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# Pinning of a polymer on a disordered line with finite range correlations: the annealed critical curve

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**Abstract:** This paper focuses on directed polymers pinned at a disordered and correlated interface. We assume that the disorder sequence is a  $q$ -order moving average and show that the critical curve of the annealed model can be expressed in terms of the Perron-Frobenius eigenvalue of an explicit transfer matrix, which generalizes the annealed bound of the critical curve for i.i.d. disorder. We provide explicit values of the annealed critical curve for  $q = 1, 2$  and a weak disorder asymptotic in the general case. Following the renewal theory approach of pinning, the processes arising in the study of the annealed model are particular Markov renewal processes.

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## 1. Introduction

Polymers are macromolecules which are modeled by self-avoiding or directed random walks. Take for instance  $S = (S_n)_{n \geq 0}$  a random walk on  $\mathbb{Z}$  starting at 0 and such that  $|S_{n+1} - S_n| \leq 1$ . By polymer of dimension 1+1 and size  $N$  we will mean a realization of the directed random walk  $\{(n, S_n)\}_{0 \leq n \leq N}$ , where each segment  $[(n, S_n), (n+1, S_{n+1})]$  stands for a constitutive unit, called monomer.

Suppose now that a reward  $h$  is given to a configuration  $\{(n, S_n)\}_{0 \leq n \leq N}$  each time it touches the interface, i.e. each time  $S_n = 0$ . One can then consider a distribution on polymers of size  $N$  whose density with respect to the initial distribution is equal, up to a renormalizing constant, to the Boltzmann factor

$$\exp(h \times \text{Card}\{n \in \{1, \dots, N\} | S_n = 0\}).$$

Depending on the sign of  $h$ , this distribution favors or penalizes polymers pinned to the interface, and letting  $N$  go to infinity, the model, called homogeneous pinning model, undergoes a localization/delocalization transition.

Pinning models can also be used to study the interaction between two polymers, since the difference of two random walks is still a random walk. One can think for example of the two complementary strands of a DNA molecule: in this

case, the values of  $n$  for which  $S_n = 0$  are the sites where the two strands are pinned, and the delocalization transition corresponds to DNA denaturation (or melting). One could argue that the binding strength between the two strands actually depends on the base pair, which is A-T or G-C. This corresponds to looking at a disordered model, i.e. a model in which the reward is  $n$ -dependent. An assumption usually made is that the reward at site  $n$  writes

$$h_n = h + \beta\omega_n$$

where  $h \in \mathbb{R}$ ,  $\beta \geq 0$  and  $\omega = (\omega_n)_{n \geq 0}$  is a frozen realization of a sequence of independent standard gaussian random variables. The space of parameters is then partitioned in localized and delocalized phases, separated by a critical curve  $\beta \mapsto h_c(\beta)$ . The presence of disorder has important consequences on the model. For example, one can show that there is localization for  $h < 0$  provided that disorder is strong enough (i.e.  $\beta$  large enough). If we consider the annealed model (i.e. the model in which the Boltzmann factor is averaged over disorder), we have the following lower bound:

$$h_c(\beta) \geq -\log \mathbb{P}(\tau_1 < +\infty) - \frac{\beta^2}{2} \quad (1)$$

where  $\tau_1$  is the first return time of  $S$  to 0. In the last few years, many rigorous results were given on relevance of disorder, which in particular answer the following question: when is (1) an equality? For these questions, as well as classical results on homogeneous and disordered pinning models, we refer to [6], [7], [13] and references therein.

In this paper we remove the independence assumption on  $\omega$  and study the effect of correlations on the right-hand side of (1), i.e on the annealed critical curve. This is partly motivated by the long-range correlations in DNA sequence, see [4] and [10] on this topic. We also mention [1] and [9] where the effect of sequence correlation is investigated, in somewhat different contexts. In [1], the authors study the effect of a pulling force applied to the extremity of a DNA strand on the number of broken base pairs (unzipping of DNA) in two correlated scenarii: integrable and nonintegrable correlations. In [9], the authors consider the effect of sequence correlation on the bubble size distribution: by bubbles we mean broken base pairs, and if we keep in mind the analogy with pinning models, it corresponds to the excursions of the directed random walk between two visits at 0.

The disorder sequence in our model is a finite-order moving average of an i.i.d sequence, which is the simplest correlated sequence one can look at, and the reason for this choice will be clearer further in the text. This will be defined in Section 2, as well as the renewal sequence  $\tau = (\tau_n)_{n \geq 0}$  (the contact points) and the polymer measures. In Section 3, we introduce classical notions for these models: the free energy, the phase diagram and the (quenched and annealed) critical curve of the model. In the proof of Theorem 3.1, a new homogeneous model emerges, whose hamiltonian does not only depends on the number of renewal points but also on their mutual distances. Section 4 is dedicated to our

results and their proofs. We show in Theorem 4.1 that the difference between the annealed critical curve in the correlated case and the annealed critical curve in the i.i.d. case can be expressed in terms of the Perron-Frobenius eigenvalue of an explicit transfer matrix, and we give a weak disorder asymptotic of this quantity in Theorem 4.2. Note that the appearance of Perron-Frobenius eigenvalues is reminiscent of results on periodic copolymers, see [3].

## 2. The model

### 2.1. Contact points between the polymer and the line

We follow the renewal theory approach of pinning. Let  $\tau$  be a discrete renewal process such that  $\tau_0 = 0$  and  $\tau_n = \sum_{k=0}^{n-1} T_k$ , where the inter-arrival times (or jumps)  $T_k$  are i.i.d. random variables taking values in  $\overline{\mathbb{N}}^*$ . Furthermore,  $K(n) = \mathbb{P}(T_0 = n) = \frac{L(n)}{n^{1+\alpha}}$  where  $\alpha \geq 0$  and  $L$  is a slowly varying function. If  $K(\infty) := 1 - \sum_{n \geq 1} K(n) > 0$  (resp.  $= 0$ ), we say that the renewal process is transient (resp. recurrent). If  $\tau$  is recurrent, we can distinguish between positive recurrence ( $\alpha > 1$  or  $\alpha = 1$  and  $L$  is such that  $\sum_{n \geq 1} L(n)/n < +\infty$ ) and null recurrence ( $\alpha \in [0, 1)$  or  $\alpha = 1$  and  $L$  is such that  $\sum_{n \geq 1} L(n)/n = +\infty$ ). We will denote by  $\delta_n$  the indicator of the event  $\{n \in \tau\} = \bigcup_{k \geq 0} \{\tau_k = n\}$  so that if  $\iota_N := \sup\{k \geq 0 | \tau_k \leq N\}$  is the number of renewal points before  $N$ , then  $\iota_N = \sum_{n=1}^N \delta_n$ .

We also suppose that for all  $n \geq 1$ ,  $K(n) > 0$  (which implies aperiodicity). This assumption seems quite restrictive, but will be necessary in Section 4.2. If this condition on  $K$  were not fulfilled, we would simply have to reduce the state space of the matrices defined in Section 4 to  $\{n \geq 1 | K(n) > 0\}^q$  (and assume that  $K$  is aperiodic).

### 2.2. Finite range correlations

Let  $(\varepsilon_n)_{n \in \mathbb{Z}}$  be a collection of independent standard gaussian random variables (independent from  $\tau$ ),  $q \geq 1$  a fixed integer, and  $(a_0, \dots, a_q) \in \mathbb{R}^{q+1}$  such that  $a_0^2 + \dots + a_q^2 = 1$ . Define the disorder sequence  $\omega = (\omega_n)_{n \geq 0}$  by the  $q$ -order moving average  $\omega_n = a_0 \varepsilon_n + \dots + a_q \varepsilon_{n-q}$ . Then  $\omega$  is a stationary centered gaussian process and its covariance function  $\rho_n := \text{Cov}(\omega_0, \omega_n)$  satisfies  $\rho_0 = 1$  and  $n > q \Rightarrow \rho_n = 0$ . From now, the notations  $\mathbb{P}_\omega$  and  $\mathbb{E}_\omega$  will be associated to disorder.

We mention that our results are adaptable to non gaussian laws, provided they have exponential moments.

### 2.3. The quenched and annealed polymer measures

We define the (free and constraint) quenched polymer measures, which depend on two parameters, the averaged pinning reward  $h \in \mathbb{R}$  and the amplitude of disorder  $\beta \geq 0$ :

$$\frac{d\mathbb{P}_{N,\beta,h,\omega}^{\mathbf{a}}}{d\mathbb{P}} = \frac{1}{Z_{N,\beta,h,\omega}^{\mathbf{a}}} \exp\left(\sum_{n=1}^N (\beta\omega_n + h)\delta_n\right) (\mathbf{1}_{\{\mathbf{a}=\mathbf{f}\}} + \delta_N \mathbf{1}_{\{\mathbf{a}=\mathbf{c}\}})$$

where

$$Z_{N,\beta,h,\omega}^{\mathbf{a}} = \mathbb{E} \left( \exp\left(\sum_{n=1}^N (\beta\omega_n + h)\delta_n\right) (\mathbf{1}_{\{\mathbf{a}=\mathbf{f}\}} + \delta_N \mathbf{1}_{\{\mathbf{a}=\mathbf{c}\}}) \right)$$

is the partition function. We also define their annealed counterparts:

$$\frac{d(\mathbb{P} \otimes \mathbb{P}_\omega)_{N,\beta,h}^{\mathbf{a}}}{d(\mathbb{P} \otimes \mathbb{P}_\omega)} = \frac{1}{Z_{N,\beta,h}^{\text{ann},\mathbf{a}}} \exp\left(\sum_{n=1}^N (\beta\omega_n + h)\delta_n\right) (\mathbf{1}_{\{\mathbf{a}=\mathbf{f}\}} + \delta_N \mathbf{1}_{\{\mathbf{a}=\mathbf{c}\}})$$

where

$$Z_{N,\beta,h}^{\text{ann},\mathbf{a}} = \mathbb{E}_\omega Z_{N,\beta,h,\omega}^{\mathbf{a}}.$$

### 3. Generalities

#### 3.1. Free energy, phase diagram, and critical curve

We give some results which are well-known for i.i.d. disorder, and which can be generalized to ergodic disorder (see [6, Thm 4.6, p.96]).

**Proposition 3.1.** *For all  $h \in \mathbb{R}$  and all  $\beta \geq 0$ , there exists a nonnegative constant  $F(\beta, h)$  such that for both  $\mathbf{a} = \mathbf{c}$  and  $\mathbf{a} = \mathbf{f}$ ,*

$$F(\beta, h) = \lim_{N \rightarrow +\infty} \frac{1}{N} \log Z_{N,\beta,h,\omega}^{\mathbf{a}}$$

$\mathbb{P}_\omega$ -almost surely and in  $L^1(\mathbb{P}_\omega)$ .

*Proof.* If  $\mathbf{a} = \mathbf{c}$ , we use the Markov property as in [6, Prop 4.2, p.91] or [7, (3.1), p.12] to write

$$\log Z_{N+M,\beta,h,\omega}^{\mathbf{c}} \geq \log Z_{M,\beta,h,\omega}^{\mathbf{c}} + \log Z_{N,\beta,h,\theta^M \omega}^{\mathbf{c}}$$

where  $\theta$  is the shift operator. We then use Kingman's subadditive theorem (see [12]) ( $\omega$  is ergodic since  $\rho_n \xrightarrow{n \rightarrow \infty} 0$ , see [5, Chp 14, §.2, Thm 2]). If  $\mathbf{a} = \mathbf{f}$ , we use the same remark as in the proof of [6, Thm 4.1, p.94].  $\square$

The phase diagram  $\mathbb{R}_+ \times \mathbb{R}$  is then divided into a localized phase

$$\mathcal{L} = \{(\beta, h) | F(\beta, h) > 0\}$$

and a delocalized one

$$\mathcal{D} = \{(\beta, h) | F(\beta, h) = 0\}.$$

For all  $\beta$ , define the critical point  $h_c(\beta) := \sup\{h \in \mathbb{R} | F(\beta, h) = 0\}$ . By convexity of  $F$  (as the limit of convex functions),  $\mathcal{D}$  is convex so the critical curve  $\beta \mapsto h_c(\beta)$  is concave. Moreover, it is nonincreasing and  $h_c(0) = -\log(1 - K(\infty))$ . For detailed arguments, we refer to [6].

### 3.2. Annealed free energy and annealed bound

The first difference that occurs when dealing with correlated disorder is that integrating on  $\omega$  the Boltzmann factor does not yield a classical homogeneous model (see (3) below). As a consequence, we will need an additional argument to define the annealed free energy.

**Lemma 3.1** (Hammersley's approximate subadditivity [8]). *Let  $h : \mathbb{N} \mapsto \mathbb{R}$  be such that for all  $n, m \geq 1$ ,*

$$h(n+m) \leq h(n) + h(m) + \Delta(n+m),$$

with  $\Delta$  a non decreasing sequence satisfying:

$$\sum_{r=1}^{\infty} \frac{\Delta(r)}{r(r+1)} < \infty.$$

Then,  $\lim_{n \rightarrow +\infty} \frac{h(n)}{n}$  exists and is finite.

**Theorem 3.1.** *For all  $h \in \mathbb{R}$  and all  $\beta \geq 0$ , there exists a nonnegative constant  $F^{\text{ann}}(\beta, h)$  such that for both  $\mathbf{a} = \mathbf{c}$  and  $\mathbf{a} = \mathbf{f}$ ,*

$$F^{\text{ann}}(\beta, h) = \lim_{N \rightarrow +\infty} \frac{1}{N} \log Z_{N, \beta, h}^{\text{ann}, \mathbf{a}}.$$

Moreover, if  $h_c^{\text{ann}}(\beta) := \sup\{h \in \mathbb{R} \mid F^{\text{ann}}(\beta, h) = 0\}$  then

$$h_c(\beta) \geq h_c^{\text{ann}}(\beta). \quad (2)$$

*Proof.* First, we compute the variance (with respect to  $\omega$ ) of  $\sum_{n=1}^N \omega_n \delta_n$ . For every realization of  $\tau$ , we have:

$$\text{Var} \left( \sum_{n=1}^N \omega_n \delta_n \right) = \sum_{i,j=1}^N \text{Cov}(\omega_i, \omega_j) \delta_i \delta_j = \sum_{n=1}^N \delta_n + 2 \sum_{1 \leq i < j \leq N} \rho_{j-i} \delta_i \delta_j. \quad (3)$$

Then,  $Z_{N, \beta, h}^{\text{ann}, \mathbf{c}} = \mathbb{E} \left( \exp \left( \left( h + \frac{\beta^2}{2} \right) \sum_{n=1}^N \delta_n + \beta^2 \sum_{i=1}^{N-1} \sum_{k=1}^{N-i} \rho_k \delta_i \delta_{i+k} \right) \delta_N \right)$ . Now, we want some sort of superadditivity for the annealed partition function. For a polymer of size  $N+M$ , observe that

$$\begin{aligned} \sum_{1 \leq i < j \leq N+M} \rho_{j-i} \delta_i \delta_j &= \sum_{1 \leq i < j \leq N} \rho_{j-i} \delta_i \delta_j + \sum_{N+1 \leq i < j \leq N+M} \rho_{j-i} \delta_i \delta_j \\ &\quad + \sum_{1 \leq i \leq N < j \leq N+M} \rho_{j-i} \delta_i \delta_j. \end{aligned}$$

Conditioned on the event  $\{N \in \tau\}$ , the second term has the same law as  $\sum_{1 \leq i < j \leq M} \rho_{j-i} \delta_i \delta_j$ . Moreover, the third term is greater than a constant  $C$  only

depending only  $\rho$  and  $q$ . We can then write

$$\begin{aligned} & Z_{N+M,\beta,h}^{\text{ann},c} \\ & \geq \mathbb{E} \left( \exp \left( \left( h + \frac{\beta^2}{2} \right) \sum_{n=1}^{N+M} \delta_n + \beta^2 \sum_{1 \leq i < j \leq N+M} \rho_{j-i} \delta_i \delta_j \right) \delta_N \delta_{N+M} \right) \\ & \geq e^{C\beta^2} Z_{N,\beta,h}^{\text{ann},c} Z_{M,\beta,h}^{\text{ann},c} \end{aligned}$$

and we conclude (for  $\mathbf{a} = \mathbf{c}$ ) by using Lemma 3.1 to  $-\log Z_{N,\beta,h}^{\text{ann},c}$  with  $\Delta(n) = -C\beta^2$ . For  $\mathbf{a} = \mathbf{f}$ , we use the same remark as in [6, Thm 4.1, p.94] (observe that for a fixed value of the parameters  $h$  and  $\beta$ , the amount of energy necessary to complete an excursion is bounded). As in [6, Prop 5.1] we use Jensen's inequality to prove that  $F(\beta, h) \leq F^{\text{ann}}(\beta, h)$ , which in turn yields the annealed bound (2).  $\square$

When disorder is i.i.d, (2) becomes  $h_c(\beta) \geq h_c(0) - \beta^2/2 := h_c^{\text{ann}}(\beta)$  and the question of knowing whether this is an equality was studied in several papers and monographs (for example, [7], [13], [6] and references therein) where we learn that the answer depends on the values of  $\alpha$  and  $\beta$ .

In the next subsection, we will study the effect of correlations on  $h_c^{\text{ann}}$  in our model.

#### 4. The annealed critical curve

##### 4.1. The result for $q = 1$ and the reason why the technique used does not apply to $q > 1$

**Proposition 4.1.** *If  $q = 1$  then we have*

$$h_c^{\text{ann}}(\beta) = h_c(0) - \frac{\beta^2}{2} - \log \left( 1 + \frac{K(1)}{1 - K(\infty)} \left( e^{\rho_1 \beta^2} - 1 \right) \right)$$

*Proof.* If  $q = 1$ , equality (3) gives:

$$Z_{N,\beta,h}^{\text{ann},c} = \mathbb{E} \left( \exp \left( \left( h + \frac{\beta^2}{2} \right) \iota_N + \rho_1 \beta^2 \sum_{n=1}^{N-1} \delta_n \delta_{n+1} \right) \delta_N \right).$$

The energetic contribution of a jump can only take two values:  $h + (2\rho_1 + 1)\beta^2/2$  if the jump has size 1 and  $h + \beta^2/2$  otherwise. The rest of the proof is a slight modification of the proof of [6, Prop 1.1], except we must consider  $K_{(q=1)}$  with  $K_{(q=1)}(1) := e^{\rho_1 \beta^2} K(1)$  and  $K_{(q=1)}(n) := K(n)$  if  $n > 1$ .  $\square$

If  $q \geq 2$ , the situation is more complicated because in this case we must consider the energetic contribution of a  $q$ -tuple of jumps instead of one of a single jump. For example, if  $q = 2$ , the energetic contribution of a jump of size 1 can be  $h + (1 + 2\rho_1)\beta^2/2$  or  $h + (1 + 2\rho_1 + 2\rho_2)\beta^2/2$ , depending on the value of the jump just before. This idea of looking at the sequence of  $q$ -tuples of consecutive inter-arrival times is developed in the next section.

#### 4.2. An auxiliary Markov chain and the transfer matrix

From now we assume  $q \geq 2$ . We will denote by  $\bar{t} = (t_0, \dots, t_{q-1})$  a  $q$ -tuple in  $(\mathbb{N}^*)^q$  and if  $(t_n)_{n \geq 0}$  is a sequence, then  $\bar{t}_n := (t_n, \dots, t_{n+q-1})$ . The projection on the first coordinate  $\bar{t} \mapsto t_0$  will be denoted by  $\pi_0$ . Let  $G$  be a function defined on such  $q$ -tuples by  $G(\bar{t}) = \sum_{k=0}^{q-1} \rho_{t_0+\dots+t_k}$ , and which should be interpreted like this: if  $\bar{t}$  is the  $q$ -tuple of the inter-arrival times of  $q+1$  consecutive renewal points on the interface, then  $G(\bar{t})$  gives the total contribution of correlations between disorder at these points.

Notice that when we compute the value of  $G$  for some  $q$ -tuple of inter-arrival times, any inter arrival time strictly greater than  $q$  "does not count". To put it more precisely, we can consider a "cemetery state", denoted by  $\star$ , and for all  $t \in \mathbb{N}^*$  and  $\bar{t}$  we define  $t^* := t\mathbf{1}_{\{t \leq q\}} + \star\mathbf{1}_{\{t > q\}}$  and  $\bar{t}^* = (t_0^*, \dots, t_{q-1}^*)$ . Then  $G$  can be considered as a function of  $\bar{t}^*$  instead of  $\bar{t}$ , if we adopt the following natural conventions:  $\rho_\star = 0$  and for all  $t \in \{1, \dots, q, \star\}$ ,  $\star + t = t + \star = \star$ . From now we will use the following notations:  $E = \{1, \dots, q, \star\}$  and  $K(\star) = \sum_{n > q} K(n)$ .

We will now make the following remark: the sequence of  $q$ -tuples  $(\bar{T}_n)_{n \geq 0}$  is a Markov chain on a countable state space, and its transition probability from state  $\bar{s} = (s_0, \dots, s_{q-1})$  to state  $\bar{t} = (t_0, \dots, t_{q-1})$  writes

$$Q(\bar{s}, \bar{t}) := K(t_{q-1}) \prod_{k=1}^{q-1} \mathbf{1}_{\{s_i = t_{i-1}\}}.$$

Note that  $Q$  is irreducible because of the positiveness of the  $K(n)$ 's. We now define the nonnegative matrices  $Q_\beta$  and  $Q_\beta^*$ , which will play the role of transfer matrices, by

$$Q_\beta(\bar{s}, \bar{t}) = e^{\beta^2 G(\bar{t})} \frac{K(t_{q-1})}{1 - K(\infty)} \prod_{i=1}^{q-1} \mathbf{1}_{\{s_i = t_{i-1}\}}$$

and

$$Q_\beta^*(\bar{s}^*, \bar{t}^*) = e^{\beta^2 G(\bar{t}^*)} \frac{K(t_{q-1}^*)}{1 - K(\infty)} \prod_{i=1}^{q-1} \mathbf{1}_{\{s_i^* = t_{i-1}^*\}}.$$

Since  $Q_\beta^*$  is an irreducible nonnegative matrix on the finite state space  $E^q$ , we know by the Perron-Frobenius theorem that there exists a Perron-Frobenius eigenvalue  $\lambda(\beta)$  and an associated right eigenvector  $\nu^* = (\nu^*(\bar{t}^*))$  with positive components (see [11]).

#### 4.3. Statement of the results

We are now ready to state our main results. The first one expresses the annealed critical curve in terms of the Perron-Frobenius eigenvalue of the transfer matrix  $Q_\beta^*$ .



**Theorem 4.1.** For all  $\alpha \geq 0$ , for all  $\beta \geq 0$ ,

$$h_c^{\text{ann}}(\beta) = h_c(0) - \Lambda(\beta)$$

where

$$\Lambda(\beta) := \frac{\beta^2}{2} + \log \lambda(\beta).$$

It seems difficult to give a nice explicit expression of  $\Lambda(\beta)$ , since it is the Perron-Frobenius eigenvalue of a matrix of size  $(q+1)^q$ . For  $q = 2$ , we have computed

$$\Lambda(\beta) = \frac{\beta^2}{2} + \log \phi(\beta) + \log \left( \frac{1 + \sqrt{1 - \frac{\psi(\beta)}{\phi(\beta)^2}}}{2} \right)$$

where

$$\begin{aligned} \phi(\beta) &= 1 + \frac{K(1)}{1 - K(\infty)} (e^{(\rho_1 + \rho_2)\beta^2} - 1) + \frac{K(2)}{1 - K(\infty)} (e^{\rho_2\beta^2} - 1) \\ \psi(\beta) &= 4 \frac{K(1)}{1 - K(\infty)} \left( 1 - \frac{K(1)}{1 - K(\infty)} \right) e^{\rho_1\beta^2} (e^{\rho_2\beta^2} - 1) \\ &\quad \times \left( 1 + \frac{K(2)}{1 - K(\infty) - K(1)} (e^{\rho_2\beta^2} - 1) \right). \end{aligned}$$

In the general case, the asymptotic behaviour of the annealed critical curve for weak disorder can be explicitated:

**Proposition 4.2.** Let  $\hat{\tau}$  be the renewal process starting from 0 with inter-arrival time law  $\hat{K} := \frac{K}{1 - K(\infty)}$ . Then we have

$$h_c(0) - h_c^{\text{ann}}(\beta) \stackrel{\beta \rightarrow 0}{\sim} \left( 1 + 2 \sum_{n=1}^q \rho_n \mathbb{P}(n \in \hat{\tau}) \right) \frac{\beta^2}{2}.$$

By a classical argument (see [7, p.6] or [6, Rmk 1.19, p.40]) we will restrict ourselves in what follows to  $\tau$  recurrent, i.e.  $K(\infty) = 0$ ,  $h_c(0) = 0$  and  $\hat{\tau} = \tau$ .

Before going into details, we outline the proof of Theorem 4.1. First, we introduce in Lemma 4.1 new Markov transition kernels built from the transfer matrices and an eigenvector associated to  $\lambda(\beta)$ . From these we give a new law for the sequence of  $q$ -tuples of consecutive inter-arrival times, to which we associate what could be called a “ $q$ -correlated” renewal process. This process is in fact a particular Markov renewal process (these are processes in which the return times are not necessarily i.i.d., but driven by a Markov chain, see [2] on this subject). With Lemma 4.4, we link the annealed free energy of our initial model to the homogeneous free energy of the new “ $q$ -correlated” renewal process. This will be the starting point of the proof of Theorem 4.1. Note that for positive recurrent renewal processes we give a shorter proof than in the general case.

#### 4.4. A "q-correlated" renewal process related to the model

For all q-tuples  $\bar{t}$ , define  $\nu(\bar{t}) = \nu^*(\bar{t}^*)$ .

**Lemma 4.1.**  $\tilde{Q}_\beta(\bar{s}, \bar{t}) := \frac{Q_\beta(\bar{s}, \bar{t})\nu(\bar{t})}{\lambda(\beta)\nu(\bar{s})}$  and  $\tilde{Q}_\beta^*(\bar{s}^*, \bar{t}^*) := \frac{Q_\beta^*(\bar{s}^*, \bar{t}^*)\nu^*(\bar{t}^*)}{\lambda(\beta)\nu^*(\bar{s}^*)}$  are Markov transition kernels.

*Proof.* For  $\tilde{Q}_\beta^*$ , the result is a direct consequence of the relation  $Q_\beta^*\nu^* = \lambda(\beta)\nu^*$  and of the positiveness of  $\lambda(\beta)$  and  $\nu$ . For  $\tilde{Q}_\beta$ , we write for all  $\bar{s} = (s_0, \dots, s_{q-1})$ ,

$$\begin{aligned} \sum_{\bar{t}} Q_\beta(\bar{s}, \bar{t})\nu(\bar{t}) &= \sum_{t \geq 1} e^{\beta^2 G(s_1, \dots, s_{q-1}, t)} K(t)\nu(s_1, \dots, s_{q-1}, t) \\ &= \sum_{t \geq 1} e^{\beta^2 G(s_1^*, \dots, s_{q-1}^*, t^*)} K(t)\nu^*(s_1^*, \dots, s_{q-1}^*, t^*) \\ &= \sum_{t^* \in E} e^{\beta^2 G(s_1^*, \dots, s_{q-1}^*, t^*)} K(t^*)\nu^*(s_1^*, \dots, s_{q-1}^*, t^*) \\ &= \lambda(\beta)\nu^*(\bar{s}^*) \\ &= \lambda(\beta)\nu(\bar{s}). \end{aligned}$$

The result follows in the same way as for  $\tilde{Q}_\beta^*$ .  $\square$

Since  $\tilde{Q}_\beta^*$  is a finite irreducible transition matrix (it has the same incidence matrix as  $Q_\beta^*$ , which is irreducible), it has a unique invariant probability measure that we denote by  $\mu^*$ . If we define  $\mu$  a measure on  $(\mathbb{N}^*)^q$  by  $\mu(\bar{t}) = \frac{K(t_0)}{K(t_0^*)} \dots \frac{K(t_{q-1})}{K(t_{q-1}^*)} \mu^*(\bar{t}^*)$ , then

**Lemma 4.2.**  $\mu$  is the invariant probability of  $\tilde{Q}_\beta$ .

*Proof.* By a direct computation of  $\sum_{\bar{t}} \mu(\bar{t})$ ,  $\mu$  is a probability. Now we prove that it is invariant. For all  $\bar{t} \in (\mathbb{N}^*)^q$ , we have

$$\begin{aligned} &\sum_{\bar{s}} \mu(\bar{s})\tilde{Q}_\beta(\bar{s}, \bar{t}) \\ &= \lambda(\beta)^{-1} e^{\beta^2 G(\bar{t})} \nu(\bar{t}) K(t_{q-1}) \sum_{s \geq 1} \frac{\mu(s, t_0, \dots, t_{q-2})}{\nu(s, t_0, \dots, t_{q-2})} \\ &= \lambda(\beta)^{-1} e^{\beta^2 G(\bar{t}^*)} \nu^*(\bar{t}^*) K(t_{q-1}) \\ &\times \sum_{s \geq 1} \frac{K(s)K(t_0) \dots K(t_{q-2})}{K(s^*)K(t_0^*) \dots K(t_{q-2}^*)} \frac{\mu^*(s^*, t_0^*, \dots, t_{q-2}^*)}{\nu^*(s^*, t_0^*, \dots, t_{q-2}^*)} \\ &= \lambda(\beta)^{-1} e^{\beta^2 G(\bar{t}^*)} \nu^*(\bar{t}^*) K(t_{q-1}^*) \frac{\mu(\bar{t})}{\mu^*(\bar{t}^*)} \sum_{s \geq 1} \frac{K(s)}{K(s^*)} \frac{\mu^*(s^*, t_0^*, \dots, t_{q-2}^*)}{\nu^*(s^*, t_0^*, \dots, t_{q-2}^*)} \\ &= (\lambda(\beta)^{-1} e^{\beta^2 G(\bar{t}^*)} \nu^*(\bar{t}^*) K(t_{q-1}^*) \frac{\mu(\bar{t})}{\mu^*(\bar{t}^*)}) \sum_{s^* \in E^q} \frac{\mu^*(s^*, t_0^*, \dots, t_{q-2}^*)}{\nu^*(s^*, t_0^*, \dots, t_{q-2}^*)} \\ &= \mu(\bar{t}) \end{aligned}$$

where for the last equality we use the fact that  $\mu^*$  is the invariant probability of  $\tilde{Q}_\beta^*$ .  $\square$

Let  $(\tilde{T}_k)_{k \geq 0}$  be a process of inter-arrival times in  $\mathbb{N}^*$  defined by the initial distribution  $\mathbb{P}(\tilde{T}_0 = t_0, \dots, \tilde{T}_{q-1} = t_{q-1}) = \prod_{k=0}^{q-1} \frac{K(t_k)}{1-K(\infty)}$  and the relation  $\mathbb{P}(\tilde{T}_k = t_q | \tilde{T}_{k-q} = t_0, \dots, \tilde{T}_{k-1} = t_{q-1}) := \tilde{Q}_\beta((t_0, \dots, t_{q-1}), (t_1, \dots, t_{q-1}, t_q))$  for all  $k \geq q$ . From this we define the process  $\tilde{\tau}$  by  $\tilde{\tau}_0 = 0$  and for all  $n \geq 1$ ,  $\tilde{\tau}_n = \sum_{k=0}^{n-1} \tilde{T}_k$ . We generalize the definitions and notations of Section 2.1 to the process  $\tilde{\tau}$ :  $\tilde{\mathbb{P}}$ ,  $\tilde{z}_N$ , and so on. The process  $\tilde{\tau}$  is what we have called a “ $q$ -correlated” renewal process, since the law of one inter-arrival time depends on the value of the  $q$  previous ones.

**Lemma 4.3.** *For all  $h \in \mathbb{R}$  and all  $\beta \geq 0$ ,*

$$F^{\text{ann}}(\beta, h) = \lim_{N \rightarrow +\infty} \frac{1}{N} \log \mathbb{E} \left( e^{(h + \frac{\beta^2}{2})\iota_N + \beta^2 \sum_{n=1}^{\iota_N} G(\bar{T}_n)} \right)$$

(the sum in the right hand side is void on the event  $\{\iota_N = 0\}$ ).

*Proof.* On one hand, we have by integrating over disorder the partition function:

$$\begin{aligned} Z_{N,\beta,h}^{\text{ann}} &= \mathbb{E} \left( \exp \left( \iota_N \left( h + \frac{\beta^2}{2} \right) + \beta^2 \sum_{1 \leq i < j \leq N} \rho_{j-i} \delta_i \delta_j \right) \right) \\ &= \mathbb{E} \left( \exp \left( \iota_N \left( h + \frac{\beta^2}{2} \right) + \beta^2 \sum_{i=1}^{N-1} \sum_{k=1}^{N-i} \rho_k \delta_i \delta_{i+k} \right) \right). \end{aligned}$$

On the other hand,  $\sum_{n=1}^{\iota_N} G(\bar{T}_n) = \sum_{i=1}^{\iota_N} \sum_{k=1}^q \rho_k \delta_i \delta_{i+k}$ . We prove the lemma by showing that there exists a constant  $C(\rho, q)$  such that

$$\left| \sum_{n=1}^{\iota_N} G(\bar{T}_n) - \sum_{i=1}^{N-1} \sum_{k=1}^{N-i} \rho_k \delta_i \delta_{i+k} \right| \leq C(\rho, q).$$

Indeed,

$$\begin{aligned} \sum_{i=1}^N \sum_{k=1}^q \rho_k \delta_i \delta_{i+k} &= \sum_{i=1}^{N-1} \sum_{k=1}^q \rho_k \delta_i \delta_{i+k} + \sum_{k=1}^q \rho_k \delta_N \delta_{N+k} \\ &= \sum_{i=1}^{N-1} \sum_{k=1}^{N-i} \rho_k \delta_i \delta_{i+k} + \sum_{k=1}^q \rho_k \delta_N \delta_{N+k} \\ &\quad + \sum_{i=N-q+1}^{N-1} \sum_{k=N-i+1}^q \rho_k \delta_i \delta_{i+k} \end{aligned}$$

where the second term is bounded in absolute value by  $q \times \max_{i=1 \dots q} |\rho_i|$  and the third term by  $\frac{q(q+1)}{2} \times \max_{i=1 \dots q} |\rho_i|$ .  $\square$

**Lemma 4.4.** For all  $h \in \mathbb{R}$  and all  $\beta \geq 0$ ,

$$F^{ann}(\beta, h) = \lim_{N \rightarrow +\infty} \frac{1}{N} \log \tilde{\mathbb{E}} \left( e^{(h+\Lambda(\beta))\iota_N} \right).$$

*Proof.* Remember that we suppose  $K(\infty) = 0$ . By decomposing on the possible values of  $\iota_N$ , we have on one hand:

$$\begin{aligned} & \mathbb{E} \left( e^{(h+\frac{\beta^2}{2})\iota_N + \beta^2 \sum_{n=1}^N G(\bar{T}_n)} \right) \\ &= \mathbb{P}(\iota_N = 0) + \sum_{n=1}^N e^{(h+\frac{\beta^2}{2})n} \\ & \times \sum_{\substack{\bar{t}_0, \dots, \bar{t}_n \\ t_0 + \dots + t_{n-1} \leq N \\ t_0 + \dots + t_n > N}} e^{\beta^2 \sum_{k=1}^n G(\bar{t}_k)} Q(\bar{t}_0, \bar{t}_1) \dots Q(\bar{t}_{n-1}, \bar{t}_n) K^{\otimes q}(\bar{t}_0) \\ &= \sum_{t > N} K(t) + \sum_{n=1}^N \frac{e^{(h+\Lambda(\beta))n}}{\lambda(\beta)^n} \sum_{\substack{\bar{t}_0, \dots, \bar{t}_n \\ t_0 + \dots + t_{n-1} \leq N \\ t_0 + \dots + t_n > N}} Q_\beta(\bar{t}_0, \bar{t}_1) \dots Q_\beta(\bar{t}_{n-1}, \bar{t}_n) K^{\otimes q}(\bar{t}_0) \\ &= \sum_{t > N} K(t) + \sum_{n=1}^N e^{(h+\Lambda(\beta))n} \\ & \times \sum_{\substack{\bar{t}_0, \dots, \bar{t}_n \\ t_0 + \dots + t_{n-1} \leq N \\ t_0 + \dots + t_n > N}} \frac{\nu(\bar{t}_0)}{\nu(\bar{t}_n)} \tilde{Q}_\beta(\bar{t}_0, \bar{t}_1) \dots \tilde{Q}_\beta(\bar{t}_{n-1}, \bar{t}_n) K^{\otimes q}(\bar{t}_0) \end{aligned}$$

and on the other hand,

$$\begin{aligned} & \tilde{\mathbb{E}} \left( e^{(h+\Lambda(\beta))\iota_N} \right) \\ &= \sum_{t > N} K(t) + \sum_{n=1}^N e^{(h+\Lambda(\beta))n} \sum_{\substack{\bar{t}_0, \dots, \bar{t}_n \\ t_0 + \dots + t_{n-1} \leq N \\ t_0 + \dots + t_n > N}} \tilde{Q}_\beta(\bar{t}_0, \bar{t}_1) \dots \tilde{Q}_\beta(\bar{t}_{n-1}, \bar{t}_n) K^{\otimes q}(\bar{t}_0) \end{aligned}$$

Since  $\nu(\bar{t}) = \nu^*(\bar{t}^*)$  and  $\nu^*$  is a finite vector with positive components, there exists  $c$  and  $C$  two positive constants such that for all  $\bar{t}_0, \bar{t}_n$ ,  $c \leq \frac{\nu(\bar{t}_0)}{\nu(\bar{t}_n)} \leq C$ . We conclude by using this remark and Lemma 4.3.  $\square$

#### 4.5. A short proof of Theorem 4.1 in the positive recurrent case

In accordance with Lemma 4.4, we will work on the homogeneous pinning model of the process  $\tilde{\tau}$ . In the positive recurrent case, a renewal-type lemma is obtained, which allows us to conclude.

**Lemma 4.5.** *If  $\alpha > 1$ , or if  $\alpha = 1$  and  $L$  is such that  $\sum_{n \geq 1} L(n)/n < \infty$  then  $\frac{i_N}{N}$  tends almost surely and in  $L^1$  to a positive constant.*

*Proof.* From Lemma 4.2, the sequence of  $q$ -tuples  $(\tilde{T}_k, \dots, \tilde{T}_{k+q-1})_{k \geq 0}$  is a positive recurrent Markov chain, with invariant probability measure  $\mu$ . If the previous conditions on  $\alpha$  are satisfied,  $\pi_0 : \bar{t} \rightarrow t_0$  (the projection on the first coordinate) is  $\mu$ -integrable. As a consequence,

$$\frac{\tilde{\tau}_N}{N} = \frac{1}{N} \sum_{k=0}^{N-1} \pi_0(\tilde{T}_k, \dots, \tilde{T}_{k+q-1}) \xrightarrow{a.s.} c := \sum_{t_1, \dots, t_{q-1} \geq 1} t_0 \times \mu(t_0, \dots, t_{q-1}) < \infty.$$

We deduce from this that  $\frac{i_N}{N} \xrightarrow{a.s.} \frac{1}{c} > 0$  by using the inequality  $\tau_{i_N} \leq N \leq \tau_{i_N+1}$ . The convergence in  $L^1$  follows from the Dominated Convergence Theorem.  $\square$

From Lemma 4.4,  $h \leq -\Lambda(\beta)$  implies that  $F^{\text{ann}}(\beta, h) = 0$ . Suppose now that  $h = -\Lambda(\beta) + \epsilon$  with  $\epsilon > 0$ . By Jensen's inequality, we have that

$$\frac{1}{N} \log \tilde{\mathbb{E}} \left( e^{(h+\Lambda(\beta))i_N} \right) \geq \epsilon \frac{\tilde{\mathbb{E}}(i_N)}{N}.$$

We conclude that  $F^{\text{ann}}(\beta, h) > 0$  by using Lemma 4.5 and Lemma 4.4.

#### 4.6. Proof of Theorem 4.1 in the general case

We now give a proof without any assumption on  $\alpha$ . The starting point is Lemma 4.4 and we will actually identify the free energy of the pinning model associated to  $\tilde{\tau}$ . Let's fix  $\epsilon > 0$ . We introduce the matrices

$$\tilde{Q}_{\beta, F}(\bar{s}, \bar{t}) = e^{\epsilon - F t_{q-1}} \tilde{Q}_{\beta}(\bar{s}, \bar{t})$$

and

$$\tilde{Q}_{\beta, F}^*(\bar{s}^*, \bar{t}^*) = e^{\epsilon - F \phi_F(t_{q-1}^*)} \tilde{Q}_{\beta}^*(\bar{s}^*, \bar{t}^*)$$

where  $\phi_F(s^*) = s^*$  if  $s^* \in \{1, \dots, q\}$  and

$$\phi_F(\star) = -\frac{1}{F} \log \frac{\sum_{t > q} e^{-Ft} K(t)}{K(\star)}$$

i.e.  $\phi_F(\star)$  verifies

$$e^{-F \phi_F(\star)} K(\star) = \sum_{t > q} e^{-Ft} K(t). \quad (4)$$

We will denote by  $\lambda_F$  the Perron-Frobenius eigenvalue of the irreducible matrix  $\tilde{Q}_{\beta, F}^*$ .

**Lemma 4.6.** *There is a unique positive real denoted by  $\tilde{F}(\epsilon)$  such that*

$$\lambda_{\tilde{F}(\epsilon)} = 1.$$

*Proof.* Elementwise,  $\tilde{Q}_{\beta,F}^*$  is smooth and strictly decreasing with respect to  $F$ . Since  $\lambda_F$  is a simple root of the characteristic equation of  $\tilde{Q}_{\beta,F}^*$  (see [11, Thm 1.1]),  $\lambda_F$  is also a smooth function of  $F$  by the Implicit Function Theorem. From the formula (see [11])

$$\lambda_F = \max_{\substack{v \geq 0 \\ \sum_{E^q} v_i = 1}} \min_{j: v_j > 0} \frac{(\tilde{Q}_{\beta,F}^* v)_j}{v_j}$$

one also obtains (see [6, Appendix A.8]) that  $\lambda_F$  is strictly decreasing in  $F$  and that  $\lambda_F \rightarrow 0$  as  $F \rightarrow \infty$ . Since  $\lambda_0 = \exp \epsilon > 1$ , the result follows.  $\square$

Let  $\tilde{v}^*$  be a Perron-Frobenius right eigenvector of  $\tilde{Q}_{\beta,\tilde{F}(\epsilon)}^*$ . We define  $\tilde{v}$  by

$$\tilde{v}(\bar{t}) = \tilde{v}^*(\bar{t}^*). \quad (5)$$

**Lemma 4.7.** *The matrices*

$$P(\bar{s}, \bar{t}) := \tilde{Q}_{\beta,\tilde{F}(\epsilon)}(\bar{s}, \bar{t}) \frac{\tilde{v}(\bar{t})}{\tilde{v}(\bar{s})} \quad (6)$$

and

$$P^*(\bar{s}^*, \bar{t}^*) := \tilde{Q}_{\beta,\tilde{F}(\epsilon)}^*(\bar{s}^*, \bar{t}^*) \frac{\tilde{v}^*(\bar{t}^*)}{\tilde{v}^*(\bar{s}^*)} \quad (7)$$

are stochastic and irreducible matrices. Furthermore, if we denote by  $l^*$  the invariant probability measure of  $P^*$ , then  $l$  defined by

$$l(\bar{s}) := l^*(\bar{s}^*) \prod_{j=0}^{q-1} \frac{K(s_j) e^{-\tilde{F}(\epsilon) s_j}}{K(s_j^*) e^{-\tilde{F}(\epsilon) \phi_{\tilde{F}(\epsilon)}(s_j^*)}} \quad (8)$$

is the invariant probability measure of  $P$ .

*Proof.* The proof is left to the reader. It consists in straightforward computations very similar to Lemma 4.1 and Lemma 4.2. We use Lemma 4.6 to prove (7), and (4), (5), (7) to prove (6).  $\square$

Note that, like  $Q_\beta$  and  $\tilde{Q}_\beta$ ,  $P$  satisfies the ‘‘consistency’’ condition

$$P(\bar{s}, \bar{t}) \neq 0 \Leftrightarrow s_i = t_{i-1} \forall i \in 1, \dots, q-1.$$

We define the process  $\tau^{(F)}$  by  $\tau_0^{(F)} = 0$  and

$$\tau_n^{(F)} = \sum_{k=0}^{n-1} T_k^{(F)} = \sum_{k=0}^{n-1} \pi_0(\bar{T}_k^{(F)})$$

where  $(T_n^{(F)})_{n \geq 0}$  is a Markov chain of transition  $P$  and initial distribution  $l$ .

**Lemma 4.8.** *There exists two constants  $C \geq c > 0$  such that*

$$ce^{\tilde{F}(\epsilon)N} \mathbb{P}(N \in \tau^{(F)}) \leq \tilde{\mathbb{E}} \left( \exp(\epsilon \tilde{\lambda}_N) \tilde{\delta}_N \right) \leq Ce^{\tilde{F}(\epsilon)N} \mathbb{P}(N \in \tau^{(F)})$$

*Proof.* Decomposing the partition function and using (6) we get

$$\begin{aligned} & \tilde{\mathbb{E}} \left( \exp(\epsilon \tilde{\lambda}_N) \tilde{\delta}_N \right) \\ &= \sum_{n=1}^N \sum_{\substack{\bar{t}_0, \dots, \bar{t}_{n-1} \\ t_0 + \dots + t_{n-1} = N}} e^{\epsilon n} \tilde{Q}_\beta(\bar{t}_0, \bar{t}_1) \dots \tilde{Q}_\beta(\bar{t}_{n-q-1}, \bar{t}_{n-q}) K^{\otimes q}(\bar{t}_0) \\ &= e^{\tilde{F}(\epsilon)N} \sum_{n=1}^N \sum_{\substack{\bar{t}_0, \dots, \bar{t}_{n-q} \\ t_0 + \dots + t_{n-1} = N}} P(\bar{t}_0, \bar{t}_1) \dots P(\bar{t}_{n-q-1}, \bar{t}_{n-q}) l(\bar{t}_0) \\ & \quad \times \left( \frac{\tilde{\nu}(\bar{t}_0)}{\tilde{\nu}(\bar{t}_{n-q})} \frac{K^{\otimes q}(\bar{t}_0) e^{-\tilde{F}(\epsilon)(t_0 + \dots + t_{q-1})}}{l(\bar{t}_0)} e^{\epsilon q} \right) \end{aligned}$$

and, from (5), (8) and the finiteness of  $E^q$ , the term in parenthesis is uniformly bounded by two positive constants  $C$  and  $c$ .  $\square$

From this we deduce:

**Lemma 4.9.** *For all  $\epsilon > 0$ ,*

$$\lim \frac{1}{N} \log \tilde{\mathbb{E}} \left( \exp(\epsilon \tilde{\lambda}_N) \tilde{\delta}_N \right) = \tilde{F}(\epsilon) > 0.$$

*Proof.* Since  $\mathbb{P}(N \in \tau^{(F)}) \leq 1$ , it will be sufficient to prove that

$$\liminf_{N \rightarrow \infty} \mathbb{P}(N \in \tau^{(F)}) > 0.$$

We use an argument that has been already used in the study of Markov renewal processes arising in the study of periodic pinning (see [2, Chp VII.4], [3] or [6, Chp 3]). We choose arbitrarily the state  $\mathbb{1} = (1, \dots, 1) \in (\mathbb{N}^*)^q$ . Consider  $(\theta_n)_{n \geq 0}$  the following sequence of stopping times:

$$\theta_0 = \inf\{n \geq 0 \mid \bar{T}_n^{(F)} = \mathbb{1}\}$$

$$\theta_{k+1} = \inf\{n > \theta_k \mid \bar{T}_n^{(F)} = \mathbb{1}\}$$

Since  $(\bar{T}_n^{(F)})_{n \geq 0}$  is positive recurrent, these stopping times are finite almost surely. If we now define the process  $\tau^\theta$  by  $\tau_n^\theta := \tau_{\theta_n}^{(F)}$  then it is clear that

$$\mathbb{P}(N \in \tau^{(F)}) \geq \mathbb{P}(N \in \tau^\theta)$$

By the strong Markov property,  $\tau^\theta$  is a (delayed) renewal process whose inter-arrival times are on average equal to

$$\begin{aligned}
m &:= \mathbb{E}_{\mathbb{1}}(T_0^{(F)} + \dots + T_{\theta_0-1})^F = \mathbb{E}_{\mathbb{1}} \sum_{n=0}^{\infty} \pi_0(\bar{T}_n^{(F)}) \mathbf{1}_{\{\theta_0 > n\}} \\
&= \sum_{\bar{t}} \pi_0(\bar{t}) \mathbb{E}_{\mathbb{1}} \sum_{n=0}^{\infty} \mathbf{1}_{\{\bar{T}_n^{(F)} = \bar{t}, \theta_0 > n\}} \\
&= \sum_{\bar{t}} \pi_0(\bar{t}) \frac{l(\bar{t})}{l(\mathbb{1})} \\
&= \frac{\sum_{k \geq 1} k(l \circ \pi_0^{-1})(k)}{l(\mathbb{1})} < \infty.
\end{aligned}$$

By the Renewal Theorem, we have

$$\mathbb{P}(N \in \tau^\theta) \xrightarrow{N \rightarrow \infty} 1/m > 0$$

and the proof is complete.  $\square$

Theorem 4.1 is now a direct consequence of Lemma 4.4 combined with Lemma 4.9.

#### 4.7. The weak disorder asymptotic: proof of Proposition 4.2

We now give some lemmas which will be useful for the proof of Proposition 4.2. If  $I \subset E^q$  then we will denote by  $Q_0^{*,I}$  the matrix with entries  $Q_0^{*,I}(\bar{s}^*, \bar{t}^*) = Q_0^*(\bar{s}^*, \bar{t}^*) \mathbf{1}_{\{\bar{t}^* \in I\}}$ . If  $P$  is an  $n$  by  $n$  matrix then  $\text{Com}(P)$  is the matrix of the cofactors of  $P$ , i.e.  $\text{Com}(P)(i, j) = (-1)^{i+j} \det M_{i,j}$  where  $M_{i,j}$  is the  $n-1$  by  $n-1$  matrix obtained by deleting the  $i$ -th line and the  $j$ -th column of  $P$ .

**Lemma 4.10.**  $Q_0^*$  is primitive and its invariant probability measure is  $K^{\otimes q}(\bar{s}^*) = K(s_0^*) \dots K(s_{q-1}^*)$ .

*Proof.* For all  $\bar{t}^* \in E^q$ ,

$$\begin{aligned}
(K^{\otimes q} Q_0^*)(\bar{t}^*) &= \sum_{\bar{s}^* \in E^q} K^{\otimes q}(\bar{s}^*) Q_0^*(\bar{s}^*, \bar{t}^*) \\
&= \sum_{s^* \in E} K^{\otimes q}(s^*, t_0^*, \dots, t_{q-2}^*) K(t_{q-1}^*) \\
&= \sum_{s^* \in E} K(s^*) K(t_0^*) \dots K(t_{q-2}^*) K(t_{q-1}^*) \\
&= K^{\otimes q}(\bar{t}^*)
\end{aligned}$$



so  $K^{\otimes q}$  is the invariant probability measure of  $Q_0^*$ . Moreover,

$$\begin{aligned} (Q_0^*)^q(\bar{s}^*, \bar{t}^*) &= \mathbb{P}(T_q^* = t_0^*, \dots, T_{2q-1}^* = t_{q-1}^* | T_0^* = s_0^*, \dots, T_{q-1}^* = s_{q-1}^*) \\ &= \mathbb{P}(T_q^* = t_0^*, \dots, T_{2q-1}^* = t_{q-1}^*) \\ &= K^{\otimes q}(\bar{t}^*) \end{aligned}$$

which is positive under the assumptions of Section 2.1. Since  $(Q_0^*)^q > 0$ ,  $Q_0^*$  is primitive.  $\square$

**Lemma 4.11.**  $\text{Tr}(\text{Com}(Q_0^* - \text{Id})) \neq 0$  and for all  $\bar{s}^* \in E^q$

$$\frac{\text{Tr}\left({}^t \text{Com}(Q_0^* - \text{Id})Q_0^*, \{\bar{s}^*\}\right)}{\text{Tr}\left({}^t \text{Com}(Q_0^* - \text{Id})\right)} = K^{\otimes q}(\bar{s}^*).$$

*Proof.* In this proof we will use the properties of the Perron-Frobenius eigenvalue of a primitive matrix, that one can find for example in [11].

We define for all  $\bar{s}^* \in E^q$ :

$$p(\bar{s}^*) := \frac{\text{Tr}\left({}^t \text{Com}(Q_0^* - \text{Id})Q_0^*, \{\bar{s}^*\}\right)}{\text{Tr}\left({}^t \text{Com}(Q_0^* - \text{Id})\right)}.$$

By Lemma 4.10, we only need to prove that  $p$  is the invariant probability measure of  $Q_0^*$ .

Since  $Q_0^*$  is stochastic, 1 is clearly a right eigenvalue of  $Q_0^*$  with associated eigenvector  $\mathbf{1}$  (the vector with 1 on all its components). Moreover,  $Q_0^*$  is primitive (Lemma (4.10)) so the Perron-Frobenius eigenvalue exists and all we have to prove is that  $|\lambda| \leq 1$  for every (possibly complex) eigenvalue of  $Q_0^*$ . Indeed, if  $v$  is an eigenvector associated with such an eigenvalue, and  $\bar{s}^* \in E^q$  is such that  $v(\bar{s}^*) = \max_{\bar{t}^* \in E^q} |v(\bar{t}^*)|$  then

$$\lambda v(\bar{s}^*) = \sum_{\bar{t}^* \in E^q} Q_0^*(\bar{s}^*, \bar{t}^*) v(\bar{t}^*)$$

so  $|\lambda| |v(\bar{s}^*)| \leq |v(\bar{s}^*)|$ , i.e  $|\lambda| \leq 1$ . This proves that 1 is the Perron-Frobenius eigenvalue of  $Q_0^*$ , with associated eigenspace  $\mathbb{R}\mathbf{1}$ .

Now, from [11, Ch. 1, Corollary 2], we have that the rows of  ${}^t \text{Com}(Q_0^* - \text{Id})$  are all equal to the same left eigenvector (for the eigenvalue 1) of  $Q_0^*$ , that we will denote by  $L$ . A first consequence is that  $\text{Tr}(\text{Com}(Q_0^* - \text{Id})) \neq 0$  because the entries of  $L$  are either all positive or all negative. Another consequence is that if we define

$$m = (m(\bar{s}^*))_{\bar{s}^* \in E^q} = \left( \text{Tr}\left({}^t \text{Com}(Q_0^* - \text{Id})Q_0^*, \{\bar{s}^*\}\right) \right)_{\bar{s}^* \in E^q}$$

then  $m(\bar{s}^*) = L(\bar{s}^*)$  for all  $\bar{s}^* \in E^q$ . Moreover, from the relation

$$(Q_0^* - \text{Id})^t \text{Com}(Q_0^* - \text{Id}) = 0$$

we deduce that  $\sum_{\bar{s}^* \in E^q} m(\bar{s}^*) = \text{Tr}({}^t \text{Com}(Q_0^* - \text{Id}))$ . Since  $p$  is simply  $m$  renormalized by  $\sum_{\bar{s}^* \in E^q} m(\bar{s}^*)$ , it is the invariant probability of  $Q_0^*$ .  $\square$

*Proof of Proposition 4.2.* In what follows, we will use the notations  $Q'_0$  and  $Q''_0$  as shortcuts for  $\frac{\partial Q_\beta^*}{\partial \beta}(0)$  and  $\frac{\partial^2 Q_\beta^*}{\partial \beta^2}(0)$ . First we will show that  $\beta \mapsto \lambda(\beta)$  is infinitely differentiable ( $\mathcal{C}^2$  would be enough). Let's define  $\phi(\beta, \lambda) = \det(Q_\beta^* - \lambda \text{Id})$  so that  $\phi(\beta, X)$  is the characteristic polynomial of  $Q_\beta^*$ , and  $\phi(\beta, \lambda(\beta)) = 0$  for all  $\beta$ . The Perron-Frobenius eigenvalue of a nonnegative primitive matrix being a simple root of its characteristic equation,  $\frac{\partial \phi}{\partial \lambda}(\beta, \lambda(\beta)) \neq 0$  for all  $\beta \geq 0$ . Since  $\phi$  is infinitely differentiable, the same holds for  $\lambda$  by the Implicit Function Theorem.

Now, a straightforward computation shows that (we use that  $\lambda(0) = 1$ )

$$\begin{aligned} \frac{\partial \Lambda}{\partial \beta}(0) &= \lambda'(0) \\ \frac{\partial^2 \Lambda}{\partial \beta^2}(0) &= 1 + \lambda''(0) - \lambda'(0)^2. \end{aligned}$$

All we need to show then is

$$\lambda'(0) = 0 \tag{9}$$

$$\lambda''(0) = 2 \sum_{n=1}^q \rho_n \mathbb{P}(n \in \tau). \tag{10}$$

By derivating the relation  $\phi(\beta, \lambda(\beta)) = 0$  we obtain

$$\frac{\partial \phi}{\partial \beta}(0, 1) + \lambda'(0) \frac{\partial \phi}{\partial \lambda}(0, 1) = 0.$$

We already know that  $\frac{\partial \phi}{\partial \lambda}(0, 1) \neq 0$  and since  $Q'_0 = 0$  then  $\frac{\partial \phi}{\partial \beta}(0, 1) = 0$ , which leads to (9).

All we have to do now is to prove (10). A Taylor expansion of  $\det(Q_\beta^* - \lambda(\beta) \text{Id})$  gives:

$$\begin{aligned} \det(Q_\beta^* - \lambda(\beta) \text{Id}) &= \det\left(Q_0^* - \text{Id} + (Q''_0 - \lambda''(0) \text{Id}) \frac{\beta^2}{2} + o(\beta^2)\right) \\ &= \text{Tr}({}^t \text{Com}(Q_0^* - \text{Id})(Q''_0 - \lambda''(0) \text{Id})) \frac{\beta^2}{2} + o(\beta^2) \end{aligned}$$

where we have used the differential of the determinant:  $\det(A + H) = \det(A) + \text{Tr}({}^t \text{Com}(A)H) + o(\|H\|)$ . But since  $\det(Q_\beta^* - \lambda(\beta) \text{Id}) = 0$  we have

$$\text{Tr}({}^t \text{Com}(Q_0^* - \text{Id})(Q''_0 - \lambda''(0) \text{Id})) = 0$$

which yields

$$\lambda''(0) = \frac{\text{Tr}({}^t \text{Com}(Q_0^* - \text{Id})Q''_0)}{\text{Tr}({}^t \text{Com}(Q_0^* - \text{Id}))}.$$

Note that  $\text{Tr}({}^t\text{Com}(Q_0^* - \text{Id})) \neq 0$  (Lemma 4.10).

Let's now consider  $Q_0''$  as a function of  $(\rho_n)_{1 \leq n \leq q}$ . Observe that

$$Q_0''(\bar{s}^*, \bar{t}^*) = 2G(\bar{t}^*)Q_0^*(\bar{s}^*, \bar{t}^*)$$

so  $Q_0''$  linearly depends on  $(\rho_n)_{1 \leq n \leq q}$ . We have then

$$Q_0''(\rho_1, \dots, \rho_q) = Q_0''(0, \dots, 0, \rho_q) + Q_0''(\rho_1, \dots, \rho_{q-1}, 0)$$

The result of the theorem is clearly true for  $q = 1$  (remember that we have an explicit expression of  $\Lambda$  in this case, see Proposition 4.1) so we can suppose that it is true for a  $(q-1)$ -order moving average and show that the result holds for  $q$ . The induction hypothesis then implies

$$\frac{\text{Tr}({}^t\text{Com}(Q_0^* - \text{Id})Q_0''(\rho_1, \dots, \rho_{q-1}, 0))}{\text{Tr}({}^t\text{Com}(Q_0^* - \text{Id}))} = 2 \sum_{n=1}^{q-1} \rho_n \mathbb{P}(n \in \tau)$$

so the only thing left to prove is that

$$\frac{\text{Tr}({}^t\text{Com}(Q_0^* - \text{Id})Q_0''(0, \dots, 0, \rho_q))}{\text{Tr}({}^t\text{Com}(Q_0^* - \text{Id}))} = 2\rho_q \mathbb{P}(q \in \tau). \quad (11)$$

Let's define  $I_q = \{\bar{s}^* \in E^q \text{ s.t. } \rho_q \text{ appears in } G(\bar{s}^*)\}$  and notice that

$$Q_0''(0, \dots, 0, \rho_q) = 2\rho_q Q_0^{*, I_q}.$$

We obtain from Lemma 4.11:

$$\begin{aligned} \frac{\text{Tr}({}^t\text{Com}(Q_0^* - \text{Id})Q_0^{*, I_q})}{\text{Tr}({}^t\text{Com}(Q_0^* - \text{Id}))} &= \sum_{\bar{t}^* \in I_q} K^{\otimes q}(\bar{t}^*) \\ &= \mathbb{P}(q \in \tau) \end{aligned}$$

which proves (11).  $\square$

## 5. Open questions

Here are possible developments and open questions we wish to discuss.

1. What happens to these results if we consider a more general moving average  $\omega_n = \sum_{k \geq 0} a_k \varepsilon_{n-k}$ , where  $\sum_{k \geq 0} a_k^2 = 1$ ? For example, it would be interesting to consider power-like decaying correlations: the difficulty is that much of the arguments used for finite order moving averages, in particular the reduction to a finite space  $E = \{1, \dots, q, \star\}$ , cannot be applied.
2. In which cases do we have  $h_c(\beta) = h_c^{\text{ann}}(\beta)$ ? In the case of i.i.d. disorder, it has been shown that for  $\alpha < 1/2$ , disorder is relevant ( $h_c(\beta) = h_c^{\text{ann}}(\beta)$  for small enough  $\beta$ ), while for  $\alpha > 1/2$ , it is irrelevant. Does the same critical value  $\alpha_c = 1/2$  hold in the correlated case?

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