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GEOMETRICALLY RATIONAL REAL CONIC BUNDLES AND VERY TRANSITIVE ACTIONS

JÉRÉMY BLANC AND FRÉDÉRIC MANGOLTE

ABSTRACT. We study very transitive groups of automorphisms of real geometrically rational surfaces with applications to the classification of real algebraic models of compact surfaces. We give an insight into the geometry of real parts which is a geometry between biregular and birational geometry's, and show several surprising facts about it.

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1. INTRODUCTION

The aim of this paper is to study the action of birational maps on the set of real points of a real algebraic variety. It is worthwhile to point out a common terminological source of confusion about the meaning of what is a *real algebraic variety*, see also the enlightening introduction of [Kol01]. From the point of view of general algebraic geometry, a real variety X is a variety defined over the real numbers, and a morphism is understood to be defined over all the geometric points. But in most of the texts in real algebraic geometry, the algebraic structure considered corresponds to the one of a neighbourhood of the real points $X(\mathbb{R})$ in the whole complex variety, which is rather the structure of a germ of an algebraic variety defined over \mathbb{R} .

From this point of view it is natural to view $X(\mathbb{R})$ as a compact submanifold of \mathbb{R}^n defined by real polynomial equations, where n is some natural integer. Likely, it is natural to say that a map $\psi: X(\mathbb{R}) \rightarrow Y(\mathbb{R})$ is an *isomorphism* if ψ is induced by a birational map $\Psi: X \dashrightarrow Y$ such that Ψ (respectively Ψ^{-1}) is regular at any point of $X(\mathbb{R})$ (respectively of $Y(\mathbb{R})$). In particular, $\psi: X(\mathbb{R}) \rightarrow Y(\mathbb{R})$ is a diffeomorphism. This notion corresponds to the notion of biregular maps defined in [BCR98, 3.2.6] for the structure of real algebraic variety commonly used in the realm of real algebraic geometry.

Conversely, let M be a compact \mathcal{C}^∞ -manifold. According to the Nash-Tognoli's theorem [Tog73], every such M is diffeomorphic to a nonsingular real algebraic subset of \mathbb{R}^m for some m . Taking the Zariski closure in \mathbb{P}^m and applying Hironaka's resolution of singularities [Hir64], we obtain that M is in fact diffeomorphic to the set of real points $X(\mathbb{R})$ of a nonsingular projective algebraic variety X defined over \mathbb{R} . Such a variety X is called an *algebraic model* of M . A natural question is, given M , to classify the algebraic models of M up to isomorphism.

There are several recent results about the question of algebraic models and their automorphism groups [BH07, HM07, HM08, KM08]. For example, when M is 2-dimensional, and admits a real rational algebraic model, then this rational algebraic model is unique [BH07]. Otherwise speaking, if X and Y are two rational real algebraic surfaces, then $X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic if and only if there are homeomorphic. Moreover, in [HM07], the proof has been shortened by showing that the automorphism group $\text{Aut}(X(\mathbb{R}))$ has a very transitive action on $X(\mathbb{R})$ for any rational real algebraic surface X .

Definition 1.0. Let G be a topological group acting continuously on a topological space M . We say that two n -tuples of distinct points (p_1, \dots, p_n) and (q_1, \dots, q_n) are *compatible* if there exists a homeomorphism $\psi: M \rightarrow M$ such that $\psi(p_i) = q_i$ for each i . Then the action of G on M is said *very transitive* if for any pair of compatible n -tuples of points (p_1, \dots, p_n) and (q_1, \dots, q_n) of M , there exists an element $g \in G$ such that $g(p_i) = q_i$ for each i .

Recall that a real projective surface is rational if it is birationally equivalent to the real projective plane, and that it is geometrically rational if its complexification is birationally equivalent to the complex projective plane. The main goal of this paper is to complete the classification of real algebraic models of surfaces by dealing with the case of geometrically rational real projective surfaces which are non-rational.

To distinguish between the Zariski topology and the topology induced by the embedding of $X(\mathbb{R})$ as a topological submanifold of \mathbb{R}^n , we will call the later the *Euclidean topology*. In the sequel, topological notions like connectedness or compactness will always refer to the Euclidean topology. We will denote by $\#M$ the number of connected components of a compact manifold M .

The number of connected components is a birational invariant. In particular, if X is a rational projective surface, $X(\mathbb{R})$ is connected.

There is one case which shares many features with the rational case.

Theorem 1.1. *Let X be a nonsingular geometrically rational real projective surface, and assume that $\#X(\mathbb{R}) = 2$. Then the action of the group $\text{Aut}(X(\mathbb{R}))$ on $X(\mathbb{R})$ is very transitive.*

When $\#X(\mathbb{R}) \geq 3$, the action of the group $\text{Aut}(X(\mathbb{R}))$ is in general far from very transitive except for the case with three components for which a slight weakening of the definition works.

Let (p_1, \dots, p_n) and (q_1, \dots, q_n) be two compatible n -tuples of points of M such that, for each i , p_i and q_i belong to the same connected component of M . We say that G is very transitive *on connected components* if for any such a pair, there exists an element $g \in G$ such that $g(p_i) = q_i$ for each i .

Theorem 1.2. *Let X be a nonsingular geometrically rational real projective surface, and assume that $\#X(\mathbb{R}) = 3$. Then $\text{Aut}(X(\mathbb{R}))$ is very transitive on connected components.*

When $\#X(\mathbb{R}) > 3$, either any element of $\text{Aut}(X(\mathbb{R}))$ preserves a conic bundle structure (Theorem 6.1), or $\text{Aut}(X(\mathbb{R}))$ is countable (Corollary 3.9); thus $\text{Aut}(X(\mathbb{R}))$ is not even 1-transitive on connected components. Indeed, we prove the following result in Section 9.

Corollary 1.3. *Let X be a nonsingular real projective surface. Then $\text{Aut}(X(\mathbb{R}))$ has a very transitive action on $X(\mathbb{R})$ if and only if the following two conditions hold:*

- (1) X is geometrically rational, and
- (2) $\#X(\mathbb{R}) \leq 2$ or $\#X(\mathbb{R}) = 3$ for a few particular X .

These results apply to the classification of algebraic models of real surfaces.

Theorem 1.4. *Let X, Y be two nonsingular geometrically rational real projective surfaces, and assume that $\#X(\mathbb{R}) \leq 2$. Then $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$ if and only if X is birational to Y and $X(\mathbb{R})$ is homeomorphic to $Y(\mathbb{R})$. This is false in general when $\#X(\mathbb{R}) \geq 3$.*

Recall that a nonsingular projective surface is minimal if any birational morphism to a nonsingular surface is an isomorphism. We have the following rigidity result about minimal geometrically rational real surfaces.

Theorem 1.5. *Let X and Y be two minimal geometrically rational real surfaces, and assume that either X or Y is non-rational. Then, the following are equivalent:*

- (1) X and Y are birational.
- (2) $X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic.

In the course of this work, we have classified the birational classes of real conic bundles and corrected an error contained in the literature (Theorem 6.1). As a consequence, we get that the only geometrically rational surfaces $X(\mathbb{R})$ for which equivalence by homeomorphism implies equivalence by isomorphism are the connected ones. In particular, this gives a converse statement to [BH07, Corollary 8.1].

Corollary 1.6. *Let M be a compact C^∞ -surface. Then M admits a unique geometrically rational model if and only if the following two conditions hold:*

- (1) M is connected, and
- (2) M is non-orientable or M is orientable with genus $g(M) \leq 1$.

For M orientable with $g(M) > 1$, there is no result close to some unicity. Thus we can ask what should be the simplest algebraic model for such an M . This question is studied in the forthcoming paper [HM09].

Let us cite some recent works concerning automorphisms of real projective surfaces.

In [RV05], it is proved that $\text{Aut}(\mathbb{P}^2(\mathbb{R}))$ is generated by linear automorphisms and certain real algebraic automorphisms of degree 5.

In [KM08] it is proved that for any rational surface X , $\text{Aut}(X(\mathbb{R})) \subset \text{Diff}(X(\mathbb{R}))$ is dense for the strong topology. For non geometrically rational surfaces, the group $\text{Aut}(X(\mathbb{R}))$ cannot be dense; as for most of the non-rational geometrically rational surfaces. The cited paper left open the question of density only for some geometrically rational surfaces with 2, 3, 4 or 5 connected components. One by-product of our results is the non-density for most of the surfaces with at least 3 connected components, see Proposition 9.2.

The paper [HM08] is devoted to the study of very transitive actions and unicity of models for some kind of singular rational surfaces.

Outline of the article. After giving some notation that will be used in the article in Section 2, we recall in Section 3 the classification of minimal geometrically rational real surfaces.

Section 4, which constitutes the technical heart of the paper, is devoted to conic bundles, and especially to the minimal ones. We provide representative elements of isomorphism classes, and explain the links between the conic bundles.

In Section 5, we investigate real surfaces which admit two conic bundles. We show in particular that these are del Pezzo surfaces, and give some description of the possible conic bundles on these surfaces. Section 6 is devoted to the proof of Theorem 1.5. We firstly correct an existing inaccuracy in the literature, by proving that if two surfaces admitting a conic bundle structure are birational, the birational map may be chosen so that it preserves the conic bundle structures. Then, we strengthen this result to isomorphisms between real parts when the surfaces are minimal, before proving Theorem 1.5.

In Section 7, we prove that if the real part of a minimal geometrically rational has 2 or 3 connected components, its automorphism group is very transitive on connected components. In Section 8, we do the same work with non-minimal surfaces. We show how to separate infinitely near points, which is certainly one of the most counter-intuitive behaviour of our geometry, and was first observed in [BH07] for rational surfaces. We also obtain the unicity of models in many cases.

Then, in Section 9, we use all the results of the previous sections, to give the proof of the main results stated in the introduction (except Theorem 1.5, proved in Section 6).

2. NOTATION

In the sequel, by a variety we will mean an algebraic variety, which may be real or complex (i.e. defined over \mathbb{R} or \mathbb{C}). If the converse is not expressively stated all our varieties will be projective and all our surfaces will be nonsingular and geometrically rational (i.e. rational over \mathbb{C}).

Recall that a real variety X may be identified with a pair (S, σ) , where S is a complex variety and σ is an anti-holomorphic involution on S ; by abuse of notation we will write $X = (S, \sigma)$. Then, $S(\mathbb{C}) = X(\mathbb{C})$ denotes the set of complex points of the variety, and $X(\mathbb{R}) = S(\mathbb{C})^\sigma$ is the set of real points. A point $p \in X$ may be real (if it belongs to $X(\mathbb{R})$), or imaginary (if it belongs to $X(\mathbb{C}) \setminus X(\mathbb{R})$). If $X(\mathbb{R})$ is non empty (which will be the case for all our surfaces), then $\text{Pic}(X) \cong \text{Pic}(S)^\sigma$, [Sil89, I.(4.5)]. As we work only with regular surfaces (i.e. $q(X) = q(S) = 0$), the Picard group is isomorphic to the Néron-Severi group, and $\rho(S)$ and $\rho(X)$ will denote respectively the rank of $\text{Pic}(S)$ and $\text{Pic}(X)$. Recall that $\rho(X) \leq \rho(S)$. We denote by $K_X \in \text{Pic}(X)$ the canonical class, which may be identified with K_S . The intersection of two divisors of $\text{Pic}(S)$ or $\text{Pic}(X)$ will always denote the usual intersection in $\text{Pic}(S)$.

We will use the classical notions of morphisms, rational maps, isomorphisms and automorphisms between real or complex varieties. Moreover, if X_1 and X_2 are two real varieties, an isomorphism *between real parts* $X_1(\mathbb{R}) \xrightarrow{\psi} X_2(\mathbb{R})$ is a birational map $\psi: X_1 \dashrightarrow X_2$ such that ψ (respectively ψ^{-1}) is regular at any point of $X_1(\mathbb{R})$ (respectively of $X_2(\mathbb{R})$). This endows $X_1(\mathbb{R})$ with a structure of a germ of algebraic variety defined over \mathbb{R} (as in [BCR98, 3.2.6]), whereas the structure of X_1 is those of an algebraic variety.

Considering geometry's on algebraic variety defined over \mathbb{R} , this notion of isomorphism between real parts gives an intermediate geometry in between the biregular geometry and the birational geometry. For example, let $\alpha: X_1(\mathbb{R}) \rightarrow X_2(\mathbb{R})$ be an isomorphism, and $\varepsilon: Y_1 \dashrightarrow X_1$, $\eta: Y_2 \dashrightarrow X_2$ be two birational maps. Then the map $\psi := \varepsilon^{-1}\alpha\eta$ is a well-defined birational map. Then ψ can be an isomorphism $Y_1(\mathbb{R}) \rightarrow Y_2(\mathbb{R})$ even if nor ε , nor η is an isomorphism between real parts. In the same vein, let $\alpha: X_1(\mathbb{R}) \rightarrow X_2(\mathbb{R})$ be an isomorphism, and let $\eta_1: Y_1 \rightarrow X_1$ and $\eta_2: Y_2 \rightarrow X_2$ be two birational morphisms which are the blow-ups of only real points (which may be proper or infinitely near points of X_1 and X_2). If α sends the points blown-up by η_1 on the points blown-up by η_2 , then $\beta = (\eta_2)^{-1}\alpha\eta_1: Y_1(\mathbb{R}) \rightarrow Y_2(\mathbb{R})$ is an isomorphism.

Using Aut and Bir to denote respectively the group of automorphisms and birational self-maps of a variety, we have the following inclusions for the groups associated to $X = (S, \sigma)$:

$$\begin{array}{ccc} \text{Aut}(S) & \subset & \text{Bir}(S) \\ \cup & & \cup \\ \text{Aut}(X) & \subset & \text{Aut}(X(\mathbb{R})) \subset \text{Bir}(X) . \end{array}$$

By \mathbb{P}^n we mean the projective n -space, which may be complex or real depending on the context. It is unique as a complex variety – written $\mathbb{P}_{\mathbb{C}}^n$. However, as a real variety, \mathbb{P}^n may either be $\mathbb{P}_{\mathbb{C}}^n$ endowed with the standard anti-holomorphic involution, written $\mathbb{P}_{\mathbb{R}}^n$, or only when n is odd, $\mathbb{P}_{\mathbb{C}}^n$ with a special involution with no real points, written $(\mathbb{P}^n, \emptyset)$. To lighten notation, and since we never speak about $(\mathbb{P}^1, \emptyset)(\mathbb{R})$ we write $\mathbb{P}^1(\mathbb{R})$ for $\mathbb{P}_{\mathbb{R}}^1(\mathbb{R})$.

3. MINIMAL SURFACES AND MINIMAL CONIC BUNDLES

The aim of this section is to reduce our study of geometrically rational surfaces to surfaces which admits a minimal conic bundle structure.

Definition 3.1. A surface X is said to be *minimal* if any birational morphism from X to a (nonsingular) surface is an isomorphism.

If X is real, this is equivalent to say that there is no real (-1) -curve and no pair of disjoint conjugate imaginary (-1) -curves on X .

Let us precise the notion of conic bundle. Since we only deal with geometrically rational surfaces, the basis of our conic bundles is always geometrically rational.

Definition 3.2. A *conic bundle* is a pair (X, π) where X is a surface and π is a morphism $X \rightarrow \mathbb{P}^1$, where any fibre of π is isomorphic to a plane conic.

Note that if (X, π) is complex, a general fibre of π is isomorphic to $\mathbb{P}_{\mathbb{C}}^1$, and a singular fibre consists of the union of two intersecting lines which are (-1) -curves of X (the double line is not allowed, the surface X being nonsingular). If (X, π) is real, a fibre over a real point of \mathbb{P}^1 may be isomorphic to $\mathbb{P}_{\mathbb{R}}^1$ or to $(\mathbb{P}^1, \emptyset)$, or to a singular fibre with two irreducible components which can be two real components, or two conjugated imaginary components intersecting in 1 real point.

We will assume in the sequel that if X is real, then the basis is $\mathbb{P}_{\mathbb{R}}^1$ (and not $(\mathbb{P}^1, \emptyset)$). This avoids certain conic bundles with no real points.

Definition 3.3. If (X, π) and (X', π') are two conic bundles, a *birational map of conic bundles* $\psi: (X, \pi) \dashrightarrow (X', \pi')$ is a birational map $\psi: X \dashrightarrow X'$ such that there exist an automorphism α of \mathbb{P}^1 with $\pi' \circ \psi = \alpha \circ \pi$.

This notion specialises to birational morphisms and to automorphisms of conic bundles. We denote by $\text{Aut}(X, \pi)$ (respectively $\text{Bir}(X, \pi)$) the group of automorphisms (respectively birational self-maps) of the conic bundle (X, π) . Observe that $\text{Aut}(X, \pi) = \text{Aut}(X) \cap \text{Bir}(X, \pi)$. Similarly, when (X, π) is real we define by $\text{Aut}(X(\mathbb{R}), \pi)$ the group $\text{Aut}(X(\mathbb{R})) \cap \text{Bir}(X, \pi)$.

Definition 3.4. A conic bundle (X, π) is said to be *minimal* if any birational morphism of conic bundles $(X, \pi) \rightarrow (X', \pi')$ is an isomorphism.

Note that a real conic bundle (X, π) is minimal if and only if the two irreducible components of any real singular fibre of π are imaginary. Compare to the complex case where (X, π) is minimal if and only if there is no singular fibre.

Any geometrically rational real surface Y is obtained by a finite sequence of blow-ups (centred at a real point or at a pair of conjugate imaginary points) from a minimal real surface X . The following classical theorem describes the possibilities for the surface X .

Recall that a surface X is a del Pezzo surface if the anti-canonical divisor $-K_X$ is ample. The same definition applies for X real or complex.

Theorem 3.5. *If X is a minimal geometrically rational real surface such that $X(\mathbb{R}) \neq \emptyset$, then one and exactly one of the following holds:*

- (1) *X is rational: it is isomorphic to $\mathbb{P}_{\mathbb{R}}^2$, to the quadric $Q_0 := \{(x : y : z : t) \in \mathbb{P}_{\mathbb{R}}^3 \mid x^2 + y^2 + z^2 = t^2\}$, or to a real Hirzebruch surface \mathbb{F}_n , $n \neq 1$;*
- (2) *X is a del Pezzo surface of degree 1 or 2 with $\rho(X) = 1$;*
- (3) *there exists a minimal conic bundle structure $\pi: X \rightarrow \mathbb{P}^1$ with an even number of singular fibres $2r \geq 4$. Moreover, $\rho(X) = 2$.*

Remark 3.6. If (S, σ) is a minimal geometrically rational real surface such that $S^\sigma = \emptyset$, then S is an Hirzebruch surface of even index.

Proof. Follows from the work of Comessatti [Com12], (see also [Mani67], [Isk79], [Sil89, Chap. V], or [Kol97]). \square

Proposition 3.7 (Topology of the real part). *In each case of the former theorem, we have:*

- (1) *X is rational if and only if $X(\mathbb{R})$ is connected. When X is moreover minimal, then $X(\mathbb{R})$ is homeomorphic to one of the following: the real projective plane, the sphere, the torus, or the Klein bottle.*
- (2) *When X is a minimal del Pezzo surface of degree 1, it satisfies $\rho(X) = 1$, and $X(\mathbb{R})$ is the disjoint union of one real projective plane and 4 spheres. If X is a minimal del Pezzo surface of degree 2 with $\rho(X) = 1$, then $X(\mathbb{R})$ is the disjoint union of 4 spheres.*
- (3) *If X is non-rational and is endowed with a minimal conic bundle with $2r$ singular fibres, then $X(\mathbb{R})$ is the disjoint union of r spheres, $r \geq 2$.*

Proof. For the first assertion, see [Sil89, Corollary VI(6.5)], for the other ones, see e.g. [Sil89, Chap. V] or [Kol97]. \square

The proofs of Corollary 1.3 and of Theorem 1.5 will split into the cases listed in Theorem 3.5. The rational case is treated in [HM07]. The next proposition states the case when X is a minimal del Pezzo surface with $\rho = 1$. The remaining part

of the paper is mainly devoted to the case when X is endowed with a minimal real conic bundle.

Proposition 3.8. *Let X, Y be two minimal geometrically rational real surfaces. Assume that X is not rational and satisfy $\rho(X) = 1$ (but $\rho(Y)$ may be equal to 1 or 2).*

- (1) *If X is a del Pezzo surface of degree 1, then any birational map $X \dashrightarrow Y$ is an isomorphism. In particular,*

$$\mathrm{Aut}(X) = \mathrm{Aut}(X(\mathbb{R})) = \mathrm{Bir}(X).$$

- (2) *If X is a del Pezzo surface of degree 2, X is birational to Y if and only if X is isomorphic to Y . Moreover, all the base-points of the elements of $\mathrm{Bir}(X)$ are real, and*

$$\mathrm{Aut}(X) = \mathrm{Aut}(X(\mathbb{R})) \subsetneq \mathrm{Bir}(X).$$

Proof. Assume the existence of a birational map $\psi: X \dashrightarrow Y$. If ψ is not an isomorphism, we decompose ψ into elementary links

$$X = X_0 \xrightarrow{\psi_1} X_1 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{n-1}} X_{n-1} \xrightarrow{\psi_n} X_n = Y$$

as in [Isk96, Theorem 2.5]. It follows from the description of the links of [Isk96, Theorem 2.6] that for any link $\psi_i: X_{i-1} \dashrightarrow X_i$, X_{i-1} and X_i are isomorphic del Pezzo surfaces of degree 2, and that ψ_i is equal to $\beta\eta\alpha\eta^{-1}$, where η is the blow-up $X' \rightarrow X_{i-1}$ of a real point of X_{i-1} , X' is a del Pezzo surface of degree 1, $\alpha \in \mathrm{Aut}(X')$ is the Bertini involution of the surface, and $\beta: X_{i+1} \rightarrow X_i$ is an isomorphism.

Therefore, Y is isomorphic to X . Moreover, if X has degree 1, ψ is an isomorphism. If X has degree 2, ψ is decomposed into conjugates of Bertini involutions, so each of its base-points is real. This proves that if $\psi \in \mathrm{Aut}(X(\mathbb{R}))$ then $\psi \in \mathrm{Aut}(X)$. Furthermore, conjugates of Bertini involutions belong to $\mathrm{Bir}(X)$ but not to $\mathrm{Aut}(X) = \mathrm{Aut}(X(\mathbb{R}))$. \square

Corollary 3.9. *Let X_0 be a minimal non-rational geometrically rational real surface with $\rho(X_0) = 1$, and let $\eta: X \rightarrow X_0$ be a birational morphism.*

Then, $\mathrm{Aut}(X(\mathbb{R}))$ is countable. Moreover, if X_0 is a del Pezzo surface of degree 1, then $\mathrm{Aut}(X(\mathbb{R}))$ is finite.

Proof. Without changing the isomorphism class of $X(\mathbb{R})$ we may assume that η is the blow-up of only real points (which may belong to X_0 as proper or infinitely near points). Since any base-point of any element of $\mathrm{Bir}(X_0)$ is real (Proposition 3.8), the same is true for any element of $\mathrm{Bir}(X)$. In particular, $\mathrm{Aut}(X(\mathbb{R})) = \mathrm{Aut}(X)$. The group $\mathrm{Aut}(X)$ acts on $\mathrm{Pic}(X) \cong \mathbb{Z}^n$, where $n = \rho(X) \geq 1$. This action gives rise to an homomorphism $\theta: \mathrm{Aut}(X) \rightarrow \mathrm{GL}(n, \mathbb{Z})$. Let us prove that θ is injective. Indeed, if $\alpha \in \mathrm{Ker}(\theta)$, then α is conjugate by η to an element of $\alpha_0 \in \mathrm{Aut}(X_0)$ which acts trivially on $\mathrm{Pic}(X_0)$. Writing S_0 the complex surface obtaining by forgetting the real structure of X_0 , S_0 is the blow-up of 7 or 8 points in general position of $\mathbb{P}_{\mathbb{C}}^2$. Thus $\alpha_0 \in \mathrm{Aut}(X_0) \subset \mathrm{Aut}(S_0)$ is the lift of an automorphism of $\mathbb{P}_{\mathbb{C}}^2$ which fixes 7 or 8 points, no 3 collinear, hence is the identity.

The morphism θ is injective, and this shows that $\mathrm{Aut}(X(\mathbb{R})) = \mathrm{Aut}(X)$ is countable. Moreover, if X_0 is a del Pezzo surface of degree 1, then $\mathrm{Bir}(X_0) = \mathrm{Aut}(X_0)$ (by Proposition 3.8). Since $\mathrm{Aut}(X_0)$ is finite, $\mathrm{Aut}(X(\mathbb{R})) \subset \mathrm{Bir}(X)$ is also finite. \square

4. EXCEPTIONAL CONIC BUNDLES

Definition 4.1. If (X, π) is a real conic bundle, $I(X, \pi) \subset \mathbb{P}^1(\mathbb{R})$ denotes the image by π of the set $X(\mathbb{R})$ of real points of X .

It is well-known that $I(X, \pi)$ is the union of a finite number of intervals (which may be \emptyset or $\mathbb{P}^1(\mathbb{R})$), and that it determines the birational class of (X, π) . (In the next section, we will prove that in fact $I(X, \pi)$ determines the birational class of X , and thus that $I(X)$ is well-defined.)

In this section, we study the real conic bundles, and especially the minimal ones. We prove that $I(X, \pi)$ also determines the equivalence class of $(X(\mathbb{R}), \pi)$ among the minimal conic bundles, and give the proof of Theorem 1.5 in the case of conic bundles (Corollary 4.17). Doing this, we will give proofs of the well-known facts on $I(X, \pi)$ cited above.

The following lemma gives some useful information about sections of real conic bundles. It is strongly inspired from the complex analogue, that can be found for example in [Bla07, Section 3].

Lemma 4.2. *Let (X, π) be a minimal real conic bundle.*

Then, the number of singular fibres of π is even, and is equal to $8 - (K_X)^2$. Denoting by $2r \geq 0$ this number, the following hold:

- (1) *if $r > 0$, then π admits no real section;*
- (2) *if s is a section distinct from its conjugate \bar{s} , then $s^2 \geq -r$ and $s \cdot \bar{s} = s^2 + r$.*

Proof. Let m be the number of singular fibres of π and denote by (S, π) the complex conic bundle obtained from (X, π) by forgetting the real structure of X .

If $m > 0$, any section of π intersects exactly one component of each singular fibre. Since (X, π) is minimal, no such component can be real, thus there is no real section.

Let s be a section, and \bar{s} be its conjugate, and assume that $s \neq \bar{s}$ (which is always true if $m > 0$). Let us denote by η the birational morphism that contracts one component in any singular fibre of π , which is the one which intersects \bar{s} . Then, η is defined over \mathbb{R} if and only if $m = 0$ (and in this case it is an isomorphism). Moreover, $\eta: (S, \pi) \rightarrow \mathbb{F}_n$ is a birational morphism of conic bundles, for some integer $n \in \mathbb{N}$. Denote by $E \in \text{Pic}(\mathbb{F}_n)$ the divisor of a section with self-intersection $-n$ (which is unique if $n \neq 0$), and by f the divisor of a general fibre, the curves $\eta(s)$ and $\eta(\bar{s})$ are respectively equivalent to $E + af$ and $E + bf$, for some integers a and b .

We compute

$$\begin{aligned} \eta(s)^2 &= (E + af)^2 &= -n + 2a, \\ \eta(\bar{s})^2 &= (E + bf)^2 &= -n + 2b, \\ \eta(s) \cdot \eta(\bar{s}) &= (E + af) \cdot (E + bf) &= -n + a + b, \end{aligned}$$

and find that $\eta(s) \cdot \eta(\bar{s}) = 1/2 \cdot (\eta(s)^2 + \eta(\bar{s})^2)$.

Because the m irreducible curves contracted by η intersect \bar{s} transversally and do not intersect s , we have $\eta(s)^2 = s^2$, $\eta(\bar{s})^2 = \bar{s}^2 + m$ and $\eta(s) \cdot \eta(\bar{s}) = s \cdot \bar{s}$. Observe that $\bar{s}^2 = s^2$, so $s \cdot \bar{s} = 1/2 \cdot (2s^2 + m)$. This implies that m is even, equal to $2r$ for some non-negative integer r and that $s \cdot \bar{s} = s^2 + r$. Since $s \neq \bar{s}$, the number $s \cdot \bar{s}$ is non-negative, so $s^2 \geq -r$.

Since η contracts exactly m curves and $(K_{\mathbb{F}_n})^2 = 8$, m equals $8 - (K_X)^2$. \square

Denoting by $-n$ the minimal self-intersection of the sections in a minimal real conic bundle with $2r$ singular fibres, the above lemma shows that $n \leq r$. According to [Bla07, Lemma 3.3], we also have $1 \leq n$, and S is obtained by blowing up some points on \mathbb{F}_1 or \mathbb{F}_0 . The simplest cases of conic bundles, which are the most general, are those for which n is equal to 1 (in particular when X is a del Pezzo surface). The most special cases are when $n = r$, which are interesting as for any given union of a finite number of intervals V , there is only one such conic bundle (X, π) such that $I(X, \pi) = V$ (this will be proved in Lemma 4.8 below).

Definition 4.3. A conic bundle (X, π) is called an *exceptional conic bundle* if π admits a section of self-intersection $-r$ where $2r$ is the number of singular fibres.

This definition was already introduced in [DI06] and [Bla08]. If (S, π) is a minimal exceptional complex conic bundle with at least 4 singular fibres, $\text{Aut}(S, \pi) = \text{Aut}(S)$ is a maximal algebraic subgroup of $\text{Bir}(S)$ ([Bla08]).

Lemma 4.4. *Let (X, π) be a minimal real conic bundle. Then, (X, π) is exceptional if and only if there exist two conjugate sections s and \bar{s} which do not intersect.*

Proof. Denote by $2r$ the number of singular fibres of π . According to Lemma 4.2, for any imaginary section s , $s \cdot \bar{s} = 0 \Leftrightarrow s^2 = -r$.

If s and \bar{s} do not intersect then $s^2 = -r$, whence (X, π) is exceptional.

Conversely, assume that s is a section of self-intersection $-r$. If s is imaginary, then $s \cdot \bar{s} = 0$. Otherwise, $r = 0$ (Lemma 4.2) and therefore $X = (\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \sigma)$ for a certain anti-holomorphic involution σ . We may thus choose another section s' which is imaginary and which has self-intersection 0. \square

Lemma 4.5. *Let (X, π) be a real conic bundle. Then, there exists a minimal exceptional real conic bundle (X', π') and a birational map of conic bundles*

$$(X, \pi) \dashrightarrow (X', \pi').$$

Proof. We may assume that (X, π) is minimal. Take a section s of π . If s intersects its conjugate \bar{s} into a real point p (respectively into a pair of imaginary points q_1 and q_2), then blow-up the point p (respectively q_1 and q_2), and contract the strict transform of the fibre of the points blown-up. Continuing this way, we obtain a birational map of conic bundles $\phi: (X, \pi) \dashrightarrow (X', \pi')$ such that (X', π') is minimal and $\phi(s)$ does not intersect its conjugate. Applying Lemma 4.4, (X', π') is an exceptional conic bundle. \square

The following construction gives a normal form for the exceptional conic bundles associated to a given finite union of intervals bounded by an even number of points. Lemma 4.7 asserts the existence, and Lemma 4.8 provides the unicity.

Construction 4.6. Let $J = (J_1, J_2)$ be a pair of two disjoint finite subsets of \mathbb{R} with the same number of elements (i.e. $J_1, J_2 \subset \mathbb{R}$, $J_1 \cap J_2 = \emptyset$ and $\#J_1 = \#J_2$).

We associate to J three homogeneous polynomials P_1, P_2, P , a set $V_J \subset \mathbb{P}^1(\mathbb{R})$, three maps $\eta_J, \sigma_J, \alpha_J$, a complex conic bundle (S_J, π_J) and a real conic bundle (X_J, π_J) . The polynomials are the following:

$$\begin{aligned} P_1(x_1, x_2) &= \prod_{a \in J_1} (x_1 - ax_2), \\ P_2(x_1, x_2) &= \prod_{a \in J_2} (x_1 - ax_2), \\ P(x_1, x_2) &= P_1(x_1, x_2) \cdot P_2(x_1, x_2). \end{aligned}$$

The set V_J equals $\{(x_1 : x_2) \mid P(x_1, x_2) \leq 0\} \subset \mathbb{P}^1(\mathbb{R})$, the map $\eta_J : S_J \rightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$, is the blow-up of the points

$$\{(x : 1), (0 : 1) \mid x \in J_1\} \cup \{(x : 1), (1 : 0) \mid x \in J_2\},$$

the morphism π_J is $\text{pr}_1 \circ \eta_J$. The maps α_J and σ_J are self-maps of S_J , which are the lifts by η_J of the following self-maps of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$.

$$\begin{aligned} \alpha'_J : ((x_1 : x_2), (y_1 : y_2)) &\dashrightarrow ((x_1 : x_2), (-y_2 \cdot P_1(x_1, x_2) : y_1 \cdot P_2(x_1, x_2))), \\ \sigma'_J : ((x_1 : x_2), (y_1 : y_2)) &\dashrightarrow ((\overline{x_1} : \overline{x_2}), (-\overline{y_2} \cdot P_1(\overline{x_1}, \overline{x_2}) : \overline{y_1} \cdot P_2(\overline{x_1}, \overline{x_2}))). \end{aligned}$$

The following lemma shows that σ_J is an anti-holomorphic involution of S_J . We write $X_J = (S_J, \sigma_J)$ the corresponding real surface.

Lemma 4.7. *Let us take the notation of Construction 4.6. Then, α_J and σ_J are respectively an holomorphic and an anti-holomorphic involution of (S_J, π_J) . The real conic bundle (X_J, π_J) (with $X_J = (S_J, \sigma_J)$) is minimal and exceptional, and satisfies $I(X_J, \pi_J) = V_J$. Moreover, the singular fibres of π_J are the fibres of $(x : 1)$, $x \in J_1 \cup J_2$.*

Proof. Denote by r the number of elements of J_1 (which is also the number of elements of J_2). The map α'_J is a birational involution of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$, which is defined over \mathbb{R} , and whose base-points are precisely the $2r$ points $\{(x : 1), (0 : 1) \mid x \in J_1\} \cup \{(x : 1), (1 : 0) \mid x \in J_2\}$ blown-up by η_J . Since α'_J is an involution and η_J is the blow-up of all of its base-points, $\alpha_J = \eta^{-1} \alpha'_J \eta$ is an automorphism of S , which belongs to $\text{Aut}(S, \pi)$.

The conic bundle (S_J, π_J) is exceptional, since the strict transform of the sections $\mathbb{P}^1 \times (0 : 1)$ and $\mathbb{P}^1 \times (1 : 0)$ have self-intersection $-r$. Moreover, its singular fibres correspond to the fibres of the points blown-up by η_J , so the fibres of the points $(x : 1) \in \mathbb{P}_{\mathbb{R}}^1$, $x \in J_1 \cap J_2$. Denote by τ the lift by η_J of the usual anti-holomorphic involution $((x_1 : x_2), (y_1 : y_2)) \mapsto ((\overline{x_1} : \overline{x_2}), (\overline{y_1} : \overline{y_2}))$. Since $\alpha_J \in \text{Aut}(S, \pi)$ commutes with τ , the map $\sigma_J = \alpha_J \tau = \tau \alpha_J$ is an anti-holomorphic involution of S . Since $X_J = (S_J, \sigma_J)$ and σ_J exchanges the two components of any singular fibre, the real conic bundle (X_J, π_J) is minimal, and is exceptional as (S_J, π_J) is.

By construction, σ_J is the lift by η_J of the map σ'_J described in Construction 4.6. Let $(x_1 : x_2)$ be a point of $\mathbb{P}_{\mathbb{R}}^1$; we want to prove that the fibre $f = \pi^{-1}((x_1 : x_2))$ contains a real point (i. e. a point fixed by σ_J) if and only if $(x_1 : x_2) \in V_J$, which is equivalent to say that $P(x_1, x_2) \leq 0$. If $(x_1 : x_2) = (a_i : 1) \in V$ for some i , then f is singular, and its unique singular point is real. Otherwise, η restricts to an isomorphism from f to $\eta(f)$. It follows that f has a real point if and only if there exists $(y_1 : y_2) \in \mathbb{P}_{\mathbb{C}}^1$ such that $(y_1 : y_2) = (-\overline{y_2} P_1(x_1, x_2) : \overline{y_1} P_2(x_1, x_2))$, which is equivalent to $y_1 \overline{y_1} P_2(x_1, x_2) = -y_2 \overline{y_2} P_1(x_1, x_2)$. This is possible if and only if $P(x_1, x_2) = P_1(x_1, x_2) \cdot P_2(x_1, x_2) \leq 0$, which means that $(x_1 : x_2) \in V_J$. This achieves to prove the equality $I(X, \pi) = V_J$. \square

In the following, Assertion (1) is well-known, whereas the other assertions (and in particular (4)) are new, and are one of the basic ingredients in the proof of the remaining results of the section.

Lemma 4.8. *Let (Y, π_Y) be a minimal exceptional real conic bundle.*

- (1) *$I(Y, \pi_Y)$ is the union of a finite union of closed intervals. The boundary points of the intervals correspond to the singular fibres of π_Y .*

- (2) If π_Y has no singular fibre, (Y, π_Y) is either isomorphic to $(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1, \text{pr}_1)$ or to $(\mathbb{P}_{\mathbb{R}}^1 \times (\mathbb{P}^1, \emptyset), \text{pr}_1)$.
- (3) If π_Y has at least one singular fibre, it is isomorphic to (X_J, π_J) , for some $J = (J_1, J_2)$, as in Construction 4.6.
- (4) If (Z, π_Z) is a minimal exceptional real conic bundle with $I(Z, \pi_Z) = I(Y, \pi_Y)$, there exists an isomorphism $\phi : (Z, \pi_Z) \rightarrow (Y, \pi_Y)$ satisfying $\pi_Z = \pi_Y \circ \phi$ (i.e. ϕ acts trivially on the basis).

Proof. Denote by $2r$ the number of singular fibres of π_Y (which is even, since (Y, π_Y) is minimal, see Lemma 4.2).

Assume first that $r = 0$, which implies that (Y, π_Y) is a real form of $(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \text{pr}_1)$, hence is isomorphic to $(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1, \text{pr}_1)$ or to $(\mathbb{P}_{\mathbb{R}}^1 \times (\mathbb{P}^1, \emptyset), \text{pr}_1)$; this proves Assertion (1) and (2).

Assume now that $r > 0$, and denote by s and \bar{s} two conjugate imaginary sections of π_Y of self-intersection $-r$. Let us write $\pi = \gamma \circ \pi_Y$, for some $\gamma \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$, such that $\pi^{-1}((1 : 0))$ is not a singular fibre. The singular fibres of π are above the points $(a_1 : 1), \dots, (a_{2r} : 1)$, where the a_i are distinct real numbers. Let $J = (J_1, J_2)$ be a partition of $\{a_1, \dots, a_{2r}\}$ into two sets of r points. Let η_Y be the birational morphism (not defined over \mathbb{R}) which contracts the component of $\pi^{-1}((a_i : 1))$ which intersects s if $a_i \in J_1$ and which intersects \bar{s} if $a_i \in J_2$. Then, the images of s and \bar{s} are two sections of self-intersection 0. Thus we may assume that η_Y is a birational morphism of conic bundles $(S, \pi) \rightarrow (\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \text{pr}_1)$, where S is the complex surface obtained by forgetting the real structure of Y , pr_1 is the projection on the first factor, and where $\eta_Y(s)$ and $\eta_Y(\bar{s})$ are equal to $\mathbb{P}_{\mathbb{C}}^1 \times (0 : 1)$ and $\mathbb{P}_{\mathbb{C}}^1 \times (1 : 0)$.

Using the notation of Construction 4.6 associated to $J = (J_1, J_2)$, η_Y is the blow-up of the $2r$ points of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$, which are exactly the points blown-up by $\eta_J : S_J \rightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$. We may therefore assume that $Y = (S_J, \sigma_Y)$, for some anti-holomorphic involution σ_Y on S_J , and that $\pi = \pi_J$ and $\eta_Y = \eta_J$.

The map $\sigma_Y \circ \sigma_J^{-1}$ belongs to $\text{Aut}(S_J, \pi_J)$ and acts trivially on the basis, since σ_J and σ_Y have the same action on the basis. Moreover, $\sigma_Y \circ \sigma_J^{-1}$ preserves any curve contracted by η_J and is therefore the lift by η_J of $\beta : ((x_1 : x_2), (y_1 : y_2)) \mapsto ((x_1 : x_2), (\mu y_1 : y_2))$ for some $\mu \in \mathbb{C}^*$. It follows that $\eta_J \circ \sigma_Y \circ \eta_J^{-1} = \beta \circ \sigma_J'$ is the map

$$((x_1 : x_2), (y_1 : y_2)) \mapsto ((\bar{x}_1 : \bar{x}_2), (-\mu \cdot \bar{y}_2 P_1(\bar{x}_1, \bar{x}_2) : \bar{y}_1 P_2(\bar{x}_1, \bar{x}_2))).$$

Since σ_Y is an involution, μ belongs to \mathbb{R} . Then, for any point $(x_1 : x_2) \in \mathbb{P}_{\mathbb{R}}^1$, the fibre $\pi^{-1}((x_1 : x_2))$ contains points fixed by σ_Y if and only if there exist $(y_1 : y_2) \in \mathbb{P}_{\mathbb{C}}^1$ such that $y_1 \bar{y}_1 P_2(x_1, x_2) = -y_2 \bar{y}_2 \cdot \mu P_1(x_1, x_2)$. Consequently, $I(Y, \pi)$ is equal to V_J if $\mu > 0$ and to the adherence of $\mathbb{P}_{\mathbb{R}}^1 \setminus V_J$ if $\mu < 0$. This proves (1).

Moreover, this also shows that $I(Y, \pi)$ is not the whole $\mathbb{P}^1(\mathbb{R})$. We could thus have chosen from the beginning the map γ so that $(0 : 1) \notin I(Y, \pi)$. This implies that $I(Y, \pi) = V_J$ and $\mu > 0$. Choosing $\lambda \in \mathbb{C}^*$ with $\lambda \cdot \bar{\lambda} = \mu$ and writing $\gamma : ((x_1 : x_2), (y_1 : y_2)) \mapsto ((x_1 : x_2), (\lambda y_1 : y_2))$, we get $\beta \circ \sigma_J' = \gamma \circ \sigma_J' \circ \gamma^{-1}$. It follows that $\eta_J^{-1} \circ \gamma \circ \eta_J \in \text{Aut}(S, \pi)$ conjugates σ_J to $\sigma_Y = \eta_J^{-1} \circ (\beta \sigma_J) \circ \eta_J$, and therefore induces an isomorphism of real conic bundles $(X_J, \pi_J) \rightarrow (Y, \pi)$, which is the identity on the basis.

In particular, (X_J, π_J) is isomorphic to (Y, π) and thus to (Y, π_Y) , and gives Assertion (3). Moreover, if (Z, π_Z) is another conic bundle with $I(Z, \pi_Z) = I(Y, \pi)$, the above description also yields the existence of an isomorphism $(X_J, \pi_J) \rightarrow (Z, \gamma \circ \pi_Z)$ which acts trivially on the basis. Therefore, there exists an isomorphism $(Z, \gamma \circ \pi_Z) \rightarrow (Y, \pi)$ which acts trivially on the basis, and thus the same is true for (Z, π_Z) and (Y, π_Y) , so Assertion (4) is proved.

It remains to prove (4) when $r = 0$. Since $I(Z, \pi_Z)$ is equal to (Y, π) , it is equal to \emptyset or $\mathbb{P}_{\mathbb{R}}^1$, hence the number of singular fibres of π_Z is 0 by (3) (or using (1)). This means that (Z, π_Z) is a real form of $(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \text{pr}_1)$. Either (Y, π) and (Z, π_Z) are both isomorphic to $(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1, \text{pr}_1)$ or to $(\mathbb{P}_{\mathbb{R}}^1 \times (\mathbb{P}^1, \emptyset), \text{pr}_1)$. Assertion (4) is obvious in both cases. \square

Lemma 4.9. *Let (X, π) be a minimal exceptional real conic bundle. The image of the homomorphism $\theta: \text{Aut}(X, \pi) \rightarrow \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$ given by the action on the base is equal to*

$$\left\{ \alpha \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1) \mid \alpha(I(X, \pi)) = I(X, \pi) \right\}.$$

Proof. Any element of $\beta \in \text{Aut}(X, \pi)$ preserves $X(\mathbb{R})$, thus $\theta(\beta)$ preserves $\pi(X(\mathbb{R})) = I(X, \pi)$.

Conversely, let $\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$ be a non-trivial element such that $\alpha(I(X, \pi)) = I(X, \pi)$. It remains to prove the existence of $\beta \in \text{Aut}(X, \pi)$ such that $\theta(\beta) = \alpha$. If π has no singular fibre, this is obvious since (X, π) is a real form of $(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \text{pr}_1)$. Otherwise, there are $2r$ singular fibres, with $r > 0$, which correspond to the $2r$ boundary points of $I(X, \pi)$ (Lemma 4.8); we denote by $F \subset I(X, \pi)$ this set of points, which is invariant by α .

Let us prove that we may decompose F into two sets of r points, each one being invariant by α . There are three kinds of non-trivial elements of $\text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$:

- (1) translations (elements with one fixed point),
- (2) rotations (elements with two fixed points, both imaginary), and
- (3) symmetries (elements with two real fixed points).

Since α preserves a finite set of at least 2 points, α is not a translation. If α is a rotation it has finite order n , and all its orbits on $\mathbb{P}^1(\mathbb{R})$ have order n . Thus, F decomposes into m orbits of n points, with $mn = 2r$. If n is even, we decompose each orbit in two sets, and if m is even, we decompose the set of orbits into two sets. The remaining case is when α is a symmetry. In well-chosen coordinates, α is the map $(x : y) \mapsto (\lambda x : y)$ for some $\lambda \in \mathbb{R}^*$. If $\lambda \neq -1$, then α has infinite order, hence $F = \{(0 : 1), (1 : 0)\}$, which decomposes into two sets of one point. If $\lambda = -1$, then $(0 : 1)$ and $(1 : 0)$ are not boundary points; we decompose F into two sets, the points $(x : y)$ with $x/y > 0$ and the others.

Now, $F = F_1 \cup F_2$, where each set F_i is invariant by α . Let us choose $\gamma \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$ such that $(1 : 0) \notin \gamma(I(X, \pi)) = I(X, \gamma\pi)$, so the set $\gamma(I(X, \pi))$ may be viewed as a subset of $\{(x : 1) \in \mathbb{P}^1(\mathbb{R}) \mid x \in \mathbb{R}\}$. The sets $\gamma(F_1)$ and $\gamma(F_2)$ correspond therefore to two sets J_1, J_2 of r real numbers. Using the notation of Construction 4.6 with $J = (J_1, J_2)$, there exists an isomorphism $\phi: (X_J, \pi_J) \rightarrow (X, \gamma\pi)$ such that $\pi_J = \gamma\pi \circ \phi$ (Lemma 4.8). Write $\delta = \gamma\alpha\gamma^{-1} = (x_1 : x_2) \rightarrow (ax_1 + bx_2 : cx_1 + dx_2)$, for some real numbers a, b, c, d with $ad - bc \neq 0$. The map δ leaves invariant the roots of $P_i(x_1, x_2) = \prod_{a \in J_i} (x_1 - ax_2)$, for each i , so $P_i(ax_1 + bx_2, cx_1 + dx_2) = \chi_i \cdot P_i(x_1, x_2)$, for some $\chi_i \in \mathbb{R}^*$. Moreover, since δ leaves V_J invariant, $\chi_1 \cdot \chi_2$ is positive. Denote

by ψ' the following automorphism of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$:

$$\psi' : ((x_1 : x_2), (y_1 : y_2)) \dashrightarrow ((ax_1 + bx_2 : cx_1 + dx_2), (\lambda y_1 : y_2)),$$

where $\lambda \in \mathbb{R}$ is such that $\lambda^2 = \chi_2/\chi_1$. Then, ψ' commutes with σ'_J . Recall that σ'_J is the following self-map of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$

$$\sigma'_J : ((x_1 : x_2), (y_1 : y_2)) \dashrightarrow ((\bar{x}_1 : \bar{x}_2), (-\bar{y}_2 \cdot P_1(\bar{x}_1, \bar{x}_2) : \bar{y}_1 \cdot P_2(\bar{x}_1, \bar{x}_2))).$$

Since $\eta_J : S_J \rightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ is the blow-up of the points $\{(x : 1), (0 : 1) \mid x \in J_1\} \cup \{(x : 1), (1 : 0) \mid x \in J_2\}$, which are preserved by ψ' , this one lifts to an automorphism $\psi = \eta_J^{-1} \psi' \eta_J$ of (S_J, π_J) . Moreover, ψ' commutes with σ'_J , so ψ belongs to $\text{Aut}(X_J, \pi_J)$. It remains to choose $\beta = \phi \psi \phi^{-1} \in \text{Aut}(X, \gamma \pi) = \text{Aut}(X, \pi)$. Since $\pi_J \psi = \delta \pi_J = (\gamma \alpha \gamma^{-1}) \circ \pi_J$ and $\pi_J = \gamma \pi \phi$, we have $\gamma \pi \phi \psi = \gamma \alpha \pi \phi$. This proves that $\pi \beta = \alpha \pi$, which means that $\theta(\beta) = \alpha$. \square

Lemma 4.10. *Let us take the notation of Construction 4.6, for some $J = (J_1, J_2)$. Denote by A the real affine hypersurface of \mathbb{R}^3 given by*

$$y^2 + z^2 = - \prod_{a \in J_1 \cup J_2} (x - a_i),$$

then the map $\psi : (x, y, z) \dashrightarrow (\eta_J)^{-1} \left(((x : 1), (y - iz : \prod_{a \in J_2} (x - a_i))) \right)$ is an embedding $A \rightarrow X_J$ defined over \mathbb{R} which induces an isomorphism

$$A(\mathbb{R}) \rightarrow X_J(\mathbb{R}).$$

Proof. Denote by $B \subset \mathbb{C}^3$ the affine hypersurface of equation $y^2 + z^2 = -P(x, 1)$ (recall that $P(x, 1) = \prod_{a \in J_1 \cup J_2} (x - a_i)$), and by $\pi_B : B \rightarrow \mathbb{C}$ the map $(x, y, z) \mapsto x$. Let $A = (B, \sigma_B)$, where σ_B sends (x, y, z) onto $(\bar{x}, \bar{y}, \bar{z})$. Denote by $\theta : B \dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ the map that sends (x, y, z) onto $((x : 1), (y - iz : P_2(x, 1)))$ if $P_2(x, 1) \neq 0$ and onto $((x : 1), (-P_1(x, 1) : y + iz))$ if $P_1(x, 1) \neq 0$. Then θ is a birational morphism, and θ^{-1} sends $((x_1 : x_2), (y_1 : y_2))$ on

$$\left(\frac{x_1}{x_2}, \frac{1}{2} \left(\frac{y_1}{y_2} P_2(x_1, x_2) - \frac{y_2}{y_1} P_1(x_1, x_2) \right), \frac{i}{2} \left(\frac{y_1}{y_2} P_2(x_1, x_2) + \frac{y_2}{y_1} P_1(x_1, x_2) \right) \right).$$

Observe that $\sigma'_J \theta = \sigma_B \theta$. In consequence, $\psi = (\eta_J)^{-1} \circ \theta$ is a real birational map $A \dashrightarrow X_J$.

Moreover, ψ is an isomorphism from A to the complement in S_J of the union of $\pi^{-1}((1 : 0))$ and the pull-back by η of $\mathbb{P}^1 \times (0 : 1)$ and $\mathbb{P}^1 \times (1 : 0)$. Indeed let $x_0 \in \mathbb{C}$. If $x_0 \in \mathbb{C}$ is such that $P(x_0, 1) \neq 0$, then θ restricts to an isomorphism from $\pi_B^{-1}(x_0)$ to $\{((x_0 : 1), (y_1 : y_2)) \in \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \mid y_1 y_2 \neq 0\} \cong \mathbb{C}^*$. If $P(x_0, 1) = 0$, then $x_0 \in J_1 \cup J_2$, and the fibre $\pi_B^{-1}(x_0)$ consists of two lines of \mathbb{C}^2 which intersect, given by $y = iz$ and $y = -iz$. If $x_0 \in J_1$, then the line $y + iz = 0$ is sent isomorphically by θ onto the fibre $\{((x_0 : 1), (y_1 : y_2)) \in \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \mid y_2 \neq 0\} \cong \mathbb{C}^*$, and the line $y - iz$ is contracted on the point $((x_0 : 1), (0 : 1))$. The map ψ sends thus isomorphically $\pi_B^{-1}(x_0)$ onto the fibre $\pi^{-1}((x_0 : 1))$ minus the two points corresponding to the two sections of self-intersection $-r$. The situation when $x_0 \in J_2$ is similar. Finally, we see that ψ induces an isomorphism between B and the complement in S_J of the two sections of self-intersection $-r$ (the strict transforms by η^{-1} of $\mathbb{P}^1 \times (0 : 1)$ and $\mathbb{P}^1 \times (1 : 0)$), and the fibre of $\pi^{-1}((1 : 0))$. Since these 3 curves do not have any real point, ψ induces an isomorphism $A(\mathbb{R}) \rightarrow X_J(\mathbb{R})$. \square

Corollary 4.11. *Let (X, π) be a minimal exceptional real conic bundle. There exists an affine real variety $A \subset X$ isomorphic to the affine surface of \mathbb{R}^3 given by*

$$y^2 + z^2 = Q(x),$$

where Q is a real polynomial with only simple roots, all real. Moreover, $I(X, \pi)$ is the closure of $\{(x : 1) \in \mathbb{P}_{\mathbb{R}}^1 \mid Q(x) \geq 0\}$, and $\pi|_A$ is the projection $(x, y, z) \mapsto x$.

Proof. Denote by $2r$ the number of singular fibres of π . If $r > 0$, then (X, π) is isomorphic to (X_J, π_J) for some $J = (J_1, J_2)$ (Lemma 4.8); the result follows then from Lemma 4.10. If $r = 0$, (X, π) is isomorphic to $(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1, \text{pr}_1)$ or to $(\mathbb{P}_{\mathbb{R}}^1 \times (\mathbb{P}^1, \emptyset), \text{pr}_1)$ (Lemma 4.8). Taking $Q(x) = 1$ or $Q(x) = -1$ gives the result. \square

We can now deduce the following important result, due to Comessatti [Com12]. See also [Kol97, Theorem 4.5].

Proposition 4.12. *Let (X, π) and (X', π') be two real conic bundles, and let $\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$. Then, the following are equivalent.*

- (1) *There exists a birational map $\varphi: X \dashrightarrow X'$ such that $\pi'\varphi = \alpha\pi$.*
- (2) $\alpha(I(X, \pi)) = I(X', \pi')$.

In particular, (X, π) and (X', π') are birational if and only if there exists an automorphism of $\mathbb{P}_{\mathbb{R}}^1$ that sends $I(X, \pi)$ on $I(X', \pi')$.

Proof. (1) \Rightarrow (2) : The map φ restricts to an isomorphism $X \setminus F \rightarrow X' \setminus F'$ where F and F' consist of finite sets of fibres of π and π' . In consequence, α sends $I(X, \pi) \setminus \pi(F)$ on $I(X', \pi') \setminus \pi(F')$. Since $\pi(F)$ and $\pi(F')$ are finite, α sends $I(X, \pi)$ on $I(X', \pi')$.

(2) \Rightarrow (1) : We may assume that the conic bundles (X, π) and (X', π') are minimal. By Lemma 4.5, both conic bundles may be supposed to be exceptional. Since $I(X, \alpha\pi) = I(X', \pi')$, Lemma 4.8 yields the existence of an isomorphism $\varphi: X \rightarrow X'$ such that $\pi'\varphi = \alpha\pi$. \square

We can now give the well-known description of $I(X, \pi)$ announced before.

Corollary 4.13. *Let (X, π) be a real conic bundle.*

- (1) *$I(X, \pi)$ is a finite union of closed intervals of $\mathbb{P}_{\mathbb{R}}^1$ (which may be \emptyset or $\mathbb{P}_{\mathbb{R}}^1$).*
- (2) *The images by π of the singular fibres of π are the boundary points of the intervals of $I(X, \pi)$.*
- (3) *The r connected components of $X(\mathbb{R})$ surject by π on r closed intervals of $\mathbb{P}^1(\mathbb{R})$.*

Proof. According to Proposition 4.12, these conditions are independent of the birational class of (X, π) . Since (X, π) is birational to a minimal exceptional conic bundle (Lemma 4.5), it suffices to prove the result when (X, π) is minimal and exceptional. This is a direct consequence of Lemmas 4.7 and 4.8. \square

The remaining part of the section is devoted to strengthen the results asserting the existence of a birational equivalence between surfaces by proving the existence of an isomorphism between real parts. The following lemma is the key ingredient of this improvement.

Lemma 4.14. *Let (X, π) be a minimal exceptional real conic bundle, with at least one singular fibre. Let q be a real point belonging to a nonsingular fibre.*

Then, there exists a section s of π whose unique real point is q , and such that s and its conjugate \bar{s} intersect transversally at q .

Proof. Denote by $r \geq 1$ the number of intervals of V . Up to an automorphism of $\mathbb{P}_{\mathbb{R}}^1$, we may assume that $(1 : 0) \notin V$. We denote by $(a_1 : 1), \dots, (a_{2r} : 1) \in \mathbb{P}_{\mathbb{R}}^1$ the boundary points of the intervals, and assume that $a_1 < a_2 < \dots < a_{2r}$. We write $p = (b : 1)$ and up to an automorphism of $\mathbb{P}_{\mathbb{R}}^1$, we can assume that $b \in]a_1, a_2[$, $a_1 = 0$, and $b = 1$. We denote by P the real polynomial $P(x) = \prod_{i=1}^{2r} (x - a_i)$. Observe that for $x \in \mathbb{R}$, $P(x) \leq 0$ if and only if $(x : 1) \in V$, and write $c = P(b) = P(1) < 0$.

We fix $Q_m(x) = c \cdot x^{2m}$ for any positive integer m . Let us prove that for some large m , we have $Q_m(x) = P(x)$ if and only if $x \in \{a_1, b\} = \{0, 1\}$. There exists m_0 such that $\frac{dQ_m}{dx}(1) < \frac{dP}{dx}(1)$ for $m \geq m_0$. Moreover, $\frac{dQ_m}{dx}(0) = 0 > \frac{dP}{dx}(0)$. Thus, there exist $\varepsilon_1, \varepsilon_2 \in]0, 1[$ such that $Q_m(x) > P(x)$ for any $m \geq m_0$ and any $x \in]0, \varepsilon_1[\cup]\varepsilon_2, 1[$. Since the sequence $(Q_m(x))_{m > 0}$ converges uniformly to 0 on the compact $[\varepsilon_1, \varepsilon_2]$, there exists $m_1 \geq m_0$ such that $Q_m(x) > P(x)$ for any $m \geq m_1$ and any $x \in]0, 1[$. Using the same argument on the interval $[1, a_2]$, we find $m_2 \geq m_1$ such that $Q_m(x) < P(x)$ for any $m \geq m_2$ and any $x \in]1, a_2]$. Then, choosing $m \geq m_2$ sufficiently big, we get $Q_m(a_2) < \min_{x \in \mathbb{R}} P(x)$, and thus $Q_m(x) < P(x)$ for any $x \in]-\infty, 0[\cup]1, \infty[$. It follows that $P(x)$ and $Q(x) = Q_m(x)$ are equal if and only if $x \in \{a_1, b\}$.

Let us use the notation of Construction 4.6 associated to $J = (J_1, J_2)$ with $J_1 = \{a_1, \dots, a_r\}$ and $J_2 = \{a_{r+1}, \dots, a_{2r}\}$. Then, since $V = V_J$, we may assume that $(X, \pi) = (X_J, \pi_J)$ (Lemma 4.8). Recall that $X_J = (S_J, \sigma_J)$, where $\eta_J : S_J \rightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ is the blow-up of the $2r$ points

$$\{(x : 1), (0 : 1) \mid x \in J_1\} \cup \{(x : 1), (1 : 0) \mid x \in J_2\},$$

and σ_J is the lift by η_J of the following self-maps of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ (written here on the affine plane $\{(x : 1), (y : 1) \in \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \mid x, y \in \mathbb{C}\}$)

$$\sigma'_J : (x, y) \dashrightarrow \left(\bar{x}, -\frac{P_1(\bar{x}, 1)}{P_2(\bar{x}, 1)} \cdot \frac{1}{\bar{y}} \right),$$

where $P_1(x, 1) \cdot P_2(x, 1) = P(x)$.

Note that none of the points at infinity (i.e. not in the affine plane described above) is real, and that η_J induces a bijection between the set of real points of S which belong to smooth fibres and the set

$$\{(x, y) \in \mathbb{C}^2 \mid x \in \mathbb{R}, x \notin \{a_1, \dots, a_{2r}\} \text{ and } y \cdot \bar{y} = -P_1(x, 1)/P_2(x, 1)\}.$$

The image by η_J of the point $q \in \pi^{-1}(p) \subset S$ is thus equal to (b, y_b) for some $y_b \in \mathbb{C}$ satisfying $y_b \cdot \bar{y}_b = -P_1(b, 1)/P_2(b, 1) = -P(b)/(P_2(b, 1))^2 = -c/P_2(b, 1)^2$. Denote by $\lambda \in \mathbb{C}[x]$ a complex polynomial such that $\lambda(x) \cdot \overline{\lambda(x)} = -Q(x)$ (always possible since $Q(x) \leq 0$ for any $x \in \mathbb{R}$). Since $\lambda(b) \cdot \overline{\lambda(b)} = -Q(b) = -c$, the complex number $\xi = \lambda(b)/(y_b \cdot P_2(b, 1))$ is such that $\xi \bar{\xi} = 1$. Replacing $\lambda(x)$ by $\lambda(x)/\xi$, we may assume that $y_b = \lambda(b)/P_2(b, 1)$.

The image of the rational map $x \mapsto \eta^{-1}\left(\left(x, \frac{\lambda(x)}{P_2(x, 1)}\right)\right)$ is a section $s \subset S$. Its conjugate $\bar{s} = \sigma(s)$ is the image of the map $x \mapsto \eta^{-1}\left(\left(\bar{x}, -\frac{P_1(\bar{x}, 1)}{\lambda(x)}\right)\right)$. The real points of s are images of a real number $x \in \mathbb{R}$ such that $\lambda(x) \cdot \overline{\lambda(x)} = -P_1(x) \cdot P_2(x) = -P(x)$. As we saw before, there are only two possibilities for x , which are $x = b$ or $x = a_1$. The image of the latter is not a real point on S (because the section cannot pass through the singular point of the fibre). The only real point of s is then $\eta^{-1}\left(\left(b, \frac{\lambda(b)}{P_2(b, 1)}\right)\right) = \eta^{-1}\left(\left(b, y_b\right)\right) = q$.

Since $Q(x) = \lambda(x) \cdot \overline{\lambda(x)}$ and $P(x)$ have distinct derivatives in $x = b$, the two sections s and \bar{s} intersect transversally at the point q . \square

Proposition 4.15. *Let (X, π) be a minimal real conic bundle with at least one singular fibre. Then, there exists a minimal exceptional real conic bundle (X', π') such that $(X(\mathbb{R}), \pi)$ and $(X'(\mathbb{R}), \pi')$ are isomorphic.*

Remark 4.16. The result is false without the assumption on the number of singular fibres. Indeed, there is no isomorphism between $(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1)(\mathbb{R})$ and $\mathbb{F}^1(\mathbb{R})$, since the first one is homeomorphic to the torus $S^1 \times S^1$ and the second one is homeomorphic to the Klein bottle.

Proof. According to Lemma 4.5, there exists a minimal exceptional real conic bundle (X', π') and a birational map of conic bundles $\varphi: (X', \pi') \dashrightarrow (X, \pi)$. Let us decompose φ into $\varphi = \phi_m \circ \cdots \circ \phi_1$, where $\phi_i: (X_{i-1}, \pi_{i-1}) \dashrightarrow (X_i, \pi_i)$ is a birational map of conic bundles, which consists of the blow-up of a real point or two imaginary conjugate points, followed by the contraction of the strict transform of the corresponding fibre(s). The real conic bundles $(X', \pi') = (X_0, \pi_0)$, (X_1, π_1) , $(X_2, \pi_2), \dots, (X_m, \pi_m) = (X, \pi)$ are all minimal.

Let us prove by induction on j that there exists a birational map of conic bundles $\psi_j: (X', \pi') \dashrightarrow (X_j, \pi_j)$ which induces an isomorphism $X'(\mathbb{R}) \dashrightarrow X_j(\mathbb{R})$. If $j = 0$, we choose the identity for ψ_0 . Assume that $j > 0$. If ϕ_j blows-up two imaginary conjugate points, it suffices to choose $\psi_j = \phi_j \circ \psi_{j-1}$. Otherwise, ϕ_j blows-up a real point $q \in X_{j-1}(\mathbb{R})$ that belongs to a smooth fibre of π_{j-1} and contracts the strict transform of its fibre. The point $(\psi_{j-1})^{-1}(q)$ is real, and belongs to a smooth fibre of $\pi_0 = \pi'$. According to Lemma 4.14, there exists a section s of X' whose unique real point is q and such that s and \bar{s} intersect transversally at $(\psi_{j-1})^{-1}(q)$. It follows that $\phi_j \psi_{j-1}(s)$ and $\phi_j \psi_{j-1}(\bar{s})$ only intersect into imaginary conjugate points. Blowing-up all of these points, and contracting the strict transforms of the fibres, and repeating the process if needed, we get a birational map of conic bundles α from (X_j, π_j) to a minimal conic bundle $(\hat{X}, \hat{\pi})$, such that $\alpha \phi_j \psi_{j-1}(s)$ and $\alpha \phi_j \psi_{j-1}(\bar{s})$ do not intersect. By Lemma 4.4, $(\hat{X}, \hat{\pi})$ is then an exceptional conic bundle. From the unicity of such conic bundles (Lemma 4.8), there exists an isomorphism of conic bundles $\beta: (\hat{X}, \hat{\pi}) \rightarrow (X', \pi')$. Since α induces an isomorphism $X_j(\mathbb{R}) \rightarrow \hat{X}(\mathbb{R})$, we may choose $\phi_j = \beta \circ \alpha$. \square

Corollary 4.17. *Let (X, π_X) and (Y, π_Y) be two minimal conic bundles, and assume that either X or Y is not rational.*

Let $\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$; for any birational map $\beta: X \dashrightarrow Y$ such that $\pi_Y \beta = \alpha \pi_X$, there exists an isomorphism $\gamma: X(\mathbb{R}) \rightarrow Y(\mathbb{R})$ such that $\pi_Y \gamma = \alpha \pi_X$.

In particular, $(X(\mathbb{R}), \pi_X)$ and $(Y(\mathbb{R}), \pi_Y)$ are isomorphic if and only if (X, π_X) and (Y, π_Y) are birational.

Proof. Since (X, π_X) is birational to (Y, π_Y) and both are minimal, the number of singular fibres of π_X and π_Y is the same, equal to $2r$ for some non-negative integer r . Moreover, X and Y being not rational, $r > 0$. Applying Proposition 4.15, we may assume that both (X, π_X) and (Y, π_Y) are minimal exceptional real conic bundles. Since $\pi_Y \gamma = \alpha \pi_X$, $\alpha(I(X, \pi_X)) = I(X, \alpha \pi_X) = I(Y, \pi_Y)$ (Proposition 4.12). In this case, Lemma 4.8 yields the existence of an isomorphism $\gamma: X \rightarrow Y$ such that $\pi_Y \gamma = \alpha \pi_X$. \square

We end this section by an easy corollary of the preceding result.

Corollary 4.18. *Let (X, π) and (Y, π_Y) be two minimal conic bundles. Then, the following are equivalent:*

- (1) $(X(\mathbb{R}), \pi)$ and $(Y(\mathbb{R}), \pi)$ are isomorphic;
- (2) (X, π) is birational to (Y, π) and $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$.

Proof. The implication (1) \Rightarrow (2) is evident. Let us prove the converse.

Since (X, π_X) is birational to (Y, π_Y) and both are minimal, the number of singular fibres of π_X and π_Y is the same, equal to $2r$ for some non-negative integer r .

Suppose that $r = 0$, which means that X is an Hirzebruch surfaces \mathbb{F}_m for some m and that $Y = \mathbb{F}_n$ for some n . Since $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$, we have $m \equiv n \pmod{2}$. It is easy to prove that $(X(\mathbb{R}), \pi)$ and $(Y(\mathbb{R}), \pi)$ are isomorphic, by taking elementary links at two imaginary distinct fibres (see for example [Mang06, Proof of Theorem 6.1]).

Suppose now that $r > 0$. Applying Proposition 4.15, we may assume that both (X, π_X) and (Y, π_Y) are minimal exceptional real conic bundles. Since (X, π_X) and (Y, π_Y) are birational, we may assume that $I(X, \pi_X) = I(Y, \pi_Y)$, up to an automorphism of $\mathbb{P}_{\mathbb{R}}^1$ (Proposition 4.12). Then, (X, π_X) and (Y, π_Y) are isomorphic (Lemma 4.8). \square

5. CONIC BUNDLES ON DEL PEZZO SURFACES

In the preceding section we studied real conic bundles structures. In this section, we focus on surfaces admitting distinct minimal conic bundles. We will see that these surfaces are necessarily del Pezzo surfaces (Lemma 5.3). We begin by the description of all possible minimal real conic bundles occurring on del Pezzo surfaces.

Lemma 5.1. *Let V be a subset of $\mathbb{P}^1(\mathbb{R})$, then the following are equivalent:*

- (1) *there exists a minimal real conic bundle (X, π) such that X is a del Pezzo surface and $I(X, \pi) = V$;*
- (2) *the set V is a union of closed intervals, and $\#V \leq 3$.*

Proof. The part (1) \Rightarrow (2) is easy. Indeed, if (X, π) is minimal, we know from Lemma 4.2 that its number of singular fibres is even, denoted $2r$, and that $2r = 8 - (K_X)^2$. Since $-K_X$ is ample, $K_X^2 \geq 1$, thus $r \leq 3$. We conclude by Corollary 4.13 which asserts that $I(X, \pi)$ is the union of r closed intervals.

Let us prove the converse. If $V = \mathbb{P}^1(\mathbb{R})$ or $V = \emptyset$, we take (X, π) to be $(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \text{pr}_1)$, where pr_1 is the projection on the first factor, endowed with the anti-holomorphic map that sends $((x_1 : x_2), (y_1 : y_2))$ onto $((\overline{x_1} : \overline{x_2}), (\pm \overline{y_2} : \overline{y_1}))$.

Assume now that V is neither empty nor equal to $\mathbb{P}^1(\mathbb{R})$. Up to automorphism of $\mathbb{P}_{\mathbb{R}}^1$, there exist $a_1 < a_2 < \dots < a_{2r} \in \mathbb{R}$ with $1 \leq r \leq 3$ such that

$$V = \bigcup_{i=1}^r \left\{ (x : 1) \mid x \in [a_{2i-1}, a_{2i}] \subset \mathbb{R} \right\}.$$

We use the notation of Construction 4.6 associated to $J = (J_1, J_2)$ with $J_1 = \{a_1, \dots, a_r\}$ and $J_2 = \{a_{r+1}, \dots, a_{2r}\}$, and observe that $V = V_J$. Let $S_0 := S_J$, $\pi_0 := \pi_J$ and $X_0 := X_J$. The real conic bundle (X_0, π_0) is exceptional, and $I(X_0, \pi_0) = V_J = V$ (Lemma 4.7).

If $r = 1$, $\eta_J: S_0 \rightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ is the blow-up of two points which do not lie on the same fibre of any projection, S_0 is thus a del Pezzo surface of degree 6; choosing $(X, \pi) = (X_0, \pi_0)$ achieves the proof.

Assume now that $2 \leq r \leq 3$. Let $\alpha_0 := \alpha_J \in \text{Aut}(S_0, \pi_0)$ and $\sigma_0 := \sigma_J$ be the holomorphic and anti-holomorphic involutions of S_0 . Recall that α_0 is the lift by η_J of the birational involution

$$\alpha'_J: ((x_1 : x_2), (y_1 : y_2)) \dashrightarrow ((x_1 : x_2), (-y_2 \prod_{i=1}^r (x_1 - a_i x_2) : y_1 \prod_{i=r+1}^{2r} (x_1 - a_i x_2)))$$

of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$. Then, α_0 acts trivially on the basis, fixes two points in each nonsingular fibre and fixes only one point in each singular fibre (the singular point). We write $\Gamma_0 \subset S_0$ the curve fixed by τ_0 , which is the strict transform of the curve of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ given by the equation

$$(y_1)^2 \cdot \prod_{i=r+1}^{2r} (x_1 - a_i x_2) + (y_2)^2 \cdot \prod_{i=1}^r (x_1 - a_i x_2) = 0.$$

The curve Γ_0 is a double covering of $\mathbb{P}_{\mathbb{C}}^1$ by means of π_0 , ramified over the $2r$ points $(a_i : 1)$. Thus $\Gamma_0(\mathbb{R}) \subset X_0(\mathbb{R})$ is the union of r ovals, which surject by π_0 on the closed intervals of $I(X_0, \pi_0) = V$.

Denote by $p_0 \in \Gamma_0(\mathbb{R})$ a real point such that $\pi_0(p_0)$ is not a boundary point. We call ψ_0 the blow-up of p_0 , followed by the contraction of the strict transform of its fibre. Then, ψ_0 is a birational map of conic bundles $(S_0, \pi_0) \dashrightarrow (S_1, \pi_1)$, where $\pi_1 = \pi_0 \psi_0^{-1}$. Since p_0 is fixed by both σ_0 and α_0 , we get an anti-holomorphic involution $\sigma_1 = \psi_0 \sigma_0 \psi_0^{-1}$ on S_1 , and an holomorphic involution $\alpha_1 = \psi_0 \alpha_0 \psi_0^{-1}$ that commutes with σ_1 . We write $X_1 = (S_1, \sigma_1)$ and choose a point $p_1 \in X_1(\mathbb{R})$ that belongs to a smooth fibre of π_1 and that does not belong to s_1 or $\overline{s_1}$ (this latter condition only avoids the base-point of ψ_0^{-1}). Similarly as before, we call ψ_1 the blow-up of p_1 , followed by the contraction of the strict transform of its fibre, and obtain a birational map of conic bundles $\psi_1: (X_1, \pi_1) \dashrightarrow (X_2, \pi_2)$ with $\pi_2 = \pi_1 \psi_1^{-1}$. Observe that $I(X_2, \pi_2) = I(X_1, \pi_1) = I(X_0, \pi_0) = V$.

We claim that if $r = 2$, then S_1 is a del Pezzo surface and that if $r = 3$ and p_1 is well chosen, then S_2 is a del Pezzo surface. Assuming this, it suffices to let $(X, \pi) := (X_i, \pi_i)$ for $i = 1$ or $i = 2$ to conclude the proof.

Let us prove the claim. Denote by s_0 and $\overline{s_0}$ the exceptional sections of π_0 , which are the strict transforms by η_J^{-1} of $\mathbb{P}_{\mathbb{C}}^1 \times (0 : 1)$ and $\mathbb{P}_{\mathbb{C}}^1 \times (1 : 0)$. Denote by s_i and $\overline{s_i}$ the strict transforms of s_0 and $\overline{s_0}$ on S_i . Since p_i does not belong to s_i or $\overline{s_i}$ for $i = 0, 1$, we have $s_i^2 = \overline{s_i}^2 = -r + i$ for $i = 1, 2$.

We prove now that any section of π_1 distinct from s_1 or $\overline{s_1}$ has self-intersection ≥ -1 , and that there are finitely many sections of self-intersection -1 . Denote by $C_1 \subset S_1$ a section of π_1 , of self-intersection $C_1^2 \leq -1$, which is distinct from s_1 or $\overline{s_1}$. The strict transform on S_0 of C_1 is written C_0 , and has self-intersection $C_1^2 \pm 1$, depending whether C_0 passes through p_0 or not. The curve $\eta_J(C_0) \subset \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ is a section of pr_1 , linearly equivalent to $f_2 + df_1$, where f_i denotes the divisor of the fibre or pr_i and d is a non-negative integer. We have $\eta_J(C_0)^2 = 2d$ and $\eta_J(C_0) \cdot \eta_J(s_0) = d$. Denote by m_1, \dots, m_{2r} the multiplicities of $\eta_J(C_0)$ at the points blown-up by η_J – where m_i corresponds to the point in the fibre $\text{pr}_1^{-1}((a_i : 1))$. Then $m_i \in \{0, 1\}$ for each i , $C_0 \cdot s_0 = d - \sum_{i=1}^r m_i \geq 0$ and $C_0 \cdot \overline{s_0} = d - \sum_{i=r+1}^{2r} m_i \geq 0$

since C_0 is distinct from s_0 and $\overline{s_0}$, whence $C_0^2 = 2d - \sum_{i=1}^{2r} m_i \geq 0$. Since $C_1^2 \leq -1$ and $C_0^2 = C_1^2 \pm 1$, the only possibility is that $C_0^2 = 0$, $C_1^2 = -1$ and that C_0 passes through p_0 . Moreover, the possibilities for the m_i are finite (at most 2^{2r}), and for each possibility, there is exactly one pencil of curves of S_0 equivalent to C_0 , and C_0 is the only curve of the pencil which passes through p_0 .

Therefore, if $r = 2$, every section of π_1 has self-intersection ≥ -1 . If $r = 3$, choosing for $p_1 \in X_1(\mathbb{R})$ a point that does not belong to any of the sections of π_1 with self-intersection -1 , every section of π_2 has self-intersection ≥ -1 .

We prove now that if $r = 2$, then S_1 is a del Pezzo surface. Recall that $s_1^2 = \overline{s_1}^2 = -1$. Denote by η_1 the contraction of the component of the singular fibres of π_1 which intersect s_1 . Then, η_1 is a birational morphism (not defined over \mathbb{R}) from S_1 to \mathbb{F}_1 , which sends $\overline{s_1}$ on the exceptional section of \mathbb{F}_1 . Composing η_1 with the contraction of this curve on a point $p_1 \in \mathbb{P}_{\mathbb{C}}^2$, we see that S_1 is the blow-up of 5 points p_1, \dots, p_5 in $\mathbb{P}_{\mathbb{C}}^2$, that the fibres of π_1 are sent on the lines passing through p_1 , and that no two of the p_i for $i \geq 2$ are collinear with p_1 . Then S_1 is a del Pezzo surface if and only if no three of the points p_i for $i \geq 2$ are collinear. This amounts to ask that no section of π_1 has self-intersection ≤ -2 , and was proved above. Then, S_1 is a del Pezzo surface.

We prove now that if $r = 3$, then S_2 is a del Pezzo surface for a well chosen p_1 . The technique is similar; recall that $s_2^2 = \overline{s_2}^2 = -1$ and denote by $\eta_2: S_2 \rightarrow \mathbb{F}_1$ the contraction of the component of each singular fibre of π_2 which intersects s_2 . Then $\eta_2(\overline{s_2})$ is the exceptional section of \mathbb{F}_1 . Contracting it on $p_1 \in \mathbb{P}_{\mathbb{C}}^2$, S_2 is the blow-up of 7 points $p_1, \dots, p_7 \in \mathbb{P}_{\mathbb{C}}^2$. Since no section of π_2 has self-intersection ≤ -2 , no three of the seven points are collinear. Then S_2 is a del Pezzo surface if and only if there is no conic passing through 6 of the 7 points. If the conic passes through p_1 , its proper transform on S_2 would be a section of self-intersection ≤ -2 . It suffices thus to avoid the possibility of a conic $D \subset \mathbb{P}_{\mathbb{C}}^2$ that passes through p_2, \dots, p_7 and not through p_1 . If such a D exists, its strict transform on S_2 is a bisection which has self-intersection -2 , and does not intersect $\overline{s_2}$, where the base-point of ψ_1^{-1} lies (the base-point belongs to $s_2 \cap \overline{s_2}$ since $p_1 \notin s_1 \cup \overline{s_1}$).

Consequently, the strict transform of D on S_1 is a curve $D_1 \subset S_1$ which has multiplicity 2 at p_1 , and has self-intersection 2. This implies that D_1 is linearly equivalent to the anti-canonical divisor $-K_{S_1}$; indeed, contracting a component of a singular fibre that touches $\overline{s_1}$, then contracting $\overline{s_1}$ and all the components of the other singular fibres that touch s_1 gives a birational morphism $S_1 \rightarrow \mathbb{P}_{\mathbb{C}}^2$, that sends D_1 on a cubic passing once through all the points blown-up. Then, the holomorphic involution α_1 acts on the linear system $|-K_{S_1}|$, which has dimension $m \geq 2$ (it corresponds to cubics of $\mathbb{P}_{\mathbb{C}}^2$ passing through 7 points) and the induced map

$$\zeta: S_1 \xrightarrow{-K_{S_1}} \mathbb{P}^m$$

is equivariant.

Since the general fibres of ζ have genus 1 and the curve Γ_1 fixed by α_1 has genus 2 (it is a double covering of \mathbb{P}^1 ramified over 6 points), then Γ_1 is not contained in a fibre, whence α_1 acts trivially on \mathbb{P}^m (in fact, the interested reader can show that $m = 2$, that ζ is a double covering and that α_0 exchanges the two points in each fibre). In consequence, D_1 is invariant by α_1 , so its singular point p_1 is fixed by α_1 . It suffices to choose p_1 not lying on Γ_1 and S_2 is a del Pezzo surface. \square

Recall the following classical result, that will be useful in the sequel.

Lemma 5.2. *Let $\pi: S \rightarrow \mathbb{P}_{\mathbb{C}}^1$ be a complex conic bundle, and assume that S is a del Pezzo surface, with $(K_X)^2 = 9 - m \leq 7$. Then, there exists a birational morphism $\eta: S \rightarrow \mathbb{P}_{\mathbb{C}}^2$ which is a blow-up of m points p_1, \dots, p_m and which sends the fibres of π onto the lines passing through p_1 . The curves of self-intersection -1 of S are*

- the exceptional curves $\eta^{-1}(p_1), \dots, \eta^{-1}(p_m)$;
- the strict transforms of the lines passing through 2 of the p_i ;
- the conics passing through 5 of the p_i ;
- the cubics passing through 8 of the p_i and being singular at one of these.

Proof. Denote by ε the contraction of one component in each singular fibre of π . Then, ε is a birational morphism of conic bundles – not defined over \mathbb{R} – from S to a del Pezzo surface which is also an Hirzebruch surface. Changing the contracted components, we may assume that ε is a map $S \rightarrow \mathbb{F}_1$. Contracting the exceptional section onto a point $p_1 \in \mathbb{P}_{\mathbb{C}}^2$, we get a birational map $\eta: S \rightarrow \mathbb{P}_{\mathbb{C}}^2$ which is the blow-up of m points p_1, \dots, p_m of $\mathbb{P}_{\mathbb{C}}^2$, and which sends the fibres of π_1 onto the lines passing through p_1 . The description of the (-1) -curves is well-known and may be found for example in [Dem76]. \square

Lemma 5.3. *Let $\pi_1: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ be a minimal real conic bundle. Then, the following conditions are equivalent:*

- 1) *There exist a real conic bundle $\pi_2: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$, such that π_1 and π_2 induce distinct foliations on $X(\mathbb{C})$.*
- 2) *Either X is isomorphic to $\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$, or X is a del Pezzo surface of degree 2 or 4.*

Moreover, if the conditions are satisfied, then the following occur:

- a) *The map π_2 is unique, up to an automorphism of $\mathbb{P}_{\mathbb{R}}^1$.*
- b) *There exist $\alpha \in \text{Aut}(X)$ and $\beta \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$ such that $\pi_1 \alpha = \beta \pi_2$. Moreover, if X is a del Pezzo surface of degree 2, α may be chosen to be the Geiser involution.*
- c) *Denoting by $f_1, f_2 \in \text{Pic}(X)$ the divisors of the general fibre of respectively π_1 and π_2 , we have $f_1 + f_2 = -cK_X$ where $c = 4/(K_X)^2 \in \mathbb{N} \cdot \frac{1}{2}$.*

Proof. Assume the existence of π_2 , and denote by f_i the divisor of the fibre of π_i for $i = 1, 2$. We have $(f_1)^2 = (f_2)^2 = 0$ and by adjunction formula $f_1 \cdot K_X = f_2 \cdot K_X = -2$, where K_X is the canonical divisor. Let us write $d = (K_X)^2$.

Since (X, π_1) is minimal, $\text{Pic}(X)$ has rank 2, hence $f_1 = aK_X + bf_2$, for some $a, b \in \mathbb{Q}$. Computing $(f_1)^2$ and $f_1 \cdot K_X$ we find respectively $0 = a^2d - 4ab = a(ad - 4b)$ and $-2 = ad - 2b$. If $a = 0$, we find $f_1 = f_2$, a contradiction. Thus, $4b = ad$ and $2b = ad + 2$, which yields $b = -1$ and $ad = -4$, so $f_1 + f_2 = -4/d \cdot K_X$. This shows that f_2 is uniquely determined by f_1 .

Denote as usual by S the complex surface associated to X . Let $C \in \text{Pic}(S)$ be an effective divisor, with reduced support, and let us prove that $C \cdot (f_1 + f_2) > 0$. Since C is effective, $C \cdot f_1 \geq 0$ and $C \cdot f_2 \geq 0$. If $C \cdot f_1 = 0$, then the support of C is contained in one fibre of π_1 . If C is a multiple of f_1 , then $C \cdot f_2 > 0$; otherwise, C is a multiple of a (-1) -curve contained in a singular fibre of f_1 , and the orbit of C by the anti-holomorphic involution is equal to a multiple of f_1 , whence $C \cdot f_2 > 0$.

Since $f_1 + f_2$ is ample, and $f_1 + f_2 = -4/d \cdot K_X$ either K_X or $-K_X$ is ample. The surface X being geometrically rational, the former cannot occur, whence $d > 0$.

If S is isomorphic to $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$, the existence of π_1, π_2 shows that X is isomorphic to $\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$. Otherwise, K_X is not a multiple in $\text{Pic}(X_{\mathbb{C}})$ and thus d is equal to 1, 2 or 4. The number of singular fibres being even and equal to $8 - (K_X)^2$ by Lemma 4.2, the only possibilities are then 2 and 4.

We have proved that 1) implies 2), a), and c).

Assume now that $X = (S, \sigma)$ is $\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$ or a del Pezzo surface of degree 2 or 4. We construct an automorphism α of X which does not belong to $\text{Aut}(X, \pi)$. Then, by taking $\pi_2 = \pi_1 \alpha$ we get assertion 1). Taking into account the unicity of π_2 , we get b).

If X is $\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$, the two conic bundles are given by the projections on each factor, and we can get for α the swap of the factors.

If X is a del Pezzo surface of degree 2, the anti-canonical map $\zeta: X \rightarrow \mathbb{P}^2$ is a double covering ramified along a smooth quartic, cf. e.g. [Dem76]. Let α be the involution associated to the double covering – α is classically called the *Geiser involution*. It fixes a smooth quartic, hence cannot preserve any conic bundle.

The remaining case is when X is a del Pezzo surface of degree 4. By Lemma 5.2, there is a birational map $\eta: S \rightarrow \mathbb{P}_{\mathbb{C}}^2$ which is the blow-up of five points p_1, \dots, p_5 of $\mathbb{P}_{\mathbb{C}}^2$, no three being collinear and which sends the fibres of π_1 on the lines passing through p_1 . There are 16 exceptional curves (curves isomorphic to $\mathbb{P}_{\mathbb{C}}^1$ of self-intersection (-1)) on S :

- $E_1 = \eta^{-1}(p_1), \dots, E_5 = \eta^{-1}(p_5)$ (5 curves);
- the strict transforms of the lines passing through p_i and p_j , denoted by L_{ij} (10 curves);
- the strict transform of the conic passing through the five points, denoted by Γ .

Note that the four singular fibres of π_1 are $E_i \cup L_{ij}$, $i = 2, \dots, 5$, and that σ exchanges thus E_i and L_{ij} for $i = 1, \dots, 5$. The intersection form being preserved, this implies that σ acts on the 16 exceptional curves as

$$(E_2 L_{12})(E_3 L_{13})(E_4 L_{14})(E_5 L_{15})(E_1 \Gamma)(L_{23} L_{45})(L_{24} L_{35})(L_{25} L_{34}).$$

After a linear change of coordinates, we may assume that $p_1 = (1 : 1 : 1)$, $p_2 = (1 : 0 : 0)$, $p_3 = (0 : 1 : 0)$, $p_4 = (0 : 0 : 1)$ and $p_5 = (a : b : c)$ for some $a, b, c \in \mathbb{C}^*$. Denote by ϕ the birational involution $(x : y : z) \dashrightarrow (ayz : bxz : cxy)$ of $\mathbb{P}_{\mathbb{C}}^2$. Since the base-points of ϕ are p_2, p_3, p_4 and since ϕ exchanges p_1 and p_5 , the map $\varphi = \eta^{-1} \phi \eta$ is an automorphism of S . Its action on the 16 exceptional curves is given by the permutation

$$(L_{23} E_4)(L_{24} E_3)(L_{34} E_2)(L_{12} L_{25})(L_{13} L_{35})(L_{14} L_{45})(\Gamma L_{15})(E_1 E_5).$$

Observe that the actions of φ and σ on the set of 16 exceptional curves commute. This means that $\varphi \sigma \varphi^{-1} \sigma^{-1}$ is an holomorphic automorphism of S which preserves any of the 16 curves. It is the lift of an automorphism of $\mathbb{P}_{\mathbb{C}}^2$ that fixes the 5 points p_1, \dots, p_5 and hence is the identity. Consequently, φ and σ commute, so $\varphi \in \text{Aut}(X)$. Since ϕ sends a general line passing through p_1 onto a conic passing through p_2, \dots, p_5 , ϕ belongs to $\text{Aut}(X) \setminus \text{Aut}(X, \pi)$. \square

Corollary 5.4. *Let X be a minimal geometrically rational real surface, which is not rational. Then, the following are equivalent:*

- (1) $\#X(\mathbb{R}) = 2$ or $\#X(\mathbb{R}) = 3$;

- (2) *There exists a geometrically rational real surface $Y(\mathbb{R})$ isomorphic to $X(\mathbb{R})$, and such that Y admits two minimal conic bundles $\pi_1: Y \rightarrow \mathbb{P}_{\mathbb{R}}^1$ and $\pi_2: Y \rightarrow \mathbb{P}_{\mathbb{R}}^1$ inducing distinct foliations on $Y(\mathbb{C})$.*

Proof. [(2) \Rightarrow (1)] By Lemma 5.3, Y is then a del Pezzo surface, which has degree 2 or 4 since Y is not rational. This implies that $\#Y(\mathbb{R}) = 2$ or $\#Y(\mathbb{R}) = 3$ by Proposition 3.7.

[(1) \Rightarrow (2)]. According to Theorem 3.5 and Proposition 3.7, (1) implies the existence of a minimal real conic bundle structure $\pi_X: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ with 4 or 6 singular fibres. This condition is equivalent to the fact that $I(X, \pi_X)$ is the union of 2 or 3 intervals (Corollary 4.13). According to Lemma 5.1, there exists a minimal real conic bundle (Y, π_1) such that Y is a del Pezzo surface and $I(Y, \pi_1) = I(X, \pi_X)$. Proposition 4.12 and Corollary 4.17 show that $(X(\mathbb{R}), \pi_X)$ and (Y, π_1) are isomorphic. Moreover Lemma 5.3 yields the existence of π_2 . \square

6. EQUIVALENCE OF SURFACES VERSUS EQUIVALENCE OF CONIC BUNDLES

This section is devoted to the proof of Theorem 1.5. It remains to solve the conic bundle case, which is done in Theorem 6.3. First of all, we correct an existing inaccuracy in the literature; in [Kol97, Exercice 5.8] or [Sil89, VI.3.5], it is asserted that all minimal real conic bundles with four singular fibres belong to a unique birational equivalence class. To the contrary, the following general result, which includes the case with four singular fibres, occurs:

Theorem 6.1. *Let $\pi_X: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ and $\pi_Y: Y \rightarrow \mathbb{P}_{\mathbb{R}}^1$ be two real conic bundles, and suppose that either X or Y is non-rational. Then, the following are equivalent:*

- (1) *The two real surfaces X and Y are birational.*
- (2) *The two real conic bundles (X, π_X) and (Y, π_Y) are birational.*
- (3) *There exists an automorphism of \mathbb{P}^1 which sends $I(X, \pi_X)$ onto $I(Y, \pi_Y)$.*

Moreover, if the number of singular fibres of π_X is at least 8, then $\text{Bir}(X) = \text{Bir}(X, \pi_X)$.

Remark 6.2. It is well-known that this result is false when X and Y are rational. Indeed, consider $(X, \pi_X) = (\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1, \text{pr}_1)$ and (Y, π_Y) be a real conic bundle with two singular fibres. The surfaces X and Y are birational, but the conic bundles (X, π_X) and (Y, π_Y) are not.

Proof. The equivalence (3) \Leftrightarrow (2) was proved in Proposition 4.12 and (2) \Rightarrow (1) is evident.

We may assume that (X, π_X) and (Y, π_Y) are minimal and that X is not rational, hence π_X has at least 4 singular fibres. Let $\psi: X \dashrightarrow Y$ a birational map, and decompose ψ into elementary links: $\psi = \psi_n \circ \dots \circ \psi_1$ (see [Isk96, Theorem 2.5]). Consider $\psi_1: X \dashrightarrow X_1$ the first link, which may be of type *II* or *IV* only by [Isk96, Theorem 2.6]. If ψ_1 is of type *II*, then ψ_1 is a birational map of conic bundles $(X, \pi_X) \dashrightarrow (X_1, \pi_1)$ for some conic bundle structure $\pi_1: X_1 \rightarrow \mathbb{P}^1$. If ψ_1 is of type *IV*, then ψ_1 is an isomorphism $X \rightarrow X_1$ and the link is precisely a change of conic bundle structure from π_X to $\pi_1: X_1 \rightarrow \mathbb{P}^1$, which induce distinct foliations on $X(\mathbb{R})$. Applying Lemma 5.3, X is a del Pezzo surfaces of degree 2 or 4, and there exist automorphisms $\alpha \in \text{Aut}(X)$ and $\beta \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$ such that $\pi_1 \psi_1 \alpha = \beta \pi_2$, whence (X, π) is isomorphic to (X_1, π_1) . We proceed by induction on the number of elementary links to conclude that (X, π_X) is birational to (Y, π_Y) . Moreover,

if π_X has at least 8 singular fibres, then no link of type *IV* may occur, so ψ is a birational map of conic bundles $(X, \pi_X) \dashrightarrow (Y, \pi_Y)$. \square

When the conic bundles are minimal, we can strengthen Theorem 6.1 to get an isomorphism between the real parts.

Theorem 6.3. *Let $\pi_X: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ and $\pi_Y: Y \rightarrow \mathbb{P}_{\mathbb{R}}^1$ be two minimal real conic bundles, and suppose that either X or Y is non-rational. Then, the following are equivalent:*

- (1) X and Y are birational.
- (2) $X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic.
- (3) $(X(\mathbb{R}), \pi_X)$ and $(Y(\mathbb{R}), \pi_Y)$ are isomorphic.

Proof. The implications (3) \Rightarrow (2) \Rightarrow (1) being evident, it suffices to prove (1) \Rightarrow (3). Since X and Y are not rational, both π_X and π_Y have at least one singular fibre. Applying Proposition 4.15, we may assume that both (X, π_X) and (Y, π_Y) are minimal exceptional real conic bundles. Then, since (X, π_X) and (Y, π_Y) are birational (Theorem 6.1), we may assume that $I(X, \pi_X) = I(Y, \pi_Y)$, up to an automorphism of $\mathbb{P}_{\mathbb{R}}^1$. Then Lemma 4.8 shows that (X, π_X) is isomorphic to (Y, π_Y) . \square

We are now able to prove Theorem 1.5 concerning minimal surfaces.

Proof of Theorem 1.5. Let X and Y be two minimal geometrically rational real surfaces, and assume that either X or Y is non-rational.

If $X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic, it is clear that X and Y are birational. Let us prove the converse.

Theorem 3.5 lists all the possibilities for X . If $\rho(X) = 1$ or $\rho(Y) = 1$, Proposition 3.8 shows that X is isomorphic to Y . Otherwise, since neither X nor Y is rational, there exist minimal conic bundle structures on X and on Y . From Theorem 6.3, we conclude that $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$. \square

To go further with non-minimal surfaces, we need to know when the group $\text{Aut}(X(\mathbb{R}))$ is very transitive for X minimal. This is done in the next sections.

7. VERY TRANSITIVE ACTIONS

Thanks to the work done in Section 4, it is easy to apply the techniques of [HM07] to prove that $\text{Aut}(X(\mathbb{R}))$ is fiberwise very transitive on a real conic bundle. After describing the transitivity of $\text{Aut}(X(\mathbb{R}))$ on the tangent space of a general point, we set the main result of that section: $\text{Aut}(X(\mathbb{R}))$ is very transitive on connected components when X is minimal and admits two conic bundle structures (Proposition 7.5). We end the section by giving a characterisation of surfaces X for which $\text{Aut}(X(\mathbb{R}))$ is able to mix the connected components of $X(\mathbb{R})$.

Lemma 7.1. *Let (X, π) be a minimal real conic bundle over $\mathbb{P}_{\mathbb{R}}^1$ with at least one singular fibre. Let (p_1, \dots, p_n) and (q_1, \dots, q_n) be two n -tuples of distinct points of $X(\mathbb{R})$, and let (b_1, \dots, b_m) be m points of $I(X, \pi)$. Assume that $\pi(p_i) = \pi(q_i)$ for each i , that $\pi(p_i) \neq \pi(p_j)$ for $i \neq j$ and that $\pi(p_i) \neq b_j$ for any i and any j .*

Then, there exists $\alpha \in \text{Aut}(X(\mathbb{R}))$ such that $\alpha(p_i) = q_i$ for every i , $\pi\alpha = \pi$ and $\alpha|_{\pi^{-1}(b_i)}$ is the identity for every i .

Remark 7.2. The same result holds for minimal real conic bundles with no singular fibre, see [BH07, 5.4]. The following proof uses *twisting maps*, see below, which were introduced in [HM07] to prove that the action of the automorphism group $\text{Aut}(S^2)$ on the quadric sphere $S^2 := \{(x : y : z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 0\}$ is very transitive.

Proof. By Proposition 4.15, we may assume that (X, π) is exceptional. Moreover, Corollary 4.11 yields the existence of an affine real surface $A \subset X$ isomorphic to the hypersurface of \mathbb{R}^3 given by

$$y^2 + z^2 = -\prod_{i=1}^{2r} (x - a_i),$$

for some $a_1, \dots, a_{2r} \in \mathbb{R}$ with $a_1 < a_2 < \dots < a_{2r}$, where $\pi|_A$ corresponds to the projection $(x, y, z) \mapsto x$, and where the inclusion $A \subset X$ induces an isomorphism $A(\mathbb{R}) \rightarrow X(\mathbb{R})$.

For $i = 1, \dots, n$, let us denote by (x_i, y_i, z_i) the coordinates of p_i in $A \subset \mathbb{R}^3$ and by (u_i, v_i, w_i) the ones of q_i . From hypothesis, we have $x_i = u_i$ for all i , thus we get $y_i^2 + z_i^2 = v_i^2 + w_i^2$ for all i . Let $\Phi_i \in \text{SO}_2(\mathbb{R})$ be the rotation sending (x_i, y_i) to (u_i, v_i) . Then by [HM07, Lemma 2.2], there exists an algebraic map $\Phi: [a_1, a_{2r}] \rightarrow \text{SO}_2(\mathbb{R})$ such that $\Phi(x_i) = \Phi_i$ for $i = 1, \dots, n$ and $\Phi(b_i)$ is the identity for $i = 1, \dots, m$. Let us recall the proof; since $\text{SO}_2(\mathbb{R})$ is isomorphic to the unit circle $S^1 := \{(x : y : z) \in \mathbb{P}^2(\mathbb{R}) \mid x^2 + y^2 = z^2\}$, it suffices to prove the statement for S^1 instead of $\text{SO}_2(\mathbb{R})$. Let Φ_0 be a point of S^1 distinct from Φ_1, \dots, Φ_n and from the identity. Since $S^1 \setminus \{\Phi_0\}$ is isomorphic to \mathbb{R} , it suffices, finally, to prove the statement for \mathbb{R} instead of $\text{SO}_2(\mathbb{R})$. The latter statement is an easy consequence of Lagrange polynomial interpolation.

Then the map defined by $\alpha: (x, y, z) \mapsto (x, (y, z) \cdot \Phi(x))$ induces an automorphism $A(\mathbb{R}) \rightarrow A(\mathbb{R})$ called the *twisting map* of π associated to Φ . Moreover, $\alpha(p_i) = q_i$, for all i , $\pi\alpha = \pi$, $\alpha|_{\pi^{-1}(b_i)}$ is the identity for every i , and π induces an automorphism $X(\mathbb{R}) \rightarrow X(\mathbb{R})$. \square

Lemma 7.3. *Let (X, π) be a minimal real conic bundle over $\mathbb{P}_{\mathbb{R}}^1$ with at least one singular fibre. Let $p \in X$ be a point in a nonsingular fibre of π , and let $\Sigma \subset I(X, \pi)$ be a finite subset, with $\pi(p) \in \Sigma$. Denote by $\eta: Y \rightarrow X$ the blow-up of p , and by $E \subset Y$ the exceptional curve. Let $q \in E$ the point corresponding to the direction of the fibre of π passing through p .*

Then, the lift of the group

$$G = \left\{ \alpha \in \text{Aut}(X(\mathbb{R})), \pi\alpha = \pi \mid \alpha|_{\pi^{-1}(\Sigma)} \text{ is the identity} \right\}$$

by η is a subgroup $\eta^{-1}G\eta \subset \text{Aut}(Y(\mathbb{R}))$ which fixes the point q , and acts transitively on $E \setminus q \cong \mathbb{A}_{\mathbb{R}}^1$.

Proof. Since G acts identically on $\pi^{-1}(\Sigma)$, it fixes p , and therefore lifts to $H = \eta^{-1}G\eta \subset \text{Aut}(Y(\mathbb{R}), \pi\eta)$, which preserves E . Moreover, G preserves the fibre of π passing through p , so H preserves its strict transform, which intersects transversally E at q , so q is fixed.

Let us prove now that the action of $\eta^{-1}G\eta$ on $E \setminus q$ is transitive. By Proposition 4.15, we may assume that (X, π) is exceptional. Then, we take an affine surface $A \subset X$, isomorphic to the hypersurface $y^2 + z^2 = P(x)$ of \mathbb{R}^3 for some polynomial

P , such that $A|_\pi$ is the projection $\text{pr}_x: (x, y, z) \mapsto x$ and the inclusion $A \subset X$ gives an isomorphism $A(\mathbb{R}) \rightarrow X(\mathbb{R})$ (Corollary 4.11). Let us write $(x_0, y_0, z_0) \in \mathbb{R}^3$ the coordinates of p . Since x is on a nonsingular fibre of π , then $P(x_0) > 0$. Up to an affine automorphism of \mathbb{R}^3 , and up to multiplication of P by some constant, we may assume that $x_0 = 0$, $P(0) = 1$, $y_0 = 0$, and $z_0 = 0$.

To any real polynomial $\lambda \in \mathbb{R}[X]$, we associate the matrix

$$\begin{pmatrix} \alpha(X) & \beta(X) \\ -\beta(X) & \alpha(X) \end{pmatrix} \in \text{SO}_2(\mathbb{R}(X)),$$

where $\alpha = \frac{1-\lambda^2}{1+\lambda^2} \in \mathbb{R}(X)$ and $\beta = \frac{2\lambda}{1+\lambda^2} \in \mathbb{R}(X)$. And corresponding to this matrix, we associate the map

$$\psi_\lambda: (x, y, z) \mapsto (x, \alpha(x) \cdot y - \beta(x) \cdot z, \beta(x) \cdot y + \alpha(x) \cdot z),$$

which belongs to $\text{Aut}(A(\mathbb{R}), \text{pr}_x)$. To impose that ψ_λ is the identity on $(\text{pr}_x)^{-1}(\Sigma)$ is the same to ask that $\lambda(x) = 0$ for each $(x : 1) \in \Sigma \subset \mathbb{P}^1(\mathbb{R})$, and in particular for $x = 0$.

Denote by $\mathcal{O} = \mathbb{R}[x, y, z]/(y^2 + z^2 - P(x))$ the ring of functions of A , by $\mathfrak{p} \subset \mathcal{O}$ the ideal of functions vanishing at p , by $\mathcal{O}_{\mathfrak{p}}$ the localisation, and by $\mathfrak{m} \subset \mathcal{O}_{\mathfrak{p}}$ the maximal ideal of $\mathcal{O}_{\mathfrak{p}}$. Then, the cotangent ring $T_{p,A}^*$ of p in A is equal to $\mathfrak{m}/\mathfrak{m}^2$, and is generated by the images $[x]$, $[y]$, $[z - 1]$ of $x, y, z - 1 \in \mathbb{R}[x, y, z]$. Since $P(0) = 1$, we may write $P(x) = 1 + xQ(x)$, for some real polynomial Q . We compute

$$[0] = [y^2 + z^2 - P(x)] = [y^2 + (z-1)^2 + 2(z-1) - xQ(x)] = [2(z-1) - xQ(0)] \in \mathfrak{m}/\mathfrak{m}^2.$$

We see that $[z - 1] = [xQ(0)/2]$, thus $\mathfrak{m}/\mathfrak{m}^2$ is generated by $[x]$ and $[y]$ as a \mathbb{R} -module. Since $\lambda(0) = 0$, we can write $\lambda(x) = x\mu(x)$, for some real polynomial μ . The linear action of ψ_λ on the cotangent space $T_{p,A}^*$ fixes $[x]$ and sends $[y]$ onto

$$\begin{aligned} [\alpha(x) \cdot y - \beta(x) \cdot z] &= \left[\frac{(1-\lambda(x)^2)y - 2\lambda(x)z}{\lambda(x)^2 + 1} \right] = [y - 2\lambda(x)(1 + xQ(0)/2)] \\ &= [y - 2\mu(0)x]. \end{aligned}$$

It suffices to change the derivative of λ at 0 (which is equal to $\mu(0)$), which may be any real number. Therefore, the action of G on the projectivisation of $T_{p,A}^*$, fixes a point (corresponding to $[x]$) but acts transitively on the complement of this point. Since E corresponds to the projectivisation of $T_{p,A}$, G acts transitively on $E \setminus q$. \square

Lemma 7.4. *Let X be a real projective surface endowed with two minimal conic bundles $\pi_1: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ and $\pi_2: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ inducing distinct foliations on $X(\mathbb{C})$. Let F_j be a real fibre of π_j , $j = 1, 2$. If $F_1(\mathbb{R}) \cap F_2(\mathbb{R}) \neq \emptyset$, then at most one of the curves F_j can be singular.*

Proof. Suppose the converse for contradiction. Then, F_i is the union of two (-1) -curves $E_{i,1}$ and $E_{i,2}$, intersecting transversally at some point p_i . Since p_i is the only real point of F_i , we have $p_1 = p_2$. Hence, $E_{2,1} \cdot F_1 \geq 2$.

According to Lemma 5.3, X is a del Pezzo surface of degree 2 or 4. Denote by S the complex surface obtained by forgetting the real structure on X , and by $\eta: S \rightarrow \mathbb{P}_{\mathbb{C}}^2$ the birational map which is the blow-up of p_1, \dots, p_m , $m = 5$ or $m = 7$, and which sends the fibres of π_1 on lines passing through p_1 (Lemma 5.2). The curves $E_{2,1}$ and $E_{2,2}$ having self-intersection -1 , these are the strict transform of lines or conics of \mathbb{P}^2 passing through 3 or 5 of the p_i . Since both curves intersect the fibres of π_1 into at least 2 points, the curves are conics not passing through p_1 .

This means that $m = 7$ and that the two conics intersect nowhere except at four of the points p_2, \dots, p_7 . This is impossible since $E_{2,1}$ and $E_{2,2}$ intersect at p . \square

We now use Lemma 7.1 to show that the action of $\text{Aut}(X(\mathbb{R}))$ is very transitive on connected components when X is a surface with two conic bundles.

Proposition 7.5. *Let X be a real projective surface, which admits two minimal conic bundles $\pi_1: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ and $\pi_2: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ inducing distinct foliations on $X(\mathbb{C})$.*

Let (p_1, \dots, p_n) and (q_1, \dots, q_n) be two n -tuples of distinct points of $X(\mathbb{R})$ such that p_i and q_i belong to the same connected component for each i . Then, there exists an element of $\text{Aut}(X(\mathbb{R}))$ which sends p_i on q_i for each i , and which sends each connected component of $X(\mathbb{R})$ on itself.

Proof. When X is rational, the result follows from [HM07, Theorem 1.4]. Thus we assume that X is non-rational, and in particular that $X(\mathbb{R})$ is non-connected.

From Lemma 7.4, any real point which is critical for one fibration is not critical for the second fibration. Otherwise speaking (recall that the fibrations are minimal) a real intersection point of a fibre F_1 of π_1 with a fibre F_2 of π_2 cannot be a singular point of F_1 and of F_2 at the same time. By Lemma 7.1 applied to (X, π_1) , and to (X, π_2) , we may assume without loss of generality that all points $p_1, \dots, p_n, q_1, \dots, q_n$ belong to smooth fibres of π_1 and to smooth fibres of π_2 . We now use Lemma 7.1 to obtain an automorphism α of $(X(\mathbb{R}), \pi_1)$ such that $\pi_2(\alpha(p_i)) \neq \pi_2(\alpha(p_j))$ and $\pi_2(\alpha(q_i)) \neq \pi_2(\alpha(q_j))$ for $i \neq j$. Hence, we may suppose that $\pi_2(p_i) \neq \pi_2(p_j)$ and $\pi_2(q_i) \neq \pi_2(q_j)$ for $i \neq j$.

Likewise, using an automorphism of $(X(\mathbb{R}), \pi_2)$ we may suppose that $\pi_1(p_i) \neq \pi_1(p_j)$ and $\pi_1(q_i) \neq \pi_1(q_j)$ for $i \neq j$.

We now show that for $i = 1, \dots, m$, there exists an element $\alpha_i \in \text{Aut}(X(\mathbb{R}))$ that sends p_i on q_i and that restricts to the identity on the sets $\cup_{j \neq i} \{p_j\}$ and $\cup_{j \neq i} \{q_j\}$. Then, the composition of the α_i will achieve the proof. Observe that $\zeta = \pi_1 \times \pi_2$ gives a finite surjective morphism $X \rightarrow \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$ which is 2-to-1 or 4-to-1 depending of the degree of X (follows from assertion (c) of Lemma 5.3). Denote by W the image of $X(\mathbb{R})$. The map $X(\mathbb{R}) \rightarrow W$ is a differential map, which has topological finite degree. Denote by W_i the connected component of W which contains both $\zeta(p_i)$ and $\zeta(q_i)$. Observe that W_i is contained in the square $I(X, \pi_1) \times I(X, \pi_2)$, and that for each point $x \in W_i$, the intersection of the horizontal and vertical lines (fibres of the two projections of $\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$) passing through x with W_i is either only $\{x\}$, when x is on the boundary of W_i , or is a bounded interval. Moreover, W_i is connected. Then, there exists a path from $\zeta(p_i)$ to $\zeta(q_i)$ which is a sequence of vertical or horizontal segments contained in W_i . We may furthermore assume that none of the segments is contained in $(\text{pr}_1)^{-1}(\pi_1(a))$ or $(\text{pr}_2)^{-1}(\pi_2(a))$ for any $a \in (\cup_{j \neq i} \{p_j\}) \cup (\cup_{j \neq i} \{q_j\})$. Denote by r_1, \dots, r_l the points of U that are sent on the singular points or ending points of the path, and by s_1, \dots, s_l some points of $X(\mathbb{R})$ which are sent by ζ on r_1, \dots, r_l respectively. Up to renumbering, $s_1 = p_i, s_l = q_i$ and two consecutive points s_j and s_{j+1} are such that $\pi_1(s_j) = \pi_1(s_{j+1})$ or $\pi_2(s_j) = \pi_2(s_{j+1})$. We construct then α_i as a composition of $l - 1$ maps, each one belonging either to $\text{Aut}(X(\mathbb{R}), \pi_1)$ or $\text{Aut}(X(\mathbb{R}), \pi_2)$ and sending s_j on s_{j+1} , and fixing the points $(\cup_{j \neq i} \{p_j\}) \cup (\cup_{j \neq i} \{q_j\})$. \square

The following proposition describes the possible mixes of connected components.

Proposition 7.6. *Let (X, π) be a minimal real conic bundle. Denote by I_1, \dots, I_r the r connected components of $I(X, \pi)$, and by M_1, \dots, M_r the r connected components of $X(\mathbb{R})$, where $I_i = \pi(M_i)$, $M_i = \pi^{-1}(I_i) \cap X(\mathbb{R})$. If $\nu \in \text{Sym}_r$ is a permutation of $\{1, \dots, r\}$, the following are equivalent:*

- (1) *there exists $\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$ such that $\alpha(I_i) = I_{\nu(i)}$ for each i ;*
- (2) *there exists $\beta \in \text{Aut}(X(\mathbb{R}), \pi)$ such that $\beta(M_i) = M_{\nu(i)}$ for each i ;*
- (3) *there exists $\beta \in \text{Aut}(X(\mathbb{R}))$ such that $\beta(M_i) = M_{\nu(i)}$ for each i ;*
- (4) *there exist two real Zariski open sets $V, W \subset X$, and $\beta \in \text{Bir}(X)$, inducing an isomorphism $V \rightarrow W$, such that $\beta(V(\mathbb{R}) \cap M_i) = W(\mathbb{R}) \cap M_{\nu(i)}$ for each i .*

Moreover, the conditions are always satisfied when $r \leq 2$, and are in general not satisfied when $r \geq 3$.

Proof. The implications (2) \Rightarrow (1) and (2) \Rightarrow (3) \Rightarrow (4) are obvious.

Let us prove (1) \Rightarrow (2). According to Proposition 4.15, we may assume that (X, π) is exceptional. The results follows from Lemma 4.9.

We prove now that if $r \leq 2$, Assertion (1) is always satisfied, hence all the conditions are equivalent (since all are true). When $r \leq 1$, take α to be the identity. When $r = 2$, we make a linear change of coordinates to the effect that $I_1 = \{(x : 1) \mid 0 \leq x \leq 1\}$ and I_2 is bounded by $(1 : 0)$ and $(\lambda : 1)$, for some $\lambda \in \mathbb{R}$, $\lambda > 1$ or $\lambda < 0$. Then, $\alpha: (x_1 : x_2) \mapsto (\lambda x_2 : x_1)$ is an involution which exchanges I_1 and I_2 .

It remains to prove the implication (4) \Rightarrow (1) for $r \geq 3$. We decompose β into elementary links

$$X = X_0 \xrightarrow{\beta_1} X_1 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-1}} X_{n-1} \xrightarrow{\beta_n} X_n = X$$

as in [Isk96, Theorem 2.5]. It follows from the description of the links of [Isk96, Theorem 2.6] that each of the links is of type *II* or *IV*, and that the links of type *II* are birational maps of conic bundles and the links of type *IV* occur on del Pezzo surfaces of degree 2.

In consequence, each of the X_i admits a conic bundle structure given by $\pi_i: X_i \rightarrow \mathbb{P}_{\mathbb{R}}^1$, where $\pi_0 = \pi_n = \pi$, and if β_i has type *II*, it is a birational map of conic bundles $(X_{i-1}, \pi_{i-1}) \dashrightarrow (X_i, \pi_i)$, and if it has type *IV*, it is an isomorphism $X_{i-1} \rightarrow X_i$ which does not send the general fibre of π_{i-1} on those of π_i . In this latter case, since π_i and $\pi_{i-1} \circ \beta_i$ have distinct general fibres, X_{i-1} and X_i are del Pezzo surfaces of degree 2, and the Geiser involution $\iota_{i-1} \in \text{Aut}(X_{i-1})$ exchanges the two general fibres (follows from [Isk96, Theorem 2.6], but also from Lemma 5.3). This means that the map $\beta_i \circ \iota_{i-1}$, that we denote by γ_i , is an isomorphism of conic bundles $(X_{i-1}, \pi_{i-1}) \rightarrow (X_i, \pi_i)$.

Now, we prove by induction on the number of links of type *IV* that β may be decomposed into compositions of elements of $\text{Bir}(X, \pi)$ and maps of the form $\psi \iota \psi^{-1}$ where ψ is a birational map of conic bundles $(X, \pi) \dashrightarrow (X', \pi')$, (X', π') is a del Pezzo surface of degree 2 and $\iota \in \text{Aut}(X')$ is the Geiser involution. If there is no link of type *IV*, β preserves the conic bundle structure given by π . Otherwise, denote by β_i the first link of type *IV*, which is an isomorphism $\beta_i: X_i \rightarrow X_{i+1}$, and write $\beta_i = \gamma_i \circ \iota_{i-1}$ as before. We write $\psi = \beta_{i-1} \circ \dots \circ \beta_1$, which is a birational map of conic bundles $\psi: (X, \pi) \dashrightarrow (X_i, \pi_i)$. Then, $\beta = (\beta_n \circ \dots \circ \beta_{i+1} \circ \gamma_i \circ \psi)(\psi^{-1} \iota_{i-1} \psi)$. Applying the induction hypothesis on the map $(\beta_n \circ \dots \circ \beta_{i+1} \circ \gamma_i \circ \psi) \in \text{Bir}(X)$, we are done.

Now, observe that when (X', π') is a minimal real conic bundle and X' is a del Pezzo surface of degree 2, the map $\zeta: X' \rightarrow \mathbb{P}_{\mathbb{R}}^2$ given by $|-K_{X'}|$ is a double covering, ramified over a smooth quartic curve $\Gamma \subset \mathbb{P}_{\mathbb{R}}^2$ (see e.g. [Dem76]). Since (X, π) is minimal, π has 6 singular fibres (Lemma 4.2), so $I(X, \pi)$ is the union of three intervals and $X(\mathbb{R})$ is the union of 3 connected components (Corollary 4.13). This implies that $\Gamma(\mathbb{R})$ is the union of three disjoint ovals. A connected component M of $X(\mathbb{R})$ is homeomorphic to a sphere, and surjects by ζ to the interior of one of the three ovals. The Geiser involution (induced by the double covering) induces an involution on M , which fixes the preimage of the oval. This means that the Geiser involution sends any connected component of $X(\mathbb{R})$ on itself. Thus, in the decomposition of β into elements of $\text{Bir}(X, \pi)$ and conjugate elements of Geiser involutions, the only relevant elements are those of $\text{Bir}(X, \pi)$. There exists thus $\beta' \in \text{Bir}(X, \pi)$ which acts on the connected components of $X(\mathbb{R})$ in the same way as β . This shows that (4) implies (1).

We finish by proving that (1) is false in general, when $r \geq 3$. This follows from the fact that if Σ is a general finite subset of $2r$ distinct points of $\mathbb{P}_{\mathbb{R}}^1$, the group $\{\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1) \mid \alpha(\Sigma) = \Sigma\}$ is trivial. Supposing this fact true, we obtain the result by applying it to the $2r$ boundary points of $I(X, \pi)$. Let us prove the fact. The set of $2r$ -tuples of $\mathbb{P}_{\mathbb{R}}^1$ is an open subset W of $(\mathbb{P}_{\mathbb{R}}^1)^{2r}$. For any non-trivial permutation $v \in \text{Sym}_{2r}$, we denote by $W_v \subset W$ the set of points $a = (a_1, \dots, a_{2r}) \in W$ such that there exists $\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$ with $\alpha(a_i) = a_{v(i)}$ for each i . Let $a \in W_v$, and take two 4-tuples Σ_1, Σ_2 of a_i 's with $\Sigma_1 \neq \Sigma_2$ and $\Sigma_2 = v(\Sigma_1)$ (this is possible since v is non-trivial). Then, the cross-ratio of the a_i 's in Σ_1 and in Σ_2 are the same. This implies a non-trivial condition on W . Consequently, W_v is contained in a closed subset of W . Doing this for all non-trivial permutations v , we obtain the result. \square

8. REAL ALGEBRAIC MODELS

The aim of this section is to go further with non-minimal surfaces with 2 or 3 connected components. We begin to show how *to separate* infinitely near points to the effect that any such a surface $Y(\mathbb{R})$ is isomorphic to a blow-up $B_{a_1, \dots, a_m} X(\mathbb{R})$ where X is minimal and a_1, \dots, a_m are distinct proper points of $X(\mathbb{R})$. Then, we replace $X(\mathbb{R})$ by an isomorphic del Pezzo model (Corollary 5.4) and we use the fact that $\text{Aut}(X(\mathbb{R}))$ is very transitive on connected components for such an X (Proposition 7.5) to prove that in many cases, if two birational surfaces Y and Z have homeomorphic real parts then $Y(\mathbb{R})$ and $Z(\mathbb{R})$ are isomorphic. As a corollary, we get that in any cases, $\text{Aut}(Y(\mathbb{R}))$ is very transitive on connected components.

Proposition 8.1. *Let X be a minimal geometrically rational real surface, with $\#X(\mathbb{R}) = 2$ or $\#X(\mathbb{R}) = 3$, and let $\eta: Y \rightarrow X$ be a birational morphism.*

Then there exists a blow-up $\eta': Y' \rightarrow X$, whose centre is a finite number of distinct real proper points of X , and such that $Y'(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$.

Moreover, we can assume that the isomorphism $Y(\mathbb{R}) \rightarrow Y'(\mathbb{R})$ induces an homeomorphism $\eta^{-1}(M) \rightarrow (\eta')^{-1}(M)$ for each connected component M of $X(\mathbb{R})$.

Proof. According to Corollary 5.4, we may assume that X admits two minimal conic bundles $\pi_1: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ and $\pi_2: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ inducing distinct foliations on $X(\mathbb{C})$. Preserving the isomorphism class of $Y(\mathbb{R})$, we may assume that the points in the centre of η are all real (such a point may be a proper point of $X(\mathbb{R})$ or an infinitely near point). Let us denote by $m (= K_X^2 - K_Y^2)$ the number of those points. We prove the result by induction on m .

The cases $m = 0$ and $m = 1$ being obvious (take $\eta' = \eta$), we assume that $m \geq 2$. We decompose η as $\eta = \theta \circ \varepsilon$, where $\varepsilon: Y \rightarrow Z$ is the blow-up of one real point $q \in Z$, and $\theta: Z \rightarrow X$ is the blow-up of $m - 1$ real points. By induction hypothesis, we may assume that θ is the blow-up of $m - 1$ proper points of X , namely $a_1, \dots, a_{m-1} \in X(\mathbb{R})$. Moreover, applying Proposition 7.5, we may move the points by an element of $\text{Aut}(X(\mathbb{R}))$, and assume that $\pi_1(a_i) \neq \pi_1(a_j)$ and $\pi_2(a_i) \neq \pi_2(a_j)$ for $i \neq j$, and that the fibre of π_1 passing through a_i and the fibre of π_2 passing through a_i are nonsingular and transverse at a_i , for each i .

If $\theta(q) \notin \{a_1, \dots, a_{m-1}\}$, then η is the blow-up of m distinct proper points of X , hence we are done. Otherwise, assume that $\theta(q) = a_1$. We write $E = \theta^{-1}(a_1) \subset Z$, and denote by $F_i \subset Z$ the strict pull-back by η of the fibre of π_i passing through a_i , for $i = 1, 2$. Then, F_1 and F_2 are two (-1) -curves which do not intersect. Hence, the point $q \in E$ belongs to at most one of the two curves, so we may assume that $q \notin F_1$. Denote by $\theta_2: Z \rightarrow X_2$ the contraction of the $m - 1$ disjoint (-1) -curves $F_1, \theta^{-1}(a_2), \dots, \theta^{-1}(a_{m-1})$. Since q does not belong to any of these curves, $\eta_2 = \theta_2 \circ \varepsilon$ is the blow-up of $m - 1$ distinct proper points of X_2 . It remains to find an isomorphism $\gamma: X_2(\mathbb{R}) \rightarrow X(\mathbb{R})$ such that for each connected component M of $X(\mathbb{R})$, $\gamma\eta_2$ sends $\eta^{-1}(M)$ on M .

Denoting $\pi' = \pi_1 \circ \theta \circ \theta_2^{-1}$, the map $\psi = \theta_2 \circ \theta^{-1}$ is a birational map of conic bundles $(X, \pi_1) \dashrightarrow (X_2, \pi')$, which factorizes as the blow-up of a_1 , followed by the contraction of the strict transform of the fibre passing through a_1 . Therefore, the conic bundle (X_2, π') is minimal. Since X is not rational and $\pi'\psi = \pi_1$, Corollary 4.17 yields the existence of an isomorphism $\gamma: X_2(\mathbb{R}) \rightarrow X(\mathbb{R})$ such that $\pi_1\gamma = \pi'$. Observe that $\gamma\eta_2 \circ \eta^{-1} = \gamma\theta_2 \circ \theta^{-1} = \gamma\psi$ is a birational map $X \dashrightarrow X$ which satisfies $\pi \circ (\gamma\eta_2 \circ \eta^{-1}) = \pi$. Consequently, for any connected component M of $X(\mathbb{R})$, which corresponds to $\pi^{-1}(V) \cap X(\mathbb{R})$, for some interval $V \subset \mathbb{P}_{\mathbb{R}}^1$, we find $\pi(\gamma\eta_2\eta^{-1}(M)) = \pi(M) = V$, thus $\gamma\eta_2$ sends $\eta^{-1}(M)$ on M . \square

Corollary 8.2. *Let X be a minimal geometrically rational real surface, such that $\#X(\mathbb{R}) = 2$ or $\#X(\mathbb{R}) = 3$, and let $\eta: Y \rightarrow X$, $\varepsilon: Z \rightarrow X$ be two birational morphisms. Denote by M_1, \dots, M_r the connected components of $X(\mathbb{R})$ ($r = 2, 3$). Then, the following are equivalent:*

- (1) $\eta^{-1}(M_i) \subset Y(\mathbb{R})$ and $\varepsilon^{-1}(M_i) \subset Z(\mathbb{R})$ are homeomorphic for each i ;
- (2) there exists an isomorphism $Y(\mathbb{R}) \rightarrow Z(\mathbb{R})$ which induces an homeomorphism $\eta^{-1}(M_i) \rightarrow \varepsilon^{-1}(M_i)$ for each i .

Proof. (2) \Rightarrow (1) being obvious, let us prove the converse. According to Proposition 8.1, we may assume that η and ε are the blow-ups of a finite number of distinct real proper points of X . Denote by Σ_η and Σ_ε these two finite sets. For each i , the fact that $\eta^{-1}(M_i) \subset Y(\mathbb{R})$ and $\varepsilon^{-1}(M_i) \subset Z(\mathbb{R})$ are homeomorphic implies that the numbers of points of $\Sigma_\eta \cap M_i$ and $\Sigma_\varepsilon \cap M_i$ coincide.

By Corollary 5.4 and Proposition 7.5, $\text{Aut}(X(\mathbb{R}))$ is very transitive on connected components of $X(\mathbb{R})$. In particular, there exists an element $\alpha \in \text{Aut}(X(\mathbb{R}))$ such that $\alpha(M_i) = M_i$ for each i and $\alpha(\Sigma_\eta) = \Sigma_\varepsilon$. Then, $\psi = \varepsilon^{-1}\alpha\eta: Y(\mathbb{R}) \rightarrow Z(\mathbb{R})$ is the wanted isomorphism. \square

Corollary 8.3. *Let Y be a geometrically rational real surface with $\#Y(\mathbb{R}) = 2$ or $\#Y(\mathbb{R}) = 3$. Let (p_1, \dots, p_n) and (q_1, \dots, q_n) be two n -tuples of distinct points of $Y(\mathbb{R})$ such that p_i and q_i belong to the same connected component for each i .*

Then, there exists an element $\alpha \in \text{Aut}(Y(\mathbb{R}))$, which fixes each connected component of $Y(\mathbb{R})$ and such that $\alpha(p_i) = q_i$ for each i .

Proof. Let $\eta: Y \rightarrow X$ be a birational morphism to a minimal real surface X ; observe that $\#X(\mathbb{R}) = \#Y(\mathbb{R})$. According to Corollary 5.4, we may assume that X admits two minimal conic bundles $\pi_1: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ and $\pi_2: X \rightarrow \mathbb{P}_{\mathbb{R}}^1$ inducing distinct foliations on $X(\mathbb{C})$. By Proposition 8.1, we can suppose that η is the blow-up of m distinct real proper points $a_1, \dots, a_m \in X$. We prove the result by induction on m .

If $m = 0$, which means that $X = Y$, the result follows from Proposition 7.5.

If $m > 0$, denote by $\eta_0: Z \rightarrow X$ the blow-up of a_1, \dots, a_{m-1} (η_0 is the identity if $m = 1$), and by $\eta_1: Y \rightarrow Z$ the blow-up of $b = \eta_0^{-1}(a_m)$.

Applying Proposition 7.5, we may assume that $\pi_1(a_i) \neq \pi_1(a_j)$ and $\pi_2(a_i) \neq \pi_2(a_j)$ for $i \neq j$, and that the fibre of π_1 passing through a_i and the fibre of π_2 passing through a_i are nonsingular and transverse at a_i , for each i . Let us denote by $E \subset Y$ the exceptional curve $\eta_1^{-1}(b)$ of η_1 and by F_i the strict transform on Y of the fibre of π_i passing through a_m , for $i = 1, 2$. Then E , F_1 and F_2 are three (-1) -curves, F_1 and F_2 do not intersect, and E intersect transversally each of the F_i . By induction hypothesis, we may use the lift of an element of $\text{Aut}(Z(\mathbb{R}))$ which fixes b to assume that no one of the points p_i belongs to $F_1 \setminus E$, $F_2 \setminus E$ or to $\eta^{-1}(a_i)$ for $i = 1, \dots, m-1$. Then the group $G = \{\alpha \in \text{Aut}(X(\mathbb{R})) \mid \pi_1 \alpha = \pi_1, \alpha \text{ fixes } a_1, \dots, a_m, \eta(p_1), \dots, \eta(p_n)\}$, acts transitively on $E \setminus F_1$ (Lemma 7.3). Lifting a well-chosen element of this group in $\text{Aut}(Y(\mathbb{R}))$, we may move the points p_i and assume that no one of the p_i belongs to F_2 (i.e. we can avoid $F_2 \cap E$). Denote by $\eta': Y \rightarrow X'$ the contraction of the disjoint (-1) -curves $F_2, \eta^{-1}(a_1), \dots, \eta^{-1}(a_{m-1})$.

Then, the birational map $\psi = \eta' \eta^{-1}: X \dashrightarrow X'$ is a birational map of conic bundles $(X, \pi_2) \dashrightarrow (X', \pi')$, where $\pi' = \pi_2 \psi^{-1}$, which consists of the blow-up of a_m , followed by the contraction of the strict transform of the fibre passing through a_m . Therefore, the conic bundle (X', π') is minimal. Since X is not rational, Corollary 4.17 yields the existence of an isomorphism $\gamma: X'(\mathbb{R}) \rightarrow X(\mathbb{R})$ such that $\pi_2 \gamma = \pi'$. Therefore, there exists an element $\beta \in \text{Aut}(X'(\mathbb{R}))$ which fixes all the points blown-up by η' , which fixes all the points $\{\eta'(p_i), p_i \notin E\}$, and which sends the points $\{\eta'(p_i), p_i \in E\}$ outside of $\eta'(E)$. Applying the lift of β on $\text{Aut}(Y(\mathbb{R}))$, we may assume that none of the points p_i belongs to E . Doing the same manipulation with the q_i , it remains to use the lift of an element of $\text{Aut}(Z(\mathbb{R}))$ which fixes b and sends $\eta_1(p_i)$ on $\eta_1(q_i)$ for each i . \square

9. PROOF OF THE MAIN RESULTS

The proof of Theorem 1.5 was given at the end of Section 5. Now, we deduce the others results stated in the introduction from the results of Sections 7 and 8. The following lemma serves to prove most of them.

Lemma 9.1. *Let (X, π) be a minimal real conic bundle, such that $I(X, \pi)$ is the union of r intervals I_1, \dots, I_r , with $r = 2$ or $r = 3$.*

Let $\eta_Y: Y \rightarrow X$ and $\eta_Z: Z \rightarrow X$ be two birational morphisms. For $i = 1, \dots, r$, we write $X_i = \pi^{-1}(I_i) \cap X(\mathbb{R})$, $Y_i = \eta_Y^{-1}(X_i) \cap Y(\mathbb{R})$ and $Z_i = \eta_Z^{-1}(X_i) \cap Z(\mathbb{R})$.

Let $p_1, \dots, p_n \in Y(\mathbb{R})$, $q_1, \dots, q_n \in Z(\mathbb{R})$ be two n -tuples of distinct points, and assume the existence of an homeomorphism $h: Y(\mathbb{R}) \rightarrow Z(\mathbb{R})$ which sends p_i on q_i

for each i , and sends Y_i on $Z_{\nu(i)}$, where $\nu \in \text{Sym}_r$ is a permutation of $\{1, \dots, r\}$. Then, the following are equivalent:

- (1) There exists an isomorphism $\beta: Y(\mathbb{R}) \rightarrow Z(\mathbb{R})$ which sends Y_i on $Z_{\nu(i)}$ for each $i \in \{1, \dots, r\}$ and sends p_j on q_j for each $j \in \{1, \dots, n\}$.
- (2) There exists an automorphism $\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1)$ which sends I_i on $I_{\nu(i)}$ for each $i \in \{1, \dots, r\}$.

Moreover, both assertions are true if $r = 2$, and false in general when $r = 3$.

Proof. Observe that the X_i (respectively the Y_i, Z_i) are the connected components of $X(\mathbb{R})$ (respectively of $Y(\mathbb{R}), Z(\mathbb{R})$).

[(1) \Rightarrow (2)] The map $\eta_Z \beta \eta_Y^{-1}$ is a birational self-map of X , which restricts to an isomorphism $\varphi: V \rightarrow W$, where V and W are two real Zariski open subsets of X . Moreover, the hypothesis on β implies that $\varphi(V(\mathbb{R}) \cap X_i) = W(\mathbb{R}) \cap X_{\nu(i)}$. The existence of α is provided by Proposition 7.6.

[(2) \Rightarrow (1)] Proposition 7.6 yields the existence of $\gamma \in \text{Aut}(X(\mathbb{R}), \pi)$ such that $\gamma(X_i) = X_{\nu(i)}$. We may thus assume that ν is the identity. According to Proposition 8.1, we may moreover suppose that η_Y and η_Z are the blow-ups of a finite set of disjoint real proper points of X . Since Y_i is homeomorphic to Z_i for each i , η_Y is the blow-up of a_1, \dots, a_m and η_Z is the blow-up of b_1, \dots, b_m , where a_j and b_j belong to the same connected component of $X(\mathbb{R})$ for each j . Then, there exists an element of $\text{Aut}(X(\mathbb{R}))$ which preserves each connected component of X and sends a_j on b_j for each j (Corollary 8.3). We may thus assume that $Y = Z$, and conclude by applying Corollary 8.3 to Y .

The fact that (2) is true when $r = 2$ and false in general when $r = 3$ was proved in Proposition 7.6. \square

Proof of Theorem 1.1. Let Y be a nonsingular geometrically rational real projective surface, with $\#Y(\mathbb{R}) = 2$. Let (p_1, \dots, p_n) and (q_1, \dots, q_n) be two n -tuples of points which are compatible. We want to prove the existence of $\alpha \in \text{Aut}(Y(\mathbb{R}))$ such that $\alpha(p_i) = q_i$ for each i .

If p_i and q_i are in the same connected component of $Y(\mathbb{R})$, the result follows from Corollary 8.3.

Otherwise, the compatibility means that the two components of $X(\mathbb{R})$ are homeomorphic and that p_i and q_i are in a distinct component for each i . Lemma 9.1 provides the existence of an element of $\text{Aut}(Y(\mathbb{R}))$ which permutes the two connected components of $Y(\mathbb{R})$. This reduces the situation to the previous case. \square

Theorem 1.2 is Corollary 8.3 applied to the case of 3 connected components.

Proof of Corollary 1.3. We prove firstly that if X is not geometrically rational, then $\text{Aut}(X(\mathbb{R}))$ is not very transitive. If X has Kodaira dimension 2, (surface of general type), it has only finitely many birational self-maps (see e.g. [Uen75].) If X has Kodaira dimension 1, every birational self-map of X preserves the elliptic fibration induced by $|K_X|$. If X has Kodaira dimension 0, and X is minimal, then $\text{Bir}(X) = \text{Aut}(X)$. The group $\text{Aut}(X)$ is an algebraic group of dimension 1 or 2 (its neutral component is an elliptic curve or an Abelian surface). Thus, $\text{Bir}(X)$ may not be 2-transitive. The case when X is not minimal is deduced from this case.

If X is a surface with Kodaira dimension $-\infty$, then X is uniruled. If furthermore, X is not geometrically rational and $X(\mathbb{R})$ is non-empty, then the Albanese map

$X \rightarrow C$ is a real ruling over a curve with genus $g(C) > 0$, see e.g. [Sil89, V.(1.8)], and the Albanese map is preserved by any birational self-map.

Assume now that X is a geometrically rational surface. When $\#X(\mathbb{R}) = 1$, X is rational; the fact that $\text{Aut}(X(\mathbb{R}))$ is very transitive is the main result of [HM07]. When $\#X(\mathbb{R}) = 2$, $\text{Aut}(X(\mathbb{R}))$ is very transitive by Theorem 1.1.

When $\#X(\mathbb{R}) > 3$, we prove now that the group $\text{Aut}(X(\mathbb{R}))$ is not transitive. Denote by $\eta: X \rightarrow X_0$ a birational morphism to a minimal real surface, and observe that $\#X_0(\mathbb{R}) = \#X(\mathbb{R}) > 3$. Let us discuss the two cases for X_0 given by Theorem 3.5. If X_0 is a del Pezzo surface with $\rho(X_0) = 1$, then $\text{Aut}(X(\mathbb{R}))$ is countable (Corollary 3.9), thus $\text{Aut}(X(\mathbb{R}))$ cannot be transitive. The other case is when $\rho(X_0) = 2$. Then, X_0 endows a real conic bundle structure (X_0, π_0) , and $\text{Bir}(X_0) = \text{Bir}(X_0, \pi_0)$ (Theorem 6.1). Since the action of $\text{Bir}(X_0, \pi_0)$ on the basis of the conic bundle is finite (there are too much boundary points), neither $\text{Aut}(X_0(\mathbb{R}))$ nor $\text{Aut}(X(\mathbb{R}))$ may be transitive.

When $\#X(\mathbb{R}) = 3$, $\text{Aut}(X(\mathbb{R}))$ is very transitive on connected components (Theorem 1.2). Thus, $\text{Aut}(X(\mathbb{R}))$ is very transitive if and only if for any homeomorphism $h: X(\mathbb{R}) \rightarrow X(\mathbb{R})$, there exists $\alpha \in \text{Aut}(X(\mathbb{R}))$ which permutes the components of $X(\mathbb{R})$ in the same way that h does. The possibilities when this occur are described by Lemma 9.1. For example, when X is minimal, it admits a minimal real conic bundle structure (X, π) (Theorem 3.5 and Proposition 3.7), where π has 6 singular fibres. Then, $\text{Aut}(X(\mathbb{R}))$ is very transitive if and only if $\{\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{R}}^1) \mid \alpha(I(X, \pi)) = I(X, \pi)\}$ acts transitively on the three intervals of $I(X, \pi)$. This is true in some special cases, but false in general. When X is not minimal, $\text{Aut}(X(\mathbb{R}))$ is very transitive for example when the three connected components of $X(\mathbb{R})$ are not homeomorphic 2-by-2, or when X is the blow-up of a minimal surface Y with a very transitive group $\text{Aut}(Y(\mathbb{R}))$. \square

Proof of Theorem 1.4. Let X, Y be two geometrically rational real surfaces, and assume that $\#X(\mathbb{R}) \leq 2$. We assume that X is birational to Y and that $X(\mathbb{R})$ is homeomorphic to $Y(\mathbb{R})$, and prove that $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$.

Remark that all geometrically rational surfaces with connected real part are birational to each others, thus in this case the statement follows from the unicity of rational models [BH07]. We may thus assume that $\#X(\mathbb{R}) = 2$. Denote by $\eta_X: X \rightarrow X_0$ and $\eta_Y: Y \rightarrow Y_0$ birational morphisms to minimal real surfaces.

Since X_0 and Y_0 are birational, $X_0(\mathbb{R})$ and $Y_0(\mathbb{R})$ are isomorphic (Theorem 1.5), so we may assume that $X_0 = Y_0$. The result now follows from Lemma 9.1. \square

Proof of Corollary 1.6. If M is connected, and M is non-orientable or M is orientable with genus $g(M) \leq 1$, then it admits a unique geometrically rational model by [BH07, Corollary 8.1]. Moreover, this model is in fact rational.

Conversely let M be a compact \mathcal{C}^∞ -surface and assume that M admits a unique geometrically rational model X . The existence of such a model implies, by Comessatti's theorem [Com14], that any connected component of M is non-orientable or is orientable with genus $g \leq 1$. The unicity means that for any geometrically rational model Y of M , then $Y(\mathbb{R})$ is isomorphic to $X(\mathbb{R})$. In particular, this implies that all geometrically rational models of M belong to a unique birational class. From Theorem 6.1 and Proposition 3.8, this means that X is rational. It remains to observe that when X is rational, $X(\mathbb{R})$ is connected, and is either non-orientable or orientable of genus ≤ 1 . When X is minimal, this follows from Proposition 3.7.

Then, blowing-up points on a surface either does nothing on the topology of the real part (if the points blown-up are imaginary), or it gives a non-orientable real part (if the points blown-up are real). \square

We finish by a result on non-density. In [KM08], it is proved that $\text{Aut}(X(\mathbb{R}))$ is dense in $\text{Diff}(X(\mathbb{R}))$ when X is a geometrically rational surface with $\#X(\mathbb{R}) = 1$ (or equivalently when X is rational). In the cited paper, it is said that $\#X(\mathbb{R}) = 2$ is probably the only other case where the density holds. The following collect the known results in this direction. The first two of them are new.

Proposition 9.2. *Let X be a geometrically rational surface.*

- *If $\#X(\mathbb{R}) \geq 5$, then $\text{Aut}(X(\mathbb{R}))$ is not dense in $\text{Diff}(X(\mathbb{R}))$;*
- *if $\#X(\mathbb{R}) = 3$ or $\#X(\mathbb{R}) = 4$, then $\text{Aut}(X(\mathbb{R}))$ is not dense in $\text{Diff}(X(\mathbb{R}))$ for a general X , but could be dense in some special cases;*
- *if $\#X(\mathbb{R}) = 1$, then $\text{Aut}(X(\mathbb{R}))$ is dense in $\text{Diff}(X(\mathbb{R}))$.*

Proof. The case $\#X(\mathbb{R}) = 1$ is the main result of [KM08]. Assume from now on that $\#X(\mathbb{R}) \geq 3$, and denote by $\eta: X \rightarrow X_0$ a birational morphism to a minimal real surface, and observe that $\#X_0(\mathbb{R}) = \#X(\mathbb{R}) \geq 3$. Let us discuss the two cases for X_0 given by Theorem 3.5.

Assume that X_0 is a del Pezzo surface with $\rho(X_0) = 1$. If the degree of X_0 is 1 then $\text{Bir}(X_0)$ is finite (Corollary 3.9), thus $\text{Aut}(X(\mathbb{R}))$ cannot be dense. If X_0 has degree 2, then $\#X_0(\mathbb{R}) = 4$ (Proposition 3.7), so $\#X(\mathbb{R}) = 4$ too. Since $\text{Aut}(X_0(\mathbb{R})) = \text{Aut}(X_0)$ is finite, $\text{Aut}(X_0(\mathbb{R}))$ cannot be dense (but maybe $\text{Aut}(X(\mathbb{R}))$ could be).

The other case is when $\rho(X_0) = 2$. Then, X_0 endows a real conic bundle structure (X_0, π_0) . If $\#X(\mathbb{R}) = \#X_0(\mathbb{R}) \geq 4$, then $\text{Bir}(X_0) = \text{Bir}(X_0, \pi_0)$ (Theorem 6.1), so $\text{Aut}(X(\mathbb{R}))$ is not dense. If $\#X_0(\mathbb{R}) = 3$, then in general $\text{Aut}(X_0(\mathbb{R}))$ does not exchange the connected component of $X_0(\mathbb{R})$. Consequently, $\text{Aut}(X_0(\mathbb{R}))$ is not dense (but maybe $\text{Aut}(X(\mathbb{R}))$ could be, if the connected components of $X(\mathbb{R})$ are not homeomorphic). \square

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