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# A Polynomial Algorithm for the Computation of Buffer Capacities with Throughput Constraint for Embedded System Design

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**Abstract**—Marked Timed Weighted Event Graphs (in short MTWEG), which are a sub-class of Petri nets, are widely used for modelling practical industrial problems. In this paper, a central practical problem for the design of streaming (e.g. multimedia or network packet processing) applications is modelled using a MTWEG. The optimization problem tackled here consists then on finding an initial marking minimizing the overall number of tokens for a minimum given throughput.

If the firings of the transitions are periodic, this problem is NP-complete and can be modelled using an Integer Linear Program. A general lower bound on the minimum overall capacity is then proved. If the initial MTWEG has a unique circuit, a polynomial time algorithm based on the resolution of a particular Diophantine equation is presented to solve it exactly. We lastly experiment it on an industrial example.

**Index Terms**—Timed Weighted Event Graphs, Periodic Schedule, Manufacturing System, Synchronous Dataflow, Buffer optimization.

## I. INTRODUCTION

Cyclic scheduling problems, in which a set of generic tasks have to be performed infinitely often, have numerous practical applications in manufacturing systems or in the design of digital signal processing. Thus, many theoretical studies were devoted to these problems (for surveys, see [1], [2]).

Marked Weighted Timed Event Graphs (in short MTWEG)  $\mathcal{G}$ , which are a subclass of Petri Nets can be used to model some of these problems. Tasks corresponds to transitions with a fixed duration. Each place  $p = (t_i, t_j)$  has exactly one input and one output transition: at the completion of a firing of  $t_i$ ,  $Z_i$  tokens are added to  $p$ . At the firing of  $t_j$ ,  $Z_j$  tokens are removed from  $p$ . If  $Z_i = 1$  for every transition  $t_i$ ,  $\mathcal{G}$  is a Marked Timed Event Graph (in short MTEG).

MTWEG and MTEG are widely used for modelling and solving practical cyclic scheduling problems. In the context of manufacturing systems, they are considered to model complex assembly lines. Workshop (*resp.* products) are usually modelled by transitions (*resp.* tokens). Between two successive transformations, products (*i.e.* tokens) have to be stored or

have to be moved from a workshop to another one. These amount of products, also called Work In Process (WIP in short), may have economical consequences. Therefore, the main problem for designers is to devise an initial configuration of WIP that allows the system to reach a given productivity and that uses the smallest amount of WIP.

MTWEG can also be considered for modelling data exchanges for streaming applications: transitions correspond to specific treatments. Places are associated with buffers. The total number of tokens of an initial marking is proportional to the overall surface of the memories. As the whole application has to be integrated on a single chip and satisfies high quality requirements, the surface minimization problem with throughput constraints is crucial for the design of these systems. However, designers of such systems usually model their system using Synchronous DataFlow Graph [3] (in short SDF) which is an equivalent formalism.

For a given MTEG or MTWEG, the two fundamental questions are the existence of a schedule and the determination of the optimal throughput.

In the case of MTEG, these two problems are polynomially solved from a long time [4], [5], [6]. Thus, the minimization of a the sum of the initial markings for a minimum given throughput is in *NP*, and many authors developed efficient heuristics and exact methods to solve it (*see.* as example [7], [8], [9]). The *NP*-completeness of this last problem was proved recently in [10].

The existence of a polynomial algorithm for the liveness and the computation of the throughput of a MTWEG (or equivalently to a SDF) is a difficult question. Up to now, the time complexity of all the algorithms developed to answer these two fundamental questions is exponential in the worst case [11], [12]. The consequence is that the optimization problems on MTWEG are possibly not in *NP*: the evaluation of the feasible solutions is not possible in polynomial time, which limits dramatically the existence of efficient algorithms. For example, Sauer [13] developed an algorithm to minimize the sum of the initial markings for a given throughput which evaluates a feasible solution using an exponential algorithm. The evaluation step of this algorithm limits significantly the size of the instances. In [14], [15], several buffer minimization problems with throughput constraint are modelled using an

Integer Linear Program with an exponential number of equations. More recently, in [16], [17] authors have dealt with this problem with throughput constraint based on a state space exploration with model checking techniques.

Another way to circumvent this problem is to reduce the set of feasible solutions. Benabid *et al.* [18] developed a polynomial time algorithm for the computation of a periodic firing of the transitions. This result can be regarded as a generalization of Reiter's result for MTEG [19]. In the case of MTWEG, the existence of a periodic firing of the transitions is clearly more restrictive than the liveness. *the asap* firing transition. The periodic scheduling policy is not necessarily optimal for the throughput criteria. However, optimization problems, such as the minimization of the initial markings are now in *NP* and efficient algorithms may be developed (even if the problem in *NP*-complete). As example, Wiggers *et al.* [20] developed a heuristic to solve it.

In this paper, we study the minimization of the overall number of initial tokens in a MTWEG for a periodic schedule with a given period. Section 2 is dedicated to basic definitions and the description of our problem. In Section 3, we show the modelling of a car radio using a MTWEG. Section 4 presents some important known basic results on periodic schedules. In Section 5, we show that our problem can be formulated using an Integer Linear Program and we show a first general lower bound on the overall places capacities. We prove in Section 6 and 7 that, if the MTWEG is a circuit, the determination of an optimal marking may be solved polynomially. In Section 8, we apply our algorithm to the example presented in Section 3. We conclude in Section 9.

## II. MODEL AND NOTATIONS

### A. Basic definitions

A Marked Timed Weighted Event Graph  $\mathcal{G} = (T, P, l, M_0)$  is defined by a set of places  $P = \{p_1, \dots, p_m\}$  and a set of transitions  $T = \{t_1, \dots, t_n\}$ . Every place  $p \in P$  is defined between two transitions  $t_i$  and  $t_j$  and is denoted by  $p = (t_i, t_j)$ . For any transition  $t_i \in T$ , we set  $\mathcal{P}^+(t_i) = \{p = (t_i, t_j) \in P, t_j \in T\}$  and  $\mathcal{P}^-(t_i) = \{p = (t_j, t_i) \in P, t_j \in T\}$ . Moreover, it is supposed that  $\mathcal{G}$  is strongly connected: for every couple of vertices  $(x, y) \in (P \cup T)^2$ , there exists a path in  $\mathcal{G}$  from  $x$  to  $y$ .

Every place  $p \in P$  is initially marked by  $M_0(p) \in \mathbb{N}$  tokens. We also suppose that every transition  $t_i$  is valued by a strictly positive integer  $Z_i$  and a processing time  $\ell(t_i)$ . If  $t_i$  is fired at time  $\tau$ ,  $Z_i$  tokens are removed from every place  $p \in \mathcal{P}^-(t_i)$ . At time  $\tau + \ell(t_i)$ ,  $Z_i$  tokens are added to every place  $p \in \mathcal{P}^+(t_i)$ .

A place  $p = (t_i, t_j)$  has a bounded capacity  $F(p) > 0$  if the number of tokens stored in  $p$  can not exceed  $F(p)$ :  $\forall \tau \geq 0, M(\tau, p) \leq F(p)$ . A MTWEG  $\mathcal{G} = (T, P, M_0, l, F)$  is said to be a bounded capacity graph if the capacity of every place  $p \in P$  is bounded by  $F(p)$ . It is proved in [21] that every place  $p = (t_i, t_j)$  with bounded capacity may be replaced by a couple of places  $(p_1 = (t_i, t_j), p_2 = (t_j, t_i))$  denoted by  $(p_1, p_2)_c$  with the initial marking  $M_0(p_1) = M_0(p)$  and  $M_0(p_2) = F(p) - M_0(p)$ . So, in this paper, we only consider

symmetric MTWEG: every place  $p = (t_i, t_j)$  is associated with a backward place  $p' = (t_j, t_i)$  modelling the limited places capacity.

It is assumed that two successive firings of the same transition cannot overlap: this is modeled by a self-loop place  $p = (t_i, t_i), \forall t_i \in T$  with  $M_0(p) = Z_i$ . For a sake of simplicity, these loops are not pictured.

The instantaneous marking of a place  $p \in P$  at time  $\tau \geq 0$  is denoted by  $M(\tau, p)$ . Clearly,  $M(0, p) = M_0(p)$ .

For any couple of integers  $(a, b) \in \mathbb{N}^2$ ,  $\gcd(a, b)$  (*resp.*  $\text{lcm}(a, b)$ ) denotes the greatest common divisor (*resp.* least common multiple) of  $a$  and  $b$ . For every couple of values  $(p, q) \in \mathbb{N} \times \mathbb{N}^*$ , we set  $\lceil p \rceil^{(q)} = \left\lceil \frac{p}{q} \right\rceil \cdot q$ .

## III. EXAMPLE

Let us consider a car-radio application described in [22]. The inputs of such systems are basically a MP3-reader and a cell phone. The output is a mixed sound from these two streams. Without any additional treatment, the output is reintroduced in the system through the cell phone, causing an echo effect. In order to obtain a pure speech in the cell phone, an additional input stream, corresponding to a microphone is added.

Figure 1 presents the streams and the main treatments. The first stream entrance, modelled by  $t_7$  is the MP3 reader.  $t_{10}$  corresponds to the entrance of the additional microphone.  $t_9$  is the output.  $t_3$  is the audio echo cancellation task.  $t_1$  mixes the two input streams.  $t_5$  produces a pure speech from the streams  $t_3$  and the cell phone.

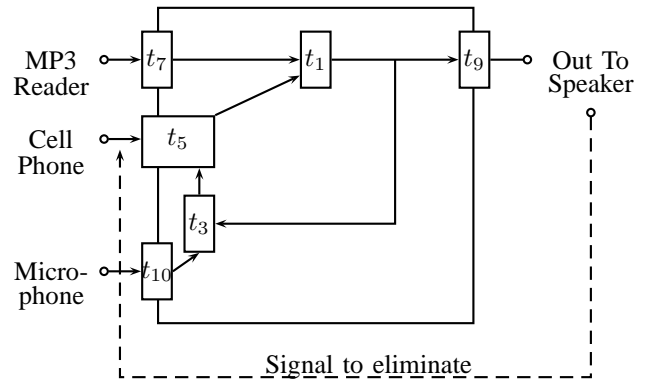


Fig. 1. Block diagram of a car-radio application

Figure 2 shows the modelling of the whole application by a MTWEG  $\mathcal{G}$ . Transitions  $t_2, t_4, t_6$  and  $t_8$  are simple rate convertors. Places model intermediate buffers of limited capacity between the components.

The processing times of the transitions are usually fixed by physical considerations and are presented by table I.

## IV. PERIODIC SCHEDULES

### A. Schedules

Let  $\mathcal{G}$  be a MTWEG. A schedule is a function  $s : T \times \mathbb{N}^* \rightarrow \mathbb{Q}^+$  which associates, with any tuple  $(t_i, q) \in T \times \mathbb{N}^*$ , the

TABLE I  
PROCESSING TIMES  $\ell(t_i)$ ,  $t_i \in T$  IN MILLISECONDS

	$t_1$	$t_2$	$t_3$	$t_4$	$t_5$	$t_6$	$t_7$	$t_8$	$t_9$	$t_{10}$
$l$	$2.3 \times 10^{-2}$	10	9.091	0.125	0.125	10	24	10	$2.3 \times 10^{-3}$	0.125

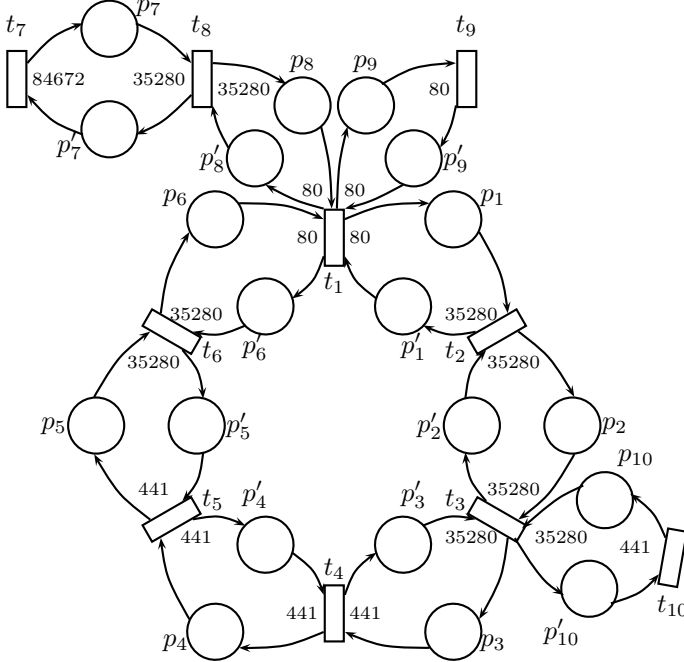


Fig. 2. A MTWEG  $\mathcal{G}$  modelling a car-radio application

starting time of the  $q$ th firing of  $t_i$ . There is a strong relationship between a schedule and the corresponding instantaneous marking. Indeed, a schedule is feasible if the number of tokens of every place  $p = (t_i, t_j)$  remains non negative at each time instant.

It has been proved in [23] that the initial marking  $M_0(p)$  of any place  $p = (t_i, t_j)$  may be replaced by  $\left\lfloor \frac{M_0(p)}{\gcd(Z_i, Z_j)} \right\rfloor \cdot \gcd(Z_i, Z_j)$  without any influence on  $s$ . Thus, we assume that the initial marking  $M_0(p)$  of every place  $p = (t_i, t_j) \in P$  is a multiple of  $\gcd(Z_i, Z_j)$ .

The throughput of a transition  $t_i$  for a schedule  $s$  is defined as

$$\lambda^s(t_i) = \lim_{q \rightarrow \infty} \frac{q}{s(t_i, q)}.$$

### B. Periodic schedules

A schedule  $s$  is periodic if there exists a vector  $w = (w_1, \dots, w_n) \in \mathbb{Q}^{+n}$  such that, for any couple  $(t_i, q) \in T \times \mathbb{N}^*$ ,  $s(t_i, q) = s(t_i, 1) + (q - 1)w_i$ .  $w_i$  is then the period of the transition  $t_i$  and  $\lambda^s(t_i) = \frac{1}{w_i}$  its throughput.

The following theorem proved in [18] characterizes the periodic schedule of a strongly connected MTWEG.

**Theorem 1.** *For any feasible periodic schedule  $s$ , there exists  $K \in \mathbb{Q}^{*+}$  such that, for any couple of transitions  $(t_i, t_j) \in T^2$ ,  $\frac{w_i}{Z_i} = \frac{w_j}{Z_j} = K$ . Moreover,  $s$  is feasible iff, for any place  $p = (t_i, t_j) \in P$ ,*

$$s(t_j, 1) - s(t_i, 1) \geq \ell(t_i) + K(Z_j - M_0(p) - \gcd_{i,j}),$$

where  $\gcd_{i,j} = \gcd(Z_i, Z_j)$ .

For our example, the throughput of the output must be equal to 44.1kHz, thus  $\frac{1}{w_9} = 44.1ms^{-1}$ . Since  $Z_9 = 80$ , we get  $K = \frac{w_9}{Z_9} = 2.83 \cdot 10^{-4}ms$ .

For any place  $p = (t_i, t_j) \in P$ , let us denote by  $H(p) = M_0(p) + \gcd_{i,j} - Z_j$  and  $L(p) = \ell(t_i)$ . For a circuit  $c$ ,  $H(c) = \sum_{p \in c} H(p)$  and  $L(c) = \sum_{p \in c} L(p)$ . Theorem 2 expresses a necessary and sufficient condition for the existence of a periodic schedule deduced from Bellman-Ford algorithm [24].

**Theorem 2.** *There exists a periodic schedule iff, for every circuit  $c$  of  $\mathcal{G}$ ,  $H(c) > 0$ .*

The minimum feasible value  $K^{opt}$  of  $K$  is then:

$$K^{opt} = \max_{c \in C(\mathcal{G})} \frac{L(c)}{H(c)} \quad (1)$$

where  $C(\mathcal{G})$  denotes the set of circuits of  $\mathcal{G}$ .

Numerous polynomial and pseudo-polynomial algorithms were developed to compute  $K^{opt}$  (see. as example [25], [26]). An experimental study of these algorithms can be found in [27].

## V. GENERAL PROBLEM

It is assumed here that  $\mathcal{G}$  is a strongly connected MTWEG. The general problem is first presented and modelled by an Integer Linear Program. A lower bound of the overall capacity is then proved.

### A. Problem Formulation

Let  $\mathcal{G} = (T, P, l, M_0)$  be a symmetric MTWEG and  $K \in \mathbb{Q}^+$  a fixed value for the period. The general problem tackled here is to find an initial marking  $M_0(p)$ ,  $p \in P$  such that:

- 1) The overall capacity  $\sum_{p \in P} F(p) = \sum_{p \in P} M_0(p)$  is minimum.
- 2) There exists a periodic schedule with a period at most equal to  $K$ .

The problem may be formulated by the following Integer Linear Program  $\Pi(K)$ :

$$\min \left( \sum_{p \in P} M_0(p) \right)$$

$$\begin{cases} \forall p = (t_i, t_j) \in P, & s(t_j, 1) - s(t_i, 1) \geq \ell(t_i) + \\ & K \cdot (Z_j - M_0(p) - \gcd_{i,j}) \\ \forall p = (t_i, t_j) \in P, & M_0(p) = k_{ij} \cdot \gcd_{i,j} \\ \forall p = (t_i, t_j) \in P & k_{ij} \in \mathbb{N} \\ \forall t_i \in T, & s(t_i, 1) \geq 0 \end{cases}$$

The first inequality expresses the necessary and sufficient condition associated with a place  $p$  on the first starting times of

a feasible periodic schedule following Theorem 1. The second equality comes from the restriction of  $M_0(p)$ ,  $p = (t_i, t_j) \in P$  to multiples of  $\gcd_{i,j} = \gcd(Z_i, Z_j)$ .

*B. A general lower bound on the overall capacity*

**Lemma 1.** *Let  $(p, p')_c$  be a couple of place with  $p = (t_i, t_j)$  and  $p' = (t_j, t_i)$ . Let also the value*

$$F_K^*(p, p') = \frac{\ell(t_i) + \ell(t_j)}{K} - 2\gcd_{i,j} + (Z_i + Z_j).$$

*Then, for every feasible solution  $M_0^*$  of  $\Pi(K)$ ,  $M_0^*(p) + M_0^*(p') \geq \lceil F_K^*(p, p') \rceil^{(\gcd_{i,j})}$ .*

*Proof:* Let the circuit  $c = (t_i, p, t_j, p', t_i)$ . Then,  $L(c) - KH(c) = \ell(t_i) + \ell(t_j) - K(Z_i + Z_j - M_0^*(p) - M_0^*(p') - 2\gcd_{i,j})$ . If  $M_0^*$  is feasible, we get  $L(c) - KH(c) \leq 0$  and thus  $M_0^*(p) + M_0^*(p') \geq \frac{\ell(t_i) + \ell(t_j)}{K} - 2\gcd_{i,j} + (Z_i + Z_j)$ . Since  $M_0^*(p)$  and  $M_0^*(p')$  are divisible by  $\gcd_{i,j}$  we get the result. ■

For every couple of places  $(p, p')_c \in P^2$ ,  $(p', p)_c$  is also a circuit. Theorem 3 is a simple outcome of Lemma 1:

**Theorem 3.**  $B = \sum_{(p,p')_c \in P^2, p=(t_i,t_j)} \frac{1}{2} \lceil F_K^*(p, p') \rceil^{(\gcd_{i,j})}$  is a lower bound on the overall capacity of a MTWEG  $\mathcal{G}$  for a maximum fixed period  $K \in \mathbb{Q}^+$ .

## VI. A POLYNOMIAL SPECIAL CASE

Let us consider here that  $\mathcal{G}$  is a double circuits of  $n$  transitions defined as  $c = (t_1, p_1, t_2, p_2, \dots, t_n, p_n, t_{n+1})$  with  $t_1 = t_{n+1}$  and  $c' = (t_{n+1}, p'_n, t_n, \dots, t_2, p'_1, t_1)$ . It is also assumed that  $\gcd(Z_1, \dots, Z_n) = 1$ . This assumption is not restrictive: if it is not true, it is proved in [23] that the integers  $Z_i$ ,  $t_i \in T$  can be replaced by  $\frac{Z_i}{\gcd(Z_1, \dots, Z_n)}$  without any influence on the existence and the period of a periodic schedule.

We first present a simplification of the ILP presented in the last section by eliminating the starting times of the first firings of the transitions. Then, we improve the lower bound presented previously. Lastly, we introduce a new system  $\mathcal{S}$  and we show that, every solution of  $\mathcal{S}$  gives an optimal solution. The resolution of  $\mathcal{S}$  is detailed in the next section.

### A. Simplification of the Linear Program

By Bellmann-Ford algorithm, there exists a solution  $M_0(p)$ ,  $p \in P$  for  $\Pi(K)$  if every circuit  $c$  verifies  $L(c) - KH(c) \leq 0$ . Thus, the system may be simplified by eliminating the starting times of the first firing of the transitions as follow:

1) For the first circuit  $c$ ,

$$L(c) - KH(c) = \sum_{i=1}^n \ell(t_i) + K \sum_{i=1}^n (Z_i - M_0(p_i) - \gcd_{i,i+1}) \leq 0.$$

Thus,

$$\sum_{i=1}^n M_0(p_i) \geq \frac{1}{2} \sum_{i=1}^n F_K^*(p_i, p'_i)$$

and then, since the numbers of tokens are integer values,

$$\sum_{i=1}^n M_0(p_i) \geq \left\lceil \frac{1}{2} \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rceil.$$

2) Similarly, we get for the circuit  $c'$ ,

$$\sum_{i=1}^n M_0(p'_i) \geq \left\lceil \frac{1}{2} \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rceil.$$

3) By Lemma 1, circuits  $(t_i, p_i, t_{i+1}, p'_i, t_i)$ ,  $i \in \{1, \dots, n\}$  induces

$$M_0(p_i) + M_0(p'_i) \geq \lceil F_K^*(p_i, p'_i) \rceil^{(\gcd_{i,i+1})}.$$

So, the system  $\Sigma(K)$  to solve for a symmetric circuit is:

$$\min \left( \sum_{p \in P} M_0(p) \right) \begin{cases} \sum_{i=1}^n M_0(p_i) \geq \left\lceil \frac{1}{2} \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rceil \\ \sum_{i=1}^n M_0(p'_i) \geq \left\lceil \frac{1}{2} \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rceil \\ \forall i \in \{1, \dots, n\}, M_0(p_i) + M_0(p'_i) \geq \lceil F_K^*(p_i, p'_i) \rceil^{(\gcd_{i,i+1})} \\ \forall p = (t_i, t_j) \in P, M_0(p) = k_{ij} \cdot \gcd_{i,j} \\ \forall p = (t_i, t_j) \in P, k_{ij} \in \mathbb{N} \end{cases}$$

### B. A Lower bound on the overall capacity

From the two first equations, we get

$$\sum_{i=1}^n M_0(p_i) + \sum_{i=1}^n M_0(p'_i) \geq A,$$

with  $A = 2 \times \left\lceil \frac{1}{2} \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rceil$ . So  $A$  is a lower bound of the overall capacity. Moreover,  $B = \sum_{i=1}^n \lceil F_K^*(p_i, p'_i) \rceil^{(\gcd_{i,i+1})}$  is also a lower bound by Theorem 3.

So,  $\max(A, B)$  is a lower bound of the overall capacity. However, this bound may be improved if  $A > B$ :

**Lemma 2.** *If  $A > B$ , then  $A = B + 1$ .*

*Proof:* Clearly,

$$A = 2 \times \left\lceil \frac{1}{2} \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rceil \leq \left\lceil \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rceil + 1.$$

and

$$\left\lceil \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rceil \leq \sum_{i=1}^n \lceil F_K^*(p_i, p'_i) \rceil^{(\gcd_{i,i+1})} = B.$$

so, lemma holds. ■

**Theorem 4.** *Let the index  $j \in \{1, \dots, n\}$  such that  $\gcd_{j,j+1}$  is minimum. If  $A > B$ , then the minimal overall capacity for a period  $K$  is equal to or greater than  $B + \gcd_{j,j+1}$ .*

*Proof:* If the overall capacity of any couple of places  $(p_i, p'_i)$  is exactly  $\lceil F_K^*(p_i, p'_i) \rceil^{(\gcd_{i,i+1})}$ , then the overall capacity is  $B$ . If  $A > B$ , this solution is then not feasible. So, there is at least a couple  $(p_i, p'_i)_c$  with  $i \in \{1, \dots, n\}$  such that  $M_0(p_i) + M_0(p'_i) > \lceil F_K^*(p_i, p'_i) \rceil^{(\gcd_{i,i+1})}$ . So a new lower bound of the capacity is  $B + \gcd_{j,j+1}$ . By Lemma 2,  $B + \gcd_{j,j+1} \geq B + 1 = A$ , and theorem follows. ■

### C. Building another linear system

The idea here is to build a simpler system  $\mathcal{S}$  and to prove that an optimum solution for  $\Sigma(K)$  can be deduced from every solution of  $\mathcal{S}$ .

Let us define the sequence  $A_i, i \in \{1, \dots, n\}$  as follows:

- If  $B \geq A$ , we set  $A_i = \lceil F_K^*(p_i, p'_i) \rceil^{(\gcd_{i,i+1})}$  for every  $i \in \{1, \dots, n\}$ ;
- Else, let  $j \in \{1, \dots, n\}$  such that  $\gcd_{j,j+1}$  is minimum.

We set  $A_j = \lceil F_K^*(p_j, p'_j) \rceil^{(\gcd_{j,j+1})} + \gcd_{j,j+1}$  and  $A_i = \lceil F_K^*(p_i, p'_i) \rceil^{(\gcd_{i,i+1})}$  for every  $i \in \{1, \dots, n\} - \{j\}$ .

Let

$$Q = \sum_{i=1}^n A_i$$

be the value of the overall capacity and  $C = \lfloor \frac{Q}{2} \rfloor$ . We also note, for every  $i \in \{1, \dots, n\}$ ,  $a_i = \gcd_{i,i+1}$ . It is proved in the next section that the following system  $\mathcal{S}$  can be solved by a polynomial-time algorithm.

$$\begin{cases} C = \sum_{i=1}^n a_i x_i \\ \forall i \in \{1, \dots, n\}, x_i \in \mathbb{N} \\ \forall i \in \{1, \dots, n\}, 0 \leq a_i x_i \leq A_i \end{cases}$$

**Theorem 5.** Let  $x_i^*, i \in \{1, \dots, n\}$  be a solution of  $\mathcal{S}$ . Then, the initial marking  $M_0^*$  defined as,  $\forall i \in \{1, \dots, n\}$ ,  $M_0^*(p_i) = a_i x_i^*$  and  $M_0^*(p'_i) = A_i - a_i x_i^*$  is an optimum solution of  $\Sigma(K)$ .

*Proof:* For every  $i \in \{1, \dots, n\}$ ,  $M_0^*(p_i)$  and  $M_0^*(p'_i)$  are clearly divisible by  $\gcd_{i,i+1}$ . Moreover,  $M_0^*(p_i) + M_0^*(p'_i) = A_i \geq \lceil F_K^*(p_i, p'_i) \rceil^{(\gcd_{i,i+1})}$ . Thus, the third inequality of  $\Sigma(K)$  is fulfilled for every couple of places.

Two subcases must be considered:

- 1) If  $B \geq A$ , then  $\frac{Q}{2} = \frac{B}{2} \geq \frac{A}{2}$ . Thus, since  $\frac{A}{2}$  is an integer value,  $C = \lfloor \frac{Q}{2} \rfloor \geq \frac{A}{2}$ . Now,

$$\sum_{i=1}^n M_0^*(p_i) = C \geq \frac{A}{2} = \left\lfloor \frac{1}{2} \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rfloor$$

and the first inequality of  $\Sigma(K)$  is fulfilled.

On the same way,

$$\sum_{i=1}^n M_0^*(p'_i) = \left\lfloor \frac{Q}{2} \right\rfloor \geq \left\lfloor \frac{1}{2} \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rfloor$$

and the second inequality of  $\Sigma(K)$  is also verified.

Lastly, the overall capacity is  $Q = \sum_{i=1}^n A_i = B$ , thus it is minimum.

- 2) Let us suppose now that  $B < A$ . Then, by Theorem 4,  $A = B + 1$  and

$$\frac{Q}{2} = \frac{B + a_j}{2} = \frac{a_j - 1}{2} + \frac{A}{2}.$$

Since  $a_j \geq 1$  and  $\frac{A}{2}$  is an integer,

$$C = \left\lfloor \frac{Q}{2} \right\rfloor \geq \left\lfloor \frac{1}{2} \sum_{i=1}^n F_K^*(p_i, p'_i) \right\rfloor,$$

Since  $\sum_{i=1}^n M_0^*(p_i) = C$  and  $\sum_{i=1}^n M_0^*(p'_i) = \left\lfloor \frac{Q}{2} \right\rfloor$ , the two first inequalities of  $\Sigma(K)$  are verified.

Lastly, the overall capacity equals  $Q = B + a_i$  and is minimum by Theorem 4, which completes the proof.  $\blacksquare$

## VII. RESOLUTION OF $\mathcal{S}$

In this section, a polynomial time-algorithm is developed to solve the system  $\mathcal{S}$ . We first present two technical lemmas expressing inequalities on  $C$  and the sequence  $A_i, i \in \{1, \dots, n\}$ . Then, a three steps algorithm is detailed to solve  $\mathcal{S}$ .

### A. Technical properties

The two following lemmas express important technical properties on  $A_i, i \in \{1, \dots, n\}$  and  $C$ :

**Lemma 3.**  $\forall i \in \{1, \dots, n\}, A_i > Z_i + Z_{i+1} - 2a_i$ .

**Lemma 4.**  $\sum_{i=1}^n (Z_i - a_i) \leq C \leq \sum_{i=1}^n A_i - \sum_{i=1}^n (Z_i - a_i)$ .

### B. Step 1 for solving $\mathcal{S}$

The system  $\sum_{i=1}^n a_i x_i = \gcd(a_1, \dots, a_n)$  with  $x_i \in \mathbb{Z}$  is a linear diophantine equation and can be solved by a generalization of the extended euclidean algorithm by a time-complexity algorithm bounded by  $\mathcal{O}(n \log(\max(a_1, \dots, a_n)^2))$ .

For our problem, we have  $\gcd(a_1, \dots, a_n) = \gcd(Z_1, \dots, Z_n) = 1$ . So, a solution to  $C = \sum_{i=1}^n a_i x_i$  with  $x_i \in \mathbb{Z}$  can be easily obtained. Let us denote it by  $\hat{X} = (\hat{x}_1, \dots, \hat{x}_n)$ .

### C. Step 2 for solving $\mathcal{S}$

The aim now is to build, from  $\hat{X}$ , another solution  $\bar{X}$  to the equality  $C = \sum_{i=1}^n a_i x_i$  with  $X \in \mathbb{N}^n$ . Let us build the sequence of integers  $\Delta_k, k \in \{0, \dots, n\}$  as follows:

- 1)  $\Delta_0 = 0, \Delta_n = 0$ ;
- 2) for any  $i \in \{1, \dots, n-1\}$ ,  $\Delta_i$  must be divisible by  $\text{lcm}(a_i, a_{i+1})$  and the inequalities  $0 \leq a_i \bar{x}_i - \Delta_{i-1} + \Delta_i < Z_{i+1}$  must be true.

Observe that, since  $Z_{i+1}$  is divisible by  $\text{lcm}(a_i, a_{i+1})$ , there are at least  $\text{lcm}(a_i, a_{i+1})$  values in the integers interval  $[0, Z_{i+1}[$  and the sequence  $\Delta_i$  exists. Moreover, since  $a_i \hat{x}_i, \Delta_i$  and  $\Delta_{i-1}$  are all divisible by  $a_i$ ,  $a_i \hat{x}_i - \Delta_{i-1} + \Delta_i \leq Z_{i+1} - a_i$ .

We set, for every  $i \in \{1, \dots, n-1\}$ ,  $\bar{x}_i = \hat{x}_i - \frac{\Delta_{i-1}}{a_i} + \frac{\Delta_i}{a_i}$ .

**Lemma 5.**  $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{N}^n$  and verifies  $C = \sum_{i=1}^n a_i \bar{x}_i$ .

*Proof:* By definition of  $\bar{X}$ ,

$$\sum_{i=1}^n a_i \bar{x}_i = \sum_{i=1}^n a_i \hat{x}_i - \sum_{i=1}^n \Delta_{i-1} + \sum_{i=1}^n \Delta_i.$$

Now, since  $\Delta_0 = \Delta_n = 0$ ,

$$\sum_{i=1}^n a_i \bar{x}_i = \sum_{i=1}^n a_i \hat{x}_i - \sum_{i=1}^{n-1} \Delta_i + \sum_{i=1}^{n-1} \Delta_i$$

and thus,  $C = \sum_{i=1}^n a_i \bar{x}_i$ .

Clearly,  $\bar{X} \in \mathbb{Z}^n$ . So, we must check that, for every  $i \in \{1, \dots, n\}$ ,  $\bar{x}_i \geq 0$ .

- 1) This is true by definition of  $\Delta_i$  for  $i \in \{1, \dots, n-1\}$ .
- 2) Now, for every  $i \in \{1, \dots, n-1\}$ ,  $a_i \bar{x}_i \leq Z_{i+1} - a_i$  and

$$a_n \bar{x}_n = C - \sum_{i=1}^{n-1} a_i \bar{x}_i \geq C - \sum_{i=1}^{n-1} (Z_{i+1} - a_i).$$

By Lemma 4,

$$C - \sum_{i=1}^{n-1} (Z_{i+1} - a_i) \geq Z_1 - a_n \geq 0$$

and thus  $\bar{x}_n \geq 0$ . ■

### D. Step 3 for solving $\mathcal{S}$

We compute now from  $\bar{X}$  a solution  $X^*$  for system  $\mathcal{S}$ . Let us build the sequence of positive integers  $\Phi_k$ ,  $k \in \{1, \dots, n+1\}$  as follows:

- 1)  $\Phi_{n+1} = 0$ ,  $\Phi_1 = 1$ ;
- 2) for any  $i \in \{2, \dots, n\}$ , if  $a_i \bar{x}_i + \Phi_{i+1} > A_i$ , then compute  $\Phi_i$  such that  $A_i - Z_i < a_i \bar{x}_i + \Phi_{i+1} - \Phi_i \leq A_i$  and  $\Phi_i$  is divisible by  $\text{lcm}(a_{i-1}, a_i)$ . Otherwise, set  $\Phi_i = 0$ .

As previously, the sequence  $\Phi_i$  exists since  $Z_i$  is divisible by  $\text{lcm}(a_{i-1}, a_i)$ , so there are at least  $\text{lcm}(a_{i-1}, a_i)$  values in the integer interval  $]A_i - Z_i + 1, A_i]$ . Moreover,  $A_i - Z_i$  and  $a_i \bar{x}_i + \Phi_{i+1} - \Phi_i$  are divisible by  $a_i$ , so the first inequality becomes  $A_i - Z_i - a_i \leq a_i \bar{x}_i + \Phi_{i+1} - \Phi_i \leq A_i$ .

We set, for every  $i \in \{1, \dots, n-1\}$ ,  $x_i^* = \bar{x}_i + \frac{\Phi_{i+1}}{a_i} - \frac{\Phi_i}{a_i}$ .

The proof of Theorem 6 is similar to Lemma 5.

**Theorem 6.**  $X^* = (x_1^*, \dots, x_n^*) \in \mathbb{N}^n$  is a solution to system  $\mathcal{S}$ .

## VIII. APPLICATION TO THE CAR-RADIO

Table II summarizes the values obtained for our example.

TABLE II  
OPTIMAL INITIAL MARKINGS FOR THE MTWEG PICTURED BY FIGURE 2

Buffers	$a_i$	$F_K^*$	$[F_K^*]^{(a_i)}$	$M_0^*(p_i)$	$M_0^*(p'_i)$
$(p_1, p'_1)$	80	70560	$882a_1$	$113a_1$	$769a_1$
$(p_2, p'_2)$	35280	67353,048	$2a_2$	0	$2a_2$
$(p_3, p'_3)$	441	67353,048	$153a_3$	$55a_3$	$98a_3$
$(p_4, p'_4)$	441	882	$2a_4$	$2a_4$	0
$(p_5, p'_5)$	441	70560	$160a_5$	$160a_5$	0
$(p_6, p'_6)$	80	70560	$882a_6$	$882a_6$	0
$(p_7, p'_7)$	7056	225792	$32a_7$	0	$32a_7$
$(p_8, p'_8)$	80	70560	$882a_8$	0	$882a_8$
$(p_9, p'_9)$	80	160	$2a_9$	0	$2a_9$
$(p_{10}, p'_{10})$	441	67353,048	$153a_{10}$	0	$153a_{10}$

The initial marking for the places from the circuit  $c = (t_1, p_1, t_2, \dots, t_6, p_6, t_1)$  was computed using the algorithm developed here. We obtained for the lower bounds  $B = 350595$  and  $A = 347270$ . Since  $B > A$ ,  $Q = B = 350595$

and  $C = \left\lfloor \frac{Q}{2} \right\rfloor = 175297$ . The vectors obtained for the three steps are  $\hat{X} = (215 \cdot C, 0, -39 \cdot C, 0, 0, 0)$ ,  $\bar{X} = (113, 0, 0, 0, 57, 1764)$  and  $X^* = (113, 0, 55, 2, 160, 882)$ .

For any couple of places  $(p_i, p'_i) \in P$  which are not in  $c$ , the minimum capacity of the buffer is  $[F_K^*(p_i, p'_i)]^{(a_i)}$ . These buffers are initially empty, so  $M_0^*(p_i) = 0$ . If we set  $M_0^*(p'_i) = [F_K^*(p_i, p'_i)]^{(a_i)}$ , we obtain a feasible solution for the system  $\Sigma(K)$ . Thus, it is an optimal initial marking.

If we compare our numerical results to [22], our results are slightly better. As example, for the couple  $(p_3, p'_3)$ , they get a capacity of  $158a_3$ , which is not minimum. Moreover, they supposed that a buffer is either initially full or empty, which limits solutions space and allows them to cut circuits. Lastly, time complexity of their algorithm is unknown.

## IX. CONCLUSION

We have developed in this paper a polynomial time algorithm for the minimization of the overall number of tokens for a minimum throughput. We proved that this problem can be modelled using an Integer Linear Program. A pertinent lower bound of the overall number of tokens is easily deduced from this formulation. We also proved then that if the initial MTWEG has a unique circuit, the problem considered is equivalent to a specific Linear Diophantine problem solvable by a polynomial time algorithm. This last algorithm was considered to solve exactly a practical application.

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