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# Edge-partitions of sparse graphs and their applications to game coloring

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## Abstract

In this note, we prove that every graph with maximum average degree less than  $\frac{32}{13}$  (resp.  $\frac{30}{11}$ ,  $\frac{32}{11}$ ,  $\frac{70}{23}$ ) admits an edge-partition into a forest and a subgraph of maximum degree 1 (resp. 2, 3, 4). This implies that these graphs have game coloring number at most 5, 6, 7, 8, respectively.

## 1 Introduction

Let  $G$  be a simple graph. The *game coloring number* of  $G$  is defined through a two-person graph ordering game. Alice and Bob take turns choosing vertices from the set of unchosen vertices of  $G$ . This defines a linear order  $L$  of the vertices of  $G$  with  $x < y$ , if and only if,  $x$  is chosen before  $y$ . The *back degree* of a vertex  $x$  with respect to  $L$  is the number of its neighbors  $y$  in  $G$  such that  $y < x$ . The back degree of  $L$  is the maximum back degree of a vertex of  $G$  with respect to  $L$ . Alice's goal is to minimize the back degree of  $L$  and Bob's goal is to maximize it. The *game coloring number*  $col_g(G)$  of  $G$  is defined to be  $k + 1$ , where  $k$  is the minimum integer such that Alice has a strategy for the graph ordering game to ensure that the back degree of  $L$  is at most  $k$ . Equivalently,  $k$  is the maximum integer such that Bob has a strategy for the graph ordering game to ensure that the back degree of  $L$  is at least  $k$ . This notion was first formally defined in [5] as a tool to find bounds to the game chromatic number [1].

Recently, Zhu [6] proved that the game coloring number of every planar graph is at most 17. This result was improved in the case of planar graphs with large girth, by Borodin *et al.* [2] and He *et al.* [4]. These results are based on some structural properties of planar graphs with large girth:

**Theorem 1 (Borodin *et al.* [2] + He *et al.* [4])** *Let  $G$  be a planar graph with girth at least  $g$ .*

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1. If  $g \geq 9$ , then  $G$  admits an edge-partition into a forest and a matching [2].
2. If  $g \geq 7$ , then  $G$  admits an edge-partition into a forest and a graph with maximum degree 2 [4].
3. If  $g \geq 5$ , then  $G$  admits an edge-partition into a forest and a graph with maximum degree 4 [4].

Zhu established this upper bound of the game coloring number:

**Lemma 1 (Zhu [5])** *Suppose that the graph  $G$  has an edge-partition into two subgraphs  $G_1$  and  $G_2$ , then  $col_g(G) \leq col_g(G_1) + \Delta(G_2)$ .*

Faigle *et al.* studied the game coloring number of a forest:

**Lemma 2 (Faigle *et al.* [3])** *Let  $T$  be a forest. Then  $col_g(T) \leq 4$ .*

Hence combining these two lemmas with Theorem 1, we have

**Corollary 1 ([2] + [4])** *Every planar graph with girth at least 9 (resp. 7, 5) has game coloring number at most 5 (resp. 6, 8).*

In this note, we study edge-partitions of sparse graphs, in the meaning of small maximum average degree, and derive bounds on the game coloring number.

The maximum average degree of  $G$ , denoted by  $Mad(G)$  is:

$$Mad(G) = \max\{2|E(H)|/|V(H)|, H \subseteq G\}$$

Our main result is:

**Theorem 2** *Let  $G$  be a simple graph.*

1. If  $Mad(G) < \frac{32}{13}$ , then  $G$  admits an edge-partition into a forest and a matching.
2. If  $Mad(G) < \frac{30}{11}$ , then  $G$  admits an edge-partition into a forest and graph with maximum degree at most 2.
3. If  $Mad(G) < \frac{32}{11}$ , then  $G$  admits an edge-partition into a forest and graph with maximum degree at most 3.
4. If  $Mad(G) < \frac{70}{23}$ , then  $G$  admits an edge-partition into a forest and graph with maximum degree at most 4.

In contrary to Theorem 1, Theorem 2 is not restricted to planar graphs. We note however that we can not infer Theorem 1 from Theorem 2, by using the usual inequality  $Mad(G) \leq 2g/(g-2)$  for every planar graph  $G$  of girth at least  $g$ .

Combining with Lemmas 1 and 2, we get:

**Corollary 2** *Let  $G$  be a simple graph.*

1. If  $Mad(G) < \frac{32}{13}$ , then  $col_g(G) \leq 5$ .
2. If  $Mad(G) < \frac{30}{11}$ , then  $col_g(G) \leq 6$ .
3. If  $Mad(G) < \frac{32}{11}$ , then  $col_g(G) \leq 7$ .
4. If  $Mad(G) < \frac{70}{23}$ , then  $col_g(G) \leq 8$ .

Section 2 is dedicated to the proof of Theorem 2. Section 3 contains some final remarks.

## 2 Proof of Theorem 2

Let  $G$  be a simple graph. Let  $d(x)$  denote the degree of  $x$  in  $G$ . A vertex of degree  $k$  (resp. at least  $k$ , at most  $k$ ) is called a  $k$ -vertex (resp.  $\geq k$ -vertex,  $\leq k$ -vertex). An  $(a, b)$ -alternating cycle is an even cycle  $x_1x_2x_3 \dots x_{2l}x_1$  such that  $d(x_i) = a$  if  $i$  is even and  $d(x_i) = b$  otherwise. An  $k_l$ -vertex is a vertex of degree  $k$  adjacent to exactly  $l$  2-vertices.

Let  $G$  be a counterexample of Theorem 2, i.e. a graph that does not admit an edge-partition into a forest and a subgraph with maximum degree  $k$  ( $k = 1, 2, 3, 4$ ), minimizing  $\sigma(G) = |V(G)| + |E(G)|$ .

### 2.1 Structural properties of $G$

**Claim 1** *The counterexample  $G$  does not contain:*

1. 1-vertices,
2. two adjacent  $\leq k + 1$ -vertices,
3.  $(k + 2, 2)$ -alternating cycles.

**Proof**

1. By contradiction, assume that  $G$  contains an 1-vertex  $v$  adjacent to  $u$ . By minimality, of  $G$ , the graph  $H = G \setminus u$  admits an edge-partition into a forest  $F$  and a subgraph  $D$  with maximum degree  $k$ . We can extend this edge-partition to  $G$  by adding the edge  $uv$  into  $F$ , a contradiction.
2. Assume that  $G$  contains two adjacent  $\leq k + 1$ -vertices, say  $u$  and  $v$ . By minimality, of  $G$ , the graph  $H = G \setminus uv$  admits an edge-partition into a forest  $F$  and a subgraph  $D$  with maximum degree  $k$ . If at least one of  $u$  and  $v$  is incident to  $k$  edges in  $D$ , then add  $uv$  in  $F$ ; otherwise, add  $uv$  into  $D$ . This extends the edge-partition to  $G$ , a contradiction.
3. Assume that  $G$  contains a  $(k + 2, 2)$ -alternating cycle  $C = x_1x_2x_3 \dots x_{2l}x_1$  with  $d(x_i) = k + 2$  if  $i$  is even and  $d(x_i) = 2$  otherwise. By minimality of  $G$ , the graph  $H = G \setminus \{x_1x_2, x_2x_3, \dots, x_{2l-1}x_{2l}, x_{2l}x_1\}$  admits an edge-partition into a forest  $F$  and a subgraph  $D$  with maximum degree  $k$ . We may assume that  $x_{2j}$  is incident to at least one edge of  $F$ , for otherwise we can add an arbitrary edge incident to  $x_{2j}$  into  $F$ . Now by adding the edges  $x_{2j}x_{2j+1}$  into  $D$ , adding  $x_{2j}x_{2j-1}$  into  $F$ , we obtain a required edge-partition of  $G$ , a contradiction.

□

### 2.2 Discharging procedures

In what follows, we will define an additional structure, called *bank*, which is a subgraph of  $G$  composed of maximal connected components, called *agencies*. In fact, we will show that each bank is a forest and each agency a tree. These structures will be used, during the discharging procedure, to transfer charges. Usually, the discharging rules operate locally; agencies will allow us to transfer charges non locally. In our discharging procedures, the vertices adjacent to an agency  $C$  will give their excess charge to  $C$  which will redistribute this excess charge to the vertices of  $C$  which does not have enough charges.

First we assign to each vertex  $v$  a charge  $\omega(v)$  equal to its degree, i.e.  $\forall v \in V(G), \omega(v) = d(v)$ . Moreover we assign to each agency  $C$  (that will be defined later) a charge  $\omega(C) = 0$ . We define then discharging rules and redistribute charges accordingly. Once the discharging is finished, a new charge function  $\omega^*$  is produced. However, the total sum of charges is kept

fixed when the discharging is in process. Nevertheless, we can show that  $\omega^*(v) \geq \frac{32}{13}$  (resp.  $\frac{30}{11}, \frac{32}{11}, \frac{70}{23}$ ) for all  $v \in V(G)$  and  $\omega^*(C) \geq 0$  for all agency  $C$  of  $G$ . Hence the following equation follows:

$$\frac{32}{13}|V(G)| \leq \sum_{v \in V(G)} \omega^*(v) + \sum_{C \text{ agency of } G} \omega^*(C) = \sum_{v \in V(G)} \omega(v) + \sum_{C \text{ agency of } G} \omega(C) = \sum_{v \in V(G)d(v)} = 2|E(G)|$$

This leads to the following obvious contradiction:

$$\frac{32}{13} = \frac{\frac{32}{13}|V(G)|}{|V(G)|} \leq \frac{2|E(G)|}{|V(G)|} \leq Mad(G) < \frac{32}{13}$$

and hence demonstrates that no such counterexample can exist (as well for  $Mad(G) < \frac{30}{11}, \frac{32}{11}, \frac{70}{23}$ ).

### 2.2.1 Graphs with $Mad < \frac{32}{13}$

Here, the *bank* of  $G$  is the subgraph of  $G$  defined as follows: its set of vertices contains all the  $3_2$ -vertices,  $3_3$ -vertices and the 2-vertices adjacent to  $3_2$ -vertices, or  $3_3$ -vertices ; its set of edges is the set of edges between the 2-vertices and the  $3_2$ -vertices,  $3_3$ -vertices. By Claim 1.3, an agency is a tree whose each leaf is a 2-vertex.

We say that a vertex, which does not belong to an agency, is *adjacent to an agency* if it is adjacent to a 2-vertex belonging to an agency.

The discharging rules are defined as follows:

- R1.** Every  $\geq 3$ -vertex gives  $\frac{3}{13}$  to each adjacent 2-vertex.
- R2.** Every  $\geq 3$ -vertex not belonging to an agency gives  $\frac{2}{13}$  to each adjacent agency.
- R3.** Each agency gives  $\frac{2}{13}$  to each of its own  $3_3$ -vertices.

Let us check first that for each vertex  $v$ ,  $\omega^*(v) \geq \frac{32}{13}$ . Let  $v$  be a  $k$ -vertex ( $k \geq 2$  by Claim 1.1).

Case  $k = 2$  Initially,  $\omega(v) = 2$ . The vertex  $v$  receives  $\frac{3}{13}$  from of each of its neighbors (which are  $\geq 3$ -vertices by Claim 1.2). Hence,  $\omega^*(v) = 2 + 2 \cdot \frac{3}{13} = \frac{32}{13}$ .

Case  $k = 3$  Initially,  $\omega(v) = 3$ . If  $v$  is adjacent to at most one 2-vertex, then  $\omega^*(v) \geq 3 - \frac{3}{13} - \frac{2}{13} \geq \frac{34}{13}$ . If  $v$  is an  $3_2$ -vertex, then  $v$  belongs to an agency and gives two times  $\frac{3}{13}$  by R1 and nothing by R2. Hence  $\omega^*(v) = 3 - 2 \cdot \frac{3}{13} = \frac{33}{13}$ . Finally assume that  $v$  is a  $3_3$ -vertex. The vertex  $v$  gives three times  $\frac{3}{13}$  by R1 and receives  $\frac{2}{13}$  by R3. Hence  $\omega^*(v) = 3 - 3 \cdot \frac{3}{13} + \frac{2}{13} = \frac{32}{13}$ .

Case  $k \geq 4$  Initially,  $\omega(v) = k$ . The vertex  $v$  is adjacent to at most  $k$  2-vertices and to at most  $k$  agencies. Hence by R1 and R2,  $\omega^*(v) \geq k - k \cdot \frac{3}{13} - k \cdot \frac{2}{13} = \frac{8k}{13} \geq \frac{32}{13}$  if  $k \geq 4$ .

It remains to prove that the charge remaining on each agency is non-negative. Let  $C$  be an agency. Let  $n_{3_3}(C)$ , and  $n_l(C)$  be the number of  $3_3$ -vertices, and leaves of  $C$  respectively.

Observe that:

$$n_l(C) \geq n_{3_3}(C) \tag{1}$$

By R2, the agency  $C$  receives  $\frac{2}{13}$  from its adjacent vertices (i.e. it receives  $n_l(C) \cdot \frac{2}{13}$ ), and gives  $\frac{2}{13}$  to each of its own  $3_3$ -vertices by R3 (i.e. it gives  $n_{3_3}(C) \cdot \frac{2}{13}$ ). Hence,  $\omega^*(C) = n_l(C) \cdot \frac{2}{13} - n_{3_3}(C) \cdot \frac{2}{13} \geq n_{3_3}(C) \cdot \frac{2}{13} - n_{3_3}(C) \cdot \frac{2}{13} \geq 0$  by Equation (1). This completes the proof of Theorem 2.1.

### 2.2.2 Graphs with $Mad < \frac{30}{11}$

Here, the *bank* of  $G$  is the subgraph of  $G$  defined as follows: its set of vertices contains all the  $4_3$ -vertices,  $4_4$ -vertices and the 2-vertices adjacent to  $4_3$ -vertices, or  $4_4$ -vertices ; its set of edges is the set of edges between the 2-vertices and the  $4_3$ -vertices,  $4_4$ -vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

- R1.** Every  $\geq 4$ -vertex gives  $\frac{4}{11}$  to each adjacent 2-vertex.
- R2.** Every  $\geq 4$ -vertex not belonging to an agency gives  $\frac{1}{11}$  to each adjacent agency.
- R3.** Each agency gives  $\frac{2}{11}$  to each of its own  $4_4$ -vertices.

Let us check first that for each vertex  $v$ ,  $\omega^*(v) \geq \frac{30}{11}$ . Let  $v$  be a  $k$ -vertex.

Case  $k = 2$  Initially,  $\omega(v) = 2$ . The vertex  $v$  receives  $\frac{4}{11}$  from of each of its neighbors (which are  $\geq 4$ -vertices by Claim 1.2). Hence,  $\omega^*(v) = 2 + 2 \cdot \frac{4}{11} = \frac{30}{11}$ .

Case  $k = 3$  Initially,  $\omega(v) = 3$ . The vertex  $v$  is not affected by the discharging procedure ; hence  $\omega^*(v) = \omega(v) = 3 \geq \frac{30}{11}$ .

Case  $k = 4$  Initially,  $\omega(v) = 4$ . Suppose that  $v$  is a 4-vertex adjacent to at most two 2-vertices. Then  $v$  gives at most two times  $\frac{4}{11}$  to its adjacent 2-vertices and two times  $\frac{1}{11}$  to its adjacent agencies. Hence  $\omega^*(v) \geq 4 - 2 \cdot \frac{4}{11} - 2 \cdot \frac{1}{11} = \frac{34}{11} \geq \frac{30}{11}$ . Assume that  $v$  is a  $4_3$ -vertex. Then  $v$  belongs to an agency ; hence  $v$  gives three times  $\frac{4}{11}$  to its adjacent 2-vertices by R1 and nothing by R2. So  $\omega^*(v) = 4 - 3 \cdot \frac{4}{11} = \frac{32}{11} \geq \frac{30}{11}$ . Finally, assume that  $v$  is a  $4_4$ -vertex. The vertex  $v$  gives four times  $\frac{4}{11}$  to its incident 2-vertices by R1 and receives  $\frac{2}{11}$  from its agency by R3. Hence  $\omega^*(v) = 4 - 4 \cdot \frac{4}{11} + \frac{2}{11} = \frac{30}{11}$ .

Case  $k \geq 5$  Initially,  $\omega(v) = k$ . The vertex  $v$  is adjacent to at most  $k$  2-vertices and to at most  $k$  agencies. Hence by R1 and R2,  $\omega^*(v) \geq k - k \cdot \frac{4}{11} - k \cdot \frac{1}{11} = \frac{6k}{11} \geq \frac{30}{11}$  if  $k \geq 5$ .

It remains to prove that the charge remaining on each agency is non-negative. Let  $C$  be an agency. Let  $n_{4_4}(C)$ ,  $n_{4_3}(C)$ , and  $n_l(C)$  be the number of  $4_4$ -vertices,  $4_3$ -vertices, and leaves of  $C$  respectively.

Observe that:

$$n_l(C) \geq n_{4_3}(C) + 2 \cdot n_{4_4}(C) \quad (2)$$

By R2, the agency  $C$  receives  $\frac{1}{11}$  from its adjacent vertices (i.e. it receives  $n_l(C) \cdot \frac{1}{11}$ ), and gives  $\frac{2}{11}$  to each of its own  $4_4$ -vertices by R3 (i.e. it gives  $n_{4_4}(C) \cdot \frac{2}{11}$ ). Hence,  $\omega^*(C) = n_l(C) \cdot \frac{1}{11} - n_{4_4}(C) \cdot \frac{2}{11} \geq 2 \cdot n_{4_4}(C) \cdot \frac{1}{11} - n_{4_4}(C) \cdot \frac{2}{11} \geq 0$  by Equation (2). This completes the proof of Theorem 2.2.

### 2.2.3 Graphs with $Mad < \frac{32}{11}$

Here, the *bank* of  $G$  is the subgraph of  $G$  defined as follows: its set of vertices contains all the  $5_4$ -vertices,  $5_5$ -vertices and the 2-vertices adjacent to  $5_4$ -vertices, or  $5_5$ -vertices ; its set of edges is the set of edges between the 2-vertices and the  $5_4$ -vertices,  $5_5$ -vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

- R1.** Every  $\geq 5$ -vertex gives  $\frac{5}{11}$  to each adjacent 2-vertex.
- R2.** Every  $\geq 5$ -vertex not belonging to an agency gives  $\frac{2}{33}$  to each adjacent agency.
- R3.** Each agency gives  $\frac{2}{11}$  to each of its own  $5_5$ -vertices.

Let us check first that for each vertex  $v$ ,  $\omega^*(v) \geq \frac{32}{11}$ . Let  $v$  be a  $k$ -vertex.

Case  $k = 2$  Initially,  $\omega(v) = 2$ . The vertex  $v$  receives  $\frac{5}{11}$  from of each of its neighbors (which are  $\geq 5$ -vertices by Claim 1.2). Hence,  $\omega^*(v) = 2 + 2 \cdot \frac{5}{11} = \frac{32}{11}$ .

Case  $k = 3, 4$  The vertex  $v$  is not affected by the discharging procedure ; hence  $\omega^*(v) = \omega(v) \geq \frac{32}{11}$ .

Case  $k = 5$  Initially,  $\omega(v) = 5$ . Assume that  $v$  is a 5-vertex adjacent to at most three 2-vertices. Then by R1 and R2  $\omega^*(v) \geq 5 - 3 \cdot \frac{5}{11} - 3 \cdot \frac{2}{33} = \frac{38}{11}$ . Assume now that  $v$  is a 5<sub>4</sub>-vertex. Then  $v$  belongs to an agency and gives nothing by R2. So  $\omega^*(v) = 5 - 4 \cdot \frac{5}{11} = \frac{35}{11}$ . Assume finally that  $v$  is 5<sub>5</sub>-vertex. By R1 and R3  $\omega^*(v) = 5 - 5 \cdot \frac{5}{11} + \frac{2}{11} = \frac{32}{11}$ .

Case  $k \geq 6$  Initially,  $\omega(v) = k$ . The vertex  $v$  is adjacent to at most  $k$  2-vertices and to at most  $k$  agencies. Hence by R1 and R2,  $\omega^*(v) \geq k - k \cdot \frac{5}{11} - k \cdot \frac{2}{33} = \frac{16k}{33} \geq \frac{32}{11}$  if  $k \geq 6$ .

It remains to prove that the charge remaining on each agency is non-negative. Let  $C$  be an agency. Let  $n_{5_5}(C)$  and  $n_l(C)$  be the number of 5<sub>5</sub>-vertices and leaves of  $C$  respectively. Observe that:

$$n_l(C) \geq 3 \cdot n_{5_5}(C) \quad (3)$$

By R2, the agency  $C$  receives  $\frac{2}{33}$  from its adjacent vertices (i.e. it receives  $n_l(C) \cdot \frac{2}{33}$ ), and gives  $\frac{2}{11}$  to each of its own 5<sub>5</sub>-vertices by R3 (i.e. it gives  $n_{5_5}(C) \cdot \frac{2}{11}$ ). Hence,  $\omega^*(C) = n_l(C) \cdot \frac{2}{33} - n_{5_5}(C) \cdot \frac{2}{11} \geq 3 \cdot n_{5_5}(C) \cdot \frac{2}{33} - n_{5_5}(C) \cdot \frac{2}{11} \geq 0$  by Equation (3). This completes the proof of Theorem 2.3.

#### 2.2.4 Graphs with $Mad < \frac{70}{23}$

Here, the *bank* of  $G$  is the subgraph of  $G$  defined as follows: its set of vertices contains all the 6<sub>6</sub>-vertices and the 2-vertices adjacent to 6<sub>6</sub>-vertices ; its set of edges is the set of edges between the 2-vertices and the 6<sub>6</sub>-vertices. By Claim 1.3, an *agency* is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

**R1.** Every  $\geq 6$ -vertex gives  $\frac{12}{23}$  to each adjacent 2-vertex, and  $\frac{1}{69}$  to each adjacent 3-vertex.

**R2.** Every  $\geq 6$ -vertex not belonging to an agency gives  $\frac{1}{23}$  to each adjacent agency.

**R3.** Each agency gives  $\frac{4}{23}$  to each of its own 6<sub>6</sub>-vertices.

Let us check first that for each vertex  $v$ ,  $\omega^*(v) \geq \frac{70}{23}$ . Let  $v$  be a  $k$ -vertex.

Case  $k = 2$  Initially,  $\omega(v) = 2$ . The vertex  $v$  receives  $\frac{12}{23}$  from of each of its neighbors (which are  $\geq 6$ -vertices by Claim 1.2). Hence,  $\omega^*(v) = 2 + 2 \cdot \frac{12}{23} = \frac{70}{23}$ .

Case  $k = 3$  The vertex  $v$  is adjacent to  $\geq 6$ -vertices by Claim 1.2. Then  $v$  receives  $\frac{1}{69}$  from each of its neighbors ; hence  $\omega^*(v) = 3 + 3 \cdot \frac{1}{69} = \frac{70}{23}$ .

Case  $k = 4, 5$  The vertex  $v$  is not affected by the discharging procedure ; hence  $\omega^*(v) = \omega(v) \geq \frac{70}{23}$ .

Case  $k = 6$  Initially,  $\omega(v) = 6$ . Assume that  $v$  is adjacent to at most five 2-vertices. Then by R1 an R2  $\omega^*(v) \geq 6 - 5 \cdot \frac{12}{23} - 5 \cdot \frac{1}{23} = \frac{73}{23}$ . Assume finally that  $v$  is a 6<sub>6</sub>-vertex. The vertex  $v$  belongs to an agency. By R1 and R3  $\omega^*(v) = 6 - 6 \cdot \frac{12}{23} + \frac{4}{23} = \frac{70}{23}$ .

Case  $k \geq 7$  Initially,  $\omega(v) = k$ . The vertex  $v$  is adjacent to at most  $k \leq 3$ -vertices and to at most  $k$  agencies. Hence by R1 and R2,  $\omega^*(v) \geq k - k \cdot \frac{12}{23} - k \cdot \frac{1}{23} = \frac{10k}{23} \geq \frac{70}{23}$  if  $k \geq 7$ .

It remains to prove that the charge remaining on each agency is non-negative. Let  $C$  be an agency. Let  $n_{6_6}(C)$  and  $n_l(C)$  be the number of  $6_6$ -vertices and leaves of  $C$  respectively. Observe that:

$$n_l(C) \geq 4 \cdot n_{6_6}(C) \quad (4)$$

By R2, the agency  $C$  receives  $\frac{1}{23}$  from its adjacent vertices (i.e. it receives  $n_l(C) \cdot \frac{1}{23}$ ), and gives  $\frac{4}{23}$  to each of its own  $6_6$ -vertices by R3 (i.e. it gives  $n_{6_6}(C) \cdot \frac{4}{23}$ ). Hence,  $\omega^*(C) = n_l(C) \cdot \frac{1}{23} - n_{6_6}(C) \cdot \frac{4}{23} \geq 4 \cdot n_{6_6}(C) \cdot \frac{1}{23} - n_{6_6}(C) \cdot \frac{4}{23} \geq 0$  by Equation (4). This completes the proof of Theorem 2.4.

### 3 Conclusion

In this note, we established that for every simple graph  $G$ , if  $Mad(G) \leq \frac{32}{13}$  (resp.  $\frac{30}{11}$ ,  $\frac{32}{11}$ ,  $\frac{70}{23}$ ) then  $G$  admits an edge-partition into a forest and a graph with maximum degree at most 1 (resp. 2, 3, 4), hence the game chromatic of  $G$  is at most 5 (resp. 6, 7, 8). In order to study the tightness of Theorem 2, we introduce a function  $f : \mathbb{N} \rightarrow \mathbb{R}$  defined by  $f(k) = \inf\{Mad(H) \mid H \text{ does not admit an edge-partition into a forest and a subgraph with maximum degree } k\}$ . It is easy to observe that the complete bipartite graph  $K_{2,2k+2}$  has  $Mad(K_{2,2k+2}) = \frac{4(k+1)}{k+2}$  and does not admit an edge-partition into a forest and a subgraph with maximum degree at most  $k$ . Hence,

$$\begin{aligned} 2,461\dots &= \frac{32}{13} \leq f(1) \leq \frac{8}{3} = 2,666\dots \\ 2,727\dots &= \frac{30}{11} \leq f(2) \leq 3 \\ 2,909\dots &= \frac{32}{11} \leq f(3) \leq \frac{16}{5} = 3,2 \\ 3,043\dots &= \frac{70}{23} \leq f(4) \leq \frac{10}{3} = 3,333\dots \end{aligned}$$

We conclude with the following problem:

**Problem 1** For every  $k$ , what are (if any) the graphs which do not admit an edge-partition into a forest and a subgraph with maximum degree at most  $k$ , but such that every graph with smaller maximal average degree does?

### References

- [1] H.L. Bodlaender. On the complexity of some coloring games. *Interat. J. Found. Comput. Sci.*, 2:133–147, 1991.
- [2] O.V. Borodin, A.V. Kostochka, N.N. Sheikh, and G. Yu. Decomposing a planar graph with girth 9 into a forest and a matching. *European Journal of Combinatorics*, 29(5):1235–1241, 2008.
- [3] U. Faigle, U. Kern, H.A. Kierstead, and W.T. Trotter. On the game chromatic number of some classes of graphs. *Ars. Combin.* 35:143–150, 1993.
- [4] W. He, X. Hou, K.-W. Lih, J. Shao, W. Wang, and X. Zhu. Edge-partitions of planar graphs and their game coloring numbers. *Journal of Graph Theory*, 41:307–317, 2002.
- [5] X. Zhu. The game coloring number of planar graphs. *Journal of Combinatorial Theory, Series B*, 75:245–258, 1999.



- [6] X. Zhu. Refined activation strategy for the marking game. *Journal of Combinatorial Theory, Series B*, 98: 1–18, 2008.