# Edge-partitions of sparse graphs and their applications to game coloring

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**Abstract**

In this note, we prove that every graph with maximum average degree less than $\frac{32}{13}$ (resp. $\frac{30}{11}$, $\frac{32}{11}$, $\frac{70}{23}$) admits an edge-partition into a forest and a subgraph of maximum degree 1 (resp. 2, 3, 4). This implies that these graphs have game coloring number at most 5, 6, 7, 8, respectively.

## 1 Introduction

Let $G$ be a simple graph. The *game coloring number* of $G$ is defined through a two-person graph ordering game. Alice and Bob take turns choosing vertices from the set of unchosen vertices of $G$. This defines a linear order $L$ of the vertices of $G$ with $x < y$, if and only if, $x$ is chosen before $y$. The *back degree* of a vertex $x$ with respect to $L$ is the number of its neighbors $y$ in $G$ such that $y < x$. The back degree of $L$ is the maximum back degree of a vertex of $G$ with respect to $L$. Alice's goal is to minimize the back degree of $L$ and Bob's goal is to maximize it. The *game coloring number* $col_g(G)$ of $G$ is defined to be $k+1$, where $k$ is the minimum integer such that Alice has a strategy for the graph ordering game to ensure that the back degree of $L$ is at most $k$. Equivalently, $k$ is the maximum integer such that Bob has a strategy for the graph ordering game to ensure that the back degree of $L$ is at least $k$. This notion was first formally defined in [5] as a tool to find bounds to the game chromatic number [1].

Recently, Zhu [6] proved that the game coloring number of every planar graph is at most 17. This result was improved in the case of planar graphs with large girth, by Borodin *et al.* [2] and He *et al.* [4]. These results are based on some structural properties of planar graphs with large girth:

**Theorem 1 (Borodin *et al.* [2] + He *et al.* [4])** *Let $G$ be a planar graph with girth at least $g$.*

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1. *If $g \geq 9$, then $G$ admits an edge-partition into a forest and a matching [2].*
2. *If $g \geq 7$, then $G$ admits an edge-partition into a forest and a graph with maximum degree 2 [4].*
3. *If $g \geq 5$, then $G$ admits an edge-partition into a forest and a graph with maximum degree 4 [4].*

Zhu established this upper bound of the game coloring number:

**Lemma 1 (Zhu [5])** *Suppose that the graph $G$ has an edge-partition into two subgraphs $G_1$ and $G_2$, then $col_g(G) \leq col_g(G_1) + \Delta(G_2)$.*

Faigle *et al.* studied the game coloring number of a forest:

**Lemma 2 (Faigle *et al.* [3])** *Let $T$ be a forest. Then $col_g(T) \leq 4$.*

Hence combining these two lemmas with Theorem 1, we have

**Corollary 1 ([2] + [4])** *Every planar graph with girth at least 9 (resp. 7, 5) has game coloring number at most 5 (resp. 6, 8).*

In this note, we study edge-partitions of sparse graphs, in the meaning of small maximum average degree, and derive bounds on the game coloring number.

The maximum average degree of $G$, denoted by $Mad(G)$ is:

$$Mad(G) = \max\{2|E(H)|/|V(H)|, H \subseteqq G\}$$

Our main result is:

**Theorem 2** *Let $G$ be a simple graph.*

1. *If $Mad(G) < \frac{32}{13}$, then $G$ admits an edge-partition into a forest and a matching.*
2. *If $Mad(G) < \frac{30}{11}$, then $G$ admits an edge-partition into a forest and graph with maximum degree at most 2.*
3. *If $Mad(G) < \frac{32}{11}$, then $G$ admits an edge-partition into a forest and graph with maximum degree at most 3.*
4. *If $Mad(G) < \frac{70}{23}$, then $G$ admits an edge-partition into a forest and graph with maximum degree at most 4.*

In contrary to Theorem 1, Theorem 2 is not restricted to planar graphs. We note however that we can not infer Theorem 1 from Theorem 2, by using the usual inequality $Mad(G) \leq 2g/(g-2)$ for every planar graph $G$ of girth at least $g$.

Combining with Lemmas 1 and 2, we get:

**Corollary 2** *Let $G$ be a simple graph.*

1. *If $Mad(G) < \frac{32}{13}$, then $col_g(G) \leq 5$.*
2. *If $Mad(G) < \frac{30}{11}$, then $col_g(G) \leq 6$.*
3. *If $Mad(G) < \frac{32}{11}$, then $col_g(G) \leq 7$.*
4. *If $Mad(G) < \frac{70}{23}$, then $col_g(G) \leq 8$.*

Section 2 is dedicated to the proof of Theorem 2. Section 3 contains some final remarks.

# 2 Proof of Theorem 2

Let $G$ be a simple graph. Let $d(x)$ denote the degree of $x$ in $G$. A vertex of degree $k$ (resp. at least $k$, at most $k$) is called a $k$-*vertex* (resp. $^{\geq}k$-*vertex*, $^{\leq}k$-*vertex*). An $(a, b)$-*alternating cycle* is an even cycle $x_1x_2x_3 \dots x_{2k}x_1$ such that $d(x_i) = a$ if $i$ is even and $d(x_i) = b$ otherwise. An $k_l$-vertex is a vertex of degree $k$ adjacent to exactly $l$ 2-vertices.

Let $G$ be a counterexample of Theorem 2, i.e. a graph that does not admit an edge-partition into a forest and a subgraph with maximum degree $k$ ($k = 1, 2, 3, 4$), minimizing $\sigma(G) = |V(G)| + |E(G)|$.

## 2.1 Structural properties of $G$

**Claim 1** *The counterexample $G$ does not contain:*

1. 1-*vertices,*
2. *two adjacent* $^{\leq}k + 1$-*vertices,*
3. $(k + 2, 2)$-*alternating cycles.*

**Proof**

1. By contradiction, assume that $G$ contains an 1-vertex $v$ adjacent to $u$. By minimality, of $G$, the graph $H = G \setminus u$ admits an edge-partition into a forest $F$ and a subgraph $D$ with maximum degree $k$. We can extend this edge-partition to $G$ by adding the edge $uv$ into $F$, a contradiction.
2. Assume that $G$ contains two adjacent $^{\leq}k+1$-vertices, say $u$ an $v$. By minimality, of $G$, the graph $H = G \setminus uv$ admits an edge-partition into a forest $F$ and a subgraph $D$ with maximum degree $k$. If at least one of $u$ and $v$ is incident to $k$ edges in $D$, then add $uv$ in $F$ ; otherwise, add $uv$ into $D$. This extends the edge-partition to $G$, a contradiction.
3. Assume that $G$ contains a $(k + 2, 2)$-alternating cycle $C = x_1x_2x_3 \dots x_{2l}x_1$ with $d(x_i) = k + 2$ if $i$ is even and $d(x_i) = 2$ otherwise. By minimality of $G$, the graph $H = G \setminus \{x_1x_2, x_2x_3, \dots, x_{2l-1}x_{2l}, x_{2l}x_1\}$ admits an edge-partition into a forest $F$ and a subgraph $D$ with maximum degree $k$. We may assume that $x_{2j}$ is incident to at least one edge of $F$, for otherwise we can add an arbitrary edge incident to $x_{2j}$ into $F$. Now by adding the edges $x_{2j}x_{2j+1}$ into $D$, adding $x_{2j}x_{2j-1}$ into $F$, we obtain a required edge-partition of $G$, a contradiction.

□

## 2.2 Discharging procedures

In what follows, we will define an additional structure, called *bank*, which is a subgraph of $G$ composed of maximal connected components, called *agencies*. In fact, we will show that each bank is a forest and each agency a tree. These structures will be used, during the discharging procedure, to transfer charges. Usually, the discharging rules operate locally ; agencies will allow us to transfer charges non locally. In our discharging procedures, the vertices adjacent to an agency $C$ will give their excess charge to $C$ which will redistribute this excess charge to the vertices of $C$ which does not have enough charges.

First we assign to each vertex $v$ a charge $\omega(v)$ equal to its degree, i.e. $\forall v \in V(G), \omega(v) = d(v)$. Moreover we assign to each agency $C$ (that will be defined later) a charge $\omega(C) = 0$. We define then discharging rules and redistribute charges accordingly. Once the discharging is finished, a new charge function $\omega^*$ is produced. However, the total sum of charges is kept

fixed when the discharging is in process. Nevertheless, we can show that $\omega^*(v) \geq \frac{32}{13}$ (resp. $\frac{30}{11}$, $\frac{32}{11}$, $\frac{70}{23}$) for all $v \in V(G)$ and $\omega^*(C) \geq 0$ for all agency $C$ of $G$. Hence the following equation follows:

$$\frac{32}{13}|V(G)| \leq \sum_{v \in V(G)} \omega^*(v) + \sum_{C \text{ agency of G}} \omega^*(C) = \sum_{v \in V(G)} \omega(v) + \sum_{C \text{ agency of G}} \omega(C) = \sum_{v \in V(G) d(v)} = 2|E(G)|$$

This leads to the following obvious contradiction:

$$\frac{32}{13} = \frac{\frac{32}{13}|V(G)|}{|V(G)|} \leq \frac{2|E(G)|}{|V(G)|} \leq \mathit{Mad}(G) < \frac{32}{13}$$

and hence demonstrates that no such counterexample can exist (as well for $\mathit{Mad}(G) < \frac{30}{11}, \frac{32}{11}, \frac{70}{23}$).

### 2.2.1 Graphs with $\mathit{Mad} < \frac{32}{13}$

Here, the *bank* of $G$ is the subgraph of $G$ defined as follows: its set of vertices contains all the $3_2$-vertices, $3_3$-vertices and the 2-vertices adjacent to $3_2$-vertices, or $3_3$-vertices ; its set of edges is the set of edges between the 2-vertices and the $3_2$-vertices, $3_3$-vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

We say that a vertex, which does not belong to an agency, is *adjacent to an agency* if it is adjacent to a 2-vertex belonging to an agency.

The discharging rules are defined as follows:

**R1.** Every $\geq$3-vertex gives $\frac{3}{13}$ to each adjacent 2-vertex.

**R2.** Every $\geq$3-vertex not belonging to an agency gives $\frac{2}{13}$ to each adjacent agency.

**R3.** Each agency gives $\frac{2}{13}$ to each of its own $3_3$-vertices.

Let us check first that for each vertex $v$, $\omega^*(v) \geq \frac{32}{13}$. Let $v$ be a $k$-vertex ($k \geq 2$ by Claim 1.1).

Case $k = 2$ Initially, $\omega(v) = 2$. The vertex $v$ receives $\frac{3}{13}$ from of each of its neighbors (which are $\geq$3-vertices by Claim 1.2). Hence, $\omega^*(v) = 2 + 2 \cdot \frac{3}{13} = \frac{32}{13}$.

Case $k = 3$ Initially, $\omega(v) = 3$. If $v$ is adjacent to at most one 2-vertex, then $\omega^*(v) \geq 3 - \frac{3}{13} - \frac{2}{13} \geq \frac{34}{13}$. If $v$ is an $3_2$-vertex, then $v$ belongs to an agency and gives two times $\frac{3}{13}$ by R1 and nothing by R2. Hence $\omega^*(v) = 3 - 2 \cdot \frac{3}{13} = \frac{33}{13}$. Finally assume that $v$ is a $3_3$-vertex. The vertex $v$ gives three times $\frac{3}{13}$ by R1 and receives $\frac{2}{13}$ by R3. Hence $\omega^*(v) = 3 - 3 \cdot \frac{3}{13} + \frac{2}{13} = \frac{32}{13}$.

Case $k \geq 4$ Initially, $\omega(v) = k$. The vertex $v$ is adjacent to at most $k$ 2-vertices and to at most $k$ agencies. Hence by R1 and R2, $\omega^*(v) \geq k - k \cdot \frac{3}{13} - k \cdot \frac{2}{13} = \frac{8k}{13} \geq \frac{32}{13}$ if $k \geq 4$.

It remains to prove that the charge remaining on each agency is non-negative. Let $C$ be an agency. Let $n_{3_3}(C)$, and $n_l(C)$ be the number of $3_3$-vertices, and leaves of $C$ respectively.
Observe that:

$$n_l(C) \geq n_{3_3}(C) \tag{1}$$

By R2, the agency $C$ receives $\frac{2}{13}$ from its adjacent vertices (i.e. it receives $n_l(C) \cdot \frac{2}{13}$), and gives $\frac{2}{13}$ to each of its own $3_3$-vertices by R3 (i.e. it gives $n_{3_3}(C) \cdot \frac{2}{13}$). Hence, $\omega^*(C) = n_l(C) \cdot \frac{2}{13} - n_{3_3}(C) \cdot \frac{2}{13} \geq n_{3_3}(C) \cdot \frac{2}{13} - n_{3_3}(C) \cdot \frac{2}{13} \geq 0$ by Equation (1). This completes the proof of Theorem 2.1.

### 2.2.2 Graphs with *Mad* $< \frac{30}{11}$

Here, the *bank* of $G$ is the subgraph of $G$ defined as follows: its set of vertices contains all the $4_3$-vertices, $4_4$-vertices and the 2-vertices adjacent to $4_3$-vertices, or $4_4$-vertices ; its set of edges is the set of edges between the 2-vertices and the $4_3$-vertices, $4_4$-vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

**R1.** Every $\geq 4$-vertex gives $\frac{4}{11}$ to each adjacent 2-vertex.

**R2.** Every $\geq 4$-vertex not belonging to an agency gives $\frac{1}{11}$ to each adjacent agency.

**R3.** Each agency gives $\frac{2}{11}$ to each of its own $4_4$-vertices.

Let us check first that for each vertex $v$, $\omega^*(v) \geq \frac{30}{11}$. Let $v$ be a $k$-vertex.

Case $k = 2$ Initially, $\omega(v) = 2$. The vertex $v$ receives $\frac{4}{11}$ from of each of its neighbors (which are $\geq 4$-vertices by Claim 1.2). Hence, $\omega^*(v) = 2 + 2 \cdot \frac{4}{11} = \frac{30}{11}$.

Case $k = 3$ Initially, $\omega(v) = 3$. The vertex $v$ is not affected by the discharging procedure ; hence $\omega^*(v) = \omega(v) = 3 \geq \frac{30}{11}$.

Case $k = 4$ Initially, $\omega(v) = 4$. Suppose that $v$ is a 4-vertex adjacent to at most two 2-vertices. Then $v$ gives at most two times $\frac{4}{11}$ to its adjacent 2-vertices and two times $\frac{1}{11}$ to its adjacent agencies. Hence $\omega^*(v) \geq 4 - 2 \cdot \frac{4}{11} - 2 \cdot \frac{1}{11} = \frac{34}{11} \geq \frac{30}{11}$. Assume that $v$ is a $4_3$-vertex. Then $v$ belongs to an agency ; hence $v$ gives three times $\frac{4}{11}$ to its adjacent 2-vertices by R1 and nothing by R2. So $\omega^*(v) = 4 - 3 \cdot \frac{4}{11} = \frac{32}{11} \geq \frac{30}{11}$. Finally, assume that $v$ is a $4_4$-vertex. The vertex $v$ gives four times $\frac{4}{11}$ to its incident 2-vertices by R1 and receives $\frac{2}{11}$ from its agency by R3. Hence $\omega^*(v) = 4 - 4 \cdot \frac{4}{11} + \frac{2}{11} = \frac{30}{11}$.

Case $k \geq 5$ Initially, $\omega(v) = k$. The vertex $v$ is adjacent to at most $k$ 2-vertices and to at most $k$ agencies. Hence by R1 and R2, $\omega^*(v) \geq k - k \cdot \frac{4}{11} - k \cdot \frac{1}{11} = \frac{6k}{11} \geq \frac{30}{11}$ if $k \geq 5$.

It remains to prove that the charge remaining on each agency is non-negative. Let $C$ be an agency. Let $n_{4_4}(C)$, $n_{4_3}(C)$, and $n_l(C)$ be the number of $4_4$-vertices, $4_3$-vertices, and leaves of $C$ respectively.

Observe that:

$$n_l(C) \geq n_{4_3}(C) + 2 \cdot n_{4_4}(C) \tag{2}$$

By R2, the agency $C$ receives $\frac{1}{11}$ from its adjacent vertices (i.e. it receives $n_l(C) \cdot \frac{1}{11}$), and gives $\frac{2}{11}$ to each of its own $4_4$-vertices by R3 (i.e. it gives $n_{4_4}(C) \cdot \frac{2}{11}$). Hence, $\omega^*(C) = n_l(C) \cdot \frac{1}{11} - n_{4_4}(C) \cdot \frac{2}{11} \geq 2 \cdot n_{4_4}(C) \cdot \frac{1}{11} - n_{4_4}(C) \cdot \frac{2}{11} \geq 0$ by Equation (2). This completes the proof of Theorem 2.2.

### 2.2.3 Graphs with *Mad* $< \frac{32}{11}$

Here, the *bank* of $G$ is the subgraph of $G$ defined as follows: its set of vertices contains all the $5_4$-vertices, $5_5$-vertices and the 2-vertices adjacent to $5_4$-vertices, or $5_5$-vertices ; its set of edges is the set of edges between the 2-vertices and the $5_4$-vertices, $5_5$-vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

**R1.** Every $\geq 5$-vertex gives $\frac{5}{11}$ to each adjacent 2-vertex.

**R2.** Every $\geq 5$-vertex not belonging to an agency gives $\frac{2}{33}$ to each adjacent agency.

**R3.** Each agency gives $\frac{2}{11}$ to each of its own $5_5$-vertices.

Let us check first that for each vertex $v$, $\omega^*(v) \geq \frac{32}{11}$. Let $v$ be a $k$-vertex.

Case $k = 2$ Initially, $\omega(v) = 2$. The vertex $v$ receives $\frac{5}{11}$ from of each of its neighbors (which are $\geq$5-vertices by Claim 1.2). Hence, $\omega^*(v) = 2 + 2 \cdot \frac{5}{11} = \frac{32}{11}$.

Case $k = 3, 4$ The vertex $v$ is not affected by the discharging procedure ; hence $\omega^*(v) = \omega(v) \geq \frac{32}{11}$.

Case $k = 5$ Initially, $\omega(v) = 5$. Assume that $v$ is a 5-vertex ajacent to at most three 2-vertices. Then by R1 and R2 $\omega^*(v) \geq 5 - 3 \cdot \frac{5}{11} - 3 \cdot \frac{2}{33} = \frac{38}{11}$. Assume now that $v$ is a $5_4$-vertex. Then $v$ belongs to an agency and gives nothing by R2. So $\omega^*(v) = 5 - 4 \cdot \frac{5}{11} = \frac{35}{11}$. Assume finally that $v$ is $5_5$-vertex. By R1 and R3 $\omega^*(v) = 5 - 5 \cdot \frac{5}{11} + \frac{2}{11} = \frac{32}{11}$.

Case $k \geq 6$ Initially, $\omega(v) = k$. The vertex $v$ is adjacent to at most $k$ 2-vertices and to at most $k$ agencies. Hence by R1 and R2, $\omega^*(v) \geq k - k \cdot \frac{5}{11} - k \cdot \frac{2}{33} = \frac{16k}{33} \geq \frac{32}{11}$ if $k \geq 6$.

It remains to prove that the charge remaining on each agency is non-negative. Let $C$ be an agency. Let $n_{5_5}(C)$ and $n_l(C)$ be the number of $5_5$-vertices and leaves of $C$ respectively. Observe that:

$$n_l(C) \geq 3 \cdot n_{5_5}(C) \tag{3}$$

By R2, the agency $C$ receives $\frac{2}{33}$ from its adjacent vertices (i.e. it receives $n_l(C) \cdot \frac{2}{33}$), and gives $\frac{2}{11}$ to each of its own $5_5$-vertices by R3 (i.e. it gives $n_{5_5}(C) \cdot \frac{2}{11}$). Hence, $\omega^*(C) = n_l(C) \cdot \frac{2}{33} - n_{5_5}(C) \cdot \frac{2}{11} \geq 3 \cdot n_{5_5}(C) \cdot \frac{2}{33} - n_{5_5}(C) \cdot \frac{2}{11} \geq 0$ by Equation (3). This completes the proof of Theorem 2.3.

### 2.2.4 Graphs with *Mad* $< \frac{70}{23}$

Here, the *bank* of $G$ is the subgraph of $G$ defined as follows: its set of vertices contains all the $6_6$-vertices and the 2-vertices adjacent to $6_6$-vertices ; its set of edges is the set of edges between the 2-vertices and the $6_6$-vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

**R1.** Every $\geq$6-vertex gives $\frac{12}{23}$ to each adjacent 2-vertex, and $\frac{1}{69}$ to each adjacent 3-vertex.

**R2.** Every $\geq$6-vertex not belonging to an agency gives $\frac{1}{23}$ to each adjacent agency.

**R3.** Each agency gives $\frac{4}{23}$ to each of its own $6_6$-vertices.

Let us check first that for each vertex $v$, $\omega^*(v) \geq \frac{70}{23}$. Let $v$ be a $k$-vertex.

Case $k = 2$ Initially, $\omega(v) = 2$. The vertex $v$ receives $\frac{12}{23}$ from of each of its neighbors (which are $\geq$6-vertices by Claim 1.2). Hence, $\omega^*(v) = 2 + 2 \cdot \frac{12}{23} = \frac{70}{23}$.

Case $k = 3$ The vertex $v$ is adjacent to $\geq$6-vertices by Claim 1.2. Then $v$ receives $\frac{1}{69}$ from each of its neighbors ; hence $\omega^*(v) = 3 + 3 \cdot \frac{1}{69} = \frac{70}{23}$.

Case $k = 4, 5$ The vertex $v$ is not affected by the discharging procedure ; hence $\omega^*(v) = \omega(v) \geq \frac{70}{23}$.

Case $k = 6$ Initially, $\omega(v) = 6$. Assume that $v$ is adjacent to at most five 2-vertices. Then by R1 an R2 $\omega^*(v) \geq 6 - 5 \cdot \frac{12}{23} - 5 \cdot \frac{1}{23} = \frac{73}{23}$. Assume finally that $v$ is a $6_6$-vertex. The vertex $v$ belongs to an agency. By R1 and R3 $\omega^*(v) = 6 - 6 \cdot \frac{12}{23} + \frac{4}{23} = \frac{70}{23}$.

Case $k \geq 7$ Initially, $\omega(v) = k$. The vertex $v$ is adjacent to at most $k$ $\leq$3-vertices and to at most $k$ agencies. Hence by R1 and R2, $\omega^*(v) \geq k - k \cdot \frac{12}{23} - k \cdot \frac{1}{23} = \frac{10k}{23} \geq \frac{70}{23}$ if $k \geq 7$.

It remains to prove that the charge remaining on each agency is non-negative. Let $C$ be an agency. Let $n_{6_6}(C)$ and $n_l(C)$ be the number of $6_6$-vertices and leaves of $C$ respectively.

Observe that:

$$n_l(C) \geq 4 \cdot n_{6_6}(C) \tag{4}$$

By R2, the agency $C$ receives $\frac{1}{23}$ from its adjacent vertices (i.e. it receives $n_l(C) \cdot \frac{1}{23}$), and gives $\frac{4}{23}$ to each of its own $6_6$-vertices by R3 (i.e. it gives $n_{6_6}(C) \cdot \frac{4}{23}$). Hence, $\omega^*(C) = n_l(C) \cdot \frac{1}{23} - n_{6_6}(C) \cdot \frac{4}{23} \geq 4 \cdot n_{6_6}(C) \cdot \frac{1}{23} - n_{6_6}(C) \cdot \frac{4}{23} \geq 0$ by Equation (4). This completes the proof of Theorem 2.4.

# 3 Conclusion

In this note, we established that for every simple graph $G$, if $Mad(G) \leq \frac{32}{13}$ (resp. $\frac{30}{11}$, $\frac{32}{11}$, $\frac{70}{23}$) then $G$ admits an edge-partition into a forest and a graph with maximum degree at most 1 (resp. 2, 3, 4), hence the game chromatic of $G$ is at most 5 (resp. 6, 7, 8). In order to study the tightness of Theorem 2, we introduce a function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(k) = \inf\{Mad(H) \mid H$ does not admit an edge-partition into a forest and a subgraph with maximum degree $k\}$. It is easy to observe that the complete bipartite graph $K_{2,2k+2}$ has $Mad(K_{2,2k+2}) = \frac{4(k+1)}{k+2}$ and does not admit an edge-partition into a forest and a subgraph with maximum degree at most $k$. Hence,

$$\begin{aligned}
2,461... = \frac{32}{13} &\leq f(1) \leq \frac{8}{3} = 2,666... \\
2,727... = \frac{30}{11} &\leq f(2) \leq 3 \\
2,909... = \frac{32}{11} &\leq f(3) \leq \frac{16}{5} = 3,2 \\
3,043... = \frac{70}{23} &\leq f(4) \leq \frac{10}{3} = 3,333...
\end{aligned}$$

We conclude with the following problem:

**Problem 1** *For every $k$, what are (if any) the graphs which do not admit an edge-partition into a forest and a subgraph with maximum degree at most $k$, but such that every graph with smaller maximal average degree does?*

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