unicity up-to-effects property. If there is a pair of graph homomorphisms $\times, \bowtie : K^2 \rightarrow K$ compatible with \times , that map C^2 to C and that satisfy the left and right sequential product properties, then this pair is the unique pair of sequential products on $C \subseteq K$ compatible with \times .

Proof. First let us look at the restrictions of \times, \bowtie to semi-pure arguments: according to proposition 2.27 they satisfy the semi-pure product properties (definition 2.8) hence by lemma 3.12 they are uniquely defined. Then, the sequential product properties (definition 2.26) and lemma 3.12 prove that $\times, \bowtie : K^2 \rightarrow K$ also are uniquely defined. \square

3.5 Examples

3.5.1 Partiality

As in section 2.7.1, we consider the effect category $\mathcal{Set} \subseteq \mathcal{Part}$ with the usual partial order relation as its consistency relation: for all $f, f' : X \rightarrow Y$,

$$f \preceq f' \iff \mathcal{D}(f) \subseteq \mathcal{D}(f') \text{ and } f, f' \text{ agree on } \mathcal{D}(f).$$

Then the symmetrization \smile of \preceq is the usual consistency of partial functions: for all $f, f' : X \rightarrow Y$,

$$f \smile f' \iff f, f' \text{ agree on } \mathcal{D}(f) \cap \mathcal{D}(f').$$

The singleton $1 = \{*\}$ is a terminal object in \mathcal{Set} , so that for every partial function $f : X \rightarrow Y$ the effect of f is the partial function $\langle \rangle_Y \circ f : X \rightarrow \{*\}$, with domain of definition $\mathcal{D}(f)$ and value $f(x) = *$ for all $x \in \mathcal{D}(f)$. Hence the same-effect relation \approx is having the same domain of definition: for all $f : X \rightarrow Y$, $f' : X \rightarrow Y'$,

$$f \approx f' \iff \mathcal{D}(f) = \mathcal{D}(f').$$

Then clearly the complementarity property holds, so that \preceq is an up-to-effects consistency with respect to \approx . Moreover, the unicity up-to-effects property is satisfied.

3.5.2 State

With the same notations as in section 2.7.2, on the effect category $C_S \subseteq K_S$ the consistency relation is symmetric, it means that both functions return the same result, whatever the change of state:

$$f \preceq_S f' \iff f \smile_S f' \iff \pi_Y \circ [f] = \pi_Y \circ [f'] .$$

The singleton $1 = \{*\}$ is a terminal object in C_S and $S \times \{*\}$ may be identified to S , so that the morphism $\langle \rangle_X : X \rightsquigarrow \{*\}$ stands for the projection $\sigma_X : S \times X \rightarrow S$ and the effect of a morphism $f : X \rightarrow Y$ stands for $\sigma_Y \circ [f] : S \times X \rightarrow S$. Hence the same-effect relation \approx_S means that both functions update the state in the same way:

$$f \approx_S f' \iff \sigma_Y \circ [f] = \sigma_Y \circ [f'] .$$

Then for all $f, f' : X \rightarrow Y$ in K_S , if $f \preceq_S f'$ and $f \approx_S f'$ then $[f], [f'] : S \times X \rightarrow S \times Y$ agree when they are projected onto Y and when they are projected onto S , so that $[f] = [f']$, which means that $f = f'$. Therefore, the complementarity property holds, which means that \preceq_S is an up-to-effects consistency with respect to \approx_S . Here also, the unicity up-to-effects property is satisfied.

4 Comparisons

In this section we compare our approach with other categorical semantics for a language with effects: Freyd-categories in section 4.1, strong monads in section 4.3 and Arrows in section 4.2; Examples are given in section 4.4.

4.1 Premonoidal categories and Freyd-categories

It is now easy to prove that cartesian effect categories are Freyd-categories, as in [PR97, PT99, Pow06].

Definition 4.1. A *binoidal category* is a category K together with two functors $\otimes : |K| \times K \rightarrow K$ and $\otimes : K \times |K| \rightarrow K$ which coincide on $|K|^2$ (so that the notation \otimes is not ambiguous). The functors \otimes can be extended as two graph homomorphisms $\times_{\text{Fr}}, \bowtie_{\text{Fr}} : K^2 \rightarrow K$, as follows. For all $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ in K , let:

$$\begin{cases} f_1 \times_{\text{Fr}} f_2 = (\text{id}_{Y_1} \otimes f_2) \circ (f_1 \otimes \text{id}_{X_2}) : X_1 \otimes X_2 \rightarrow Y_1 \otimes Y_2 \\ f_1 \bowtie_{\text{Fr}} f_2 = (f_1 \otimes \text{id}_{Y_2}) \circ (\text{id}_{X_1} \otimes f_2) : X_1 \otimes X_2 \rightarrow Y_1 \otimes Y_2 \end{cases}$$

A morphism $f_1 : X_1 \rightarrow Y_1$ is *central* if for all $f_2 : X_2 \rightarrow Y_2$, $f_1 \times_{\text{Fr}} f_2 = f_1 \bowtie_{\text{Fr}} f_2$ and symmetrically $f_2 \times_{\text{Fr}} f_1 = f_2 \bowtie_{\text{Fr}} f_1$. Let $t : \Phi \Rightarrow \Psi$ be a natural transformation between two functors $\Phi, \Psi : K' \rightarrow K$, then t is *central* if every component of t is central.

In theorem 4.3 the graph homomorphisms $\times_{\text{Fr}}, \bowtie_{\text{Fr}}$ will be related to the sequential products \times, \bowtie from section 2. In the next definition, “natural” means natural in each component separately [Sel01].

Definition 4.2. A *symmetric premonoidal category* is a binoidal category K together with an object I of K and central natural isomorphisms with components $a_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, $l_X : X \otimes I \rightarrow X$, $r_X : I \otimes X \rightarrow X$ and $c_{X,Y} : X \otimes Y \rightarrow X \otimes Y$, subject to the usual coherence equations for symmetric monoidal categories [Mac97, Sel01]. Note that every symmetric monoidal category, hence every category with finite products, is symmetric premonoidal. A *symmetric premonoidal functor* between two symmetric premonoidal categories is a functor that preserves the partial functor \otimes , the object I and the natural isomorphisms a, l, r, c . It is *strict* if in addition it maps central morphisms to central morphisms. A *Freyd-category* is an identity-on-objects functor $J : C \rightarrow K$ where the category C has finite products, the category K is symmetric premonoidal and the functor J is strict symmetric premonoidal.

The following result states that each cartesian effect category with a pure terminal object is a Freyd-category. It is an easy consequence of the results in section 2.

Theorem 4.3. *Let $C \subseteq K$ be a cartesian effect category with a pure terminal object 1. Let a, l, r, c be the natural isomorphisms on C defined as in section 2.5.3. Let $J : C \rightarrow K$ be the inclusion, let $\otimes : |K| \times K \rightarrow K$ and $\otimes : K \times |K| \rightarrow K$ be the restrictions of \times and \times , respectively, and let $I = 1$. This forms a Freyd-category, where \times_{Fr} and \times_{Fr} coincide with \times and \times , respectively.*

Proof. It follows from proposition 2.9 that C is a category with finite products. The graph homomorphisms $\otimes : |K| \times K \rightarrow K$ and $\otimes : K \times |K| \rightarrow K$ are functors by corollary 2.10 and they coincide on $|K|^2$ by remark 2.15, hence K with \otimes is a binoidal category. Definition 2.11 states that the graph homomorphisms $\times_{\text{Fr}}, \times_{\text{Fr}}$ are the sequential products \times, \times , so that both notions of central morphism (definitions 2.14 and 4.1) coincide. The fact that the transformations a, l, r, c are natural, in the sense of symmetric premonoidal categories, is a consequence of theorem 2.25. All the components of a, l, r, c are defined from the symmetric monoidal category C . It follows that they are isomorphisms, that they satisfy the coherence equations, and since pure morphisms are central (theorem 2.16) it follows also that they are central. Hence K with \otimes, I and a, l, r, c is a symmetric premonoidal category. Clearly the inclusion functor $J : C \rightarrow K$ is symmetric premonoidal, and strictly because of theorem 2.16. \square

Remark 4.4. It is well-known that the center of a symmetric premonoidal category is a symmetric monoidal category [PR97]. For effect categories with sequential products this result can also be deduced from remark 2.23.

4.2 Arrows

In the functional language Haskell, in order to deal with effects one can use strong monads [Wad92] (see section 4.3), which have been generalized as Arrows [Hug00]. A categorical model of Arrows is presented in [Atk08]. Here we follow [Pat01], and we build a bridge from cartesian effect categories to Arrows.

Definition 4.5. An *Arrow type* is a binary type constructor A of the form:

```
class Arrow A where
  arr :: (X → Y) → A X Y
  (>>>) :: A X Y → A Y Z → A X Z
  first :: A X Y → A (X, Z) (Y, Z)
```

satisfying the following equations:

(1)	$\text{arr id} \ggg f$	$= f$
(2)	$f \ggg \text{arr id}$	$= f$
(3)	$(f \ggg g) \ggg h$	$= f \ggg (g \ggg h)$
(4)	$\text{arr } (w.v)$	$= \text{arr } v \ggg \text{arr } w$
(5)	$\text{first } (\text{arr } v)$	$= \text{arr } (v \times \text{id})$
(6)	$\text{first } (f \ggg g)$	$= \text{first } f \ggg \text{first } g$
(7)	$\text{first } f \ggg \text{arr } (\text{id} \times v)$	$= \text{arr } (\text{id} \times v) \ggg \text{first } f$
(8)	$\text{first } f \ggg \text{arr fst}$	$= \text{arr fst} \ggg f$
(9)	$\text{first } (\text{first } f) \ggg \text{arr assoc}$	$= \text{arr assoc} \ggg \text{first } f$

where the functions (\times) , fst and assoc are defined as:

$(\times) :: (X \rightarrow X') \rightarrow (Y \rightarrow Y') \rightarrow (X, Y) \rightarrow (X', Y')$ such that $(f \times g)(x, y) = (f x, g y)$
 $\text{fst} :: (X, Y) \rightarrow X$ such that $\text{fst}(x, y) = x$
 $\text{assoc} :: ((X, Y), Z) \rightarrow (X, (Y, Z))$ such that $\text{assoc}((x, y), z) = (x, (y, z))$

Let C_H denote the category of Haskell types and ordinary functions, so that the Haskell notation $(X \rightarrow Y)$ represents $C_H(X, Y)$, made of the Haskell ordinary functions from X to Y . An arrow \mathbf{A} constructs a type $\mathbf{A} X Y$ for all types X and Y . We slightly modify the definition of Arrows by allowing $(X \rightarrow Y)$ to represent $C(X, Y)$ for any cartesian category C and by requiring that $\mathbf{A} X Y$ is a set rather than a type: more on this issue can be found in [Atk08]. In addition, we use categorical notations instead of Haskell syntax. So, from now on, for any cartesian category C , an *Arrow* A on C associates to each objects X, Y of C a set $A(X, Y)$, together with three operations:

$$\begin{aligned} \mathbf{arr} &: C(X, Y) \rightarrow A(X, Y) \\ \mathbf{>>>} &: A(X, Y) \rightarrow A(Y, Z) \rightarrow A(X, Z) \\ \mathbf{first} &: A(X, Y) \rightarrow A(X \times Z, Y \times Z) \end{aligned}$$

that satisfy the equations (1)–(9). Basically, the correspondence between a cartesian effect category $C \subseteq K$ and an Arrow A on C identifies $K(X, Y)$ with $A(X, Y)$ for all types X and Y . This is stated more precisely in proposition 4.6.

Proposition 4.6. *Every cartesian effect category $C \subseteq K$ gives rise to an Arrow A on C , according to the following table:*

Cartesian effect categories	Arrows
$K(X, Y)$	$A(X, Y)$
$C(X, Y) \subseteq K(X, Y)$	$\mathbf{arr} : C(X, Y) \rightarrow A(X, Y)$
$f \mapsto (g \mapsto g \circ f)$	$\mathbf{>>>} : A(X, Y) \rightarrow A(Y, Z) \rightarrow A(X, Z)$
$f \mapsto f \times \text{id}$	$\mathbf{first} : A(X, Y) \rightarrow A(X \times Z, Y \times Z)$

Proof. The first and second line in the table say that $A(X, Y)$ is made of the morphisms from X to Y in K and that \mathbf{arr} is the conversion from pure morphisms to arbitrary morphisms. The third and fourth lines say that $\mathbf{>>>}$ is the (reverse) composition of morphisms and that \mathbf{first} is the right semi-product with the identity. The following table proves that A is an Arrow, by translating each property (1)–(9) in terms of cartesian effect categories and giving the argument for its proof. Note that \mathbf{fst} is the common name for projections like p_1, q_1, \dots (in section 2) and that \mathbf{assoc} is the natural isomorphism a as in section 2.5.3.

(1)	$f \circ \text{id} = f$	identity in K
(2)	$\text{id} \circ f = f$	identity in K
(3)	$h \circ (g \circ f) = (h \circ g) \circ f$	associativity in K
(4)	$w \circ v \text{ in } C = w \circ v \text{ in } K$	$C \subseteq K$ is a functor
(5)	$v \times \text{id} \text{ in } C = v \times \text{id} \text{ in } K$	proposition 2.9
(6)	$(g \circ f) \times \text{id} = (g \times \text{id}) \circ (f \times \text{id})$	lemma 2.20
(7)	$(\text{id} \times v) \circ (f \times \text{id}) = (f \times \text{id}) \circ (\text{id} \times v)$	corollary 2.18
(8)	$q_1 \circ (f \times \text{id}) = f \circ p_1$	definition 2.8
(9)	$a \circ ((f \times \text{id}) \times \text{id}) = (f \times \text{id}) \circ a$	lemma 2.24

□

The Arrow combinators \mathbf{second} , $(**)$ and $(\&\&)$ can be derived from \mathbf{arr} , $(\mathbf{>>>})$ and \mathbf{first} , see e.g [Hug00, Pat01]. The correspondence in proposition 4.6 is easily extended to these functions. We denote $\langle \text{id}_X, \text{id}_X \rangle : X \rightsquigarrow X \times X$ the diagonal in C and $\langle f_1, f_2 \rangle_l = (f_1 \times f_2) \circ \langle \text{id}, \text{id} \rangle : X \rightarrow Y_1 \times Y_2$ (where the subscript l stands for “left”) for all $f_1 : X \rightarrow Y_1$ and $f_2 : X \rightarrow Y_2$ in K . The natural isomorphism c is defined as in section 2.5.3, it corresponds to \mathbf{swap} .

Cartesian effect categories	Arrows
$(\text{id} \times f) = c \circ (f \times \text{id}) \circ c$	$\mathbf{second} f = \mathbf{arr} \mathbf{swap} \mathbf{>>>} \mathbf{first} f \mathbf{>>>} \mathbf{arr} \mathbf{swap}$
$f_1 \times f_2 = (\text{id} \times f_2) \circ (f_1 \times \text{id})$	$f_1 ** f_2 = \mathbf{first} f_1 \mathbf{>>>} \mathbf{second} f_2$
$\langle f_1, f_2 \rangle_l = (f_1 \times f_2) \circ \langle \text{id}, \text{id} \rangle$	$f_1 \&\& f_2 = \mathbf{arr}(\lambda x \rightarrow (x, x)) \mathbf{>>>} (f_1 ** f_2)$

For instance in [Hug00, §4.1] it is stated that $\&\&\&$ is not a categorical product since in general $(f_1 \&\&\& f_2) \gg \mathbf{arr\ fst}$ is different from f_1 : “*there is no reason to expect Haskell’s pair type, $\&\&\&$, to be a categorical product in the category of arrows, or indeed to expect any categorical product to exist*”. We can state this more precisely in a cartesian effect category, where $(f_1 \&\&\& f_2) \gg \mathbf{arr\ fst}$ corresponds to $q_1 \circ \langle f_1, f_2 \rangle_l$. Indeed, according to theorem 2.29, $q_1 \circ (f_1 \times f_2) \preceq f_1 \circ p_1$, so that, by substitution, $q_1 \circ \langle f_1, f_2 \rangle_l \preceq f_1 \circ p_1 \circ \langle \text{id}, \text{id} \rangle$. We thus get the following result, which of course is weaker than an equality:

$$q_1 \circ \langle f_1, f_2 \rangle_l \preceq f_1 .$$

4.3 Strong monads

The use of strong monads for dealing with computational effects is introduced in [Mog89, Mog91]. As in examples 2.2 and 3.6, let C_0 be a category with a monad (M, μ, η) , that satisfies the mono requirement, let K be the Kleisli category of M and let C denote the image of C_0 in K by the faithful functor $J : C_0 \rightarrow K$ associated with M , so that $C \subseteq K$. A morphism $f : X \rightarrow Y$ in K stands for $[f] : X \rightarrow M(Y)$ in C_0 , and a pure morphism $v : X \rightsquigarrow Y$ stands for $[v] = \eta_Y \circ v_0 : X \rightarrow M(Y)$ for the unique $v_0 : X \rightarrow Y$ in C_0 such that $v = J(v_0)$. The composition $g \circ f$ of $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ stands for $[g \circ f] = [g]^* \circ [f]$ where $[g]^* = \mu_Z \circ M([g])$. It follows that when $v : X \rightsquigarrow Y$ and $w : Y \rightsquigarrow Z$, one gets $[g \circ v] = [g] \circ v_0$, $[w \circ f] = M(w_0) \circ [f]$ and $[w \circ v] = \eta_Z \circ w_0 \circ v_0$.

It has been seen in example 3.6 that if there is a terminal object 1 in C_0 then the effect of a morphism $f : X \rightarrow Y$ of K , in the sense of remark 3.3, stands for $[\langle \rangle_Y \circ f] = M(\langle \rangle_Y) \circ [f] : X \rightarrow M(1)$ in C_0 , so that for all $f : X \rightarrow Y$ and $f' : X \rightarrow Y'$:

$$f \approx f' \iff M(\langle \rangle_Y) \circ [f] = M(\langle \rangle_{Y'}) \circ [f'] : X \rightarrow M(1) .$$

Now, let us assume that C_0 , hence C , has binary products. In [Mog89], it is explained why the monad (M, μ, η) and the product \times are not sufficient for dealing with several variables: there is a type mismatch from $Y_1 \times M(Y_2)$ to $M(Y_1 \times Y_2)$. This issue is solved by introducing a *strength* t for the monad (M, μ, η) , which means, a natural transformation with components $t_{Y_1, Y_2} : Y_1 \times M(Y_2) \rightarrow M(Y_1 \times Y_2)$ satisfying four axioms [Mog89]. One of these axioms is that for all X , $r_{M(X)} = M(r_X) \circ t_{1, X} : 1 \times M(X) \rightarrow M(X)$, where the natural isomorphism r is made of the projections $r_X : 1 \times X \rightarrow X$ as in section 2.5.3. A *strong monad* is a monad with a strength. In K , let $v : X_1 \rightsquigarrow Y_1$ and $f : X_2 \rightarrow Y_2$. In order to form a kind of product of v and f , the usual method using strong monads is to compose in C_0 the product $v_0 \times [f] : X_1 \times X_2 \rightarrow Y_1 \times M(Y_2)$ with the strength $t_{Y_1, Y_2} : Y_1 \times M(Y_2) \rightarrow M(Y_1 \times Y_2)$.

Definition 4.7. For all $v : X_1 \rightsquigarrow Y_1$ and $f : X_2 \rightarrow Y_2$ in K , the *left Kleisli product* of v and f is $v \times_{\text{Kl}} f : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ defined by:

$$[v \times_{\text{Kl}} f] = t_{Y_1, Y_2} \circ (v_0 \times [f]) : X_1 \times X_2 \rightarrow M(Y_1 \times Y_2) .$$

In proposition 4.11, we will prove that the left Kleisli product is a left semi-pure product, under some assumption about the strength.

Lemma 4.8. *Let C_0 be a category with binary products and with a strong monad (M, μ, η, t) . For all Y_1, Y_2 in C_0 (with the projections $q_2 : Y_1 \times Y_2 \rightarrow Y_2$ and $q'_2 : Y_1 \times M(Y_2) \rightarrow M(Y_2)$):*

$$M(q_2) \circ t_{Y_1, Y_2} = q'_2 .$$

Proof. The projection q_2 can be decomposed as $q_2 = r_2 \circ (\langle \rangle_{Y_1} \times Y_2)$, where $r_2 = r_{Y_2} : 1 \times Y_2 \rightarrow Y_2$ is the projection. Hence on the one hand $M(q_2) = M(r_2) \circ M(\langle \rangle_{Y_1} \times Y_2)$, and on the other hand $q'_2 = r'_2 \circ (\langle \rangle_{Y_1} \times M(Y_2))$ where $r'_2 = r_{M(Y_2)} : 1 \times M(Y_2) \rightarrow M(Y_2)$ is the projection. In the following diagram, the square on the top is commutative because t is natural, and the square on the bottom is commutative

because of the property of the strength with respect to r . Hence the large square is commutative, and the result follows

$$\begin{array}{ccc}
 Y_1 \times M(Y_2) & \xrightarrow{t_{Y_1, Y_2}} & M(Y_1 \times Y_2) \\
 \downarrow \langle \rangle \times M(\text{id}) & = & \downarrow M(\langle \rangle \times \text{id}) \\
 1 \times M(Y_2) & \xrightarrow{t_{1, Y_2}} & M(1 \times Y_2) \\
 \downarrow r'_2 & = & \downarrow M(r_2) \\
 M(Y_2) & \xrightarrow{\text{id}} & M(Y_2)
 \end{array}$$

q'_2 (left arrow) and $M(q_2)$ (right arrow)

□

Definition 4.9. Let C_0 be a category with binary products, with a strong monad (M, μ, η, t) and with a consistency \preceq on $C \subseteq K$. Let \preceq_0 denote the relation between parallel morphisms of C_0 with codomain of the form $M(Y)$ defined by: $\forall f, f' : X \rightarrow Y$ in K ,

$$[f] \preceq_0 [f'] \iff f \preceq f'.$$

The strength is *consistent* with the unit if for all Y_1, Y_2 (with the projections $q_1 : Y_1 \times Y_2 \rightarrow Y_1$ and $q'_1 : Y_1 \times M(Y_2) \rightarrow Y_1$):

- $M(q_1) \circ t_{Y_1, Y_2} \preceq_0 \eta_{Y_1} \circ q'_1$.

Remark 4.10. Roughly speaking, the fact that the strength is consistent with the unit means that “the strength is similar to the unit, except for the fact that the strength may have effects”. In order to check whether this consistency property is satisfied, one can first look at the case when $Y_2 = 1$, so that $M(q_1)$ is an isomorphism.

Proposition 4.11. Let C_0 be a category with binary products, with a strong monad (M, μ, η, t) and with a consistency \preceq on $C \subseteq K$ such that the strength is consistent with the unit. Then the left Kleisli product \times_{Kl} is a left semi-pure product on $C \subseteq K$.

Proof. In the diagram below, the squares on the left handside illustrate the binary product property of $v_0 \times [f]$ and the squares on the right handside illustrate lemma 4.8 (bottom right) and the assumption that the strength is consistent with the unit (top right).

$$\begin{array}{ccccc}
 X_1 & \xrightarrow{v_0} & Y_1 & \xrightarrow{\eta} & M(Y_1) \\
 p_1 \uparrow & & q'_1 \uparrow & \preceq_0 & \uparrow M(q_1) \\
 X_1 \times X_2 & \xrightarrow{v_0 \times [f]} & Y_1 \times M(Y_2) & \xrightarrow{t} & M(Y_1 \times Y_2) \\
 p_2 \downarrow & & q'_2 \downarrow & = & \downarrow M(q_2) \\
 X_2 & \xrightarrow{[f]} & M(Y_2) & \xrightarrow{\text{id}} & M(Y_2)
 \end{array}$$

$[v]$ (top arrow) and $[f]$ (bottom arrow)

Let us give the same name to the projections in C_0 and to their images by J in C . It follows immediately from the bottom squares that $M(q_2) \circ [v \times_{\text{Kl}} f] = [f] \circ p_2$, i.e. that $[q_2 \circ (v \times_{\text{Kl}} f)] = [f \circ p_2]$, which means that:

$$q_2 \circ (v \times_{\text{Kl}} f) = f \circ p_2.$$

Now let us look at the top squares. Let $Y = Y_1 \times M(Y_2)$ and let $g, g' : Y \rightarrow Y_1$ in K be defined by $[g] = M(q_1) \circ t_{Y_1, Y_2}$ and $[g'] = \eta_{Y_1} \circ q'_1$. The assumption that the strength is consistent with the unit may be stated as $g \preceq g'$. Let $X = X_1 \times X_2$ and let $w_0 = v_0 \times [f] : X \rightarrow Y$ in C_0 and $w = J(w_0) : X \rightsquigarrow Y$ in K . It follows from $g \preceq g'$, by the substitution property of \preceq , that $g \circ w \preceq g' \circ w$. On the one hand $[g \circ w] = [g] \circ w_0 = M(q_1) \circ t_{Y_1, Y_2} \circ (v_0 \times [f]) = M(q_1) \circ [v \times_{\text{Kl}} f] = [q_1 \circ (v \times_{\text{Kl}} f)]$. On the other hand $[g' \circ w] = [g'] \circ w_0 = \eta_{Y_1} \circ q'_1 \circ (v_0 \times [f])$. With the equality $q'_1 \circ (v_0 \times [f]) = v_0 \circ p_1$ the latter yields $[g' \circ w] = \eta_{Y_1} \circ v_0 \circ p_1 = [v \circ p_1]$. Hence:

$$q_1 \circ (v \times_{\text{Kl}} f) \preceq v \circ p_1 .$$

So that \times_{Kl} satisfies the left semi-pure product property. \square

Now \times_{Kl} will be denoted simply \times .

4.4 Examples

Let us look at three strong monads implemented in Haskell [Has].

4.4.1 Partiality: the Maybe monad

Partiality has been studied in examples 2.7.1 and 3.5.1. On the other hand, the **Maybe** monad in Haskell deals with computations which may return **Nothing** (which means that they fail to return a value): in a chain of computations, if a step returns **Nothing** then the chain does return **Nothing**. This corresponds to the monad with endofunctor $M(X) = X + \{e\}$ on $\mathcal{S}et$, where the symbol e stands for the error (like **Nothing** in Haskell). A function $f : X \rightarrow Y + \{e\}$ can be seen as a partial function from X to Y with domain $\mathcal{D}(f) = \{x \in X \mid f(x) \neq e\}$. The strength associated to this monad has as components the functions $t_{Y_1, Y_2} : Y_1 \times (Y_2 + \{e\}) \rightarrow (Y_1 \times Y_2) + \{e\}$ that map $\langle y_1, y_2 \rangle$ to $\langle y_1, y_2 \rangle$ and $\langle y_1, e \rangle$ to e . The strength is consistent with the unit: with the same notations as in definition 4.9, both functions $M(q_1) \circ t_{Y_1, Y_2}$ and $\eta_{Y_1} \circ q'_1$ return the same value y_1 on any argument $\langle y_1, y_2 \rangle$, while $M(q_1) \circ t_{Y_1, Y_2}(\langle y_1, e \rangle) = e$ and $\eta_{Y_1} \circ q'_1(\langle y_1, e \rangle) = y_1$, so that indeed $M(q_1) \circ t_{Y_1, Y_2}$ is smaller than $\eta_{Y_1} \circ q'_1$, as partial functions.

4.4.2 State: the State monad

Side-effects due to a hidden state have been studied in examples 2.7.2 and 3.5.2 in a “linear” way. This effect can equivalently be studied with the help of the **State** monad with endofunctor $M(X) = (S \times X)^S$ on $\mathcal{S}et$. The strength associated to the **State** monad has as components the functions $t_{Y_1, Y_2} : Y_1 \times (S \times Y_2)^S \rightarrow (S \times Y_1 \times Y_2)^S$ that map (y_1, φ_2) to φ such that $\varphi(s) = \langle s_2, y_1, y_2 \rangle$, where $\varphi_2(s) = \langle s_2, y_2 \rangle$, for each $s \in S$. The strength is consistent with the unit: with the same notations as in definition 4.9, the function $M(q_1) \circ t_{Y_1, Y_2}$ maps (y_1, φ_2) to φ_1 such that $\varphi_1(s) = \langle s_2, y_1 \rangle$ where $\varphi_2(s) = \langle s_2, y_2 \rangle$ for some y_2 , and the function $\eta_{Y_1} \circ q'_1$ maps (y_1, φ_2) to φ'_1 such that $\varphi'_1(s) = \langle s, y_1 \rangle$. Hence indeed $M(q_1) \circ t_{Y_1, Y_2}$ and $\eta_{Y_1} \circ q'_1$ differ only by their effect on the state.

4.4.3 Non-determinism: the List monad

Non-determinism can be expressed by the **List** monad \mathcal{L} on $C_0 = \mathcal{S}et$. For each set X , $\mathcal{L}(X)$ is the set of lists of elements of X , and for each function $f : X \rightarrow Y$ the function $\mathcal{L}(f) : \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$ maps (x_1, \dots, x_n) to $(f(x_1), \dots, f(x_n))$. The multiplication μ flattens each list of lists and the unit η maps each element x to the list (x) . The strength t associated to the **List** monad has as components the functions $t_{Y_1, Y_2} : Y_1 \times \mathcal{L}(Y_2) \rightarrow \mathcal{L}(Y_1 \times Y_2)$ that map each pair $\langle y_1, (y_{2,1}, \dots, y_{2,n}) \rangle$ to the list of pairs $(\langle y_1, y_{2,1} \rangle, \dots, \langle y_1, y_{2,n} \rangle)$. For all morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in the Kleisli category K , the composition $g \circ f : X \rightarrow Z$ is

defined by $[g \circ f](x) = (z_{1,1}, \dots, z_{1,p_1}, \dots, z_{n,1}, \dots, z_{n,p_n})$ for all $x \in X$ such that $[f](x) = (y_1, \dots, y_n)$ and $[g](y_i) = (z_{i,1}, \dots, z_{i,p_i})$ for each $i \in \{1, \dots, n\}$.

The singleton $1 = \{*\}$ is terminal in $\mathcal{S}et$, so that a list in $\mathcal{L}(1)$ can be identified to its length in \mathbb{N} . For each morphism $f : X \rightarrow Y$ in K , let $|f| : X \rightarrow \mathbb{N}$ denote the function that maps each $x \in X$ to the length of the list $[f](x)$. According to example 3.6, the effect of $f : X \rightarrow Y$ stands for $[\langle \rangle_Y \circ f] = \mathcal{L}(\langle \rangle_Y) \circ [f] : X \rightarrow \mathcal{L}(1)$, which gets identified with $|f| : X \rightarrow \mathbb{N}$. So, the same-effect relation is the relation “having the same length”:

$$\forall f : X \rightarrow Y, f' : X \rightarrow Y', f \approx f' \iff |f| = |f'|,$$

and a morphism f is effect-free when $|f|$ is the constant function 1. As explained in section 3.2, the relation \approx is an equivalence relation that satisfies substitution and such that pure morphisms are effect-free. In addition, here, every effect-free morphism is pure:

$$\forall f : X \rightarrow Y, f \approx \text{id}_X \iff |f| = 1 \iff f \text{ is pure.}$$

On the other hand, for each $f : X \rightarrow Y$ in K and each $k \in \mathbb{N}$, let $f^{(k)} : X \rightarrow Y$ in K denote the morphism defined by $[f^{(k)}](x) = (y_1^k, \dots, y_n^k)$ for each $x \in X$ such that $[f](x) = (y_1, \dots, y_n)$, where y_i^k denotes the replication of k times y_i . Then clearly for all $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in K and for all $k, l \in \mathbb{N}$:

$$|f^{(k)}| = k|f|, f^{(1)} = f, (f^{(k)})^{(l)} = f^{(kl)}, h^{(l)} = f^{(kl)} \Rightarrow h = f^{(k)}, g^{(l)} \circ f^{(k)} = (g \circ f)^{(kl)}.$$

Let us define the relation \preceq as follows:

$$\forall f, f' : X \rightarrow Y, f \preceq f' \iff \exists k \in \mathbb{N}, f = f'^{(k)}.$$

For all objects X and Y , the *empty* morphism $e : X \rightsquigarrow Y$ is defined as the unique morphism with constant length 0: it is such that $[e]$ maps every $x \in X$ to the empty list in $\mathcal{L}(Y)$.

Proposition 4.12. *With the relations \approx and \preceq defined as above:*

1. *the symmetrization \smile of \preceq is such that for all $f, f' : X \rightarrow Y$, $f \smile f'$ if and only if either f is empty, or f' is empty, or there are positive integers n, n' such that $f^{(n)} = f'^{(n')}$;*
2. *$C \subseteq K$ is an effect category where in addition \preceq satisfies replacement and pure morphisms are maximal for \preceq ;*
3. *\preceq is an up-to-effects relation with respect to \approx ;*
4. *the unicity up-to-effects property is satisfied, with respect to the cartesian product;*
5. *$C \subseteq K$ with its semi-pure products defined as the Kleisli products is a cartesian effect category.*

Proof.

1. The empty morphism $e : X \rightarrow Y$ is such that $e \preceq f$ for every $f : X \rightarrow Y$, because $e = f^{(0)}$. Hence $e \smile f$ (and $f \smile e$) for every $f : X \rightarrow Y$. Now let us assume that neither f nor f' is empty. Let $f \smile f'$, then there are g and positive k, k' such that $f = g^{(k)}$ and $f' = g'^{(k')}$, so that $f^{(k')} = g^{(kk')} = f'^{(k)}$, as required. If there are n, n' positive such that $f^{(n)} = f'^{(n')}$, then let $h = f^{(n)} = f'^{(n')}$ and $D = \text{lcm}(n, n')$, and let us check that there is some g such that $h = g^{(D)}$. If $h = v^{(\langle h \rangle)}$ for a pure morphism v this is clear, otherwise let us consider some $x \in X$ with at least two distinct values in the list $h(x) = (y_1, \dots, y_p)$, and let i be an index such that $y_{i-1} \neq y_i$. Since $h = f^{(n)}$ the index i is a multiple of n , and similarly i it is a multiple of n' , so that i is a multiple of D . This proves that $h = g^{(D)}$ for some g . Now let $k = D/n$ and $k' = D/n'$, we have $f^{(n)} = h = g^{(D)}$ so that $f = g^{(k)}$, and similarly $f' = g'^{(k')}$, hence $f \smile f'$.

2. The relation \preceq is reflexive because $f^{(1)} = f$ and it is transitive because $(f^{(k)})^{(l)} = f^{(kl)}$. It satisfies substitution because $g^{(l)} \circ f = g^{(l)} \circ f^{(1)} = (g \circ f)^{(l)}$ and replacement because $g \circ f^{(k)} = g^{(1)} \circ f^{(k)} = (g \circ f)^{(k)}$, hence it satisfies pure replacement. If $v \preceq f$ with v pure, then $v = f^{(k)}$ for some $k \in \mathbb{N}$ so that $1 = k|f|$,

hence $k = 1$ and $f = v$. This proves that pure morphisms are maximal for \preceq , from which it follows that \preceq is the equality on pure morphisms. So, \preceq is a consistency relation.

3. In order to prove the complementarity of \preceq and \approx , let $f, f' : X \rightarrow Y$ be such that $f \approx f'$ (which means, $|f| = |f'|$) and $f \smile f'$. If f or f' is empty then so is the other, because both must have length 0, hence $f = f'$. Now let us assume that neither f nor f' is empty, so that (using (1)) $f^{(n)} = f'^{(n')}$ for some n, n' positive. Then $n|f| = n'|f'|$, which together with $|f| = |f'|$ yields $n = n'$, hence $f = f'$.

4. Let $f, f' : X \rightarrow Y_1 \times Y_2$ be such that $q_i \circ f \smile q_i \circ f'$ for $i \in \{1, 2\}$. Note that $|q_i \circ f| = |f|$ and $|q_i \circ f'| = |f'|$ because the projections are pure. If $q_i \circ f$ is empty for some i then so is f , hence $f \smile f'$, and similarly if $q_i \circ f'$ is empty for some i then $f \smile f'$. Now let us assume that neither $q_i \circ f$ nor $q_i \circ f'$ is empty, so that (using (1)) $q_i \circ f^{(n_i)} = q_i \circ f'^{(n'_i)}$ for some n_i, n'_i positive, for $i = 1$ and for $i = 2$. It follows that $n_i|f| = n'_i|f'|$ for $i \in \{1, 2\}$, so that $n_1 n'_2 = n'_1 n_2$. Let $k = n_1 n_2$ and $k' = n_1 n'_2 = n'_1 n_2$. Let $F = f^{(k)}$ and $F' = f'^{(k')}$. Then $q_1 \circ F = (q_1 \circ f)^{(k)} = (q_1 \circ f)^{(n_1)(n_2)}$ and $q_1 \circ F' = (q_1 \circ f')^{(k')} = (q_1 \circ f')^{(n'_1)(n_2)}$ so that $q_1 \circ F = q_1 \circ F'$. And similarly $q_2 \circ F = (q_2 \circ f)^{(k)} = (q_2 \circ f)^{(n_2)(n_1)}$ and $q_2 \circ F' = (q_2 \circ f')^{(k')} = (q_2 \circ f')^{(n_2)(n'_1)}$ so that $q_2 \circ F = q_2 \circ F'$. The equalities $q_i \circ F = q_i \circ F'$ in K mean that $\mathcal{L}(q_i) \circ [F] = \mathcal{L}(q_i) \circ [F']$ in C_0 , so that $[F] = [F']$, which means that $F = F'$. So, $f^{(k)} = f'^{(k')}$, which proves (using (1)) that $f \smile f'$.

5. Let Y_1, Y_2 be sets. For all $y_1 \in Y_1$ and $(y_{2,1}, \dots, y_{2,n}) \in \mathcal{L}(Y_2)$, the pair $\langle y_1, (y_{2,1}, \dots, y_{2,n}) \rangle \in Y_1 \times \mathcal{L}(Y_2)$ is mapped by $\eta_{Y_1} \circ s_{1,0}$ to the list $(y_1) \in \mathcal{L}(Y_1)$ and by $M(q_{1,0}) \circ t_{Y_1, Y_2}$ to the list $(y_1, \dots, y_1) \in \mathcal{L}(Y_1)$ of length n . Hence $M(q_{1,0}) \circ t_{Y_1, Y_2} \preceq_0 \eta_{Y_1} \circ s_{1,0}$, so that the result follows from proposition 4.11. \square

Here is the diagram illustrating the left semi-pure product of $v : X_1 \rightsquigarrow Y_1$ and $f : X_2 \rightarrow Y_2$. This diagram is in the category C_0 of sets, with $y_1 = v_0(x_1)$ and $(y_{2,1}, \dots, y_{2,n}) = [f](x_2)$.

$$\begin{array}{ccc}
X_1 & \xrightarrow{[v]} & \mathcal{L}(Y_1) \\
p_1 \uparrow & \nearrow \approx_0 & \uparrow \mathcal{L}(q_1) \\
X_1 \times X_2 & \xrightarrow{[v \times f]} & \mathcal{L}(Y_1 \times Y_2) \\
p_2 \downarrow & \searrow = & \downarrow \mathcal{L}(q_2) \\
X_2 & \xrightarrow{[f]} & \mathcal{L}(Y_2)
\end{array}
\quad
\begin{array}{ccc}
x_1 & \xrightarrow{\quad} & (y_1) \neq (y_1, \dots, y_1) \\
\uparrow & & \uparrow \\
\langle x_1, x_2 \rangle & \xrightarrow{\quad} & \langle \langle y_1, y_{2,1} \rangle, \dots, \langle y_1, y_{2,n} \rangle \rangle \\
\downarrow & & \downarrow \\
x_2 & \xrightarrow{\quad} & (y_{2,1}, \dots, y_{2,n})
\end{array}$$

Then the definition of the left sequential product (definition 2.11) yields, for all $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$, with $(y_{1,1}, \dots, y_{1,n_1}) = [f_1](x_1)$ and $(y_{2,1}, \dots, y_{2,n_2}) = [f_2](x_2)$:

$$\forall (x_1, x_2) \in X_1 \times X_2, [f_1 \times f_2](x_1, x_2) = (\langle y_{1,1}, y_{2,1} \rangle, \dots, \langle y_{1,1}, y_{2,n_2} \rangle, \dots, \langle y_{1,n_1}, y_{2,1} \rangle, \dots, \langle y_{1,n_1}, y_{2,n_2} \rangle) .$$

And symmetrically:

$$\forall (x_1, x_2) \in X_1 \times X_2, [f_1 \times f_2](x_1, x_2) = (\langle y_{1,1}, y_{2,1} \rangle, \dots, \langle y_{1,n_1}, y_{2,1} \rangle, \dots, \langle y_{1,1}, y_{2,n_2} \rangle, \dots, \langle y_{1,n_1}, y_{2,n_2} \rangle) .$$

It follows that $f_1 \times f_2$ and $f_1 \times f_2$ are usually distinct.

Remark 4.13. This example can easily be adapted to *multisets* instead of lists, then $f_1 \times f_2$ and $f_1 \times f_2$ are equal, so that all morphisms are central.

References

- [Atk08] Robert Atkey. What is a Categorical Model of Arrows? Mathematically Structured Functional Programming (MSFP'08), 2008.
- [DDR07] Jean-Guillaume Dumas, Dominique Duval, Jean-Claude Reynaud. Sequential products in effect categories. <http://www.citebase.org/abstract?id=oai:arXiv.org:0707.1432>, 2007.
- [DR05] Dominique Duval, Jean-Claude Reynaud. Diagrammatic logic and exceptions: an introduction. Mathematics, Algorithms, Proofs (MAP'05), Dagstuhl Seminar 05021, 2005.
- [Has] Haskell. All About Monads. A comprehensive guide to the theory and practice of monadic programming in Haskell. http://www.haskell.org/all_about_monads/.
- [HJ06] Chris Heunen, Bart Jacobs. Arrows, like Monads, are Monoids. *Electronic Notes in Theoretical Computer Science*, p. 219–236, 2006.
- [Hug00] John Hughes. Generalising monads to arrows. *Science of Computer Programming* 37 (1–3), p. 67–111, 2000.
- [HLPP07] Martin Hyland, Paul Blain Levy, Gordon D. Plotkin, John Power. Combining algebraic effects with continuations. *Theoretical Computer Science* 375 (1–3), p. 20–40, 2007.
- [HPP06] Martin Hyland, Gordon Plotkin, John Power. Combining Effects: Sum and Tensor. *Theoretical Computer Science* 357, p. 70–99, 2006.
- [Mac97] Saunders Mac Lane. Categories for the Working Mathematician, 2nd edition. Graduate Texts in Mathematics Vol. 5, Springer Verlag, 1997.
- [Mog89] Eugenio Moggi. Computational lambda-calculus and monads. Logic In Computer Science (LICS), IEEE Press, p. 14–23, 1989.
- [Mog91] Eugenio Moggi. Notions of Computation and Monads. *Information and Computation* 93, p. 55–92, 1991.
- [Pat01] Ross Paterson. A New Notation for Arrows. International Conference on Functional Programming, ACM, p. 229–240, 2001.
- [Pow06] John Power. Generic models for computational effects. *Theoretical Computer Science* 364 (2), p. 254–269, 2006.
- [PR97] John Power, Edmund Robinson. Premonoidal Categories and Notions of Computation. *Mathematical Structures in Computer Science* 7 (5), p. 453–468, 1997.
- [PT99] John Power, Hayo Thielecke. Closed Freyd- and κ -Categories. Int. Coll. on Automata, Languages and Programming, (ICALP'99). Springer Verlag, LNCS 1644, p. 625–634, 1999.
- [Sel01] Peter Selinger. Control categories and duality: on the categorical semantics of the lambda-mu calculus. *Mathematical Structures in Computer Science* 11 (2), p. 207–260, 2001.
- [Wad92] Philip Wadler. Monads for functional programming. Program Design Calculi: Proceedings of the 1992 Marktoberdorf International Summer School. Springer Verlag, LNCS 925, 1995.