HAL
open science

Cartesian effect categories are Freyd-categories

Jean-Guillaume Dumas, Dominique Duval, Jean-Claude Reynaud

To cite this version:

Jean-Guillaume Dumas, Dominique Duval, Jean-Claude Reynaud. Cartesian effect categories are Freyd-categories. Journal of Symbolic Computation, 2011, 46 (3), pp.272-293. 10.1016/j.jsc.2010.09.008 . hal-00369328v3

HAL Id: hal-00369328

https://hal.science/hal-00369328v3

Submitted on 12 Jun 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Cartesian effect categories are Freyd-categories

Jean-Guillaume Dumas * Dominique Duval † Jean-Claude Reynaud ‡

June 12., 2009

Abstract

Most often, in a categorical semantics for a programming language, the substitution of terms is expressed by composition and finite products. However this does not deal with the order of evaluation of arguments, which may have major consequences when there are side-effects. In this paper Cartesian effect categories are introduced for solving this issue, and they are compared with strong monads, Freyd-categories and Haskell's Arrows. It is proved that a Cartesian effect category is a Freyd-category where the premonoidal structure is provided by a kind of binary product, called the sequential product. The universal property of the sequential product provides Cartesian effect categories with a powerful tool for constructions and proofs. To our knowledge, both effect categories and sequential products are new notions.

Keywords. Categorical logic, computational effects, monads, Freyd-categories, premonoidal categories, Arrows, sequential product, effect categories, Cartesian effect categories.

1 Introduction

A categorical semantics for a programming language usually associates an object to each type, a morphism to each term, and uses composition and finite products for dealing with the substitution of terms. This framework behaves very well in a simple equational setting, but it has to be adapted as soon as there is some kind of computational effects, for instance non-termination or state updating in an imperative language. Then there are two kinds of terms: the general terms may cause effects while the *pure* terms are effect-free. Following (Moggi, 1991), a general term may be seen as a *program* that returns a *value* which is pure. In this paper we focus on the following *sequentiality* issue: the categorical products do not deal with the order of evaluation of the arguments, although this order may have major consequences when there are side-effects. For solving this sequentiality issue, we introduce *Cartesian effect categories* as an alternative for Cartesian categories.

Other approaches include strong monads (Moggi, 1989), Freyd-categories (Power and Robinson, 1997) and Arrows (Hughes, 2000). These frameworks are quite similar from several points of view (Heunen and Jacobs, 2006; Atkey, 2008), while our framework is more precise. A first draft for Cartesian effect categories can be found in (Dumas et al., 2007), and a similar approach in (Duval and Reynaud, 2005).

A category is called Cartesian if it has finite products, and a subcategory C of a category K is called wide if it has the same objects as K. A *Freyd-category* is a generalization of a Cartesian category that consists essentially in a category K with a wide subcategory C, such that C is Cartesian (hence C is symmetric monoidal) and K is symmetric premonoidal. A *Cartesian effect category*, as defined in this paper, is more precise and more homogeneous than a Freyd-category: like the symmetric monoidal structure on C derives

*Laboratoire Jean Kuntzmann, Université de Grenoble, 38041 Grenoble, France – Jean-Guillaume.Dumas@imag.fr – http://ljk.imag.fr/membres/Jean-Guillaume.Dumas

†Laboratoire Jean Kuntzmann, Université de Grenoble, 38041 Grenoble, France – Dominique.Duval@imag.fr – http://ljk.imag.fr/membres/Dominique.Duval

‡Malhivert, 38640 Claix, France – Jean-Claude.Reynaud@imag.fr

from its product, in a Cartesian effect category the symmetric premonoidal structure on K derives from some kind of product, called a *sequential product*, which extends the product of C and generalizes the usual categorical product. In fact, there are two steps in our definition. First an *effect category* is defined, without mentioning any kind of product: it is made of a category K with a wide subcategory C and with a relation $\lhd$ called *consistency* between morphisms. Then a *Cartesian effect category* is defined as an effect category with a binary product on C extended by a *sequential product* on K, which itself is defined thanks to a universal property that generalizes the categorical product property and involves the consistency relation. Like every universal property, this provides a powerful tool for constructions and proofs in a Cartesian effect category.

Let us look at two basic examples of effect categories (two morphisms in a category are called parallel if they share the same domain and the same codomain).

The *non-termination* effect involves partial functions. As usual, two partial functions are called *consistent* when they coincide on the intersection of their domains of definition. Thus, on the one hand, two partial functions f and f' are consistent if and only if there is a total function v such that v is consistent both with f and with f'. On the other hand, let us say that two partial functions have the *same effect* if they have the same domain of definition. Then clearly, two partial functions have the same effect and are consistent if and only if they are equal.

In an imperative programming language, there are side-effects due to the modification of the *state*, since the functions in the sense of the programming language, in addition to have arguments and a return value, are allowed to use the state and to modify it. A function is called *pure* if it neither use nor modify the state, and the side-effects are due to the non-pure functions. Let us say that a function f is *consistent* with a pure function v when both return the same value when they are given the same arguments. Then two arbitrary functions are called *consistent* when they are consistent with a common pure function, which means that both return the same value when they are given the same arguments and that in addition this value does not depend on the state. It should be noted that this consistency relation is not reflexive. Therefore, if two functions have the same effect and are consistent then they are equal, but the converse is false.

More generally, an *effect category* is a category K with a wide subcategory C and with a consistency relation $\lhd$ between parallel morphisms, the first one in K and the second one in C, satisfying a form of compatibility with the composition. The morphisms in C are called *pure* and are denoted with $\rightsquigarrow$. Two morphisms in K are called *consistent* when there is a pure morphism v such that $f \lhd v$ and $f' \lhd v$; this is denoted $f \lhd\rhd f'$, and the properties of consistency are such that the relation $\lhd\rhd$ extends $\lhd$. Let 1 be a terminal object in C, the *effect* of a morphism f is defined as the morphism $\mathcal{E}(f) = \langle\,\rangle_Y \circ f$ where $\langle\,\rangle_Y$ is the unique pure morphism $\langle\,\rangle_Y : Y \rightsquigarrow 1$. It is assumed that the following *complementarity* property holds, which means that the consistency relation is a kind of "up-to-effects" relation: *if two morphisms have the same effect and are consistent, then they are equal.*

This notion of consistency coincides with the usual one for partial functions, but to our knowledge it is new in the general setting of computational effects. For instance, we will see in section 2.6 that it is fairly different from the notion of *having the same result* that is defined in (Moggi, 1995) in the framework of evaluation logic. Let us look more closely at the complementarity property (for some fixed domain and codomain). On the one hand, to have the same effect is an equivalence relation $\approx$ with one distinguished equivalence class, the class of the morphisms without effect, which contains all the pure morphisms. On the other hand, to be consistent is a symmetric relation $\lhd\rhd$, with each maximal clique made of a unique pure morphism and all the morphisms that are consistent with it. The complementarity property asserts that there is at most one morphism in the intersection of a given equivalence class for $\approx$ and a given maximal clique for $\lhd\rhd$.

A binary product on a category C provides a bifunctor $\times$ on C such that for all $v_1 : X_1 \to Y_1$ and $v_2 : X_2 \to Y_2$, the morphism $v_1 \times v_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is characterized by the following diagram, where the p_i's and q_i's are the projections. This property is symmetric in v_1 and v_2. When C is the category of sets, this means that $(v_1 \times v_2)(x_1, x_2) = \langle v_1(x_1), v_2(x_2)\rangle$.

$$\begin{tikzcd}
X_1 \arrow[r, "v_1"] & Y_1 \\
X_1 \times X_2 \arrow[u, "p_1"] \arrow[d, "p_2"'] \arrow[r, "v_1 \times v_2"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "="] & Y_1 \times Y_2 \arrow[u, "q_1"'] \arrow[d, "q_2"] \\
X_2 \arrow[r, "v_2"] & Y_2
\end{tikzcd}$$

A *Cartesian effect category* is defined as an effect category with a binary product on C, extended by two symmetric *semi-pure products* $v \ltimes f$ and $f \rtimes v$ where v is pure. The left semi-pure product $v \ltimes f$ is characterized by the following diagram, which means that $q_1 \circ (v \ltimes f) \lhd v \circ p_1$ and $q_2 \circ (v \ltimes f) = f \circ p_2$ (the right semi-pure product is characterized by a symmetric diagram).

$$\begin{tikzcd}
Y_1 \arrow[r, squiggly, "v"] & Y_1 \\
Y_1 \times X_2 \arrow[u, squiggly, "p_1"] \arrow[d, squiggly, "p_2"'] \arrow[r, "v \ltimes f"] \arrow[ur, phantom, "\triangledown"] \arrow[dr, phantom, "="] & Y_1 \times Y_2 \arrow[u, squiggly, "q_1"'] \arrow[d, squiggly, "q_2"] \\
X_2 \arrow[r, "f"] & Y_2
\end{tikzcd}$$

This property means that the effect of $v \ltimes f$ is the effect of f, and that "up to effects" $v \ltimes f$ looks like an ordinary binary product. Then the left *sequential product* of two arbitrary morphisms f_1 and f_2 is easily obtained by composing two semi-pure products: $f_1 \ltimes f_2 = (\mathrm{id}_1 \ltimes f_2) \circ (f_1 \rtimes \mathrm{id}_2)$ where id_1 and id_2 denote the identities of Y_1 and X_2, respectively. This definition formalizes the notion of *sequentiality*: "first f_1, then f_2". The right sequential product is defined in a symmetric way. We will check that the sequential product extends the semi-pure product, so that there is no ambiguity in using the same symbols $\ltimes$ and $\rtimes$ for both. This approach, to our knowledge, is completely new. It can be summarized as follows: *while the universal property of a binary product consists in two equalities, the universal property of a semi-pure product consists in one equality and one consistency.*

For instance, in the category of sets with partial functions, $v \ltimes f$ is the partial function such that $(v \ltimes f)(x_1, x_2) = \langle y_1, y_2 \rangle$ where $y_1 = v(x_1)$ and $y_2 = f(x_2)$ whenever $f(x_2)$ is defined, otherwise $(v \ltimes f)(x_1, x_2)$ is not defined. When side-effects are due to the updating of the state, $v \ltimes f$ is such that for each state s, $(v \ltimes f)(s, x_1, x_2) = \langle s_2, y_1, y_2 \rangle$ where $\langle s, y_1 \rangle = v(s, x_1)$ and $\langle s_2, y_2 \rangle = f(s, x_2)$.

The properties of the sequential product imply that a Cartesian effect category is a Freyd-category. On the other hand, each strong monad defines a Freyd-category (Power and Robinson, 1997). We prove that a Freyd-category defined from a strong monad is a weak Cartesian effect category if and only if, roughly speaking: *the strength of the monad is consistent with the identity.*

Section 2 is devoted to effect categories and section 3 to Cartesian effect categories. Then Cartesian effect categories are related to Freyd-categories, Arrows and strong monads in section 4. Several examples are considered in sections 2.5, 3.8 and 4.4.

2 Effect categories

2.1 Pure morphisms

Definition 2.1. A subcategory C of a category K is *wide* if it has the same objects as K; this is denoted $C \subseteqq K$. Given $C \subseteqq K$, a morphism of K is called *pure* if it is in C; then it is denoted with "$\rightsquigarrow$". An object 1 is a *pure terminal* object in $C \subseteqq K$ if it is terminal in C, then for each object X the unique pure morphism from X to 1 is denoted $\langle\,\rangle_X : X \rightsquigarrow 1$.

Remark 1. **Pure morphisms in a Kleisli category.** Let C_0 be a category (called the base category) with a monad (M, μ, η) (or simply M) and let K_M be the Kleisli category of M. Then K_M has the same objects as C_0 and for all objects X and Y there is a bijection between $C_0(X, MY)$ and $K_M(X, Y)$. In this paper, for each morphism $f : X \to Y$ in K_M the corresponding morphism in C_0 is denoted $[f] : X \to MY$, and we

say that f *stands for* $[f]$, and for each morphism $\varphi : X \to MY$ in C_0 the corresponding morphism in K_M is denoted $]\varphi[: X \to Y$. So, $](\,[f]\,)[= f$ for every f in K_M and $[(]\varphi[)] = \varphi$ for every φ in C_0 with codomain MY for some Y. Let $J : C_0 \to K_M$ denote the functor associated with M and let $C_M = J(C_0)$. Then J is the identity on objects, so that C_M is a wide subcategory of K_M. A pure morphism $v : X \rightsquigarrow Y$ in K_M is a morphism $v = J(v_0)$ for some $v_0 : X \to Y$ in C_0; this means that $[v] = \eta_Y \circ v_0 : X \to MY$ in C_0. Each identity id_X in K_M henceforth stands for $[\mathrm{id}_X] = \eta_X$ and the composition $g \circ f$ of $f : X \to Y$ and $g : Y \to Z$ stands for $[g \circ f] = [g]^* \circ [f]$ where $[g]^* = \mu_Z \circ M[g]$. It follows that when $v : X \rightsquigarrow Y$ and $w : Y \rightsquigarrow Z$, then $[g \circ v] = [g] \circ v_0$, $[w \circ f] = Mw_0 \circ [f]$ and $[w \circ v] = \eta_Z \circ w_0 \circ v_0$. It should be noted that it does not make sense to say that a morphism in C_0 is pure or not. Indeed, each morphism $\varphi : X \to MY$ in C_0 gives rise in K_M both to a pure morphism $v = J(\varphi) : X \to MY$ and to a morphism $f =]\varphi[: X \to Y$, related by $[v] = \eta_{MY} \circ [f]$ in C_0.

C_0	$X \xrightarrow{[f]} MY$	$\begin{tikzcd} X \arrow[r, "{[v]}"] \arrow[dr, "v_0"'] & MY \arrow[d, phantom, "="] \\ & Y \arrow[u, "\eta_Y"'] \end{tikzcd}$	$\begin{tikzcd} & M^2Y \\ X \arrow[ur, "{[J(\varphi)]}"] \arrow[r, "\varphi"'] & MY \arrow[u, "\eta_{MY}"'] \arrow[ul, phantom, "="] \end{tikzcd}$
K_M	$X \xrightarrow{f} Y$	$X \overset{v=J(v_0)}{\rightsquigarrow} Y$	$\begin{tikzcd} & MY \\ X \arrow[ur, rightsquigarrow, "J(\varphi)"] \arrow[r, "]\varphi["'] & Y \end{tikzcd}$

In addition, the functor $J : C_0 \to K_M$ has a right adjoint, which means that for each object X there is an object $X^\dagger$ called the *lifting* of X, with an isomorphism $K_M(X, Y) \cong C_0(X, Y^\dagger)$ natural in X and Y. Let us assume that the *mono requirement* is satisfied by the monad, which means that η_X is a mono for every object X, or equivalently that the functor J is faithful, so that it defines an isomorphism from C_0 to C_M.

2.2 Effects

In this section we define the effect of a morphism f as a kind of measure of how far f is from being pure: pure morphisms are effect-free and the effect of $v \circ f$, when v is pure, is the same as the effect of f.

Definition 2.2. Let K be a category with a wide subcategory C and with a pure terminal object 1. *The effect* of a morphism $f : X \to Y$ is the morphism $\mathcal{E}(f) = \langle\,\rangle_Y \circ f : X \to 1$. We denote $f \approx f'$ when $f : X \to Y$ and $f' : X \to Y'$ have the same effect:

$$\forall f : X \to Y\,,\, \forall f' : X \to Y'\,,\; f \approx f' \iff \langle\,\rangle_Y \circ f = \langle\,\rangle_{Y'} \circ f'\,.$$

A morphism $f : X \to Y$ is *effect-free* if $\mathcal{E}(f) = \mathcal{E}(\mathrm{id}_X)$, which means that $\mathcal{E}(f) = \langle\,\rangle_X$.

The following properties are easily derived from the definition.

Proposition 1. *The same-effect relation $\approx$ is an equivalence relation between morphisms with the same domain that satisfies:*

- *Pure morphisms are effect-free.* $\forall v : X \rightsquigarrow Y\,,\; v \approx \mathrm{id}_X$.
- *Substitution.* $\forall f : X \to Y\,,\, \forall g : Y \to Z\,,\, \forall g' : Y \to Z'\,,\; g \approx g' \implies g \circ f \approx g' \circ f$.
- *Pure wiping.* $\forall f : X \to Y\,,\, \forall w : Y \rightsquigarrow Z\,,\; w \circ f \approx f$.

Remark 2. **Effects in a Kleisli category.** Within the same framework as in remark 1, let us assume that there is a terminal object 1 in C_0, or equivalently in C_M. For each object X, the pure morphism $\langle\,\rangle_X : X \rightsquigarrow 1$ stands for $[\langle\,\rangle_X] = \eta_1 \circ \langle\,\rangle_X : X \to M1$ in C_0, and for each morphism $f : X \to Y$ in K_M the effect $\mathcal{E}(f)$ of f stands for $[\langle\,\rangle_Y \circ f] = M\langle\,\rangle_Y \circ [f] : X \to M1$ in C_0. Let $\approx_0$ denote the relation between morphisms in C_0 defined by $[f] \approx_0 [f']$ if and only if $f \approx f'$. Then in C_0:

$$\forall \varphi : X \to MY\,,\, \forall \varphi' : X \to MY'\,,\; \varphi \approx_0 \varphi' \iff M\langle\,\rangle_Y \circ \varphi = M\langle\,\rangle_{Y'} \circ \varphi'\,.$$

2.3 Consistency

Now we define a consistency relation between two parallel morphisms.

Definition 2.3. Let K be a category with a wide subcategory C. A *consistency* relation $\lhd$ is a relation between parallel morphisms, the second one being pure, which satisfies:

- *Pure reflexivity.* $\forall v : X \rightsquigarrow Y\,,\ v \lhd v$.
- *Compatibility with composition.* $\forall f : X \to Y\,, \forall g : Y \to Z\,, \forall u : Y \rightsquigarrow Y'\,, \forall v : X \rightsquigarrow Y'\,, \forall w : Y' \rightsquigarrow Z\,,$ $(u \circ f \lhd v) \wedge (g \lhd w \circ u) \implies g \circ f \lhd w \circ v$.

$$\begin{tikzcd} X \arrow[r, squiggly, "v"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\nabla"] & Y' \arrow[r, squiggly, "w"] & Z \\ & Y \arrow[u, squiggly, "u"] \arrow[ur, "g"'] \arrow[u, phantom, "\wedge"] \arrow[u, phantom, "\nabla"] & \end{tikzcd} \implies \begin{tikzcd} X \arrow[rr, squiggly, bend left, "w\circ v"] \arrow[rr, bend right, "g\circ f"'] \arrow[rr, phantom, "\nabla"] & & Z \end{tikzcd}$$

Two parallel morphisms f and f' are called *consistent* when $f \lhd v \rhd f'$ for some pure morphism v, this is denoted $f \lhd\rhd f'$.

The following properties are easily derived from the definition.

Proposition 2. *Let K be a category with a wide subcategory C and with a consistency relation $\lhd$. Then:*

- *Preservation by composition.* $\forall f : X \to Y\,, \forall v : X \rightsquigarrow Y\,, \forall g : Y \to Z\,, \forall w : Y \rightsquigarrow Z\,,$ $(f \lhd v) \wedge (g \lhd w) \implies g \circ f \lhd w \circ v$.

$$\begin{tikzcd} X \arrow[r, squiggly, "v"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\nabla"] & Y \arrow[r, squiggly, "w"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\nabla"] & Z \end{tikzcd} \implies \begin{tikzcd} X \arrow[rr, squiggly, bend left, "w\circ v"] \arrow[rr, bend right, "g\circ f"'] \arrow[rr, phantom, "\nabla"] & & Z \end{tikzcd}$$

- *Pure substitution.* $\forall v : X \rightsquigarrow Y\,, \forall g : Y \to Z\,, \forall w : Y \rightsquigarrow Z\,, g \lhd w \implies g \circ v \lhd w \circ v$.
- *Pure replacement.* $\forall f : X \to Y\,, \forall v : X \rightsquigarrow Y\,, \forall w : Y \rightsquigarrow Z\,, f \lhd v \implies w \circ f \lhd w \circ v$.

Definition 2.4. An *effect category* $(C \subseteqq K, \lhd)$ is made of a category K and a wide subcategory C of K, with a pure terminal object 1 and the same-effect relation $\approx$ as in definition 2.2, together with a consistency relation $\lhd$ which satisfies:

- *Complementarity with* $\approx$. $\forall f, f' : X \to Y\,, (f \approx f') \wedge (f \lhd\rhd f') \implies f = f'$.

In essence, the complementarity property can be stated as follows: *if two morphisms have the same effect and are consistent, then they are equal.*

The following properties are easily derived.

Proposition 3. *Let $(C \subseteqq K, \lhd)$ be an effect category. Then:*

- *Consistency on effects.* $\forall f : X \to Y\,, (\exists v\,,\ f \lhd v) \implies \mathcal{E}(f) \lhd \langle\,\rangle_X$.
- *Consistency on pure morphisms.* $\forall v, v' : X \rightsquigarrow Y\,, v \lhd v' \iff v = v'$.
- *Consistency is unambiguous.* $\forall f : X \to Y\,, \forall v : X \rightsquigarrow Y\,, f \lhd\rhd v \iff f \lhd v$.

Remark 3. It follows that a pure morphism v is consistent with itself and with no other pure morphism. In general a morphism f may be consistent with no pure morphism or with several ones. The relation $\lhd\rhd$ is symmetric but in general it is not reflexive.

Remark 4. Let K be a category with a wide subcategory C and with a pure terminal object 1. Then the same-effect relation $\approx$ is uniquely defined, and there is a "trivial" consistency relation: the equality of pure morphisms. But neither the existence nor the unicity of a non-trivial consistency relation $\lhd$ is guaranteed.

2.4 Extended consistency

The consistency $\lhd$ is a relation between two morphisms, the second one being pure. It can be extended to pairs of arbitrary morphisms.

Definition 2.5. In an effect category $(C \subseteqq K, \lhd)$, *an extended consistency* is a relation $\blacktriangleleft$ between parallel morphisms such that:

- *Extension.* $\forall f : X \to Y\,,\ \forall v : X \rightsquigarrow Y\,,\ f \lhd v \implies f \blacktriangleleft v$.
- *Substitution.* $\forall f : X \to Y\,,\ \forall g, g' : Y \to Z\,,\ g \blacktriangleleft g' \implies g \circ f \blacktriangleleft g' \circ f$.

The symmetric relation $\blacktriangleleft\!\blacktriangleright$ is defined by $f \blacktriangleleft\!\blacktriangleright f'$ if and only if there is a morphism f'' such that $f \blacktriangleleft f'' \blacktriangleright f'$. This relation $\blacktriangleleft\!\blacktriangleright$ is weaker than the relation $\lhd\!\rhd$.

It follows easily that $\blacktriangleleft$ is reflexive and that $f \blacktriangleleft f'$ implies $f \blacktriangleleft\!\blacktriangleright f'$.

Remark 5. It is easy to check that in an effect category $(C \subseteqq K, \lhd)$ there is a smallest extended consistency $\blacktriangleleft$, which is defined as follows: $\forall h, h' : X \to Y\,,$

$$h \blacktriangleleft h' \iff \exists f : X \to Y\,,\ \exists g : Y \to Z\,,\ \exists w : Y \rightsquigarrow Z\,,\ (h = g \circ f) \wedge (h' = w \circ f) \wedge (g \lhd w)$$

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \arrow[r, squiggly, "w"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\triangledown"] & Z \end{tikzcd} \iff \begin{tikzcd} X \arrow[r, squiggly, "w \circ f"] \arrow[r, bend right, "g \circ f"'] \arrow[r, phantom, "\blacktriangledown"] & Z \end{tikzcd}$$

In addition, this relation $\blacktriangleleft$ satisfies pure replacement:
$\forall f, f' : X \to Y\,,\ \forall w : Y \rightsquigarrow Z\,,\ f \blacktriangleleft f' \implies w \circ f \blacktriangleleft w \circ f'$.

2.5 Examples of effect categories

Several examples are introduced in this section. For each example, the same-effect relation $\approx$ is described, then a consistency relation $\lhd$ is chosen in such a way that we get an effect category, and the smallest extended consistency relation $\blacktriangleleft$ is described. It will be checked in sections 3.8 and 4.4 that in each example the chosen consistency relation gives rise to a Cartesian effect category. The examples about errors, lists, finite multisets and finite sets are provided directly by a monad M, then K_M and C_M are defined as in remark 1. States could be treated with monads, at the cost of using an extra adjunction, but this would not be possible for partiality over an arbitrary base category.

Errors. Let C_0 be a category with an initial object 0 and with a distinguished object E (for "errors"), hence with a unique morphism $!_E : 0 \to E$. Let us assume that there are coproducts of the form $X + E$ that *behave well* in the sense of *extensivity* (Carboni et al., 1993): for every $\varphi : X \to Y + E$, there is a coproduct $X = \mathcal{D}_\varphi + \overline{\mathcal{D}}_\varphi$ with two morphisms $\varphi_Y : \mathcal{D}_\varphi \to Y$ and $\varphi_E : \overline{\mathcal{D}}_\varphi \to E$ such that $\varphi = \varphi_Y + \varphi_E$. The *error monad* on C_0 has $MX = X + E$ as endofunctor and the coprojection $\eta_X : X \to X + E$ as unit. A morphism $f : X \to Y$ in the Kleisli category K_M stands for a morphism $[f] : X \to Y + E$ in C_0, such that $[f] = [f]_Y + [f]_E$ as explained above. A pure morphism $v = J(v_0) : X \rightsquigarrow Y$ in K_M stands for $[v] = \eta_Y \circ v_0 : X \to Y + E$ in C_0, such that $[v] = v_0 + !_E : X \to Y + E$ in C_0. Let us assume that C_0 has a terminal object 1. For each morphism $f : X \to Y$ in K_M, the effect $\mathcal{E}(f) = \langle\,\rangle_Y \circ f : X \to 1$ is such that $[\mathcal{E}(f)] = (\langle\,\rangle_Y + \mathrm{id}_E) \circ [f] = \langle\,\rangle_{\mathcal{D}_{[f]}} + [f]_E$. All this can be illustrated as follows in C_0, first for a pure

morphism v then for a morphism f and finally for the effect $\mathcal{E}(f)$; the vertical arrows are the coprojections:

$$\begin{tikzcd}
X \arrow[r, "v_0"] \arrow[d, "\mathrm{id}_X"'] \arrow[dr, phantom, "="] & Y \arrow[d] \\
X \arrow[r, "{[v]}"] & Y+E \\
0 \arrow[r, "!_E"] \arrow[u, "!_X"] \arrow[ur, phantom, "="] & E \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}_{[f]} \arrow[r, "{[f]_Y}"] \arrow[d] \arrow[dr, phantom, "="] & Y \arrow[d] \\
X \arrow[r, "{[f]}"] & Y+E \\
\overline{\mathcal{D}}_{[f]} \arrow[r, "{[f]_E}"] \arrow[u] \arrow[ur, phantom, "="] & E \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}_{[f]} \arrow[r, "{[f]_Y}"] \arrow[rr, bend left, "\langle\,\rangle_{\mathcal{D}_{[f]}}", "="'] \arrow[d] \arrow[dr, phantom, "="] & Y \arrow[r, "\langle\,\rangle_Y"] \arrow[d, phantom, "="] & 1 \arrow[d] \\
X \arrow[rr, "{[\mathcal{E}(f)]}"] & & 1+E \\
\overline{\mathcal{D}}_{[f]} \arrow[r, "{[f]_E}"] \arrow[u] \arrow[rr, bend right, "{[f]_E}"', "="] \arrow[ur, phantom, "="] & E \arrow[r, "\mathrm{id}_E"] & E \arrow[u]
\end{tikzcd}$$

Let $i_{[f]} : \mathcal{D}_{[f]} \to X$ denote the coprojection and let $\overset{\simeq}{\to}$ denote an isomorphism in C_0.

- $\forall f : X \to Y\,,\ \forall f' : X \to Y'\,,\ f \approx f' \iff \exists i : \overline{\mathcal{D}}_{[f]} \overset{\simeq}{\to} \overline{\mathcal{D}}_{[f']}\,,\ [f]_E = [f']_E \circ i.$
- $\forall f : X \to Y\,,\ \forall v = J(v_0) : X \rightsquigarrow Y\,,\ f \lhd v \iff [f]_Y = v_0 \circ i_{[f]}.$

When C_0 is the category of sets, we say that $\mathcal{D}_\varphi$ is *the domain of definition of* φ and that φ *raises the error* e *at* x whenever $\varphi(x) = e \in E$, so that a morphism v is pure if and only if $[v]$ does not raise any error. Then, $f \approx f'$ means that $[f]$ and $[f']$ raise the same errors for the same arguments, hence they have the same domain of definition. Furthermore, $f \lhd v$ means that $[f]$ coincides with $[v]$ on $\mathcal{D}_{[f]}$, hence $f \lhd\rhd f'$ means that $[f]$ and $[f']$ coincide on $\mathcal{D}_{[f]} \cap \mathcal{D}_{[f']}$. Then the smallest extended consistency relation is such that for all $f, f' : X \to Y$, $f \blacktriangleleft f'$ if and only if $\mathcal{D}_{[f]} \subseteq \mathcal{D}_{[f']}$ and $[f]$ coincides with $[f']$ on $\mathcal{D}_{[f]}$ and also on $\overline{\mathcal{D}}_{[f']}$. It follows that $\blacktriangleleft$ is transitive and that $\blacktriangleleft\blacktriangleright$ is the same relation as $\lhd\rhd$.

Partiality. A *category of partial morphisms* is defined here, as in (Curien and Obtulowitz, 1989), as a category K with a wide subcategory C such that the category K is enriched with a partial order $\leq$ and every pure arrow is maximal for $\leq$. Then the morphisms in K are called the *partial functions* and the morphisms in C the *total functions*, as in the fundamental situation of sets. In addition, let us assume that there is a pure terminal object 1, and wherefore the effect of a morphism $f : X \to Y$ is the morphism $\langle\,\rangle_Y \circ f$ (in (Curien and Obtulowitz, 1989) this morphism is called the *domain of definition of* f).

- $\forall f : X \to Y\,,\ \forall f' : X \to Y'\,,\ f \approx f' \iff \langle\,\rangle_Y \circ f = \langle\,\rangle_Y \circ f'.$
- $\forall f : X \to Y\,,\ \forall v = J(v_0) : X \rightsquigarrow Y\,,\ f \lhd v \iff f \leq v.$
- $\forall f, f' : X \to Y\,,\ f \blacktriangleleft f' \iff f \leq f'.$

We add, as a new axiom, the complementarity of $\approx$ and $\lhd$.

On sets, with the usual notion of partial function, the inclusion of C in K has a right adjoint with lifting $X^\dagger = X + 1$, so that the partial functions from X to Y can be identified to the (total) functions from X to $Y + 1$ and the partial order $\leq$ corresponds to the inclusion of the domains of definition (in their usual sense, as subsets). Then both points of view (partiality and error) are equivalent.

State. Let C_0 be a category with a distinguished object S (for "states") and with products of the form $S \times X$. For each set X let $\sigma_X : S \times X \to S$ and $\pi_X : S \times X \to X$ denote the projections. Let K be the category with the the same objects as C_0 and with a morphism $f : X \to Y$ for each $[f] : S \times X \to S \times Y$ in C_0; we say that f in K *stands for* $[f]$ in C_0. Let C be the wide subcategory of K with the pure morphisms $v = J(v_0) : X \rightsquigarrow Y$ standing for $[v] = \mathrm{id}_S \times v_0 : S \times X \to S \times Y$. Let us assume that C_0 has a terminal object 1. We may identify $S \times 1$ with S, so that the morphism $\langle\,\rangle_X : X \rightsquigarrow 1$ stands for the projection $\sigma_X : S \times X \to S$ and the effect of a morphism $f : X \to Y$ stands for $\sigma_Y \circ [f] : S \times X \to S$.

- $\forall f : X \to Y\,,\ \forall f' : X \to Y'\,,\ f \approx f' \iff \sigma_Y \circ [f] = \sigma_{Y'} \circ [f'].$

- $\forall f : X \to Y$, $\forall v = J(v_0) : X \rightsquigarrow Y$, $f \lhd v \iff \pi_Y \circ [f] = v_0 \circ \pi_X$.
- $\forall f, f' : X \to Y$, $f \blacktriangleleft f' \iff \pi_Y \circ [f] = \pi_Y \circ [f']$.

It follows that $\blacktriangleleft$ is an equivalence relation, so that $\blacktriangleleftright$ is the same as $\blacktriangleleft$.

On sets, $f \approx f'$ means that $[f]$ and $[f']$ modify the state in the same way, and $f \lhd v$ means that $[f]$ always returns the same value as v_0, so that $f \lhd\rhd f'$ means that $[f]$ and $[f']$ both always return the same value, which in addition does not depend on the state, while $f \blacktriangleleft f'$ (as well as $f \blacktriangleleftright f'$) means that $[f]$ and $[f']$ both always return the same value, which may depend on the state.

Lists. Let us consider the *list monad* with endofunctor $\mathcal{L}$ on the category of sets. The unit η maps each x to (x) and the multiplication μ flattens each list of lists. Since 1 is a singleton, a list ℓ in $\mathcal{L}(1)$ may be identified to its length $\mathrm{len}(()\ell)$ in $\mathbb{N}$, and the effect of a morphism $f : X \to Y$ to $\mathrm{len}(\circ)f : X \to \mathbb{N}$. Then, a morphism f is effect-free when $\mathrm{len}(\circ)f$ is the constant function 1. For each $x \in X$ and $k \in \mathbb{N}$, we denote by $(x)^k$ the list $(x, \ldots, x)$ where x is repeated k times. More generally, for each list $\underline{x} = (x_1, \ldots, x_n) \in \mathcal{L}(X)$ and each list of naturals $\underline{k} = (k_1, \ldots, k_n)$ with the same length as $\underline{x}$, we denote by $\underline{x}^{\underline{k}}$ the list $(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n)$ where each x_i is repeated k_i times.

- $\forall f : X \to Y$, $\forall f' : X \to Y'$, $f \approx f' \iff \forall x \in X$, $\mathrm{len}(()f(x)) = \mathrm{len}(()f'(x))$.
- $\forall f : X \to Y$, $\forall v = J(v_0) : X \rightsquigarrow Y$, $f \lhd v \iff \forall x \in X$, $\exists k \in \mathbb{N}$, $[f](x) = (v_0(x))^k$.
- $\forall f, f' : X \to Y$, $f \blacktriangleleft f' \iff \forall x \in X$, $\exists \underline{k} \in \mathcal{L}(\mathbb{N})$, $[f](x) = [f'](x)^{\underline{k}}$.

It follows that $f \lhd\rhd f'$ if and only if for each $x \in X$ there is some $y \in Y$ that is the unique element (if any) in the lists $[f](x)$ and $[f'](x)$, and that $f \blacktriangleleftright f'$ as soon as f and f' are parallel.

Finite (multi)sets. The example of lists can easily be adapted to the *finite multiset monad* and to the *finite set monad* on the category of sets. For the finite multiset monad, $\mathcal{M}_{\mathrm{fin}}(1)$ can be identified to $\mathbb{N}$ and the effect of a morphism to the cardinal of its image.

- $\forall f : X \to Y$, $\forall f' : X \to Y'$, $f \approx f' \iff \forall x \in X$, $\mathrm{card}(()f(x)) = \mathrm{card}(()f'(x))$.
- $\forall f : X \to Y$, $\forall v = J(v_0) : X \rightsquigarrow Y$, $f \lhd v \iff \forall x \in X$, $[f](x) \subseteq \{v_0(x)\}$.
- $\forall f, f' : X \to Y$, $f \blacktriangleleft f' \iff \forall x \in X$, $[f](x) \subseteq [f'](x)$.

For the finite set monad, the definitions of $\lhd$ and $\blacktriangleleft$ are similar, but $\approx$ is different. Since $\mathcal{P}_{\mathrm{fin}}(1)$ has only two elements $\emptyset$ and 1, we get $f \approx f'$ if and only if for all $x \in X$ either both $f(x)$ and $f'(x)$ are empty or both are non-empty.

2.6 Results in evaluation logic

In (Moggi, 1995), within the framework of *evaluation logic* and with respect to a strong monad satisfying some extra properties, Moggi defines the relation $c \Downarrow a$, which means that the value a is a *result* of the computation c. With the same notations as in remark 1, $c : 1 \to MX$ and $a : 1 \to X$ are morphisms in C_0, or equivalently $c = [f]$ for a morphism $f : 1 \to X$ in K_M and $a = v_0 : X \to Y$ yields a pure morphism $v = J(v_0) : 1 \rightsquigarrow X$. Then it may happen that f is *consistent* with v in the sense of this paper. The following table compares both notions for several monads on sets.

Monad MY	Results (Moggi, 1995) $c \Downarrow a$	Consistency (this paper) $f \lhd v$
$Y + E$	$c = a$ (thus, c is total)	$c \in Y \implies c = a$
$(Y \times S)^S$	$\exists s \in S$, $\exists s' \in S$, $c(s) = (a, s')$	$\forall s \in S$, $\exists s' \in S$, $c(s) = (a, s')$
$\mathcal{L}(Y)$	$a \in c$	$\exists k \in \mathbb{N}$, $c = (a)^k$
$\mathcal{P}_{\mathrm{fin}}(Y)$	$a \in c$	$c = \{a\}$ or $c = \emptyset$

From this table we see that in general $f \lhd v \not\Rightarrow c \Downarrow a$ and $c \Downarrow a \not\Rightarrow f \lhd v$. It can easily be seen from the example of the state monad that having the same results is not a consistency relation in general, since two different morphisms may have the same effect and the same results. Therefore, the notion of result in evaluation logic does not easily fit with our notion of consistency.

3 Cartesian effect categories

3.1 Cartesian categories

In this paper a *Cartesian category* is a category with chosen finite products. We denote by 1 the terminal object, $\times$ for the products and $p, q, r, s, t, \ldots$ (with indices) for the projections. The binary product defines a functor $\times : C^2 \to C$ such that for all $v_1 : X_1 \to Y_1$ and $v_2 : X_2 \to Y_2$, the morphism $v_1 \times v_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is the unique morphism that satisfies the *binary product property*:

$$q_1 \circ (v_1 \times v_2) = v_1 \circ p_1$$

$$q_2 \circ (v_1 \times v_2) = v_2 \circ p_2$$

$$\begin{tikzcd}
X_1 \arrow[r, "v_1"] \arrow[d, phantom, "="] & Y_1 \\
X_1 \times X_2 \arrow[u, "p_1"] \arrow[r, "v_1 \times v_2"] \arrow[d, "p_2"'] & Y_1 \times Y_2 \arrow[u, "q_1"'] \arrow[d, "q_2"] \\
X_2 \arrow[r, "v_2"] & Y_2
\end{tikzcd}$$

In a Cartesian category C, the *swap* natural transformation c, with components $c_{X_1,X_2} : X_1 \times X_2 \to X_2 \times X_1$, is defined from the projections $p_i : X_1 \times X_2 \to X_i$ and $p'_i : X_2 \times X_1 \to X_i$ by $p'_i \circ c_{X_1,X_2} = p_i$ for $i = 1, 2$. It follows that $c_{X_2,X_1} = c^{-1}_{X_1,X_2}$.

Now, Cartesian products in a category are generalized, first as semi-pure products, then as sequential products, in an effect category.

3.2 Semi-pure products

Let us consider an effect category $(C \subseteqq K, \lhd)$ where C is a Cartesian category. We define the *semi-pure products* as two graph homomorphisms $\ltimes : C \times K \to K$ and $\rtimes : K \times C \to K$ that extend $\times$ and that satisfy some generalization of the binary product property involving the consistency relation $\lhd$. *while the universal property of a binary product consists in two equalities, the universal property of a semi-pure product consists in one equality and one consistency.*

Definition 3.1. Let $(C \subseteqq K, \lhd)$ be an effect category with a binary product $\times$ on C. A graph homomorphism $\ltimes : C \times K \to K$ is *the left semi-pure product* on $(C \subseteqq K, \lhd, \times)$ if it extends $\times$ and satisfies the *left semi-pure product property*: for all $v_1 : X_1 \rightsquigarrow Y_1$ and $f_2 : X_2 \to Y_2$, the morphism $v_1 \ltimes f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is the unique morphism such that:

$$q_1 \circ (v_1 \ltimes f_2) \lhd v_1 \circ p_1$$

$$q_2 \circ (v_1 \ltimes f_2) = f_2 \circ p_2$$

$$\begin{tikzcd}
X_1 \arrow[r, squiggly, "v_1"] \arrow[d, phantom, "\triangledown"] & Y_1 \\
X_1 \times X_2 \arrow[u, squiggly, "p_1"] \arrow[r, "v_1 \ltimes f_2"] \arrow[d, squiggly, "p_2"'] \arrow[d, phantom, shift right=4ex, "="] & Y_1 \times Y_2 \arrow[u, squiggly, "q_1"'] \arrow[d, squiggly, "q_2"] \\
X_2 \arrow[r, "f_2"] & Y_2
\end{tikzcd}$$

Symmetrically, a graph homomorphism $\rtimes : K \times C \to K$ is *the right semi-pure product* on $(C \subseteqq K, \lhd, \times)$ if it extends $\times$ and satisfies the *right semi-pure product property*: for all $f_1 : X_1 \to Y_1$ and $v_2 : X_2 \rightsquigarrow Y_2$, the

morphism $f_1 \rtimes v_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is the unique morphism such that:

$$q_1 \circ (f_1 \rtimes v_2) = f_1 \circ p_1$$

$$q_2 \circ (f_1 \rtimes v_2) \lhd v_2 \circ p_2$$

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "p_1"', "\wr"] \arrow[dr, phantom, "="] & Y_1 \arrow[d, "q_1", "\wr"'] \\
X_1 \times X_2 \arrow[r, "f_1 \rtimes v_2"] \arrow[d, "p_2"', "\wr"] \arrow[dr, phantom, "\triangle"] & Y_1 \times Y_2 \arrow[d, "q_2", "\wr"'] \\
X_2 \arrow[r, squiggly, "v_2"] & Y_2
\end{tikzcd}$$

A *Cartesian effect category* is an effect category $(C \subseteqq K, \lhd)$ with a binary product $\times$ on C and with semi-pure products $\ltimes$ and $\rtimes$ (for short, it may be denoted $C \subseteqq K$ or simply K).

A straightforward consequence of definition 3.1 is that the right semi-pure product can be determined from the left one, as follows. Consequently, from now on, we generally omit the right semi-pure products.

Proposition 4. *In a Cartesian effect category. for all $f_1 : X_1 \to Y_1$ and $v_2 : X_2 \rightsquigarrow Y_2$:*

$$(f_1 \rtimes v_2) = c_{Y_2,Y_1} \circ (v_2 \ltimes f_1) \circ c_{X_1,X_2} \ .$$

In a binary product $v_1 \times v_2$, obviously the first projection $q_1 \circ (v_1 \times v_2)$ does not depend on v_2, and symmetrically the second projection $q_2 \circ (v_1 \times v_2)$ does not depend on v_1. For a left semi-pure product $v_1 \ltimes f_2$, this remains true for the second projection but not for the first one. However, a consequence of the complementarity of $\lhd$ with $\approx$ is that $q_1 \circ (v_1 \ltimes f_2)$ depends on f_2 precisely through its effect $\mathcal{E}(f_2)$, as stated in the next proposition.

Proposition 5. *In a Cartesian effect category, for all $v_1 : X_1 \rightsquigarrow Y_1$, $f_2 : X_2 \to Y_2$ and $f_2' : X_2 \to Y_2$, $\mathcal{E}(q_1 \circ (v_1 \ltimes f_2)) = \mathcal{E}(v_1 \ltimes f_2) = \mathcal{E}(f_2 \circ p_2)$ and:*

$$\mathcal{E}(f_2) = \mathcal{E}(f_2') \implies q_1 \circ (v_1 \ltimes f_2) = q_1 \circ (v_1 \ltimes f_2') \ .$$

Proof. The first result derives from the pure wiping property of the effect. For the second result, let $h = v_1 \ltimes f_2$ and $h' = v_1 \ltimes f_2'$. The left semi-pure product property implies that $q_1 \circ h \lhd \rhd q_1 \circ h'$ and $q_2 \circ h = q_2 \circ h'$. The latter implies that $q_2 \circ h \approx q_2 \circ h'$, and thus by pure wiping we have also $q_1 \circ h \approx q_1 \circ h'$. The result now follows from the complementarity of $\lhd$ with $\approx$. □

The next proposition follows from the fact that the restriction of $\ltimes$ to C^2 coincides with the binary product functor $\times$ on C.

Proposition 6. *In a Cartesian effect category, for all objects X_1 and X_2:*

$$\mathrm{id}_{X_1} \ltimes \mathrm{id}_{X_2} = \mathrm{id}_{X_1} \times \mathrm{id}_{X_2} = \mathrm{id}_{X_1 \times X_2} \ .$$

Remark 6. Let us assume that the following *unicity condition* holds:

$$\forall h, h' : X \to Y_1 \times Y_2 \, , \, (q_1 \circ h \lhd \rhd q_1 \circ h') \wedge (q_2 \circ h = q_2 \circ h') \implies h = h' \ .$$

In this case, if there is a graph homomorphism $\ltimes : C \times K \to K$ extending $\times$ and satisfying the left semi-pure product property, then $\ltimes$ is the left semi-pure product.

3.3 Sequential products

In accordance with the intended meaning of "sequential", we define sequential products as composed from two consecutive semi-pure products.

Definition 3.2. In a Cartesian effect category, the pair of *sequential products* composed from the semi-products $\ltimes$, $\rtimes$ is made of the graph homomorphisms $\ltimes_{\mathrm{seq}}, \rtimes_{\mathrm{seq}} : K^2 \to K$ (the *left* and *right* sequential products, respectively) defined as follows:

- for all $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$:

$$f_1 \ltimes_{\mathrm{seq}} f_2 = (\mathrm{id}_{Y_1} \ltimes f_2) \circ (f_1 \rtimes \mathrm{id}_{X_2})$$

- for all $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$:

$$f_1 \rtimes_{\mathrm{seq}} f_2 = (f_1 \rtimes \mathrm{id}_{Y_2}) \circ (\mathrm{id}_{X_1} \ltimes f_2)$$

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, squiggly, "p_1"'] \arrow[dr, phantom, "="] & Y_1 \arrow[r, squiggly, "\mathrm{id}"] \arrow[d, squiggly, "r_1"] \arrow[dr, phantom, "\nabla"] & Y_1 \arrow[d, squiggly, "q_1"] \\
X_1 \times X_2 \arrow[r, "f_1 \rtimes \mathrm{id}"] \arrow[d, squiggly, "p_2"'] \arrow[dr, phantom, "\triangle"] & Y_1 \times X_2 \arrow[r, "\mathrm{id} \ltimes f_2"] \arrow[d, squiggly, "r_2"] \arrow[dr, phantom, "="] & Y_1 \times Y_2 \arrow[d, squiggly, "q_2"] \\
X_2 \arrow[r, squiggly, "\mathrm{id}"] & X_2 \arrow[r, "f_2"] & Y_2
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \arrow[r, squiggly, "\mathrm{id}"] \arrow[d, squiggly, "p_1"'] \arrow[dr, phantom, "\nabla"] & X_1 \arrow[r, "f_1"] \arrow[d, squiggly, "s_1"] \arrow[dr, phantom, "="] & Y_1 \arrow[d, squiggly, "q_1"] \\
X_1 \times X_2 \arrow[r, "\mathrm{id} \ltimes f_2"] \arrow[d, squiggly, "p_2"'] \arrow[dr, phantom, "="] & X_1 \times Y_2 \arrow[r, "f_1 \rtimes \mathrm{id}"] \arrow[d, squiggly, "s_2"] \arrow[dr, phantom, "\triangle"] & Y_1 \times Y_2 \arrow[d, squiggly, "q_2"] \\
X_2 \arrow[r, "f_2"] & Y_2 \arrow[r, squiggly, "\mathrm{id}"] & Y_2
\end{tikzcd}$$

It follows easily from proposition 4 that the right sequential product can be determined from the left one, as follows. Consequently, from now on, we generally omit the right sequential products.

Proposition 7. *In a Cartesian effect category, for all $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$:*

$$(f_1 \rtimes_{\mathrm{seq}} f_2) = c_{Y_2,Y_1} \circ (f_2 \ltimes_{\mathrm{seq}} f_1) \circ c_{X_1,X_2} \ .$$

Proposition 8. *In a Cartesian effect category, the left sequential product $\ltimes_{\mathrm{seq}}$ extends the left semi-pure product $\ltimes$.*

Proof. Let $v : X_1 \rightsquigarrow Y_1$ and $f : X_2 \to Y_2$. Since $v \ltimes_{\mathrm{seq}} f = (\mathrm{id}_{Y_1} \ltimes f) \circ (v \rtimes \mathrm{id}_{X_2})$ and since $\rtimes$ extends the binary product $\times$ on C^2:

$$v \ltimes_{\mathrm{seq}} f = (\mathrm{id}_{Y_1} \ltimes f) \circ (v \times \mathrm{id}_{X_2}) \ .$$

The left semi-pure product property yields:

$$q_1 \circ (\mathrm{id}_{Y_1} \ltimes f) \lhd r_1 \ \text{ and } \ q_2 \circ (\mathrm{id}_{Y_1} \ltimes f) = f \circ r_2$$

so that by pure substitution:

$$q_1 \circ (v \ltimes_{\mathrm{seq}} f) \lhd r_1 \circ (v \times \mathrm{id}_{X_2}) \ \text{ and } \ q_2 \circ (v \ltimes_{\mathrm{seq}} f) = f \circ r_2 \circ (v \times \mathrm{id}_{X_2})$$

hence from the binary product property we get:

$$q_1 \circ (v \ltimes_{\mathrm{seq}} f) \lhd v \circ p_1 \ \text{ and } \ q_2 \circ (v \ltimes_{\mathrm{seq}} f) = f \circ p_2$$

which is the left semi-pure product property. □

Remark 7. It follows from proposition 8 that we may drop the subscript "seq".

Definition 3.3. In a Cartesian effect category, for all $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ the *left pairing* of f_1 and f_2 is $\langle f_1, f_2 \rangle_l = (f_1 \ltimes f_2) \circ \langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \to Y_1 \times Y_2$ and the *right pairing* of f_1 and f_2 is $\langle f_1, f_2 \rangle_r = (f_1 \rtimes f_2) \circ \langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \to Y_1 \times Y_2$.

Remark 8. Another point of view on sequential products, as "direct" generalizations of binary products (independently from any *a priori* semi-pure products) is given in section 3.7.

3.4 Pure morphisms are central

The next definition is similar to the definition of central morphisms in a binoidal category, see section 4.1.

Definition 3.4. In a Cartesian effect category, a morphism k_1 is *central* if for each morphism f_2:

$$k_1 \ltimes f_2 = k_1 \rtimes f_2 \ .$$

Then it follows from proposition 7 that $f_2 \ltimes k_1 = f_2 \rtimes k_1$. The *center* C_K of K is made of the objects of K together with the central morphisms, we will prove in theorem 12 that C_K is a subcategory of K.

Remark 9. According to definition 3.2, in a Cartesian effect category a morphism $k_1 : X_1 \to Y_1$ is central if and only if for each morphism $f_2 : X_2 \to Y_2$:

$$(k_1 \rtimes \mathrm{id}_{Y_2}) \circ (\mathrm{id}_{X_1} \ltimes f_2) = (\mathrm{id}_{Y_1} \ltimes f_2) \circ (k_1 \rtimes \mathrm{id}_{X_2}) \ .$$

Remark 10. It follows from definition 3.2 and proposition 6 that the identities are central. Theorem 9 now proves that this is valid for all pure morphisms.

Theorem 9. *In a Cartesian effect category, every pure morphism is central.*

Proof. Given $v : X_1 \rightsquigarrow Y_1$ and $f : X_2 \to Y_2$, let us prove that the left semi-pure product $v \ltimes f$ is equal to the right sequential product $v \rtimes f$. Let:

$$h = v \rtimes f = (v \rtimes \mathrm{id}_{Y_2}) \circ (\mathrm{id}_{X_1} \ltimes f) = (v \times \mathrm{id}_{Y_2}) \circ (\mathrm{id}_{X_1} \ltimes f) \ .$$

Using the binary product property:

$$q_1 \circ h = v \circ s_1 \circ (\mathrm{id}_{X_1} \ltimes f) \ \text{ and } \ q_2 \circ h = s_2 \circ (\mathrm{id}_{X_1} \ltimes f)$$

then the left semi-pure product property:

$$s_1 \circ (\mathrm{id}_{X_1} \ltimes f) \lhd p_1 \ \text{ and } \ s_2 \circ (\mathrm{id}_{X_1} \ltimes f) = f \circ p_2$$

we get by pure replacement:

$$q_1 \circ h \lhd v \circ p_1 \ \text{ and } \ q_2 \circ h = f \circ p_2$$

which means that the left semi-pure product property is satisfied: $h = v \ltimes f$, as required. □

Remark 11. In view of theorem 9 there would be no ambiguity in denoting $\times$ for the semi-pure products $\ltimes$ and $\rtimes$, however we will not use this opportunity, in order to keep in mind that the semi-pure products are not real products.

3.5 Functoriality properties

As reminded in section 3.1, the binary product in a Cartesian category is a functor. In this section it is proved that similarly the semi-pure products in a Cartesian effect category are functors.

Lemma 10. *In a Cartesian effect category, for all X_1, $f_2 : X_2 \to Y_2$ and $g_2 : Y_2 \to Z_2$:*

$$(\mathrm{id}_{X_1} \ltimes g_2) \circ (\mathrm{id}_{X_1} \ltimes f_2) = \mathrm{id}_{X_1} \ltimes (g_2 \circ f_2) \ .$$

Proof. The proof is easily obtained by chasing the following diagram and using the compatibility of consistency with composition.

$$\begin{tikzcd}
X_1 \arrow[r, squiggly, "\mathrm{id}"] & X_1 \arrow[r, squiggly, "\mathrm{id}"] & X_1 \\
X_1 \times X_2 \arrow[u, squiggly, "p_1"] \arrow[r, "\mathrm{id} \ltimes f_2"] \arrow[ur, phantom, "\nabla"] & X_1 \times Y_2 \arrow[u, squiggly, "s_1"'] \arrow[r, "\mathrm{id} \ltimes g_2"] \arrow[ur, phantom, "\nabla"] & X_1 \times Z_2 \arrow[u, squiggly, "s'_1"'] \\
X_2 \arrow[u, squiggly, "p_2"] \arrow[r, "f_2"] \arrow[ur, phantom, "="] & Y_2 \arrow[u, squiggly, "s_2"'] \arrow[r, "g_2"] \arrow[ur, phantom, "="] & Z_2 \arrow[u, squiggly, "s'_2"']
\end{tikzcd}$$

□

Lemma 11. *In a Cartesian effect category, for all $f_1 : X_1 \to Y_1$, $k_1 : Y_1 \to Z_1$, $f_2 : X_2 \to Y_2$ and $g_2 : Y_2 \to Z_2$ with k_1 central:*

$$(k_1 \ltimes g_2) \circ (f_1 \ltimes f_2) = (k_1 \circ f_1) \ltimes (g_2 \circ f_2)$$

Proof. According to definition 3.2:

$$(k_1 \ltimes g_2) \circ (f_1 \ltimes f_2) = (\mathrm{id}_{Z_1} \ltimes g_2) \circ (k_1 \rtimes \mathrm{id}_{Y_2}) \circ (\mathrm{id}_{Y_1} \ltimes f_2) \circ (f_1 \rtimes \mathrm{id}_{X_2}) \ .$$

Since k_1 is central, this is equal to $(\mathrm{id}_{Z_1} \ltimes g_2) \circ (\mathrm{id}_{Z_1} \ltimes f_2) \circ (k_1 \rtimes \mathrm{id}_{X_2}) \circ (f_1 \rtimes \mathrm{id}_{X_2})$. The result now follows from lemma 10 and definition 3.2 again. □

Theorem 12. *In a Cartesian effect category $C \subseteqq K$, the center C_K is a wide subcategory of K that contains C, and the restrictions of the sequential products are functors $\ltimes : C_K \times K \to K$ and $\rtimes : K \times C_K \to K$.*

Proof. The central morphisms form a subcategory of K: this comes from remark 10 for identities and from lemma 11 and its symmetric version for composition. The center C_K is wide by definition, and it contains C because of theorem 9. The restrictions of the left sequential product is a functor: by proposition 6 for identities and lemma 11 for composition. Symmetrically, the restrictions of the right sequential product is a functor. □

3.6 Naturality properties

As reminded in section 3.1, a Cartesian category C with $\times : C^2 \to C$ and 1 forms a symmetric monoidal category, which means that the projections can be combined in order to get natural isomorphisms a, r, l, c with components:

- $a_X = a_{X_1,X_2,X_3} : (X_1 \times X_2) \times X_3 \to X_1 \times (X_2 \times X_3)$,
- $r_X : 1 \times X \to X$, $l_X : X \times 1 \to X$,
- $c_X = c_{X_1,X_2} : X_1 \times X_2 \to X_2 \times X_1$,

which satisfy the symmetric monoidal coherence conditions (Mac Lane, 1997). In this section we prove that in a Cartesian effect category $C \subseteqq K$, the natural isomorphisms a, r, l, c that are defined from C satisfy more general naturality conditions, involving the sequential products $\ltimes, \rtimes$. The verification of the next result is straightforward from the definitions.

Lemma 13. *In a Cartesian effect category, for all f_1, f_2, f_3 and pure v_1, v_2, v_3:*

$$\begin{cases} a_Y \circ (f_1 \rtimes (v_2 \rtimes v_3)) = ((f_1 \rtimes v_2) \rtimes v_3) \circ a_X \\ a_Y \circ (v_1 \ltimes (f_2 \rtimes v_3)) = ((v_1 \ltimes f_2) \rtimes v_3) \circ a_X \\ a_Y \circ (v_1 \ltimes (v_2 \ltimes f_3)) = ((v_1 \ltimes v_2) \ltimes f_3) \circ a_X \end{cases}$$

Theorem 14. *In a Cartesian effect category, for all $f : X \to Y$, $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ and $f_3 : X_3 \to Y_3$:*

$$\begin{cases} r_Y \circ (\mathrm{id}_1 \ltimes f) = f \circ r_X \\ l_Y \circ (f \rtimes \mathrm{id}_1) = f \circ l_X \\ c_Y \circ (f_1 \rtimes f_2) = (f_2 \ltimes f_1) \circ c_X \\ a_Y \circ (f_1 \ltimes (f_2 \ltimes f_3)) = ((f_1 \ltimes f_2) \ltimes f_3) \circ a_X \\ a_Y \circ (f_1 \rtimes (f_2 \rtimes f_3)) = ((f_1 \rtimes f_2) \rtimes f_3) \circ a_X \end{cases}$$

Proof. Since r_X and l_X are the projections, the first two lines comes from the definition of semi-pure products. Since c_X is the swap morphism from section 3.1, the third line is proposition 7. As for the fourth line, let us use the definition of sequential products:

$$f_1 \ltimes (f_2 \ltimes f_3) = (\mathrm{id} \ltimes (f_2 \ltimes f_3)) \circ (f_1 \rtimes \mathrm{id}) \quad \text{and} \quad f_2 \ltimes f_3 = (\mathrm{id} \ltimes f_3) \circ (f_2 \rtimes \mathrm{id})$$

hence by lemma 10:

$$\mathrm{id} \ltimes (f_2 \ltimes f_3) = (\mathrm{id} \ltimes (\mathrm{id} \ltimes f_3)) \circ (\mathrm{id} \ltimes (f_2 \rtimes \mathrm{id}))$$

and finally:

$$f_1 \ltimes (f_2 \ltimes f_3) = (\mathrm{id} \ltimes (\mathrm{id} \ltimes f_3)) \circ (\mathrm{id} \ltimes (f_2 \rtimes \mathrm{id})) \circ (f_1 \rtimes \mathrm{id}) \ .$$

In a symmetric way:

$$(f_1 \ltimes f_2) \ltimes f_3 = (\mathrm{id} \ltimes f_3) \circ ((\mathrm{id} \ltimes f_2) \rtimes \mathrm{id}) \circ ((f_1 \rtimes \mathrm{id}) \rtimes \mathrm{id}) \ .$$

Hence the result follows from the three lines of lemma 13, together with proposition 6 for dealing with identities. □

3.7 The sequential product properties

Sequential products also satisfy the *left and right sequential product properties*, as defined below, which generalize the binary product property. We use an extended consistency $\blacktriangleleft$, as defined in section 2.4.

Definition 3.5. Let $(C \subseteqq K, \lhd)$ be an effect category with an extended consistency relation $\blacktriangleleft$ and with a pair of graph homomorphisms $\ltimes', \rtimes' : K^2 \to K$ extending $\times$. Then the *left sequential product property* states that for all $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, the morphism $f_1 \ltimes' f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ satisfies:

$$q_1 \circ (f_1 \ltimes' f_2) \blacktriangleleft f_1 \circ p_1$$

$$q_2 \circ (f_1 \ltimes' f_2) = f_2 \circ r_2 \circ (f_1 \rtimes' \mathrm{id}_{X_2})$$

$$\begin{tikzcd}
X_1 \arrow[rr, "f_1"] & & Y_1 \\
X_1 \times X_2 \arrow[u, squiggly, "p_1"] \arrow[rr, "f_1 \ltimes' f_2"] \arrow[dr, bend right, "f_1 \rtimes' \mathrm{id}"'] \arrow[rr, phantom, shift right=4ex, "\blacktriangledown"] & & Y_1 \times Y_2 \arrow[u, squiggly, "q_1"'] \arrow[d, squiggly, "q_2"] \\
& Y_1 \times X_2 \arrow[r, phantom, "="] \arrow[d, squiggly, "r_2"] & \\
& X_2 \arrow[r, "f_2"] & Y_2
\end{tikzcd}$$

Symmetrically, the *right sequential product property* says that for all $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, the morphism $f_1 \rtimes' f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ satisfies:

$$q_1 \circ (f_1 \rtimes' f_2) = f_1 \circ s_1 \circ (\mathrm{id}_{X_1} \ltimes' f_2)$$

$$q_2 \circ (f_1 \rtimes' f_2) \blacktriangleleft f_2 \circ p_2$$

$$\begin{tikzcd}
& X_1 \arrow[r, "f_1"] & Y_1 \\
& X_1 \times Y_2 \arrow[u, squiggly, "s_1"] \arrow[r, phantom, "="] & \\
X_1 \times X_2 \arrow[ur, bend left, "\mathrm{id} \ltimes' f_2"] \arrow[r, "f_1 \rtimes' f_2"] \arrow[d, squiggly, "p_2"'] \arrow[r, phantom, shift right=4ex, "\blacktriangle"] & & Y_1 \times Y_2 \arrow[uu, squiggly, "q_1"'] \arrow[d, squiggly, "q_2"] \\
X_2 \arrow[r, "f_2"] & & Y_2
\end{tikzcd}$$

Proposition 15. *In a Cartesian effect category, the sequential products $\ltimes, \rtimes$ satisfy the sequential product properties.*

Proof. The left sequential product is defined as $f_1 \ltimes f_2 = (\mathrm{id}_{Y_1} \ltimes f_2) \circ (f_1 \rtimes \mathrm{id}_{X_2})$. Since $\blacktriangleleft$ extends $\lhd$, the left semi-pure product property yields:

$$q_1 \circ (\mathrm{id}_{Y_1} \ltimes f_2) \blacktriangleleft r_1 \;\text{ and }\; q_2 \circ (\mathrm{id}_{Y_1} \ltimes f_2) = f_2 \circ r_2$$

so that by the substitution property of $\blacktriangleleft$:

$$q_1 \circ (f_1 \ltimes f_2) \blacktriangleleft r_1 \circ (f_1 \rtimes \mathrm{id}_{X_2}) \;\text{ and }\; q_2 \circ (f_1 \ltimes f_2) = f_2 \circ r_2 \circ (f_1 \rtimes \mathrm{id}_{X_2})\,.$$

The right semi-pure product property implies that $r_1 \circ (f_1 \rtimes \mathrm{id}_{X_2}) = f_1 \circ p_1$, hence:

$$q_1 \circ (f_1 \ltimes f_2) \blacktriangleleft f_1 \circ p_1 \;\text{ and }\; q_2 \circ (f_1 \ltimes f_2) = f_2 \circ r_2 \circ (f_1 \rtimes \mathrm{id}_{X_2})$$

which is the left sequential product property. □

Remark 12. The following condition is called the *extended unicity* condition:

$$\forall h, h' : X \to Y_1 \times Y_2\,,\; (q_1 \circ h \blacktriangleleft\!\blacktriangleright q_1 \circ h') \wedge (q_2 \circ h = q_2 \circ h') \implies h = h'$$

Since $\blacktriangleleft\!\blacktriangleright$ is weaker than $\lhd\!\rhd$, the extended unicity condition implies the unicity condition of remark 6. Whenever the extended unicity condition holds, the sequential product properties can be used as a definition of the sequential products, instead of definition 3.2. In addition, although this looks like a mutually recursive definition of the left and right sequential products, this recursivity has only two steps.

Indeed, let $\ltimes, \rtimes$ be the sequential products and let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$. First let $h = f_1 \rtimes \mathrm{id}_{X_2}$. The right semi-pure product property states that $q_1 \circ h = f_1 \circ p_1$ and $q_2 \circ h \lhd v_2 \circ p_2$, thanks to the unicity condition this is a characterization of h. Now let $k = f_1 \ltimes f_2$, from proposition 15 we get $q_1 \circ k \blacktriangleleft f_1 \circ p_1$ and $q_2 \circ k = f_2 \circ r_2 \circ h$, and thanks to the extended unicity condition this is a characterization of k.

3.8 Some examples of Cartesian effect categories

In this section and in section 4.4 we check that the effect categories from section 2.5 can be seen as Cartesian effect categories. In each example, for any pure morphism v and morphism f we build a morphism $v \ltimes f$, and it is left as an exercise to check that $v \ltimes f$ actually is the left semi-pure product of v and f. In addition, it happens that the extended unicity condition is satisfied, so that the sequential products are characterized by the sequential product properties.

Errors. According to (Carboni et al., 1993), an extensive category with products is distributive. So, in the category C_0, for all X, Y, Z the canonical map from $X \times Y + X \times Z$ to $X \times (Y + Z)$ is an isomorphism. Let $v = J(v_0) : X_1 \rightsquigarrow Y_1$ and $f : X_2 \to Y_2$ in K, so that by distributivity $X_1 \times X_2$ is isomorphic to $(X_1 \times \mathcal{D}_{[f]}) + (X_1 \times \overline{\mathcal{D}}_{[f]})$. We define $v \ltimes f : X_1 \times X_2 \to Y_1 \times Y_2$ by $\mathcal{D}_{[v \ltimes f]} = X_1 \times \mathcal{D}_{[f]}$, $\overline{\mathcal{D}}_{[v \ltimes f]} = X_1 \times \overline{\mathcal{D}}_{[f]}$, $[v \ltimes f]_Y = v_0 \times [f]_Y$ and $[v \ltimes f]_E = [f]_E \circ \pi$, where $\pi : X_1 \times \overline{\mathcal{D}}_{[f]} \to \overline{\mathcal{D}}_{[f]}$ is the projection.

$$\begin{tikzcd}
\mathcal{D}_{[f]} \arrow[r, "{[f]_Y}"] \arrow[d] \arrow[dr, phantom, "="] & Y_2 \arrow[d] \\
X_2 \arrow[r, "{[f]}"] & Y_2 + E \\
\overline{\mathcal{D}}_{[f]} \arrow[r, "{[f]_E}"] \arrow[u] \arrow[ur, phantom, "="] & E \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \times \mathcal{D}_{[f]} \arrow[r, "{v_0 \times [f]_Y}"] \arrow[d] \arrow[dr, phantom, "="] & Y_1 \times Y_2 \arrow[d] \\
X_1 \times X_2 \arrow[r, "{[v \ltimes f]}"] & Y_1 \times Y_2 + E \\
X_1 \times \overline{\mathcal{D}}_{[f]} \arrow[r, "{[f]_E \circ \pi}"] \arrow[u] \arrow[ur, phantom, "="] & E \arrow[u]
\end{tikzcd}$$

On sets, as expected, this provides the left sequential product: $\forall x_1 \in X_1\,, \forall x_2 \in X_2\,,$

$$(f_1 \ltimes f_2)(x_1, x_2) = \begin{cases} \langle [f_1](x_1), [f_2](x_2) \rangle & \text{if } [f_1](x_1) \in Y_1 \;\text{ and }\; [f_2](x_2) \in Y_2 \\ [f_2](x_2) & \text{if } [f_1](x_1) \in Y_1 \;\text{ and }\; [f_2](x_2) \in E \\ [f_1](x_1) & \text{if } [f_1](x_1) \in E \end{cases}$$

When E has one element all morphisms are central, but as soon as E has more than one element there are non-central morphisms.

Partiality. Given a category of partial morphisms, if we impose the existence of sequential products and the fact that all morphisms are central, then we get a notion that is rather similar to the notion of *partial Cartesian* category of partial morphisms in (Curien and Obtulowitz, 1989).

On sets, up to adjunction, the left sequential product is the same as for the monad $X+1$: $\mathcal{D}_{(f_1 \ltimes f_2)} = \mathcal{D}_{f_1} \ltimes \mathcal{D}_{f_2}$ and

$$\forall x_1 \in \mathcal{D}_{f_1}\,,\ \forall x_2 \in \mathcal{D}_{f_2}\,,\ (f_1 \ltimes f_2)(x_1, x_2) = \langle [f_1](x_1), [f_2](x_2) \rangle\ .$$

State. Let $v = J(v_0) : X_1 \rightsquigarrow Y_1$ and $f : X_2 \to Y_2$ in K. Let us define $v \ltimes f : X_1 \times X_2 \to Y_1 \times Y_2$, up to the relevant commutations, by $[v \ltimes f] = v_0 \times [f] : S \times X_1 \times Y_1 \to S \times Y_1 \times Y_2$.

$$\begin{tikzcd}
X_1 \arrow[r, "v_0"] & Y_1 \\
S \times X_1 \times X_2 \arrow[u] \arrow[r, "{[v \ltimes f]}", "="'] \arrow[d] & S \times Y_1 \times Y_2 \arrow[u] \arrow[d] \\
S \times X_2 \arrow[r, "{[f]}", "="'] & S \times Y_2
\end{tikzcd}$$

On sets, as expected, this provides the left sequential product:

$$\forall x_1 \in X_1\,,\ \forall x_2 \in X_2\,,\ \forall s \in S\,,\ [f_1 \ltimes f_2](s, x_1, x_2) = \langle s_2, y_1, y_2 \rangle$$

where $[f_1](s, x_1) = \langle s_1, y_1 \rangle$ and $[f_2](s_1, x_2) = \langle s_2, y_2 \rangle$. The left sequential product $f_1 \ltimes f_2$ is usually distinct from the right sequential product $f_1 \rtimes f_2$.

4 Comparisons

The use of *strong monads* for dealing with computational effects has been introduced by Moggi for reasoning about programs (Moggi, 1989, 1991; Wadler, 1992). This has been generalized by Power and Robinson, who defined *Freyd-categories* and proved that a strong monad is equivalent to a Freyd-category with an adjunction (Power and Robinson, 1997; Power and Thielecke, 1999). Independently, Arrows have been introduced by Hughes for generalizing strong monads in Haskell (Hughes, 2000; Paterson, 2001); it was believed that Arrows are "essentially" equivalent to Freyd-categories, until Atkey proved that Arrows are in fact more general than Freyd categories (Atkey, 2008). In this section we directly compare each of these three frameworks to Cartesian effect categories: Freyd-categories in section 4.1, Arrows in section 4.2 and strong monads in section 4.3. Examples are considered in section 4.4.

4.1 Freyd-categories

In this section, it is proved that Cartesian effect categories are Freyd-categories (Power and Robinson, 1997; Power and Thielecke, 1999; Selinger, 2001). Let $|K|$ denote the smallest wide subcategory of K, made of the objects and identities of K.

Definition 4.1. A *binoidal category* is a category K together with two functors $\otimes : |K| \times K \to K$ and $\otimes : K \times |K| \to K$ which coincide on $|K|^2$ (so that the notation $\otimes$ is not ambiguous). The functors $\otimes$ can be extended as two graph homomorphisms $\ltimes_{\mathrm{Fr}}, \rtimes_{\mathrm{Fr}} : K^2 \to K$, as follows. For all $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ in K, let:

$$\begin{cases} f_1 \ltimes_{\mathrm{Fr}} f_2 = (\mathrm{id}_{Y_1} \otimes f_2) \circ (f_1 \otimes \mathrm{id}_{X_2}) : X_1 \otimes X_2 \to Y_1 \otimes Y_2 \\ f_1 \rtimes_{\mathrm{Fr}} f_2 = (f_1 \otimes \mathrm{id}_{Y_2}) \circ (\mathrm{id}_{X_1} \otimes f_2) : X_1 \otimes X_2 \to Y_1 \otimes Y_2 \end{cases}$$

A morphism $k_1 : X_1 \to Y_1$ is *central* if for all $f_2 : X_2 \to Y_2$, $k_1 \ltimes_{\mathrm{Fr}} f_2 = k_1 \rtimes_{\mathrm{Fr}} f_2$ and symmetrically $f_2 \ltimes_{\mathrm{Fr}} k_1 = f_2 \rtimes_{\mathrm{Fr}} k_1$. Let $t : \Phi \Rightarrow \Psi$ be a natural transformation between two functors $\Phi, \Psi : K' \to K$, then t is *central* if every component of t is central.

In theorem 16 the graph homomorphisms $\ltimes_{\mathrm{Fr}}, \rtimes_{\mathrm{Fr}}$ will be related to the sequential products $\ltimes, \rtimes$ from section 3. In the next definition, "natural" means natural in each component separately.

Definition 4.2. A *symmetric premonoidal category* is a binoidal category K together with an object I of K and central natural isomorphisms with components $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : X \otimes I \to X$, $r_X : I \otimes X \to X$ and $c_{X,Y} : X \otimes Y \to X \otimes Y$, subject to the usual coherence equations for symmetric monoidal categories (Mac Lane, 1997). Note that every symmetric monoidal category, hence every category with finite products, is symmetric premonoidal. A *symmetric premonoidal functor* between two symmetric premonoidal categories is a functor that preserves the partial functor $\otimes$, the object I and the natural isomorphisms a, l, r, c. It is *strict* if in addition it maps central morphisms to central morphisms. A *Freyd-category* is an identity-on-objects functor $J : C \to K$ where the category C has finite products, the category K is symmetric premonoidal and the functor J is strict symmetric premonoidal.

The following result states that every Cartesian effect category is a Freyd-category. It is an easy consequence of the results in section 3.

Theorem 16. *Let $C \subseteqq K$ be a Cartesian effect category. Let a, l, r, c be the natural isomorphisms on C defined as in section 3.6. Let $J : C \to K$ be the inclusion, let $\otimes : |K| \times K \to K$ and $\otimes : K \times |K| \to K$ be the restrictions of $\ltimes$ and $\rtimes$, respectively, and let $I = 1$. This forms a Freyd-category, where $\ltimes_{\mathrm{Fr}}$ and $\rtimes_{\mathrm{Fr}}$ coincide with $\ltimes$ and $\rtimes$, respectively.*

Proof. The graph homomorphisms $\otimes : |K| \times K \to K$ and $\otimes : K \times |K| \to K$ coincide on $|K|^2$, and they are functors by theorem 12, hence K with $\otimes$ is a binoidal category. Then, definitions 3.2 and 4.1 state that the graph homomorphisms $\ltimes_{\mathrm{Fr}}, \rtimes_{\mathrm{Fr}}$ are the sequential products $\ltimes, \rtimes$. It follows that both notions of central morphism (definitions 3.4 and 4.1) coincide. The fact that the transformations a, l, r, c are natural, in the sense of symmetric premonoidal categories, is an immediate consequence of theorem 14 (in fact for a it is lemma 13). Since all the components of a, l, r, c are defined from the symmetric monoidal category C, we know that they are isomorphisms and that they satisfy the coherence equations. In addition, since all pure morphisms are central by theorem 9, it follows that a, l, r, c are central. Hence K with $\otimes$, I and a, l, r, c is a symmetric premonoidal category. Clearly the inclusion functor $J : C \to K$ is symmetric premonoidal, and it is strict because of theorem 9. □

4.2 Arrows

In view of the similarities between Freyd-categories and Arrows, it can be guessed that every Cartesian effect category gives rise to an Arrow (Hughes, 2000; Paterson, 2001); this is stated in this section.

Definition 4.3. An *Arrow type* is a binary type constructor `A` of the form:

```
class Arrow A where
  arr :: (X → Y) → A X Y
  (>>>) :: A X Y → A Y Z → A X Z
  first :: A X Y → A (X, Z) (Y, Z)
```

satisfying the following equations:

$$
\begin{array}{lrcl}
(1) & \texttt{arr}\ \mathrm{id} \ggg f & = & f \\
(2) & f \ggg \texttt{arr}\ \mathrm{id} & = & f \\
(3) & (f \ggg g) \ggg h & = & f \ggg (g \ggg h) \\
(4) & \texttt{arr}\ (w.v) & = & \texttt{arr}\ v \ggg \texttt{arr}\ w \\
(5) & \texttt{first}\ (\texttt{arr}\ v) & = & \texttt{arr}\ (v \times \mathrm{id}) \\
(6) & \texttt{first}\ (f \ggg g) & = & \texttt{first}\ f \ggg \texttt{first}\ g \\
(7) & \texttt{first}\ f \ggg \texttt{arr}\ (\mathrm{id} \times v) & = & \texttt{arr}\ (\mathrm{id} \times v) \ggg \texttt{first}\ f \\
(8) & \texttt{first}\ f \ggg \texttt{arr}\ \texttt{fst} & = & \texttt{arr}\ \texttt{fst} \ggg f \\
(9) & \texttt{first}\ (\texttt{first}\ f) \ggg \texttt{arr}\ \texttt{assoc} & = & \texttt{arr}\ \texttt{assoc} \ggg \texttt{first}\ f
\end{array}
$$

where the functions $(\times)$, `fst` and `assoc` are defined as:
$(\times) :: (X \to X') \to (Y \to Y') \to (X, Y) \to (X', Y')$ such that $(f \times g)(x, y) = (f\,x, g\,y)$
$\texttt{fst} :: (X, Y) \to X$ such that $\texttt{fst}(x, y) = x$
$\texttt{assoc} :: ((X, Y), Z) \to (X, (Y, Z))$ such that $\texttt{assoc}((x, y), z) = (x, (y, z))$

Let C_H denote the category of Haskell types and ordinary functions, so that the Haskell notation $(\texttt{X} \to \texttt{Y})$ represents $C_H(X, Y)$, made of the Haskell ordinary functions from X to Y. An arrow `A` constructs a type $\texttt{A}\ X\ Y$ for all types X and Y. We slightly modify the definition of Arrows by allowing $(\texttt{X} \to \texttt{Y})$ to represent $C(X, Y)$ for any Cartesian category C and by requiring that $\texttt{A}\ X\ Y$ is a set rather than a type: more on this issue can be found in (Atkey, 2008). In addition, we use categorical notations instead of Haskell syntax. For this reason, from now on, for any Cartesian category C, an *Arrow A on C* associates to each objects X, Y of C a set $A(X, Y)$, together with three operations: $\texttt{arr} : C(X, Y) \to A(X, Y)$, $\ggg : A(X, Y) \to A(Y, Z) \to A(X, Z)$, $\texttt{first} : A(X, Y) \to A(X \times Z, Y \times Z)$, that satisfy the equations (1)–(9). Basically, the correspondence between a Cartesian effect category $C \subseteqq K$ and an Arrow A on C identifies $K(X, Y)$ with $A(X, Y)$ for all types X and Y. This is stated more precisely in proposition 17.

Proposition 17. *Every Cartesian effect category $C \subseteqq K$ gives rise to an Arrow A on C, according to the following table:*

Cartesian effect categories	*Arrows*
$K(X, Y)$	$A(X, Y)$
$C(X, Y) \subseteq K(X, Y)$	$\texttt{arr} : C(X, Y) \to A(X, Y)$
$f \mapsto (g \mapsto g \circ f)$	$\ggg : A(X, Y) \to A(Y, Z) \to A(X, Z)$
$f \mapsto f \times \mathrm{id}$	$\texttt{first} : A(X, Y) \to A(X \times Z, Y \times Z)$

Proof. The first and second line in the table say that $A(X, Y)$ is made of the morphisms from X to Y in K and that `arr` is the conversion from pure morphisms to arbitrary morphisms. The third and fourth lines say that $\ggg$ is the (reverse) composition of morphisms and that `first` is the right semi-pure product with the identity. Now we prove that A is an Arrow by translating each property (1)–(9) in terms of Cartesian effect categories and giving the argument for its proof. Note that `fst` is the common name for projections like $p_1, q_1, \ldots$ (in section 3) and that `assoc` is the natural isomorphism a as in section 3.6.

(1)	$f \circ \mathrm{id}$	$=$	f	identity in K
(2)	$\mathrm{id} \circ f$	$=$	f	identity in K
(3)	$h \circ (g \circ f)$	$=$	$(h \circ g) \circ f$	associativity in K
(4)	$w \circ v$ in C	$=$	$w \circ v$ in K	$C \subseteq K$ is a functor
(5)	$v \times \mathrm{id}$ in C	$=$	$v \times \mathrm{id}$ in K	$\times$ in K extends $\times$ in C
(6)	$(g \circ f) \times \mathrm{id}$	$=$	$(g \times \mathrm{id}) \circ (f \times \mathrm{id})$	lemma 10
(7)	$(\mathrm{id} \times v) \circ (f \times \mathrm{id})$	$=$	$(f \times \mathrm{id}) \circ (\mathrm{id} \times v)$	theorem 9
(8)	$q_1 \circ (f \times \mathrm{id})$	$=$	$f \circ p_1$	definition 3.1
(9)	$a \circ ((f \times \mathrm{id}) \times \mathrm{id})$	$=$	$(f \times \mathrm{id}) \circ a$	lemma 13

□

The Arrow combinators `second`, $(***)$ and $(\&\&\&)$ can be derived from `arr`, $(\ggg)$ and `first`, see e.g (Hughes, 2000; Paterson, 2001). The correspondence in proposition 17 is easily extended to these functions. The left pairing $\langle f_1, f_2 \rangle_l$ and the natural isomorphism c (corresponding to `swap`) are defined in section 3.3 and 3.6, respectively.

Cartesian effect categories	Arrows
$(id \times f) = c \circ (f \times id) \circ c$	$\texttt{second}\ f = \texttt{arr swap} \ggg \texttt{first}\ f \ggg \texttt{arr swap}$
$f_1 \ltimes f_2 = (\mathrm{id} \times f_2) \circ (f_1 \times \mathrm{id})$	$f_1 *** f_2 = \texttt{first}\ f_1 \ggg \texttt{second}\ f_2$
$\langle f_1, f_2 \rangle_l = (f_1 \ltimes f_2) \circ \langle \mathrm{id}, \mathrm{id} \rangle$	$f_1 \&\&\& f_2 = \texttt{arr}(\lambda x \to (x, x)) \ggg (f_1 *** f_2)$

For instance in (Hughes, 2000, §4.1) it is stated that &&& is not a categorical product since in general f_1 is different from $(f_1 \&\&\& f_2) \ggg$ `arr fst`: "*there is no reason to expect Haskell's pair type, &&&, to be a categorical product in the category of arrows, or indeed to expect any categorical product to exist*". We can state this more precisely in a Cartesian effect category, where $(f_1 \&\&\& f_2) \ggg$ `arr fst` corresponds to $q_1 \circ \langle f_1, f_2 \rangle_l$. Indeed, both morphisms are consistent: it follows from proposition 15 and pure substitution that $q_1 \circ \langle f_1, f_2 \rangle_l \blacktriangleleft f_1$.

4.3 Strong monads

Strong monads correspond to Freyd-categories $J : C \to K$ with a right adjoint for J (Power and Robinson, 1997), while Cartesian effect categories correspond to Freyd-categories with a sequential product (theorem 16). In this section, we give a condition which characterizes the strong monads such that the corresponding Freyd-category is a *weak* Cartesian effect category, which means that there are two graph homomorphisms $\ltimes : C \times K \to K$ and $\rtimes : K \times C \to K$ which satisfy the left and right semi-pure product property respectively, but which may not be unique.

We use the same notations as in remark 1. It has been seen in remark 2 that the effect of a morphism $f : X \to Y$ of K stands for $[\langle\,\rangle_Y \circ f] = M\langle\,\rangle_Y \circ [f] : X \to M1$ in C_0, so that in C_0:

$$\forall \varphi : X \to MY\,,\, \forall \varphi' : X \to MY'\,,\, \varphi \approx_0 \varphi' \iff M\langle\,\rangle_Y \circ \varphi = M\langle\,\rangle_{Y'} \circ \varphi'\,.$$

Let $\lhd$ be a consistency relation on $C \subseteqq K$, then the relation $\lhd_0$ in C_0 is defined by $[f] \lhd_0 [v] \iff f \lhd v$, or equivalently:

$$\forall \varphi, \varphi' : X \to MY \text{ in } C_0\,,\, \varphi \lhd_0 \varphi' \iff \exists v_0 : X \to Y \text{ in } C_0\,,\, (\varphi' = \eta_Y \circ v_0) \wedge (]\varphi[\lhd J(v_0))\,.$$

The pure substitution property of $\lhd$ (proposition 3) corresponds to the following *substitution property of* $\lhd_0$:

$$\forall v_0 : X \to Y\,,\, \forall w_0 : Y \to Z\,,\, \forall \psi : Y \to MZ\,,\, \psi \lhd_0 \eta_Z \circ w_0 \implies \psi \circ v_0 \lhd_0 \eta_Z \circ w_0 \circ v_0\,.$$

Now in addition let us assume that C_0, hence C, is Cartesian. In (Moggi, 1989), it is explained why the monad (M, μ, η) and the product $\times$ are not sufficient for dealing with several variables: there is a *type mismatch* from $Y_1 \times MY_2$ to $M(Y_1 \times Y_2)$. This issue is solved by adding a *strength*, i.e., a natural transformation t with components $t_{Y_1,Y_2} : Y_1 \times MY_2 \to M(Y_1 \times Y_2)$ satisfying four axioms (Moggi, 1989). One of these axioms is that for all X, $r_{MX} = Mr_X \circ t_{1,X} : 1 \times MX \to MX$, where the natural isomorphism r is made of the projections $r_X : 1 \times X \to X$ as in section 3.6. Let us assume that we are given a strength t for our monad. In K, let $v : X_1 \rightsquigarrow Y_1$ and $f : X_2 \to Y_2$; in order to form a kind of product of v and f, the usual method consists in composing in C_0 the product $v_0 \times [f] : X_1 \times X_2 \to Y_1 \times MY_2$ with the strength $t_{Y_1,Y_2} : Y_1 \times MY_2 \to M(Y_1 \times Y_2)$; we call this construction the *left Kleisli product*. The right Kleisli product is defined symmetrically.

Definition 4.4. For all $v = J(v_0) : X_1 \rightsquigarrow Y_1$ and $f : X_2 \to Y_2$ in K, the *left Kleisli product* of v and f in K is defined by:

$$[v \ltimes_{\mathrm{Kl}} f] = t_{Y_1,Y_2} \circ (v_0 \times [f]) : X_1 \times X_2 \to M(Y_1 \times Y_2) \text{ in } C_0\,.$$

Lemma 18. *The strength can be expressed as a left Kleisli product:*

$$]t_{Y_1,Y_2}[= \mathrm{id}_{Y_1} \ltimes_{\mathrm{Kl}}]\mathrm{id}_{MY_2}[\text{ in } K\,.$$

For all Y_1, Y_2, with projections $q_2 : Y_1 \times Y_2 \rightsquigarrow Y_2$ and $q_2' : Y_1 \times MY_2 \rightsquigarrow MY_2$:

$$q_2 \circ]t_{Y_1,Y_2}[= q_2' \text{ in } K\,.$$

Proof. In K, let $v = \mathrm{id}_{Y_1} : Y_1 \rightsquigarrow Y_1$ and $f =]\mathrm{id}_{MY_2}[: MY_2 \to Y_2$, so that $v_0 = \mathrm{id}_{Y_1}$ and $[f] = \mathrm{id}_{MY_2}$ in C_0. Then $v_0 \times [f] = \mathrm{id}_{Y_1 \times MY_2}$ so that $[v \ltimes_{\mathrm{Kl}} f] = t_{Y_1,Y_2}$, this is the first property. Now, for readability, we omit the subscript 0 for naming the projections in C_0. The result is equivalent to $Mq_2 \circ t_{Y_1,Y_2} = q_2'$ in C_0. The

projection q_2 can be decomposed as $q_2 = r_2 \circ (\langle\rangle_{Y_1} \times Y_2)$, where $r_2 = r_{Y_2} : 1 \times Y_2 \to Y_2$ is the projection. Hence on the one hand $Mq_2 = Mr_2 \circ M(\langle\rangle_{Y_1} \times Y_2)$, and on the other hand $q'_2 = r'_2 \circ (\langle\rangle_{Y_1} \times MY_2)$ where $r'_2 = r_{MY_2} : 1 \times MY_2 \to MY_2$ is the projection.

$$\begin{tikzcd}
Y_1 \times MY_2 \arrow[r, "t_{Y_1,Y_2}"] \arrow[d, "\langle\rangle \times M\mathrm{id}"'] \arrow[dd, bend right=60, "q'_2"'] \arrow[dr, phantom, "="] & M(Y_1 \times Y_2) \arrow[d, "M(\langle\rangle \times \mathrm{id})"] \arrow[dd, bend left=60, "Mq_2"] \\
1 \times MY_2 \arrow[r, "t_{1,Y_2}"] \arrow[d, "r'_2"'] \arrow[dr, phantom, "="] & M(1 \times Y_2) \arrow[d, "Mr_2"] \\
MY_2 \arrow[r, "\mathrm{id}"] & MY_2
\end{tikzcd}$$

In the previous diagram, the square on the top is commutative since t is natural, and the square on the bottom is commutative thanks to the property of the strength with respect to r. Hence the large square is commutative, and the result follows. □

Theorem 19. *Let C_0 be a Cartesian category with a strong monad (M, μ, η, t) and with a consistency relation $\lhd$ on $C \subseteqq K$. Then C_0 with the left and right Kleisli products is a weak Cartesian effect category if and only if for all Y_1, Y_2 (with the projections $q_1 : Y_1 \times Y_2 \to Y_1$ and $q'_1 : Y_1 \times MY_2 \to Y_1$):*

$$q_1 \circ]t_{Y_1,Y_2}[\lhd q'_1 \text{ in } K\,, \quad \text{or equivalently } \; Mq_1 \circ t_{Y_1,Y_2} \lhd_0 \eta_{Y_1} \circ q'_1 \text{ in } C_0\,.$$

If in addition $\forall \varphi, \varphi' : X \to M(Y_1 \times Y_2)$ in C_0,

$$(Mq_1 \circ \varphi \lhd \rhd_0 Mq_1 \circ \varphi') \wedge (Mq_2 \circ \varphi = Mq_2 \circ \varphi') \implies \varphi = \varphi' \text{ in } C_0\,,$$

then C_0 with the left and right Kleisli products is a Cartesian effect category.

Roughly speaking (i.e., forgetting the projections), this means that C_0 with the Kleisli products is a weak Cartesian effect category if and only if: *the strength of the monad is consistent with the identity.*

Proof. Let us consider the morphism $]t_{Y_1,Y_2}[$. By the first part of lemma 18 $]t_{Y_1,Y_2}[= \mathrm{id}_{Y_1} \ltimes_{\mathrm{Kl}}]\mathrm{id}_{MY_2}[$. Therefore, if the left Kleisli product does satisfy the left semi-pure product property, then $q_1 \circ]t_{Y_1,Y_2}[\lhd q'_1$. Now, let us assume that $q_1 \circ]t_{Y_1,Y_2}[\lhd q'_1$; this is illustrated below, together with $q_2 \circ]t_{Y_1,Y_2}[= q'_2$ (second part of lemma 18), first in K then in C_0:

$$\begin{tikzcd}
Y_1 \arrow[r, squiggly, "\mathrm{id}"] & Y_1 \\
Y_1 \times MY_2 \arrow[u, squiggly, "q'_1"] \arrow[r, "]t["] \arrow[d, squiggly, "q'_2"'] \arrow[ur, phantom, "\nabla"] \arrow[dr, phantom, "="] & Y_1 \times Y_2 \arrow[u, squiggly, "q_1"'] \arrow[d, squiggly, "q_2"] \\
MY_2 \arrow[r, "]\mathrm{id}["] & Y_2
\end{tikzcd}
\qquad
\begin{tikzcd}
Y_1 \arrow[r, "\eta"] & MY_1 \\
Y_1 \times MY_2 \arrow[u, "q'_1"] \arrow[r, "t"] \arrow[d, "q'_2"'] \arrow[ur, phantom, "\nabla_0"] \arrow[dr, phantom, "="] & M(Y_1 \times Y_2) \arrow[u, "Mq_1"'] \arrow[d, "Mq_2"] \\
MY_2 \arrow[r, "\mathrm{id}"] & MY_2
\end{tikzcd}$$

For any $v : X_1 \rightsquigarrow Y_1$ and $f : X_2 \to Y_2$, the morphism $v \ltimes_{\mathrm{Kl}} f$ in K is defined by $[v \ltimes_{\mathrm{Kl}} f] = t_{Y_1,Y_2} \circ (v_0 \times [f])$ in C_0. In the diagram below, in C_0, the left-hand side illustrates the binary product property of $v_0 \times [f]$ and

the right-hand side is as above.

$$
\begin{tikzcd}
X_1 \arrow[r, "v_0"] \arrow[rr, bend left, "{[v]}", "="'] & Y_1 \arrow[r, "\eta"] & MY_1 \\
X_1 \times X_2 \arrow[u, "p_1"] \arrow[r, "v_0 \times [f]"] \arrow[d, "p_2"'] \arrow[ur, phantom, "="] & Y_1 \times MY_2 \arrow[u, "q'_1"] \arrow[r, "t"] \arrow[d, "q'_2"'] \arrow[ur, phantom, "\nabla_0"] & M(Y_1 \times Y_2) \arrow[u, "Mq_1"'] \arrow[d, "Mq_2"] \\
X_2 \arrow[r, "{[f]}"] \arrow[rr, bend right, "="'] \arrow[ur, phantom, "="] & MY_2 \arrow[r, "\mathrm{id}"] \arrow[ur, phantom, "="] & MY_2
\end{tikzcd}
$$

It follows immediately from the bottom part of this diagram that $Mq_2 \circ [v \ltimes_{\mathrm{Kl}} f] = [f] \circ p_2$, which means that $q_2 \circ (v \ltimes_{\mathrm{Kl}} f) = f \circ p_2$ in K. Moreover, it follows from the top part, using the substitution property of $\lhd_0$, that $Mq_1 \circ [v \ltimes_{\mathrm{Kl}} f] \lhd_0 [v] \circ p_1$, which means that $q_1 \circ (v \ltimes_{\mathrm{Kl}} f) \lhd v \circ p_1$ in K. The left semi-pure product property is hence satisfied by $\ltimes_{\mathrm{Kl}}$.

Then the last part of the theorem follows immediately from remark 6. □

4.4 More examples of Cartesian effect categories

In this section we consider the effect categories in section 2.5 which are defined from a strong monad. In each example the strength is described, then it is easy to check that the conditions of theorem 19 are satisfied, so that the Kleisli category gives rise to a cartesian effect category with the Kleisli products as semi-pure products. However, for the monads of lists and of finite (multi)sets, the extended consistency relation is so weak that the sequential product properties (definition 3.5) are not sufficient for characterizing the sequential products.

Errors. The strength t_{X_1,X_2} is obtained by composing the isomorphism $X_1 \times (X_2 + E) \cong (X_1 \times X_2) + (X_1 \times E)$ with $\mathrm{id}_{X_1 \times X_2} + \sigma_{X_1} : (X_1 \times X_2) + (X_1 \times E) \to (X_1 \times X_2) + E$, where σ_{X_1} is the projection. The Kleisli products are semi-pure products from section 3.8.

Lists. The strength is such that for all $x_1 \in X_1$ and $\underline{x}_2 = (x_{2,1}, \ldots, x_{2,k}) \in \mathcal{L}(X_2)$, $t_{X_1,X_2}(x_1, \underline{x}_2) = (\langle x_1, x_{2,1} \rangle, \ldots, \langle x_1, x_{2,k} \rangle)$. It follows that $Mp_1 \circ t_{X_1,X_2}(x_1, \underline{x}_2) = (x_1)^k$ while $\eta_{X_1} \circ p'_1(x_1, \underline{x}_2) = (x_1)$. So, the left sequential product is:

$$\forall x_1 \in X_1 \,,\, \forall x_2 \in X_2 \,,\, (f_1 \ltimes f_2)(x_1, x_2) = (\langle y_1, z_1 \rangle, \ldots, \langle y_1, z_p \rangle, \ldots, \langle y_n, z_1 \rangle, \ldots, \langle y_n, z_p \rangle) \,,$$

where $f_1(x_1) = (y_1, \ldots, y_n)$ and $f_2(x_2) = (z_1, \ldots, z_p)$, so that there are non-central morphisms.

Finite (multi)sets. Finite multisets and finite sets have similar properties. For sets, the strength is such that for all $x_1 \in X_1$ and $\underline{x}_2 \in \mathcal{P}_{\mathrm{fin}}(X_2)$, $t_{X_1,X_2}(x_1, \underline{x}_2) = \{\langle x_1, x' \rangle \mid x' \in \underline{x}_2\}$, and both the left and the right sequential product are:

$$\forall x_1 \in X_1, \forall x_2 \in X_2, (f_1 \ltimes f_2)(x_1, x_2) = (f_1 \rtimes f_2)(x_1, x_2) = \{\langle y, z \rangle \mid y \in f_1(x_1) \wedge z \in f_2(x_2)\} \,.$$

5 Conclusion

This paper deals with the major issue of formalizing computational effects, especially while using multivariate functions. For this purpose, we have introduced several new features: first a *consistency* relation and the associated notion of *effect category*, then the *semi-pure* and *sequential* products for getting a *Cartesian effect category*. Thanks to the universal property of the semi-pure products, each Cartesian effect category is

endowed with a powerful tool for definitions and proofs. This has been used for proving that every Cartesian effect category is a Freyd-category and for giving conditions which ensure that a strong monad gives rise to a Cartesian effect category. We have studied several examples of effects, in each case we get a Cartesian effect category.

Since the notions of effect category and Cartesian effect category are new, there is still a large amount of work to do in order to study their applications and their limitations. For instance, in order to define some kind of closure, one could try to generalize the results of (Curien and Obtulowitz, 1989) on partiality to other effects. Further investigations include: enhancing the comparison with (Moggi, 1995) in order to clarify the relations between Cartesian effect categories and evaluation logic; fitting more examples in our framework (e.g. continuations). In addition, the issue of combining effects, as in (Hyland et al., 2006), might be revisited from the point of view of effect categories.

Acknowledgments

The authors would like to thank Eugenio Moggi for pointing out the papers (Curien and Obtulowitz, 1989) and (Moggi, 1995).

References

Atkey, R., 2008. What is a categorical model of arrows? In: Mathematically Structured Functional Programming (MSFP'08).

Carboni, A., Lack, S., Walters, R., 1993. Introduction to extensive and distributive categories. Journal of Pure and Applied Algebra 84, 145–158.

Curien, P.-L., Obtulowitz, A., 1989. Partiality, cartesian closedness and toposes. Information and Computation 80, 50–95.

Dumas, J.-G., Duval, D., Reynaud, J.-C., 2007. Sequential products in effect categories.
URL `http://arxiv.org/abs/0707.1432`

Duval, D., Reynaud, J.-C., 2005. Diagrammatic logic and exceptions: an introduction. In: Dagstuhl Seminar Proceedings - MAP05, Mathematics, Algorithms, Proofs.
URL `http://www.dagstuhl.de/05021/Materials/`

Heunen, C., Jacobs, B., 2006. Arrows, like monads, are monoids. Electronic Notes in Theoretical Computer Science 158, 219–236.

Hughes, J., 2000. Generalising monads to arrows. Science of Computer Programming 37, 67–111.

Hyland, M., Plotkin, G., Power, J., 2006. Combining effects: Sum and tensor. Theoretical Computer Science 357, 70–99.

Mac Lane, S., 1997. Categories for the Working Mathematician, 2nd edition. Vol. 5 of Graduate Texts in Mathematics. Springer Verlag.

Moggi, E., 1989. Computational lambda-calculus and monads. In: Logic In Computer Science (LICS). IEEE Press, pp. 14–23.

Moggi, E., 1991. Notions of computation and monads. Information and Computation 93, 55–92.

Moggi, E., 1995. A semantics for evaluation logic. Fundamenta Informaticae 22, 117–152.

Paterson, R., 2001. A new notation for arrows. In: International Conference on Functional Programming. ACM, pp. 229–240.

Power, J., Robinson, E., 1997. Premonoidal categories and notions of computation. Mathematical Structures in Computer Science 7, 453–468.

Power, J., Thielecke, H., 1999. Closed Freyd- and κ-categories. In: Int. Coll. on Automata, Languages and Programming (ICALP'99). Vol. 1644 of LNCS. Springer Verlag, pp. 625–634.

Selinger, P., 2001. Control categories and duality: on the categorical semantics of the lambda-mu calculus. Mathematical Structures in Computer Science 11, 207–260.

Wadler, P., 1992. The essence of functional programming. In: 19th Symposium on Principles of Programming Languages. ACM.