



**HAL**  
open science

## Some annealed bounds for renewal pinning polymer models with weakly dependent disorder.

Julien Poisat

► **To cite this version:**

Julien Poisat. Some annealed bounds for renewal pinning polymer models with weakly dependent disorder.. 2009. hal-00368656v1

**HAL Id: hal-00368656**

**<https://hal.science/hal-00368656v1>**

Preprint submitted on 17 Mar 2009 (v1), last revised 22 Feb 2011 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SOME ANNEALED BOUNDS FOR RENEWAL PINNING POLYMER MODELS WITH WEAKLY DEPENDENT DISORDER

J. POISAT

## Abstract

The aim of this paper is to provide some estimates on the critical curve of a renewal pinning polymer model in the general case of ergodic disorder. More precisely, annealed bounds are given when the disorder sequence is no longer i.i.d but has still some nice mixing properties.

Key words: Pinning polymer models, renewal processes, disorder, ergodicity, subadditivity, annealed model, weak dependence.

AMS Subject classification: 82B44, 60K37, 60K35

Author's address:

J. Poisat, Université de Lyon, Institut Camille Jordan, 43 bld du 11 novembre 1918, 69622 Villeurbanne, France, e-mail: poisat@math.univ-lyon1.fr

# 1 Introduction

Polymers are macromolecules constituted of repeating units called monomers. They can be either natural (like proteins and DNA) or synthetic (like PVC, polystyrene, silicone, ...). This paper will focus on pinning models (see [8], [9] and references therein), which are used to study the interaction between a polymer and an interface (the membrane of a cell or the interface between two solvents for instance) or between two polymers (which can be the two strands of a DNA molecule).

In this model, a polymer is represented by the graph of a random walk  $(n, S_n)_{n \geq 0}$ , where each segment  $[(n, S_n), (n+1, S_{n+1})]$  stands for a monomer unit. In a simplified way, the energy of a polymer, denoted  $\mathcal{H}$ , is the number of times the polymer touches or crosses the interface (the axis line), i.e the number of times  $S_n$  is equal to 0. The probability of a configuration  $S = (S_n)_{n \geq 0}$  is then given by the Boltzmann factor

$$\mathbb{P}_\beta(S) = \frac{\exp(\beta\mathcal{H})}{Z} \mathbb{P}(S)$$

where  $\beta$  is the inverse temperature,  $Z$  is a normalization constant (the partition function) and  $\mathbb{P}$  is the measure of the initial random walk. This new measure gives additional weight to configurations in which the polymer is localized near the interface, especially if the temperature is low. By modifying the parameter  $\beta$ , the model exhibits a localization-delocalization phenomena as the result of a competition between energy and entropy. Notice that all we need to define the energy of a polymer is the renewal sequence of the pinning points, so instead of giving the law of the random walk, we will give the law of the return time to 0, which at the same time generalizes the model.

One application of this model is the study of the denaturation (or melting) of DNA. This is the process by which a double-stranded molecule of DNA unwinds and gives two single-stranded strings; and it is due to the breaking of the hydrogen bonds between the two strands. In our model, the pinning points represent the hydrogen bonds that remain unbroken, and since the binding energy depends on the nucleotide sequence (there are two possible pairs: adenine-thymine and guanine-cytosine), we want to consider an inhomogeneous, or disordered version of our model. Moreover, it has been stated that the nucleotide sequence in some particular genes is highly correlated (see [11] and [3]) so the model we choose to study is a disordered pinning model with a correlated sequence of bases  $\omega = (\omega_n)_{n \geq 0}$ .

This paper is organized as follows. After defining the model more pre-

cisely, we will prove the existence of a quantity called free energy. Then we will prove the existence of a critical curve which separates the localized phase and the delocalized phase in the phase diagram. The end of the section will be dedicated to annealed bounds and we will have to deal with exponential moments of the form (here the integration is with respect to the disorder  $\omega$ )

$$\mathbb{E} \left( \exp \left( \beta \sum_{n=1}^N \omega_n \mathbf{1}_{\{S_n=0\}} \right) \right)$$

which is not easy when the  $\omega_n$ 's are not i.i.d. We propose three ways to tackle with this difficulty:

**First and second-order autoregressive process (Sections 5 and 6).**

In this first model, the disorder sequence is given by a gaussian process with correlation function  $\rho = (\rho_n)_{n \geq 1}$  which is null, except for  $n \leq 2$ . This may be a quite unrealistic assumption on the correlation, but the model has the advantage of being solvable. Indeed, explicit annealed bounds are given, and the analysis at high and low temperature is done. In the first-order autoregressive disorder case, two methods are used. The first one is longer but more explanatory, and it involves a modified homogeneous model. Note that we consider a gaussian disorder because the results have a more readable expression, but it can be generalized for any law.

**Weakly dependent disorder (Section 7).** In this part we follow a different approach, where we use the new dependence coefficients (see [6] for a reference) and an approximate subadditive theorem (see [10]).

**Causal shift with independent input (Section 8).** In this last model, inspired from time series, the disorder is a function (usually a series) of independent inputs. This particular structure allows us to give an annealed bound.

## 2 Notations

Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space and  $\tau = (\tau_n)_{n \geq 0}$  be a renewal process representing the contact points of the polymer with the interface, such that

$$\begin{cases} \tau_0 := 0 \\ K(n) := \mathbb{P}(\tau_1 = n) = \frac{L(n)}{n^{1+\alpha}}, n \geq 1 \end{cases}$$

where  $\alpha > 0$  and  $L$  is a slowly varying function. Sometimes we will use  $\tau$  as the (random) set of values that the process takes, i.e

$$n \in \tau \iff \exists i \geq 0 \text{ s.t } \tau_i = n.$$

We define  $K(\infty) = 1 - \sum_{n \geq 1} K(n)$  so that  $\tau$  is recurrent if and only if  $K(\infty) = 0$ . In that case, it is positive recurrent if and only if  $\alpha > 1$ . We will assume that  $K(\cdot)$  is aperiodic and that  $K(\infty) < 1$ .

For any real  $h$ , any nonnegative real  $\beta$ , any real sequence  $\omega$  and any integer  $N$  (the size of the polymer) we define the partition function :

$$Z_{N,\beta,h,\omega}^{\mathbf{a}} := \mathbb{E} \left( \exp \left( \sum_{n=1}^N (\beta \omega_n + h) \mathbf{1}_{\{n \in \tau\}} \right) \mathbf{1}_{\{\tau \in \Omega^{\mathbf{a}}\}} \right)$$

where  $\mathbf{a} = \mathbf{f}$  (free) or  $\mathbf{a} = \mathbf{c}$  (constraint) and

$$\begin{cases} \Omega^{\mathbf{f}} = \Omega \\ \Omega^{\mathbf{c}} = \{N \in \tau\}. \end{cases}$$

Sometimes we will use the notation

$$\nu_N := \sup\{k \geq 0 \mid \tau_k \leq N\} = \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}} \quad (1)$$

Let  $(E, \mathcal{A}, \mu, T)$  be **another** probability space where  $T$  is a measure-preserving transformation on  $E$  that is  $\mu$ -ergodic. We are interested in the existence of the free energy when  $\omega_n = f(T^n x)$  for  $x \in E$  and for a suitable measurable function  $f$ . In this case the partition function will be denoted by  $Z_{N,\beta,h,x}^{\mathbf{a}}$ .

**Remark 2.1.** When  $\beta = 0$  (homogeneous case), we will simply denote the partition function by  $Z_{N,0,h}$ .

We will also define a new law for the polymer with parameters  $(\beta, h, \omega)$  and density:

$$\frac{d\mathbb{P}_{N,\beta,h,\omega}^{\mathbf{a}}}{d\mathbb{P}}(\tau) = \frac{1}{Z_{N,\beta,h,\omega}^{\mathbf{a}}} \exp \left( \sum_{n=1}^N (\beta \omega_n + h) \mathbf{1}_{\{n \in \tau\}} \right) \mathbf{1}_{\{\tau \in \Omega^{\mathbf{a}}\}}$$

**Remark 2.2.** We have therefore three ways to write the integral sign:

- $\mathbb{E}$  for the initial polymer law (i.e the law of the renewal process),
- $\mathbb{E}_{N,\beta,h,\omega}^a$  (and also  $\text{Var}_{N,\beta,h,\omega}^a$ ,  $\text{Cov}_{N,\beta,h,\omega}^a$ ) for the polymer law defined above,
- $\mathbb{E}_\mu$  for the disorder.

### 3 Free energy

In [8], the existence of the free energy is first proved in the homogeneous case, then in the case of disorder given by an i.i.d sequence. The proof for the more general case of stationary ergodic sequences using Kingman's subadditive (or superadditive) theorem is presented as an alternative approach. Here we will give the full details of the proof.

**Theorem 3.1.** For every  $f$  in  $L^1(\mu)$  and every  $(\beta, h)$  in  $\mathbb{R}^+ \times \mathbb{R}$ , there exists a nonnegative constant  $F(\beta, h)$ , called **free energy**, such that for  $\mathbf{a} = \mathbf{c}$  and  $\mathbf{a} = \mathbf{f}$ ,

$$\frac{1}{N} \log(Z_{N,\beta,h,x}^{\mathbf{a}}) \rightarrow F(\beta, h) \text{ } \mu\text{-a.s and in } L^1(\mu).$$

**Remark 3.1.** The existence of the free energy is proved for a fixed couple  $(\beta, h)$  so we will write  $Z_{N,x}^{\mathbf{a}}$  instead of  $Z_{N,\beta,h,x}^{\mathbf{a}}$  to make the proof more readable. The dependence of the free energy  $F$  with respect to its parameter  $(\beta, h)$  will be studied in the next section.

*Proof.* First, we will prove the theorem in the constraint case ( $\mathbf{a} = \mathbf{c}$ ) using Kingman's subadditive theorem (see [12] for a reference). For  $M < N$  let's consider the random subsets  $\tau$  that contain M and N. It leads to the following inequality :

$$Z_{N,x}^{\mathbf{c}} \geq \mathbb{E} \left( e^{\sum_{n=1}^M (\beta f(T^n x) + h) 1_{n \in \tau}} e^{\sum_{n=M+1}^N (\beta f(T^n x) + h) 1_{n \in \tau}} 1_{M \in \tau} 1_{N \in \tau} \right)$$

Then, from Markov property,

$$Z_{N,x}^{\mathbf{c}} \geq Z_{M,x}^{\mathbf{c}} Z_{N-M, T^M x}^{\mathbf{c}}$$

We will now note  $Z_{M,N,x}^c := Z_{N-M,T^M x}^c$  and define for every  $M < N$ ,  $F_{M,N} := \log Z_{M,N}$  which is a measurable function on  $\mathbf{E}$ . The transformation  $\mathbf{T}$  preserves the measure  $\mu$  so  $(F_{M,N})$  is stationary and it is super-additive by the previous inequality. It is now enough to prove that  $\sup_N \frac{1}{N} \mathbb{E}_\mu(F_{0,N}) < \infty$  :

$$\begin{aligned} \frac{1}{N} \mathbb{E}_\mu(F_{0,N}) &\leq \frac{1}{N} \mathbb{E}_\mu \log \mathbb{E} \left( \exp \left( \sum_{n=1}^N |\beta| |f(T^n x)| + |h| \right) \right) \\ &= \frac{|\beta|}{N} \sum_{n=1}^N \mathbb{E}_\mu |f(T^n x)| + |h| \\ &= |\beta| \mathbb{E}_\mu(|f|) + |h| \end{aligned}$$

which is finite since  $f$  is integrable. By Kingman's theorem,  $\frac{\log Z_{N,x}^c}{N}$  tends (almost surely and in  $L^1(\mu)$ ) to a measurable function  $F(\beta, h)$  which is  $\mathbf{T}$ -invariant. By ergodicity of  $\mathbf{T}$ , it is a constant.

In the second part of the proof we show that this constant is nonnegative.

$$\begin{aligned} \frac{1}{N} \log Z_{N,x}^c &\geq \frac{1}{N} \log \mathbb{E} \left( \exp \left( \sum_{n=1}^N (\beta f(T^n x) + h) \mathbf{1}_{\{n \in \tau\}} \right) \mathbf{1}_{\{\tau_1 = N\}} \right) \\ &= \frac{1}{N} \log (\exp(\beta f(T^N x) + h) K(N)) \\ &= \beta \frac{f(T^N x)}{N} + \frac{h}{N} + \frac{\log K(N)}{N} \end{aligned}$$

which tends to 0  $\mu$ -a.s as  $N$  tends to infinity by Birkhoff's theorem and the assumption on  $K(\cdot)$ .

All these results still hold for  $\mathbf{a} = \mathbf{f}$  since

$$Z_{N,x}^c \leq Z_{N,x}^f \leq cN \exp(|\beta f(T^N x)| + |h|) Z_{N,x}^c \quad \mu \text{ a.s}$$

for a suitable positive constant  $c$ . Indeed, for  $j \in \{1, \dots, N\}$  we denote by  $E_j$  the event in which the last renewal point before  $N$  is  $N-j$ . We have (we use the notation  $Z_{N,x}^f(E_j) := \mathbb{E} \left( \exp \left( \sum_{n=1}^N (\beta \omega_n + h) \mathbf{1}_{\{n \in \tau\}} \right) \mathbf{1}_{\{\tau \in E_j\}} \right)$ ) by

Markov property:

$$\begin{aligned}
Z_{N,x}^f &= Z_{N,x}^c + \sum_{j=1}^N Z_{N,x}^f(E_j) \\
&= Z_{N,x}^c + \sum_{j=1}^N Z_{N-j,x}^c \mathbb{P}(\tau_1 > j) \\
&= Z_{N,x}^c + \sum_{j=1}^N Z_{N-j,x}^c \left( \sum_{n=j+1}^{\infty} K(n) \right)
\end{aligned}$$

But  $\sum_{n=j+1}^{\infty} K(n) \sim \frac{L(j)}{\alpha j^\alpha}$  (see Appendix A of [8] and references therein for this point) so there exists a positive constant  $c$  such that for every  $j$  in  $\{1, \dots, N\}$ ,  $\sum_{n=j+1}^{\infty} K(n) \leq cjK(j) \leq cNK(j)$ . It gives

$$Z_{N,x}^f \leq Z_{N,x}^c + cN \sum_{j=1}^N Z_{N-j,x}^c K(j)$$

Now notice that from Markov property,

$$Z_{N,x}^c = \sum_{j=1}^N Z_{N-j,x}^c \exp(\beta f(T^N x) + h) K(j).$$

We obtain

$$\begin{aligned}
Z_{N,x}^f &\leq Z_{N,x}^c + cN \exp(|\beta f(T^N x)| + |h|) Z_{N,x}^c \\
&\leq (1 + cN \exp(|\beta f(T^N x)| + |h|)) Z_{N,x}^c \\
&\leq CN \exp(|\beta f(T^N x)| + |h|) Z_{N,x}^c
\end{aligned}$$

for a suitable constant  $C$ .

□

## 4 The Phase Diagram

We are now interested in the representation of the free energy in the half plane  $\{(\beta, h), \beta \geq 0, h \in \mathbb{R}\}$ , called the phase diagram. We will show the existence of a concave curve  $\beta \mapsto h_c(\beta)$  called the critical curve, which separates the localized phase  $\mathcal{L} = \{(\beta, h) | F(\beta, h) > 0\}$  from the delocalized phase  $\mathcal{D} = \{(\beta, h) | F(\beta, h) = 0\}$ . The arguments, based on the convexity of the free energy, are given in [8]. Still we will give a detailed proof.



## 4.1 Existence of a critical curve

From now we will assume, unless stated otherwise, that **the disorder is centered**, i.e  $\int_E f(x)d\mu(x)(= \mathbb{E}_\mu(\omega_0)) = 0$ . We don't lose any generality, since a polymer with homogeneous charge  $h$  and a disorder  $(\omega_n)_{n \geq 0}$  with mean  $m$  is the same as a polymer with centered disorder  $(\omega_n - m)_{n \geq 0}$  and homogeneous charge  $h + m$ .

**Theorem 4.1.** *The following properties hold:*

- $(\beta, h) \mapsto F(\beta, h)$  is convex on  $\mathbb{R}^+ \times \mathbb{R}$
- $\mathcal{D}$  is convex

and if we denote  $h_c(\beta) := \sup\{h | F(\beta, h) = 0\}$  then

- $h_c(\cdot)$  is concave and continuous
- $\forall \beta \geq 0, h_c(\beta) \leq h_c(0) = -\log \sum_{n \geq 1} K(n)$

**Remark 4.1.** *We will not prove that  $h_c(0) = -\log(\sum_{n \geq 1} K(n))$  (see [9] or chapter 2 of [8] for this point, and more generally for what happens in the homogeneous case) but we will just notice that  $h_c(0) = 0$  if  $\tau$  is recurrent and  $h_c(0) > 0$  otherwise.*

**Remark 4.2.** *The last point of the theorem gives us an upper bound for  $h_c$ . This bound can actually be made strict when the charges  $(\omega_n)_{n \geq 0}$  are i.i.d (see chapter 5 of [8]), which shows that disorder has a localizing effect.*

*Proof.* To prove our first point, let's consider  $\log Z_{N,\beta,h}^c$  (we omit  $x$ ) as a function of  $\beta$  and  $h$  and show that it is convex for all  $N$ . By derivation under the integral sign, we have that

$$\begin{aligned} \frac{\partial^2}{\partial h^2} \log Z_{N,\beta,h}^c &= \text{Var}_{N,\beta,h}^c \left( \sum_{n=1}^N 1_{n \in \tau} \right) \\ \frac{\partial^2}{\partial \beta^2} \log Z_{N,\beta,h}^c &= \text{Var}_{N,\beta,h}^c \left( \sum_{n=1}^N \omega_n 1_{n \in \tau} \right) \\ \frac{\partial^2}{\partial h \partial \beta} \log Z_{N,\beta,h}^c &= \text{Cov}_{N,\beta,h}^c \left( \sum_{n=1}^N 1_{n \in \tau}, \sum_{n=1}^N \omega_n 1_{n \in \tau} \right) \end{aligned}$$

To make the proof clearer we will note  $a = \sum_{n=1}^N 1_{n \in \tau}$  and  $b = \sum_{n=1}^N \omega_n 1_{n \in \tau}$  so that the hessian of  $\log Z_{N,\beta,h}^c$  writes:

$$H = \begin{pmatrix} \text{Var}(a) & \text{Cov}(a, b) \\ \text{Cov}(a, b) & \text{Var}(b) \end{pmatrix}$$

where we have omitted the subscripts on Var and Cov. If we write  $\chi_H = X^2 - sX + p$  then

$$\begin{aligned} s &= \text{Var}(a) + \text{Var}(b) \geq 0 \\ p &= \text{Var}(a) \text{Var}(b) - \text{Cov}(a, b)^2 \geq 0 \text{ by Cauchy-Schwarz} \\ \Delta &:= s^2 - 4p = (\text{Var}(a) - \text{Var}(b))^2 + 4 \text{Cov}(a, b)^2 \geq 0 \end{aligned}$$

The hessian is positive for all  $(\beta, h)$  so the function  $(\beta, h) \mapsto \frac{1}{N} \log Z_{N,\beta,h}^c$  is convex (for every  $N$  and  $\mu$ - almost surely). Let  $t$  be in  $(0, 1)$  and  $(\beta, h) = t(\beta_0, h_0) + (1-t)(\beta_1, h_1)$ . We have for every  $\omega$

$$\frac{1}{N} \log Z_{N,\beta,h,\omega}^c \leq t \frac{1}{N} \log Z_{N,\beta_0,h_0,\omega}^c + (1-t) \frac{1}{N} \log Z_{N,\beta_1,h_1,\omega}^c$$

From Theorem 3.1 we have then

$$F(\beta, h) \leq tF(\beta_0, h_0) + (1-t)F(\beta_1, h_1)$$

which proves the convexity of  $F$ . Also it is clear that the convexity of  $F(\cdot, \cdot)$  leads to the convexity of  $\mathcal{D}$ .

Our second point is the concavity of  $h_c(\cdot)$ . For all  $(\beta_1, \beta_2)$  and  $t \in [0, 1]$  we want  $h_c(t\beta_1 + (1-t)\beta_2) \geq th_c(\beta_1) + (1-t)h_c(\beta_2)$ , ie:

$$F(t\beta_1 + (1-t)\beta_2, th_c(\beta_1) + (1-t)h_c(\beta_2)) = 0$$

which is immediate by convexity of  $\mathcal{D}$ . Continuity follows directly for  $\beta > 0$ . For the continuity at  $\beta = 0$ , notice that the free energy could be defined for every  $\beta \in \mathbb{R}$ , without changing anything to the convexity of  $F$ .

For the last point ( $h_c(\beta) \leq h_c(0)$ ), it is enough to prove that  $F(0, h) \leq F(\beta, h)$ . Since  $\beta \rightarrow \mathbb{E}_\mu \log Z_{N,\beta,h}^c$  is convex and that its derivative at  $\beta = 0$  is  $\mathbb{E}_\mu \mathbb{E}_{N,0,h}^c \left( \sum_{n=1}^N \omega_n 1_{n \in \tau} \right) = 0$  (remember that  $\omega_0 = f(\cdot)$  is centered),  $\beta \rightarrow \mathbb{E}_\mu \log Z_{N,\beta,h}^c$  is nondecreasing and by taking the limit  $N \rightarrow \infty$ , so is  $\beta \rightarrow F(\beta, h)$ .  $\square$

## 4.2 Annealed bounds

### 4.2.1 Annealed bound for the i.i.d case.

In the i.i.d case, it is easy to give a lower bound, called annealed bound (see Figure 1). We will note

$$\Lambda(\beta) = \log \mathbb{E}_\mu \left( e^{\beta \omega_0} \right) \in [0, +\infty]$$

and assume that  $\beta_0 = \sup\{\beta \geq 0 | \Lambda(\beta) < \infty\} > 0$ . Note that  $\Lambda$  is a convex function on  $[0, \beta_0)$ .

**Theorem 4.2.** *If the  $\omega_n$ 's are i.i.d, then*

$$\forall \beta \in [0, \beta_0), \quad h_c(\beta) \geq h_c^{ann, iid}(\beta) := h_c(0) - \Lambda(\beta)$$

*Proof.* From Jensen inequality we have

$$\frac{1}{N} \mathbb{E}_\mu (\log Z_{N, \beta, h, \omega}^a) \leq \frac{1}{N} \log \mathbb{E} \left( e^{h \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}}} \mathbb{E}_\mu \left( e^{\beta \sum_{n=1}^N \omega_n \mathbf{1}_{\{n \in \tau\}}} \right) \mathbf{1}_{\{\tau \in \Omega^a\}} \right) \quad (2)$$

Since the  $\omega_n$ 's are i.i.d,

$$\mathbb{E}_\mu \left( e^{\beta \sum_{n=1}^N \omega_n \mathbf{1}_{n \in \tau}} \right) = \exp \left( \Lambda(\beta) \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}} \right)$$

Then,

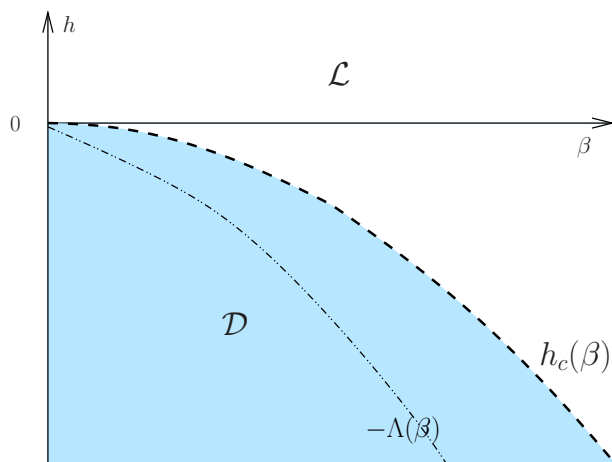
$$\frac{1}{N} \mathbb{E}_\mu (\log Z_{N, \beta, h, \omega}^a) \leq \frac{1}{N} \log Z_{N, 0, h + \Lambda(\beta)}^a$$

and by taking the limit,

$$F(\beta, h) \leq F(0, h + \Lambda(\beta))$$

We conclude by evaluating this inequality with  $h = h_c(0) - \Lambda(\beta)$ .  $\square$

Here the independency of the  $\omega_n$ 's is crucial. In the next sections we will give some annealed bounds without this assumption.



1

Figure 1: The critical curve  $\beta \mapsto h_c(\beta)$  of the phase diagram (when  $h_c(0) = 0$ ) and the annealed bound in the i.i.d case

### 4.2.2 A generalization of $\Lambda$

Let's denote by (A) the following assumptions on the disorder:

- The random variable  $\omega_0$  is  $\mu$ -integrable and centered (or, if we choose the notation of a dynamical system,  $f \in L^1(\mu)$  and  $\int_E f(x)d\mu(x) = 0$ )
- the sequence  $\omega = (\omega_n)_{n \geq 0}$  is stationary and ergodic.

In the previous subsection we obtained an annealed bound by first integrating on the disorder the annealed partition function, which yielded a homogeneous partition function with charge  $h + \Lambda(\beta)$ . The inequality we present now holds for more general disorder (i.e not necessarily i.i.d).

First we define  $\hat{\tau} = (\hat{\tau}_n)_{n \geq 0}$  a renewal process whose law is given by

$$\mathbb{P}(\hat{\tau}_1 = n) = \frac{K(n)}{1 - K(\infty)} := \hat{K}(n)$$

and

$$\hat{\tau}_0 := 0 \tag{3}$$

This new renewal process is recurrent, so that the expression  $\sum_{n=1}^N \omega_{\hat{\tau}_n}$  is well defined for any  $N$ . We can then state the following proposition.

**Theorem 4.3.** *Under assumptions (A), if*

$$\Lambda^d(\beta) := \lim \frac{1}{N} \log \mathbb{E} \mathbb{E}_\mu \left( \exp \left( \beta \sum_{n=1}^N \omega_{\hat{\tau}_n} \right) \right)$$

*exists and is finite, then*

$$h_c(\beta) \geq h_c^{\text{ann}}(\beta) := h_c(0) - \Lambda^d(\beta).$$

**Remark 4.3.** *Notice that if the  $\omega_n$ 's are i.i.d, then  $\Lambda^d = \Lambda$ , so  $\Lambda^d$  is indeed a generalization of  $\Lambda$ .*

*Proof.* We first assume  $\tau$  is recurrent, so that  $h_c(0) = 0$  and the law of  $\hat{\tau}$  is the same as the one of  $\tau$ . Recalling the notation (1), we have by Jensen inequality

$$\begin{aligned} \frac{1}{N} \mathbb{E}_\mu \log Z_{N,\beta,h}^f &\leq \frac{1}{N} \log \mathbb{E} \mathbb{E}_\mu e^{\sum_{k=1}^N (h + \beta \omega_k) 1_{k \in \tau}} \\ &= \frac{1}{N} \log \mathbb{E} \mathbb{E}_\mu e^{\sum_{k=1}^N h + \beta \omega_{\tau_k}} \end{aligned}$$

where the sum is void whenever  $\iota_N = 0$ . By partitioning on the possible values of  $\iota_N$ , we obtain

$$\begin{aligned}\mathbb{E}\mathbb{E}_\mu\left(e^{\sum_{k=1}^{\iota_N} h+\beta\omega_{\tau_k}}\right) &= \sum_{n=0}^N \mathbb{E}\mathbb{E}_\mu\left(e^{\sum_{k=1}^{\iota_N} h+\beta\omega_{\tau_k}} \mathbf{1}_{\{\iota_N=n\}}\right) \\ &= \mathbb{P}(\iota_N = 0) + \sum_{n=1}^N \mathbb{E}\mathbb{E}_\mu\left(e^{\sum_{k=1}^n h+\beta\omega_{\tau_k}} \mathbf{1}_{\{\tau_n \leq N\}} \mathbf{1}_{\{\tau_{n+1} > N\}}\right) \\ &\leq 1 + \sum_{n=1}^N \mathbb{E}\mathbb{E}_\mu\left(e^{\sum_{k=1}^n h+\beta\omega_{\tau_k}}\right)\end{aligned}$$

and since from the existence and finiteness of  $\Lambda^d(\beta)$ ,

$$\mathbb{E}\mathbb{E}_\mu\left(e^{\sum_{k=1}^n (h+\beta\omega_{\tau_k})}\right) = e^{n(h+\Lambda^d(\beta))+g(n)}$$

(where  $g(0) = 0$  and  $g(n) = o(n)$ ) we have by choosing  $h = -\Lambda^d(\beta)$

$$\frac{1}{N} \mathbb{E}_\mu \log Z_{N,\beta,-\Lambda^d(\beta)}^f \leq \frac{1}{N} \log \left( \sum_{n=0}^N e^{g(n)} \right)$$

From Theorem 3.1, the left-hand term tends to  $F(\beta, -\Lambda^d(\beta))$  and the right-hand term tends to 0 as  $N$  tends to  $+\infty$ , therefore  $F(\beta, -\Lambda^d(\beta)) = 0$ , which means that  $h_c(\beta) \geq -\Lambda^d(\beta)$ .

In fact, assuming that  $\tau$  is recurrent is not restrictive. If it is not case, we write

$$\begin{aligned}Z_{N,\beta,h,\omega}^c &= \mathbb{E}\left(e^{\sum_{n=1}^N (h+\beta\omega_n) \mathbf{1}_{\{n \in \tau\}} \mathbf{1}_{\{N \in \tau\}}}\right) \\ &= \sum_{n=1}^N \sum_{\substack{0:=t_0 < t_1 < \dots \\ < t_{n-1} < t_n := N}} \prod_{k=0}^{n-1} e^{h+\beta\omega_{t_{k+1}}} K(t_{k+1} - t_k) \\ &= \sum_{n=1}^N \sum_{\substack{0:=t_0 < t_1 < \dots \\ < t_{n-1} < t_n := N}} \prod_{k=0}^{n-1} e^{h+\log(1-K(\infty))+\beta\omega_{t_{k+1}}} \hat{K}(t_{k+1} - t_k)\end{aligned}$$

and since  $h_c(0) = -\log(1 - K(\infty))$ , we have

$$Z_{N,\beta,h,\omega}^c = \mathbb{E}\left(e^{\sum_{n=1}^N (h-h_c(0)+\beta\omega_n) \mathbf{1}_{\{n \in \hat{\tau}\}} \mathbf{1}_{\{N \in \hat{\tau}\}}}\right).$$

□

## 5 Annealed bound for a first-order autoregressive gaussian process

Let  $\varepsilon = (\varepsilon_n)_{n \in \mathbb{Z}}$  be a gaussian white noise, i.e the  $\varepsilon_n$ 's are independent standard centered gaussian random variables. Since we are able to give an annealed bound for i.i.d sequences, we will try to do so for the simplest correlated sequence derived from such sequences: the linear combination of two consecutive terms of the white noise. Let's define the disorder, for all  $n$  and for some real  $a_0, a_1$ , as

$$\omega_n = a_0 \varepsilon_n + a_1 \varepsilon_{n-1} \quad (4)$$

It is an exemple of what is called in the statistical literature a moving average (or autoregressive process). More precisely, it is a first order one, since  $\omega_n$  only depends on the present value of the noise and its value on the moment just before. A q-order moving average would be a linear combination of  $(\varepsilon_n, \varepsilon_{n-1}, \dots, \varepsilon_{n-q})$ .

Of course, the  $\omega_n$ 's are centered gaussian r.v with variance  $\sigma^2 := a_0^2 + a_1^2$ , and  $\omega$  is stationary. Its covariance function is given by

$$\rho_n := \text{Cov}(\omega_0, \omega_n) = \begin{cases} a_0 a_1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad (5)$$

Taking a covariance function with a cut-off at  $n = 1$  means that the disorder at a site only interact with its closest neighbours.

Such a sequence  $\omega$  is ergodic. Indeed, we only need that  $\rho_n$  tends to 0 as  $n$  tends to  $\infty$  for  $\omega$  to be ergodic. For this point, the reader can refer to chapter 14 of [4]. All the assumptions are now satisfied for the existence of the free energy.

**Remark 5.1.** *In this section we will assume that  $\sum_{n \geq 2} K(n) \neq 0$ , i.e the renewal process does not only make jumps of one unit. We make this assumption so that the new model that we introduce in the first subsection does not reduce to a classical homogeneous model.*

The main theorem of this section gives an annealed bound on the critical curve for this type of disorder:

**Theorem 5.1.** *If the disorder  $\omega$  is given by (4) then we have for all nonnegative  $\beta$ ,*

$$h_c(\beta) \geq h_c^{\text{ann}}(\beta) := h_c(0) - \frac{\beta^2 \sigma^2}{2} - \log \left( 1 + \frac{K(1)}{1 - K(\infty)} (e^{\rho_1 \beta^2} - 1) \right)$$

**Remark 5.2.** *As it was mentioned in the introduction, this result can be generalized for non gaussian white noise  $(\varepsilon_n)_{n \in \mathbb{Z}}$ . In the general case, we define  $\Lambda^*(\cdot) := \mathbb{E}_\mu(e^{\varepsilon_0})$  and Theorem 5.1 writes (whenever  $\Lambda^*$  is finite)*

$$h_c(\beta) \geq h_c^{\text{ann}}(\beta) := h_c(0) - \Lambda^*(a_0 \beta) - \Lambda^*(a_1 \beta) - \log \left( 1 + \frac{K(1)}{1 - K(\infty)} (e^{\Lambda^*((a_0+a_1)\beta) - \Lambda^*(a_0\beta) - \Lambda^*(a_1\beta)} - 1) \right)$$

*One can check that taking  $\Lambda^*(\beta) = \frac{\beta^2}{2}$  gives Theorem 5.1*

In a first part we will introduce a modified homogeneous pinning model whose relevance will be clearer when we integrate out the disorder. The second part is the proof of the theorem and the last part is a discussion on the result.

## 5.1 A new homogeneous model and its phase diagram

Let  $\tilde{\mathbb{P}}_{N, \tilde{\beta}, \tilde{h}}^{\mathbf{a}}$  be a new probability measure whose density with respect to the polymer measure is given by:

$$\frac{d\tilde{\mathbb{P}}_{N, \tilde{\beta}, \tilde{h}}^{\mathbf{a}}}{d\mathbb{P}}(\tau) = \frac{1}{\tilde{Z}_{N, \tilde{\beta}, \tilde{h}}^{\mathbf{a}}} \exp \left( \tilde{h} \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}} + \tilde{\beta} \sum_{n=0}^{N-1} \mathbf{1}_{\{n \in \tau\}} \mathbf{1}_{\{n+1 \in \tau\}} \right) \mathbf{1}_{\{\tau \in \Omega^{\mathbf{a}}\}}$$

where  $(\tilde{\beta}, \tilde{h}) \in \mathbb{R}^2$  and  $\tilde{Z}_{N, \tilde{\beta}, \tilde{h}}^{\mathbf{a}}$  is the partition function of this new polymer measure.

Even if it is a homogeneous model, there are two parameters, because there are two different terms in the hamiltonian: one of them is the usual pinning term  $\sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}}$  and the other one is a new term of interaction, which will give additional weight to configurations in which the polymer crosses the interface at neighbouring sites.

What we want to show is that:



**Theorem 5.2.** *There exists a nonnegative function  $(\tilde{\beta}, \tilde{h}) \mapsto \tilde{F}(\tilde{\beta}, \tilde{h})$  such that for all  $(\tilde{\beta}, \tilde{h})$  in  $\mathbb{R}^2$ ,*

- $\frac{1}{N} \log \tilde{Z}_{N, \tilde{\beta}, \tilde{h}}^c \rightarrow \tilde{F}(\tilde{\beta}, \tilde{h})$
- $\tilde{F}(\tilde{\beta}, \tilde{h}) > 0 \iff \tilde{h} > \tilde{h}_c(\tilde{\beta}) := h_c(0) - \log \left( 1 + \frac{K(1)}{\sum_{n \geq 1} K(n)} (e^{\tilde{\beta}} - 1) \right)$

**Remark 5.3.** *The function  $\tilde{F}$  is the free energy naturally associated with the new polymer measure.*

*Proof.* We will follow the ideas of [8] (Ch. 1) on the classical homogeneous pinning model and write the partition function more explicitly:

$$\tilde{Z}_{N, \tilde{\beta}, \tilde{h}}^c = \sum_{n=1}^N \sum_{\substack{0:=t_0 < t_1 < \dots \\ < t_{n-1} < t_n := N}} \prod_{k=0}^{n-1} e^{\tilde{h}} \left( 1 + (e^{\tilde{\beta}} - 1) \mathbf{1}_{\{t_{k+1} - t_k = 1\}} \right) K(t_{k+1} - t_k) \quad (6)$$

Now let's define

$$K_{\tilde{\beta}}(n) = \begin{cases} K(1) \exp(\tilde{\beta}) & \text{if } n = 1 \\ K(n) & \text{if } n > 1 \end{cases}$$

so that (6) rewrites as

$$\tilde{Z}_{N, \tilde{\beta}, \tilde{h}}^c = \sum_{n=1}^N \sum_{0:=t_0 < t_1 < \dots < t_{n-1} < t_n := N} \prod_{k=0}^{n-1} e^{\tilde{h}} K_{\tilde{\beta}}(t_{k+1} - t_k) \quad (7)$$

Let's set

$$\phi(F) = \sum_{n \geq 1} K_{\tilde{\beta}}(n) e^{-Fn}$$

which is well defined and continuous for  $F \geq 0$  (the function series is normally convergent). Since  $\phi$  is strictly decreasing and

$$\begin{aligned} \phi(0) &= \sum_{n \geq 1} K_{\tilde{\beta}}(n) = \sum_{n \geq 1} K(n) + (e^{\tilde{\beta}} - 1)K(1) \\ \lim_{\infty} \phi &= 0 \end{aligned}$$

we have that for all  $\tilde{h}$  such that

$$\exp(-\tilde{h}) \in \left( 0, \sum_{n \geq 1} K(n) + (e^{\tilde{\beta}} - 1)K(1) \right] \quad (8)$$

there exists a unique element of  $[0, +\infty)$ , denoted by  $\tilde{F}(\tilde{\beta}, \tilde{h})$  such that

$$\phi(\tilde{F}(\tilde{\beta}, \tilde{h})) = \exp(-\tilde{h}). \quad (9)$$

If condition (8) is not satisfied we set  $\tilde{F}(\tilde{\beta}, \tilde{h}) = 0$ . Note that condition (8) is equivalent to

$$\begin{aligned} \tilde{h} &\geq -\log\left(\sum_{n \geq 1} K(n) + (e^{\tilde{\beta}} - 1)K(1)\right) \\ &= h_c(0) - \log\left(1 + \frac{K(1)}{\sum_{n \geq 1} K(n)}(e^{\tilde{\beta}} - 1)\right) \end{aligned}$$

and we denote by  $\tilde{h}_c(\tilde{\beta})$  the right-hand term of the inequality, so that  $\tilde{F}$  is a nonnegative function defined on  $\mathbb{R}^2$  satisfying:

$$\tilde{F}(\tilde{\beta}, \tilde{h}) > 0 \iff \tilde{h} > \tilde{h}_c(\tilde{\beta}).$$

All we need to prove now is the first point of the theorem, i.e  $\tilde{F}$  is actually the (new) free energy. We begin by the case  $\tilde{F}(\tilde{\beta}, \tilde{h}) > 0$ . Note that equation (9) can be rewritten as

$$\sum_{n \geq 1} K_{\tilde{\beta}}(n) e^{-\tilde{F}(\tilde{\beta}, \tilde{h})n} e^{\tilde{h}} = 1$$

If we set  $K_{\tilde{\beta}, \tilde{h}}^{\text{new}}(n) := K_{\tilde{\beta}}(n) e^{-\tilde{F}(\tilde{\beta}, \tilde{h})n} e^{\tilde{h}}$ , the renewal process with interarrival law given by  $K_{\tilde{\beta}, \tilde{h}}^{\text{new}}(\cdot)$  is positive recurrent and (6) becomes

$$\begin{aligned} \tilde{Z}_{N, \tilde{\beta}, \tilde{h}}^c &= \sum_{n=1}^N \sum_{0:=t_0 < t_1 < \dots < t_{n-1} < t_n := N} \prod_{k=0}^{n-1} e^{\tilde{F}(\tilde{\beta}, \tilde{h})(t_{k+1} - t_k)} K_{\tilde{\beta}, \tilde{h}}^{\text{new}}(t_{k+1} - t_k) \\ &= \exp(\tilde{F}(\tilde{\beta}, \tilde{h})N) \mathbb{P}_{\tilde{\beta}, \tilde{h}}^{\text{new}}(N \in \tau) \end{aligned}$$

By the renewal theorem (see [1]), we now have

$$\tilde{Z}_{N, \tilde{\beta}, \tilde{h}}^c \sim \frac{1}{\sum_{n \geq 1} n K_{\tilde{\beta}, \tilde{h}}^{\text{new}}(n)} \exp(\tilde{F}(\tilde{\beta}, \tilde{h})N)$$

and we may conclude. In the case  $\tilde{F}(\tilde{\beta}, \tilde{h}) = 0$ , we have that  $\sum_{n \geq 1} K_{\tilde{\beta}, \tilde{h}}^{\text{new}}(n) = \sum_{n \geq 1} K_{\tilde{\beta}}(n) e^{\tilde{h}} \leq 1$  so that  $K_{\tilde{\beta}, \tilde{h}}^{\text{new}}(\cdot)$  is a sub-probability, or a probability on  $\overline{\mathbb{N}}^*$  if we set  $K_{\tilde{\beta}, \tilde{h}}^{\text{new}}(\infty) = 1 - \sum_{n \geq 1} K_{\tilde{\beta}, \tilde{h}}^{\text{new}}(n)$ . Therefore,

$$\tilde{Z}_{N, \tilde{\beta}, \tilde{h}}^c = \mathbb{P}_{\tilde{\beta}, \tilde{h}}^{\text{new}}(N \in \tau) \leq 1$$

Since  $\tilde{Z}_{N, \tilde{\beta}, \tilde{h}}^c \geq \exp(\tilde{h})K(N)$ , we may now conclude.  $\square$

The second point of Theorem 5.2 gives the exact expression of the critical curve that separates localized and delocalized phases in the phase diagram (see figure 2). In particular we have the following properties on  $\tilde{h}_c$ , whose proofs are left to the reader:

**Proposition 5.1.** *The function  $\tilde{\beta} \mapsto \tilde{h}_c(\tilde{\beta})$  is a concave decreasing function, and*

- $\tilde{h}_c(0) = -\log(\sum_{n \geq 1} K(n)) (= h_c(0))$
- $\lim_{\tilde{\beta} \rightarrow -\infty} \tilde{h}_c(\tilde{\beta}) = -\log(\sum_{n \geq 2} K(n))$
- $\tilde{h}_c(\tilde{\beta}) \sim_{+\infty} -\tilde{\beta}$
- $\tilde{h}'_c(0) = -\frac{K(1)}{\sum_{n \geq 1} K(n)}$

We will make a few comments on these results. First, notice that the value of  $\tilde{h}_c$  at 0 makes it consistent with the classical homogeneous model. Then, the absolute value of the derivative at  $\tilde{\beta} = 0$  is equal to

$$\frac{K(1)}{\sum_{n \geq 1} K(n)} = \mathbb{P}(\tau_1 = 1 \mid \tau_1 < \infty)$$

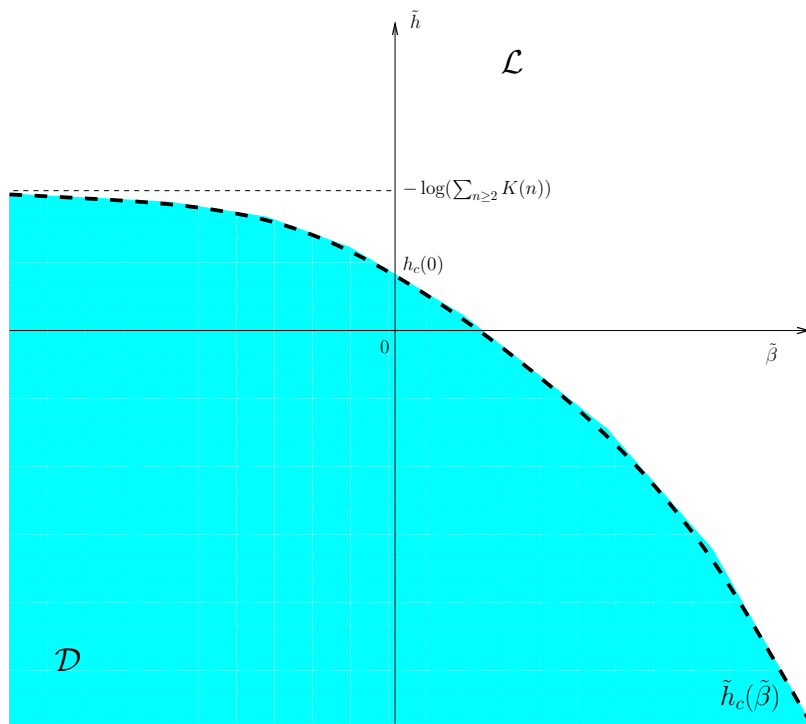
so what we have established here is a connection between the localizing effect of the interaction term for  $\tilde{\beta}$  near 0 and the weight of  $K(1)$  with respect to the sum of the  $K(n)$ 's. Another interesting point is the limit when  $\tilde{\beta} \rightarrow -\infty$ , which means that the critical point converges to the one of a classical homogeneous model where we have added the value of  $K(1)$  to  $K(\infty)$  and then put  $K(1)$  at 0.

## 5.2 Proof of Theorem 5.1

First we will show that

**Proposition 5.2.** *For every real  $h$ , and nonnegative  $\beta$ ,*

$$F(\beta, h) \leq \tilde{F}(a_0 a_1 \beta^2, h + \frac{\beta^2 \sigma^2}{2})$$



1

Figure 2: The phase diagram of the new homogeneous model

*Proof.* Let's fix a realization of  $\tau$ . Then  $\sum_{n=1}^N \omega_n \mathbf{1}_{\{n \in \tau\}}$  is a centered gaussian r.v, with variance given by

$$\text{Var}_\mu \left( \sum_{n=1}^N \omega_n \mathbf{1}_{\{n \in \tau\}} \right) = \sum_{k,l=1}^N \text{Cov}_\mu(\omega_k, \omega_l) \mathbf{1}_{\{k \in \tau\}} \mathbf{1}_{\{l \in \tau\}}$$

From (5), we have

$$\text{Var}_\mu \left( \sum_{n=1}^N \omega_n \mathbf{1}_{\{n \in \tau\}} \right) = \sigma^2 \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}} + 2a_0 a_1 \sum_{n=1}^{N-1} \mathbf{1}_{\{n \in \tau\}} \mathbf{1}_{\{n+1 \in \tau\}} \quad (10)$$

Notice that the term  $\sum_{n=1}^{N-1} \mathbf{1}_{\{n \in \tau\}} \mathbf{1}_{\{n+1 \in \tau\}}$  is similar to the term introduced in the hamiltonian of the new homogeneous model. From (2) and (10) we obtain

$$\begin{aligned} & \frac{1}{N} \mathbb{E}_\mu(\log Z_{N,\beta,h,\omega}^c) \\ & \leq \frac{1}{N} \log \mathbb{E} \left( e^{(h + \frac{\sigma^2 \beta^2}{2}) \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}}} e^{a_0 a_1 \beta^2 \sum_{n=1}^{N-1} \mathbf{1}_{\{n \in \tau\}} \mathbf{1}_{\{n+1 \in \tau\}}} \mathbf{1}_{\{N \in \tau\}} \right) \end{aligned}$$

If  $a_0 a_1 \geq 0$ , then

$$\begin{aligned} & \frac{1}{N} \mathbb{E}_\mu(\log Z_{N,\beta,h,\omega}^c) \\ & \leq \frac{1}{N} \log \mathbb{E} \left( e^{(h + \frac{\sigma^2 \beta^2}{2}) \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}}} e^{a_0 a_1 \beta^2 \sum_{n=0}^{N-1} \mathbf{1}_{\{n \in \tau\}} \mathbf{1}_{\{n+1 \in \tau\}}} \mathbf{1}_{\{N \in \tau\}} \right) \end{aligned}$$

and from Theorem 3.1 and Theorem 5.2, the left-hand term tends to  $F(\beta, h)$  while the right-hand term tends to  $\tilde{F}(a_0 a_1 \beta^2, h + \frac{\beta^2 \sigma^2}{2})$ . If  $a_0 a_1 < 0$  then, since  $1 \leq e^{a_0 a_1 \beta^2 \mathbf{1}_{\{1 \in \tau\}} - a_0 a_1 \beta^2}$ , we have

$$\begin{aligned} & \frac{1}{N} \mathbb{E}_\mu(\log Z_{N,\beta,h,\omega}^c) \\ & \leq -\frac{a_0 a_1 \beta^2}{N} + \frac{1}{N} \log \mathbb{E} \left( e^{(h + \frac{\sigma^2 \beta^2}{2}) \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}}} e^{a_0 a_1 \beta^2 \sum_{n=0}^{N-1} \mathbf{1}_{\{n \in \tau\}} \mathbf{1}_{\{n+1 \in \tau\}}} \mathbf{1}_{\{N \in \tau\}} \right) \end{aligned}$$

and we conclude by letting  $N$  tend to  $+\infty$  as in the previous case.  $\square$

To prove Theorem 5.1, we only need to apply the inequality from Proposition 5.2 with  $h := \tilde{h}_c(a_0 a_1 \beta^2) - \frac{\beta^2 \sigma^2}{2}$ . What we get is the following inequality:

$$F \left( \beta, \tilde{h}_c(a_0 a_1 \beta^2) - \frac{\beta^2 \sigma^2}{2} \right) \leq 0$$

This implies  $h_c(\beta) \geq \tilde{h}_c(a_0 a_1 \beta^2) - \frac{\beta^2 \sigma^2}{2}$ , which is precisely Theorem 5.1.

### 5.3 Limits at low and high temperatures

First note that

$$h_c^{\text{ann}}(\beta) = h_c^{\text{ann, iid}}(\beta) - \log \left( 1 + \frac{K(1)}{1 - K(\infty)} (e^{\rho_1 \beta^2} - 1) \right) \quad (11)$$

which is consistent with taking the covariance  $\rho_1$  equal to 0. The second term of the right hand term is then due to the correlation between two consecutive terms of the disorder sequence. It does not only involve the value of this correlation but also the quantity  $\frac{K(1)}{1 - K(\infty)}$ , which can write (with the notation of section 4.2.2)  $\mathbb{P}(\hat{\tau}_1 = 1)$  or  $\mathbb{P}(\tau_1 = 1 | \tau_1 < \infty)$ . In fact, this is quite natural, because if  $K(1) = 0$ , then the polymer never visits two neighbouring sites, so the correlation between  $\omega_n$  and  $\omega_{n+1}$  will have no effect.

Moreover, if the  $\omega_n$ 's are positively correlated ( $\rho_1 > 0$ ) then for  $\beta > 0$ ,  $h_c^{\text{ann}}(\beta) < h_c^{\text{ann, iid}}(\beta)$  and if they are negatively correlated ( $\rho_1 < 0$ ), then for  $\beta > 0$ ,  $h_c^{\text{ann}}(\beta) > h_c^{\text{ann, iid}}(\beta)$ . It shows that positive (resp negative) correlations tend to localize (resp delocalize) the polymer (at least as far as the annealed measure is concerned).

We will now precise the effect of correlations in the high and low temperatures limits. We assume  $K(1) > 0$ .

#### 5.3.1 High temperature limit

By high temperature limit, we mean that  $\beta$  tends to 0. A linearization of (11) gives

$$h_c^{\text{ann}}(\beta) - h_c(0) \sim_0 - \left( 1 + 2 \frac{\rho_1}{\sigma^2} \frac{K(1)}{1 - K(\infty)} \right) \frac{\sigma^2 \beta^2}{2}$$

which also writes

$$\frac{h_c^{\text{ann}}(\beta) - h_c(0)}{h_c^{\text{ann, iid}}(\beta) - h_c(0)} \sim_0 1 + 2 \frac{\rho_1}{\sigma^2} \frac{K(1)}{1 - K(\infty)}$$

where the coefficient  $(1 + 2 \frac{\rho_1}{\sigma^2} \frac{K(1)}{1 - K(\infty)})$  is nonnegative since  $\frac{K(1)}{1 - K(\infty)} \leq 1$  and  $\frac{2|\rho_1|}{\sigma^2} = \frac{2|a_0 a_1|}{a_0^2 + a_1^2} \leq 1$ . (see Figure 3) We have then obtained at high temperature a multiplier coefficient between the annealed bounds (lower bounds) of the critical curves in the cases of i.i.d disorder and first-order autoregressive disorder.

### 5.3.2 Low temperature limit

For large values of  $\beta$ , what happens depends on the sign of  $\rho_1$ . If it is positive then

$$h_c^{\text{ann}}(\beta) = -\log K(1) - \left(1 + 2\frac{\rho_1}{\sigma^2}\right) \frac{\sigma^2 \beta^2}{2} + o_{+\infty}(1)$$

so the asymptotic curve is (up to a rescaling of the temperature) the annealed critical curve of a model with i.i.d disorder, whose law of interarrival times is  $K'(n) = K(1)\mathbf{1}_{\{n=1\}}$

If  $\rho_1$  is negative, then

$$h_c^{\text{ann}}(\beta) = -\log(1 - K(1) - K(\infty)) - \frac{\sigma^2 \beta^2}{2} + o_{+\infty}(1)$$

Here, the asymptotic curve is exactly the annealed critical curve of a model with i.i.d disorder, and whose law of interarrival times is defined by  $K'(n) = K(n)\mathbf{1}_{\{n>1\}}$ .

## 5.4 Calculation of $\Lambda^d$

In this subsection, we will point out that we can obtain Theorem 5.1 by directly calculating  $\Lambda^d$  (see Section 4.2.2). We denote by  $\hat{\tau} = (\hat{\tau}_n)_{n \geq 0}$  the (recurrent) renewal process with interarrival time distribution  $\frac{K(\cdot)}{1-K(\infty)}$  (and  $\hat{\tau}_0 = 0$ ).

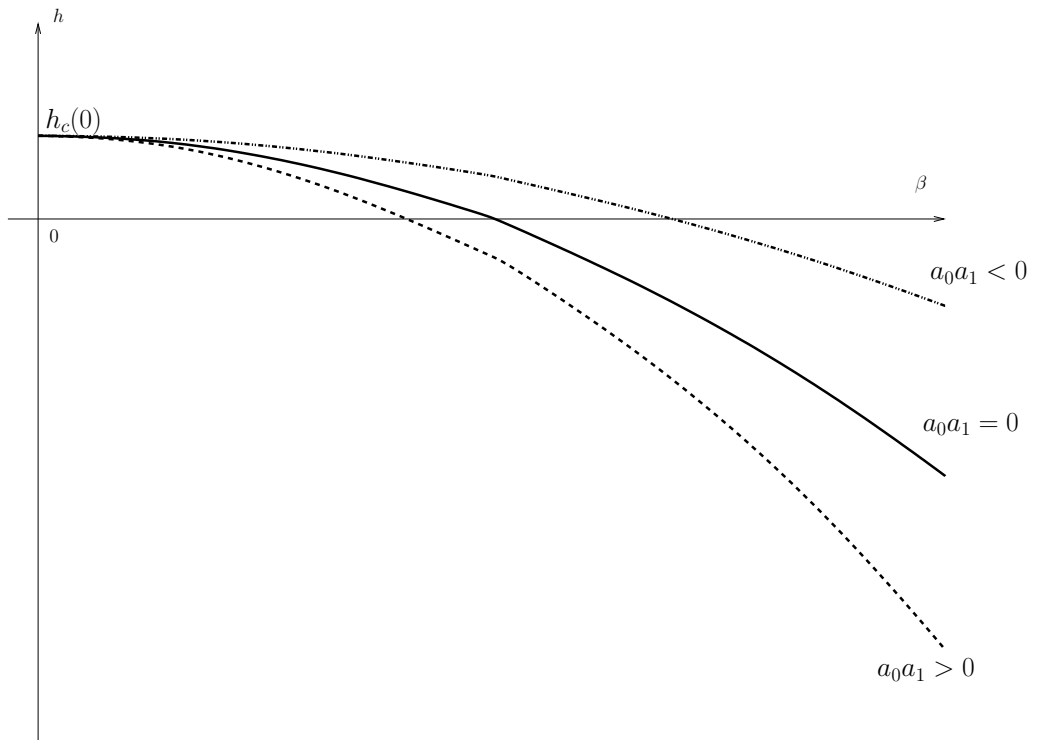
Since

$$\begin{aligned} \text{Var} \left( \sum_{n=1}^N \omega_{\hat{\tau}_n} \right) &= \sum_{k,l=1}^N \text{Cov}(\omega_{\hat{\tau}_k}, \omega_{\hat{\tau}_l}) \\ &= \sigma^2 N + 2\rho_1 \sum_{n=1}^N \mathbf{1}_{\{\hat{\tau}_{n+1} - \hat{\tau}_n = 1\}} \end{aligned}$$

we have

$$\begin{aligned} \mathbb{E}_{\mu} \left( \exp \left( \beta \sum_{n=1}^N \omega_{\hat{\tau}_n} \right) \right) &= e^{\frac{\sigma^2 \beta^2}{2} N} \mathbb{E} \left( \exp(\rho_1 \beta^2 \sum_{n=1}^N \mathbf{1}_{\{\hat{\tau}_{n+1} - \hat{\tau}_n = 1\}}) \right) \\ &= e^{\frac{\sigma^2 \beta^2}{2} N} \left( \mathbb{E}(\exp(\rho_1 \beta^2 \mathbf{1}_{\{\hat{\tau}_1 = 1\}})) \right)^N \end{aligned}$$

because the random sequence  $(\hat{\tau}_{n+1} - \hat{\tau}_n)_{n \geq 0}$  is i.i.d.



1

Figure 3: The annealed critical curve at a neighbourhood of  $\beta = 0$ , when  $\omega_n = a_0 \varepsilon_n + a_1 \varepsilon_{n-1}$



Since  $\mathbb{E}(\exp(\rho_1 \beta^2 \mathbf{1}_{\{\hat{\tau}_1=1\}})) = 1 + \frac{K(1)}{1-K(\infty)}(e^{\rho_1 \beta^2} - 1)$ , we finally get

$$\Lambda^d(\beta) = \frac{\beta^2 \sigma^2}{2} + \log \left( 1 + \frac{K(1)}{1-K(\infty)}(e^{\rho_1 \beta^2} - 1) \right)$$

which, combined with Theorem 4.3, yields the result of Theorem 5.1.

We will use the technique consisting of calculating  $\Lambda_d$  in the next section as well.

## 6 Annealed bound for a second-order autoregressive gaussian process

We continue the study of pinning polymers with correlated sequences of bases, by increasing the range of correlations from one to two, which will make another difficulty arise.

With the same definition of  $\varepsilon = (\varepsilon_n)_{n \in \mathbb{Z}}$  we define now:

$$\omega_n = a_0 \varepsilon_n + a_1 \varepsilon_{n-1} + a_2 \varepsilon_{n-2}$$

with (optional)

$$a_0^2 + a_1^2 + a_2^2 = 1$$

so that  $\omega$  is a stationary sequence of standard centered gaussian variables with correlation function given by

$$\rho_n := \text{Cov}(\omega_0, \omega_n) = \begin{cases} a_0 a_1 + a_1 a_2 & \text{if } n = 1 \\ a_0 a_2 & \text{if } n = 2 \\ 0 & \text{if } n > 2 \end{cases}$$

### 6.1 Statement of the result

The following result gives the precise value of  $\Lambda^d$  in this case. As a consequence of Theorem 4.3, it yields the annealed bound, which is  $h_c^{\text{ann}}(\beta) = h_c(0) - \Lambda^d(\beta)$ .

**Theorem 6.1.** *If  $\omega$  is defined as above, then  $\Lambda^d(\beta)$  exists and is finite for all nonnegative  $\beta$ . Moreover,*

$$\Lambda^d(\beta) = \frac{\beta^2}{2} + \log \phi(\beta) + \log \frac{1 + \sqrt{1 - \frac{\psi(\beta)}{\phi(\beta)^2}}}{2}$$

where

$$\begin{aligned} \phi(\beta) &= 1 + \frac{K(1)}{1 - K(\infty)}(e^{(\rho_1 + \rho_2)\beta^2} - 1) + \frac{K(2)}{1 - K(\infty)}(e^{\rho_2\beta^2} - 1) \\ \psi(\beta) &= 4 \frac{K(1)}{1 - K(\infty)} \left(1 - \frac{K(1)}{1 - K(\infty)}\right) e^{\rho_1\beta^2} (e^{\rho_2\beta^2} - 1) \\ &\quad \times \left(1 + \frac{K(2)}{1 - K(\infty) - K(1)}(e^{\rho_2\beta^2} - 1)\right) \end{aligned}$$

Before starting the proof, we justify the calculation of  $\Lambda^d$  (which was not the first technique we applied in the previous model -  $\rho_1 \neq 0$  and  $\rho_2 = 0$ ). First let's compute  $\text{Var} \left( \sum_{n=1}^N \omega_n \mathbf{1}_{\{n \in \tau\}} \right)$ .

$$\begin{aligned} \text{Var} \left( \sum_{n=1}^N \omega_n \mathbf{1}_{\{n \in \tau\}} \right) &= \sum_{k,l=1}^N \text{Cov}(\omega_k, \omega_l) \mathbf{1}_{\{k \in \tau\}} \mathbf{1}_{\{l \in \tau\}} \\ &= \sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}} + 2\rho_1 \sum_{n=1}^{N-1} \mathbf{1}_{\{n \in \tau\}} \mathbf{1}_{\{n+1 \in \tau\}} \\ &\quad + 2\rho_2 \sum_{n=1}^{N-2} \mathbf{1}_{\{n \in \tau\}} \mathbf{1}_{\{n+2 \in \tau\}} \end{aligned}$$

We could introduce a new homogeneous model, as in Section 5.1, with another parameter in front of a new pinning term  $\sum_{n=1}^{N-2} \mathbf{1}_{\{n \in \tau\}} \mathbf{1}_{\{n+2 \in \tau\}}$ . However, it seems difficult in this case to obtain an expression of the partition function similar to (6).

## 6.2 Proof of Theorem 6.1

In the sequel, we assume that  $\tau$  is recurrent, otherwise we would replace the interarrival time distribution  $K(\cdot)$  by  $\frac{K(\cdot)}{1 - K(\infty)}$  at the end of the proof (see Section 4.2.2 for explanations). We begin by a lemma:

**Lemma 6.1.** Let  $(Z_n)_{0 \leq n \leq N}$  be  $(N+1)$  i.i.d Bernoulli variables with parameter  $p$ , and  $\lambda$  any real. Then,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \log \mathbb{E} \left( e^{\lambda \sum_{n=0}^{N-1} Z_n Z_{n+1}} \right) = \log \varrho(\lambda, p)$$

where

$$\varrho(\lambda, p) = \frac{1 + p(e^\lambda - 1)}{2} \left( 1 + \sqrt{1 - \frac{4p(1-p)(e^\lambda - 1)}{(1 + p(e^\lambda - 1))^2}} \right)$$

This lemma is inspired from the calculation of the free energy of the Ising model, in which the spins  $Z_n$  have their values in  $\{-1, 1\}$  instead of  $\{0, 1\}$ , and  $p = \frac{1}{2}$ . The proof uses a transfer matrix method (see [7], p.107).

*Proof.* The result clearly holds in the degenerate cases  $p \in \{0, 1\}$ . We now assume  $p \notin \{0, 1\}$ . We begin by writing the partition function more explicitly:

$$\mathbb{E} \left( e^{\lambda \sum_{n=0}^{N-1} Z_n Z_{n+1}} \right) = \sum_{z_0, \dots, z_N \in \{0, 1\}} \left( \prod_{n=0}^{N-1} e^{\lambda z_n z_{n+1}} \right) \left( \prod_{n=0}^N \mathbb{P}(Z_n = z_n) \right)$$

and we define a matrix  $Q$  by  $Q(z_n, z_{n+1}) = e^{\lambda z_n z_{n+1}} \mathbb{P}(Z_n = z_n)$  so that the partition function writes

$$\begin{aligned} & \mathbb{E} \left( e^{\lambda \sum_{n=0}^{N-1} Z_n Z_{n+1}} \right) \\ &= \sum_{z_N \in \{0, 1\}} \left( \sum_{z_0, \dots, z_{N-1} \in \{0, 1\}} Q(z_0, z_1) \times \dots \times Q(z_{N-1}, z_N) \right) \mathbb{P}(Z_N = z_N) \\ &= \sum_{z_0, z_N \in \{0, 1\}} Q^N(z_0, z_N) \mathbb{P}(Z_N = z_N) \end{aligned}$$

Because of our assumption on  $p$ , the entries of the matrix

$$Q = \begin{pmatrix} 1-p & 1-p \\ p & pe^\lambda \end{pmatrix}$$

are positive, so by the Perron-Frobenius theorem,

$$\frac{1}{N} \log \sum_{z_0, z_N \in \{0, 1\}} Q^N(z_0, z_N) \mathbb{P}(Z_N = z_N) \longrightarrow \log \varrho(\lambda, p)$$

where  $\varrho(\lambda, p)$  is the Perron-Frobenius eigenvalue of  $Q$  given in the lemma.  $\square$

We can now prove Theorem 6.1.

*Proof.* We will first integrate on the disorder the annealed partition function:

$$\mathbb{E}\mathbb{E}_\mu \left( \exp \left( \beta \sum_{n=1}^N \omega_{\tau_n} \right) \right) \quad (12)$$

Since, for a fixed realization of the renewal process,

$$\begin{aligned} \text{Var}_\mu \left( \sum_{n=1}^N \omega_{\tau_n} \right) &= \sum_{k,l=1}^N \text{Cov}_\mu(\omega_{\tau_k}, \omega_{\tau_l}) \\ &= N + 2 \sum_{k<l} \text{Cov}_\mu(\omega_{\tau_k}, \omega_{\tau_l}) \\ &= N + 2\rho_1 \sum_{n=1}^{N-1} \mathbf{1}_{\{\tau_{n+1}-\tau_n=1\}} \\ &\quad + 2\rho_2 \sum_{n=1}^{N-1} \mathbf{1}_{\{\tau_{n+1}-\tau_n=2\}} + 2\rho_2 \sum_{n=1}^{N-2} \mathbf{1}_{\{\tau_{n+2}-\tau_n=2\}} \end{aligned}$$

so if we denote  $\tau_{n+1} - \tau_n$  by  $T_n$ , we have

$$\text{Var}_\mu \left( \sum_{n=1}^N \omega_{\tau_n} \right) = N + 2 \sum_{n=1}^{N-1} (\rho_1 \mathbf{1}_{\{T_n=1\}} + \rho_2 \mathbf{1}_{\{T_n=2\}}) + 2\rho_2 \sum_{n=1}^{N-2} \mathbf{1}_{\{T_n=1\}} \mathbf{1}_{\{T_{n+1}=1\}}$$

and if we set

$$X_n := \rho_1 \mathbf{1}_{\{T_n=1\}} + \rho_2 \mathbf{1}_{\{T_n=2\}}$$

which are i.i.d random variables, we have

$$\mathbb{E}\mathbb{E}_\mu \left( \exp \left( \beta \sum_{n=1}^N \omega_{\tau_n} \right) \right) = e^{\frac{\beta^2}{2}N} \mathbb{E} \left( e^{\beta^2 \left\{ \sum_{n=1}^{N-1} X_n + \rho_2 \sum_{n=1}^{N-2} \mathbf{1}_{\{T_n=1\}} \mathbf{1}_{\{T_{n+1}=1\}} \right\}} \right)$$

In Section 5.4, it was easy to conclude at this point of the proof using the independency of the  $T_n$ 's. Here the term  $\sum_{n=0}^{N-2} \mathbf{1}_{\{T_n=1\}} \mathbf{1}_{\{T_{n+1}=1\}}$  prevents us from doing so. We will isolate this term via a change of measure and use Lemma 6.1. We have

$$\begin{aligned} \Lambda_N &:= \frac{1}{N} \log \mathbb{E}\mathbb{E}_\mu \left( \exp \left( \beta \sum_{n=1}^N \omega_{\tau_n} \right) \right) \\ &= \frac{\beta^2}{2} + \frac{1}{N} \log \mathbb{E} \left( e^{\beta^2 \sum_{n=1}^{N-1} X_n + \rho_2 \beta^2 \sum_{n=1}^{N-2} \mathbf{1}_{\{T_n=1\}} \mathbf{1}_{\{T_{n+1}=1\}}} \right) \\ &= \frac{\beta^2}{2} + \frac{1}{N} \log \mathbb{E} \left( e^{\beta^2 \sum_{n=1}^{N-1} X_n} \right) + \frac{1}{N} \log \tilde{\mathbb{E}}_N \left( e^{\rho_2 \beta^2 \sum_{n=1}^{N-2} \mathbf{1}_{\{T_n=1\}} \mathbf{1}_{\{T_{n+1}=1\}}} \right) \end{aligned}$$

where the new measure  $\tilde{\mathbb{P}}_N$  is defined by

$$\frac{d\tilde{\mathbb{P}}_N}{d\mathbb{P}} = \frac{e^{\beta^2 \sum_{n=1}^{N-1} X_n}}{\mathbb{E}\left(e^{\beta^2 \sum_{n=1}^{N-1} X_n}\right)}$$

The second term in the expression of  $\Lambda_N$  above is easy to calculate because the  $X_n$ 's are i.i.d, so it is equal to:

$$\begin{aligned} \log \mathbb{E}(e^{\beta^2 X_1}) &= \log \left( 1 - K(1) - K(2) + e^{\rho_1 \beta^2} K(1) + e^{\rho_2 \beta^2} K(2) \right) \\ &= \log \left( 1 + K(1)(e^{\rho_1 \beta^2} - 1) + K(2)(e^{\rho_2 \beta^2} - 1) \right) \end{aligned}$$

We now need to precise the law of the interarrival times  $(T_n)_{1 \leq n \leq N-1}$  under the measure  $\tilde{\mathbb{P}}_N$ . Using the independency of the interarrival times (under the initial measure) we have:

$$\begin{aligned} &\tilde{\mathbb{P}}_N(T_1 = t_1, T_2 = t_2, \dots, T_{N-1} = t_{N-1}) \\ &= \frac{1}{\mathbb{E}\left(e^{\beta^2 \sum_{n=1}^{N-1} X_n}\right)} \mathbb{E}\left(\mathbf{1}_{\{T_1=t_1\}} \mathbf{1}_{\{T_2=t_2\}} \cdots \mathbf{1}_{\{T_{N-1}=t_{N-1}\}} e^{\beta^2 \sum_{n=1}^{N-1} X_n}\right) \\ &= \prod_{n=1}^{N-1} \frac{\mathbb{E}\left(\mathbf{1}_{\{T_n=t_n\}} e^{\beta^2(\rho_1 \mathbf{1}_{\{T_n=1\}} + \rho_2 \mathbf{1}_{\{T_n=2\}})}\right)}{\mathbb{E}(e^{\beta^2 X_1})} \\ &= \prod_{n=1}^{N-1} \tilde{K}(t_n) \end{aligned}$$

where

$$\tilde{K}(t) := \frac{K(t) + K(1)(e^{\rho_1 \beta^2} - 1)\mathbf{1}_{\{t=1\}} + K(2)(e^{\rho_2 \beta^2} - 1)\mathbf{1}_{\{t=2\}}}{\mathbb{E}(e^{\beta^2 X_1})}$$

so that, under  $\tilde{\mathbb{P}}_N$ , the  $T_n$ 's ( $1 \leq n \leq N-1$ ) are i.i.d with law  $\tilde{K}(\cdot)$  (it is worth noticing that it does not depend on  $N$ ). Furthermore, the variables  $Z_n := \mathbf{1}_{\{T_n=1\}}$  (for the same set of index) are i.i.d Bernoulli random variables with parameter  $\tilde{K}(1)$ . All we have to do is now to apply Lemma 6.1 with  $\lambda := \rho_2 \beta^2$  and  $p := \tilde{K}(1)$ .  $\square$

### 6.3 Limits at low and high temperatures

We now give an asymptotic study of our result.

### 6.3.1 High temperature limit

A linearization of  $\log \varrho(\lambda, p)$  for  $\lambda \rightarrow 0$  yields

$$\log \varrho(\lambda, p) = p^2 \lambda + o_0(\lambda)$$

Now recall that

$$\begin{aligned} \Lambda^d(\beta) &= \log \left( 1 + \frac{K(1)}{1 - K(\infty)} (e^{\rho_1 \beta^2} - 1) + \frac{K(2)}{1 - K(\infty)} (e^{\rho_2 \beta^2} - 1) \right) \\ &\quad + \log \varrho(\rho_2 \beta^2, \tilde{K}(1)) + \frac{\beta^2}{2} \end{aligned}$$

Since  $\tilde{K}(1) \xrightarrow{\beta \rightarrow 0} K(1)$ , a linearization gives

$$\Lambda^d(\beta) \sim_0 \left\{ 1 + 2 \left( \frac{K(1)}{1 - K(\infty)} \rho_1 + \left( \frac{K(2)}{1 - K(\infty)} + \left( \frac{K(1)}{1 - K(\infty)} \right)^2 \right) \rho_2 \right) \right\} \frac{\beta^2}{2}$$

It actually writes more simply

$$\Lambda^d(\beta) \sim_0 \{1 + 2 (\mathbb{P}(1 \in \hat{\tau}) \rho_1 + \mathbb{P}(2 \in \hat{\tau}) \rho_2)\} \frac{\beta^2}{2}$$

so

$$\frac{h_c^{\text{ann}}(\beta) - h_c(0)}{h_c^{\text{ann, iid}}(\beta) - h_c(0)} \sim_0 1 + 2 (\mathbb{P}(1 \in \hat{\tau}) \rho_1 + \mathbb{P}(2 \in \hat{\tau}) \rho_2)$$

**Conjecture:** In the general case of a  $q$ -order gaussian autoregressive process, defined by

$$\omega_n = a_0 \varepsilon_n + a_1 \varepsilon_{n-1} + \cdots + a_q \varepsilon_{n-q},$$

we have

$$\frac{h_c^{\text{ann}}(\beta) - h_c(0)}{h_c^{\text{ann, iid}}(\beta) - h_c(0)} \sim_{\beta \rightarrow 0^+} 1 + 2 \sum_{k=1}^q \mathbb{P}(k \in \hat{\tau}) \rho_k.$$

To prove this result, we have to establish the existence of  $\Lambda^d$  for  $q > 2$ . The explicit computation seems difficult, because for each  $p \in \{1, \dots, q\}$ , the effect of  $\rho_p$  involves all the possible ways for the renewal process to be in  $n$  and  $n + p$ , for  $n \in \{1, \dots, N - p\}$ . As we have seen, this can be done for small orders ( $q = 1$  or  $2$ ) but it becomes quite challenging for  $q > 2$ .

### 6.3.2 Low temperature limit

If we look at the expression of  $\phi(\beta)$  in Theorem 6.1, we see that the asymptotic behaviour at low temperature ( $\beta \rightarrow +\infty$ ) depends on the value of  $\max(0, \rho_2, \rho_1 + \rho_2)$ . Thus we define:

$$\begin{aligned}\mathcal{A}_1 &= \{\rho_1 > 0, \rho_2 > -\rho_1\} \\ \mathcal{A}_2 &= \{\rho_2 < 0, \rho_2 < -\rho_1\} \\ \mathcal{A}_3 &= \{\rho_1 < 0, \rho_2 > 0\}\end{aligned}$$

If  $(\rho_1, \rho_2) \in \mathcal{A}_1$  then

$$h_c^{\text{ann}}(\beta) = -\log K(1) - (1 + 2(\rho_1 + \rho_2))\frac{\beta^2}{2} + o_{+\infty}(1)$$

If  $(\rho_1, \rho_2) \in \mathcal{A}_2$  then

$$h_c^{\text{ann}}(\beta) = -\log(1 - K(\infty) - K(1) - K(2)) - \frac{\beta^2}{2} + o_{+\infty}(1)$$

and if  $(\rho_1, \rho_2) \in \mathcal{A}_3$ ,

$$h_c^{\text{ann}}(\beta) = -\log K(2) - (1 + 2\rho_2)\frac{\beta^2}{2} + o_{+\infty}(1)$$

We now have to precise what happens on the boundaries (notice that we already know what happens on the boundary between  $\mathcal{A}_2$  and  $\mathcal{A}_3$ ). If  $\rho_1 + \rho_2 = 0$  and  $\rho_2 < 0$  then

$$h_c^{\text{ann}}(\beta) = -\log(1 - K(\infty) - K(2)) - \frac{\beta^2}{2} + o_{+\infty}(1)$$

and if  $\rho_1 = 0$  and  $\rho_2 > 0$  then

$$h_c^{\text{ann}}(\beta) = -\log(K(1) + K(2)) - (1 + 2\rho_2)\frac{\beta^2}{2} + o_{+\infty}(1)$$

What all these different asymptotic behaviours have in common is that they write, up to a rescaling, as the annealed critical curve of an i.i.d disordered model, with a certain interarrival time law  $K'(\cdot)$ . (as usual,  $K'$  can be a sub-probability ) For example, if  $(\rho_1, \rho_2)$  is in  $\mathcal{A}_1$  then the rescaling coefficient is  $\sqrt{1 + 2(\rho_1 + \rho_2)}$  and the limit interarrival time distribution is  $K'(n) := K(1)\mathbf{1}_{\{n=1\}}$ , or if  $(\rho_1, \rho_2)$  is in  $\mathcal{A}_2$ , there is no rescaling and  $K'(n) = K(n)\mathbf{1}_{\{n>2\}}$ .

## 7 Annealed bound for super mixing disorder

In this section, we will prove the existence of  $\Lambda^d$  when the  $\omega_n$ 's satisfy a very strong condition of mixing, called  $\eta$ -weak dependence, with hyperexponential mixing rate.

In what follows, we will assume that  $\omega$  is an integrable (with mean 0) and stationary random sequence such that  $\beta_0 = \sup\{\beta \geq 0 | \Lambda(\beta) < \infty\} > 0$ .

### 7.1 Statement of the result

We will begin by some definitions.

First, we introduce the dependence coefficients:

$$\varepsilon(k) = \sup \frac{|\text{Cov}_\mu(f(\omega_{i_1}, \dots, \omega_{i_u}), g(\omega_{j_1}, \dots, \omega_{j_v}))|}{\psi(f, g)}$$

where

$$\psi(f, g) := u \text{Lip}(f) \|g\|_\infty + v \text{Lip}(g) \|f\|_\infty$$

and the supremum is taken over the indices such that

$$i_1 < \dots < i_u \leq i_u + k \leq j_1 < \dots < j_v$$

and all the functions  $f : \mathbb{R}^u \mapsto \mathbb{R}$  and  $g : \mathbb{R}^v \mapsto \mathbb{R}$  that are bounded and lipschitz. The coefficient  $\varepsilon(k)$  measures the correlation between functions depending on the past ( $f(\omega_{i_1}, \dots, \omega_{i_u})$ ) and others depending on the future ( $g(\omega_{j_1}, \dots, \omega_{j_v})$ ), where past and future are separated by a gap greater than  $k$ . These coefficients were developed by several authors, see [6] for a reference.

**Definition 7.1.** *If the sequence  $(\varepsilon(k))_{k \geq 0}$  tends to 0, then  $\omega$  is said to be  $\eta$ -weakly dependent.*

**Remark 7.1.** *Examples of such processes are finite-order regressive processes, Bernoulli shifts with independent or  $\eta$ -weakly dependent innovation process, and Markov chains. (see [6])*

If we want the free energy to be well defined, we have to make sure that this condition of weak dependence implies ergodicity. In fact we will show



that it implies mixing in the sense of [4] (which in turn implies ergodicity), that is for any (centered) functions  $f$  and  $g$  in  $L^2$ ,

$$\lim_{n \rightarrow +\infty} \mathbb{E}_\mu(f(\omega_0)g(\omega_n)) = 0 \quad (13)$$

From Definition 7.1, (13) is satisfied for lipschitz bounded functions. We will then use an argument of density. Let  $f_\epsilon$  and  $g_\epsilon$  be two centered functions in  $\mathcal{C}_c^\infty$ , the space of infinitely differentiable functions with compact support, (which are therefore bounded and lipschitz) such that  $\|f - f_\epsilon\|_2 \leq \epsilon$  and  $\|g - g_\epsilon\|_2 \leq \epsilon$ . We write

$$\begin{aligned} \mathbb{E}_\mu(f(\omega_0)g(\omega_n)) &= \mathbb{E}_\mu((f(\omega_0) - f_\epsilon(\omega_0))g(\omega_n)) + \mathbb{E}_\mu(f_\epsilon(\omega_0)g_\epsilon(\omega_n)) \\ &\quad + \mathbb{E}_\mu(f(\omega_0)(g(\omega_n) - g_\epsilon(\omega_n))) \\ &\quad + \mathbb{E}_\mu((f_\epsilon(\omega_0) - f(\omega_0))(g(\omega_n) - g_\epsilon(\omega_n))) \end{aligned}$$

and so by Cauchy-Schwarz,

$$|\mathbb{E}_\mu(f(\omega_0)g(\omega_n))| \leq \epsilon \|g\|_2 + \psi(f_\epsilon, g_\epsilon) \varepsilon(n) + \epsilon \|f\|_2 + \epsilon^2$$

We then conclude by choosing  $n$  large enough.

Until the end of the section, we will make additional assumptions on our model:

- The  $\omega_n$ 's are bounded, i.e there exists a constant  $M$  such that

$$\forall n \geq 0, |\omega_n| \leq M \text{ a.s}$$

- The  $\omega_n$ 's are  $\eta$ -weakly dependent with dependence coefficients

$$\varepsilon(n) = O(\theta^{n(\log n)^\gamma}) \quad (14)$$

where  $0 < \theta < 1$  and  $\gamma > 1$ .

We also define for all  $j \leq k$  :

$$S_j^k := \sum_{j \leq l \leq k} \omega_{\hat{\tau}_l}$$

where  $\hat{\tau}$  is the renewal process defined in Section 4.2.2.

**Theorem 7.1.** *Under the previous assumptions on the disorder  $\omega$ ,  $\Lambda^d(\beta) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \mathbb{E}_\mu (\exp(\beta S_1^N))$  exists and is finite.*

## 7.2 Proof of Theorem 7.1

The proof of the first point of Theorem 7.1 is based on a result on approximate subadditive functions, which is due to Hammersley (see [10]). The combination of hyperexponential mixing sequences and Hammersley Theorem has already been used in [2] to obtain a large deviation principle and in [5] in the context of ruin probability.

We will also note  $\tau$  instead of  $\hat{\tau}$ .

**Lemma 7.1.** *Assume  $h : \mathbb{N} \rightarrow \mathbf{R}$  be such that for all  $n, m \geq 1$ ,*

$$h(n + m) \leq h(n) + h(m) + \Delta(m + n),$$

*with  $\Delta$  a non decreasing sequence satisfying :*

$$\sum_{r=1}^{\infty} \frac{\Delta(r)}{r(r+1)} < \infty.$$

*Then,  $\lambda = \lim_{n \rightarrow \infty} \frac{h(n)}{n}$  exists and is finite. Moreover, for all  $m \geq 1$ ,*

$$\lambda \leq \frac{h(m)}{m} - \frac{\Delta(m)}{m} + 4 \sum_{r=2m}^{\infty} \frac{\Delta(r)}{r(r+1)}$$

We begin by a technical lemma

**Lemma 7.2.** *For every  $n \in \mathbb{N}^*$  and every  $\beta > 0$  there exists a bounded Lipschitz function  $\phi_n^\beta : x = (x_1, \dots, x_n) \mapsto \phi_n^\beta(x_1, \dots, x_n)$  which coincides with  $x \mapsto \exp(\beta \sum_{i=1}^n x_i)$  on the subset  $\bigcap_{i=1}^n \{|x_i| \leq M\}$ . Moreover,*

$$\|\phi_n^\beta\|_\infty = e^{n\beta M}$$

*and*

$$\text{Lip } \phi_n^\beta = \beta e^{n\beta M}$$

*Proof.* For  $n = 1$ , we define

$$\phi_1^\beta(x) := \begin{cases} e^{-\beta M} & \text{if } x < -M \\ e^{\beta x} & \text{if } -M \leq x \leq M \\ e^{\beta M} & \text{if } x > M \end{cases}$$

This function is bounded by  $e^{\beta M}$  and its Lipschitz coefficient is bounded above by the left derivative at  $x = M$ , which is  $\beta e^{\beta M}$ . Now if we consider

$\frac{|\phi_1^\beta(M-\epsilon)-\phi_1^\beta(M)|}{\epsilon}$  as  $\epsilon$  tends to 0, we see that the Lipschitz coefficient is exactly  $\beta e^{\beta M}$ .

For  $n \geq 2$ , notice that the function  $x = (x_1, \dots, x_n) \mapsto e^{\beta \sum_{i=1}^n x_i}$  is constant on the hyperplanes  $P_a = \{x \in \mathbb{R}^n, \sum_{i=1}^n x_i = a\}$  (which are all normal to the vector  $(1, \dots, 1)$ ). The hyperplanes  $P_{-nM}$  and  $P_{nM}$  determine a partition of  $\mathbb{R}^n$  in three regions on which we define

$$\phi_n^\beta(x_1, \dots, x_n) := \begin{cases} e^{-\beta nM} & \text{if } \sum_{i=1}^n x_i < -nM \\ e^{\beta \sum_{i=1}^n x_i} & \text{if } -nM \leq \sum_{i=1}^n x_i \leq nM \\ e^{\beta nM} & \text{if } \sum_{i=1}^n x_i > nM \end{cases}$$

It is clearly bounded by  $e^{\beta nM}$ . Let us compute its Lipschitz coefficient, and consider two points  $y$  and  $z$ . We denote by  $y'$  the normal projection of  $z$  on the hyperplane  $\{x \in \mathbb{R}^n, \sum_{i=1}^n x_i = \sum_{i=1}^n y_i\}$  so that

$$\frac{|\phi_n^\beta(y) - \phi_n^\beta(z)|}{\sum_{i=1}^n |y_i - z_i|} \leq \frac{|\phi_n^\beta(y') - \phi_n^\beta(z)|}{\sum_{i=1}^n |y'_i - z_i|} \leq \beta e^{\beta nM}$$

from what we know on the case  $n = 1$ . Again, to prove that it is exactly the Lipschitz coefficient, take  $y$  on  $P_{nM}$ ,  $y_\epsilon$  its normal projection on  $P_{nM-\epsilon}$ , and make  $\epsilon$  tend to 0.  $\square$

We follow with

**Lemma 7.3.** *For all  $\beta \geq 0$ , any  $n, m \geq 1$  and  $r \in \{1, \dots, \max(n, m)\}$ ,*

$$\mathbb{E}_\mu \left( e^{\beta S_1^{n+m}} \right) \leq \mathbb{E}_\mu \left( e^{\beta S_1^n} \right) \mathbb{E}_\mu \left( e^{\beta S_{n+1}^{n+m}} \right) \left( e^{2\beta r M} + (n+m)\epsilon(r)\beta e^{2\beta(n+m)M} \right)$$

$\mathbb{P}$ - almost surely.

The proof of this result was inspired by [5].

*Proof.* First observe that if  $j < k$  then

$$e^{-\beta(k-j+1)M} \leq \mathbb{E}_\mu \left( e^{\beta S_j^k} \right) \leq e^{\beta(k-j+1)M}.$$

First suppose that  $m \geq n$  and fix an integer  $0 < r \leq m$ . We have

$$\begin{aligned}
\mathbb{E}_\mu(e^{\beta S_1^{n+m}}) &= \mathbb{E}_\mu(e^{\beta S_1^n} e^{\beta S_{n+1}^{n+r}} e^{\beta S_{n+r+1}^{n+m}}) \\
&\leq e^{\beta r M} \left( \mathbb{E}_\mu(e^{\beta S_1^n}) \mathbb{E}_\mu(e^{\beta S_{n+r+1}^{n+m}}) + \text{Cov}_\mu \left( e^{\beta S_1^n}, e^{\beta S_{n+r+1}^{n+m}} \right) \right) \\
&\stackrel{(\tau_{n+r+1} - \tau_n \geq r)}{\leq} e^{\beta r M} \left( \mathbb{E}_\mu(e^{\beta S_1^n}) \mathbb{E}_\mu(e^{\beta S_{n+r+1}^{n+m}}) + (n+m)\varepsilon(r)\beta e^{\beta n M} e^{\beta(m-r)M} \right) \\
&\leq e^{2\beta r M} \mathbb{E}_\mu(e^{\beta S_1^n}) \mathbb{E}_\mu(e^{\beta S_{n+1}^{n+m}}) + (n+m)\varepsilon(r)\beta e^{2\beta(n+m)M} \mathbb{E}_\mu(e^{\beta S_1^n}) \mathbb{E}_\mu(e^{\beta S_{n+1}^{n+m}}) \\
&\leq \mathbb{E}_\mu(e^{\beta S_1^n}) \mathbb{E}_\mu(e^{\beta S_{n+1}^{n+m}}) (e^{2\beta r M} + (n+m)\varepsilon(r)\beta e^{2\beta(n+m)M}).
\end{aligned}$$

In the third line we have used the definition of the dependence coefficients and the technical lemma 7.2. By applying the same technique it is clear that the result holds if  $n > m$  (take  $r$  in  $\{1, \dots, n\}$ ).  $\square$

Now we can prove Theorem 7.1.

*Proof.* First notice that by stationarity of the  $\omega_n$ 's and the independence of  $(\tau_{n+l} - \tau_n)_{l \geq 1}$  from  $\tau_n$  we have:

$$\begin{aligned}
\mathbb{E} \left( \mathbb{E}_\mu(e^{\beta S_{n+1}^{n+m}}) | \tau_1, \dots, \tau_n \right) &= \mathbb{E} \left( \mathbb{E}_\mu(e^{\beta \sum_{k=1}^m \omega(\tau_{k+n} - \tau_n) + \tau_n}) | \tau_1, \dots, \tau_n \right) \\
&= \mathbb{E}(\mathbb{E}_\mu(e^{\beta S_1^m}))
\end{aligned}$$

so that if we integrate (with respect to the renewal process) the formula in Lemma 7.3, we obtain (for every  $r \in \{1, \dots, m\}$ ):

$$\mathbb{E} \mathbb{E}_\mu \left( e^{\beta S_1^{n+m}} \right) \leq \mathbb{E} \mathbb{E}_\mu \left( e^{\beta S_1^n} \right) \mathbb{E} \mathbb{E}_\mu \left( e^{\beta S_1^m} \right) (e^{2\beta r M} + (n+m)\varepsilon(r)\beta e^{2\beta(n+m)M}) \quad (15)$$

Now let's set for  $n, m \geq 1$ , with a view to applying Lemma 7.1:

$$h(n) := \log \mathbb{E}(\mathbb{E}_\mu(e^{\beta S_1^n}))$$

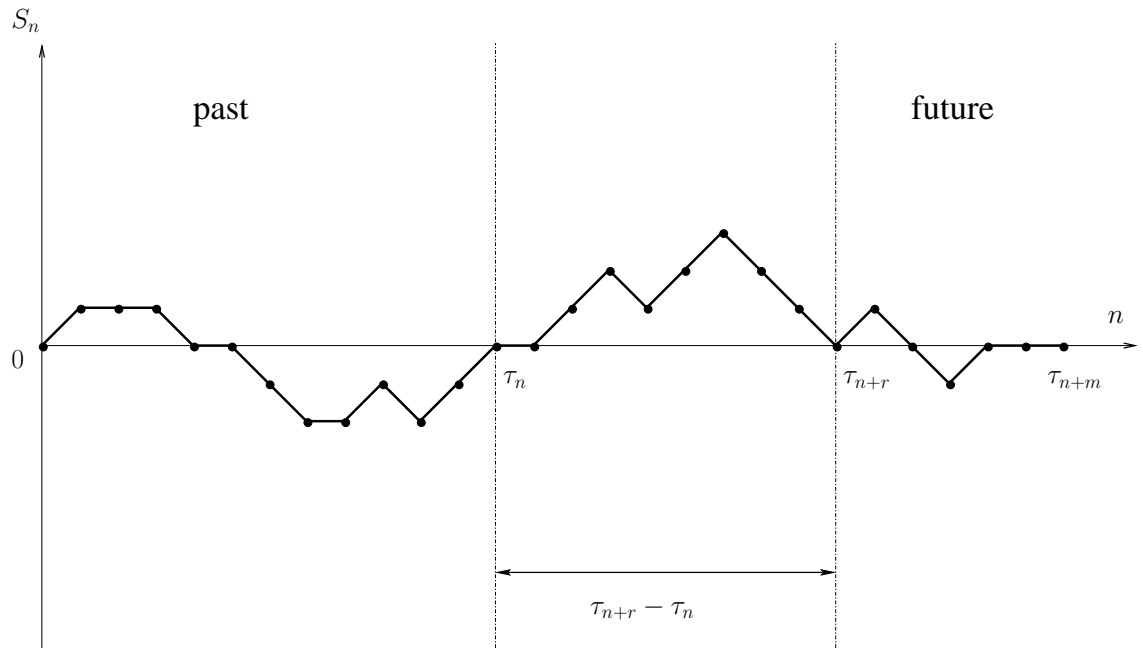
and

$$r(n, m) := \left\lceil \frac{n+m}{2(1 + \log(n+m)^\kappa)} \right\rceil (\leq \max(n, m))$$

for some  $\kappa$  such that  $1 < \kappa < \gamma$ .

Using (14), we have for  $k$  large:

$$\log \left( e^{\beta M \frac{k}{1 + \log(k)^\kappa}} + k\varepsilon \left( \frac{k}{2(1 + \log(k)^\kappa)} \right) \beta e^{2\beta k M} \right) \sim M\beta \frac{k}{\log(k)^\kappa}$$



1

Figure 4: We consider a polymer of size  $n + m$  and introduce a gap of size  $r$  which will separate the past history of the polymer and its future, so that we can use the dependence coefficients. The size of the gap depends on  $n$  and  $m$ , and must be chosen in such a way that the approximate subadditive theorem can be applied.

so there exists a constant  $C > 0$  such that

$$\log \left( e^{\beta M \frac{k}{1+\log(k)^\kappa}} + k\varepsilon \left( \frac{k}{2(1+\log(k)^\kappa)} \right) \beta e^{2\beta k M} \right) \leq CM\beta \frac{k}{\log(k)^\kappa}$$

and we set for  $k \geq 2$ ,  $\Delta(k) := CM\beta \frac{k}{\log(k)^\kappa}$ , which is non-decreasing. We conclude by taking the logarithm in (15) and by applying Lemma 7.1.

□

## 8 Causal shifts with independent inputs

In this section we are interested in another kind of disorder for which we are able to give an annealed bound. As it is called in [6],  $\omega = (\omega_n)_{n \geq 0}$  will be a causal shift with independent inputs defined by

$$\omega_n = H((\zeta_{n-i})_{i \in \mathbb{N}})$$

where  $H : \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}$  is a measurable function and  $\zeta = (\zeta_n)_{n \in \mathbb{Z}}$  is i.i.d. We also define

$$d_n := \left\| \sup_u \left| H((\zeta_{n-i})_{i \in \mathbb{N}}) - H(\zeta_n, \dots, \zeta_1, u_0, u_{-1}, \dots) \right| \right\|_\infty$$

and assume that  $\sum_{n \geq 1} d_n < \infty$ . Notice that the sequence  $(d_n)_{n \geq 1}$  is decreasing. Indeed, for every sequence  $u$  indexed by  $\mathbb{Z} \setminus \mathbb{N}^*$ , we have almost surely

$$\begin{aligned} & \left| H((\zeta_{n+1-i})_{i \geq 0}) - H(\zeta_{n+1}, \dots, \zeta_1, u_0, u_{-1}, \dots) \right| \\ & \leq \sup_v \left| H((\zeta_{n+1-i})_{i \geq 0}) - H(\zeta_{n+1}, \dots, \zeta_2, v_0, v_{-1}, \dots) \right| \end{aligned}$$

so that by taking the supremum in  $u$  and using the stationarity of  $\zeta$ , we obtain  $d_{n+1} \leq d_n$ .

Since the  $\zeta_n$ 's are i.i.d,  $\omega$  is stationary. As for ergodicity, it comes as a consequence of  $\theta$ -weak dependence, which is the same as  $\eta$ -weak dependence in Definition 7.1, except the coefficients  $\psi(f, g)$  are given by

$$\psi(f, g) := v \operatorname{Lip}(g) \|f\|_\infty$$

and the supremum is taken over bounded functions  $f$  and Lipschitz functions  $g$ . Indeed,  $\omega$  is  $\theta$ -weakly dependent with coefficients  $\varepsilon(n) = d_n$  (see [5]). Therefore, the only assumptions we need to make on  $\omega$  is integrability (with mean 0) and  $\beta_0 = \sup\{\beta \geq 0 \mid \Lambda(\beta) < \infty\} > 0$ .

**Theorem 8.1.** *Under the previous assumptions on  $\omega$  and if  $0 \leq \beta < \beta_0$ , then we have*

$$h_c(\beta) \geq h_c(0) - \tilde{\Lambda}(\beta)$$

where  $\tilde{\Lambda}(\beta) = \Lambda(\beta) + 2\beta d_1$ .

**Remark 8.1.** *This result is consistent with the i.i.d case. Indeed, saying that the  $\omega_n$ 's are independent is the same as saying that the function  $H$  only depends on its first variable, and therefore  $d_1 = 0$ .*

The idea of the proof was given in [5].

*Proof.* First let's fix an arbitrary sequence  $u = (u_0, u_{-1}, \dots, u_{-n})_{n \geq 0}$  and write

$$\begin{aligned} \sum_{n=1}^{\iota_N} \beta \omega_{\tau_n} &= \sum_{n=1}^{\iota_N} \beta H(\zeta_{\tau_n - j}, j \geq 0) \\ &= \sum_{n=1}^{\iota_N} \beta H(\zeta_{\tau_n}, \dots, \zeta_{\tau_{n-1}+1}, u_0, u_{-1}, \dots) \\ &\quad + \sum_{n=1}^{\iota_N} \beta (H(\zeta_{\tau_n - j}, j \geq 0) - H(\zeta_{\tau_n}, \dots, \zeta_{\tau_{n-1}+1}, u_0, u_{-1}, \dots)) \end{aligned}$$

We define for  $l \leq k$ ,  $U_l^k = H(\zeta_k, \dots, \zeta_l, u_0, u_{-1}, \dots)$ . Because of the stationarity of  $(\zeta_n)_{n \geq 0}$ , we have on one hand that  $U_l^k$  only depends on  $k - l$  and on the other that the second term in the last inequality is smaller than  $\sum_{n=1}^{\iota_N} \beta d_{\tau_n - \tau_{n-1}}$ . Then

$$\begin{aligned} \frac{1}{N} \mathbb{E}_\mu(\log Z_{N, \beta, h, x}^c) &\leq \frac{1}{N} \log \mathbb{E} \left( e^{h \sum_{n=1}^{\iota_N} 1_{n \in \tau}} \mathbb{E}_\mu \left( \exp \left( \sum_{n=1}^{\iota_N} \beta \omega_{\tau_n} \right) 1_{N \in \tau} \right) \right) \\ &\leq \frac{1}{N} \log \mathbb{E} \left( e^{h \sum_{n=1}^{\iota_N} 1_{n \in \tau}} e^{\beta \sum_{n=1}^{\iota_N} d_{\tau_n - \tau_{n-1}}} \mathbb{E}_\mu \left( \exp \left( \sum_{n=1}^{\iota_N} \beta U_{\tau_{n-1}+1}^{\tau_n} \right) 1_{N \in \tau} \right) \right) \end{aligned}$$

Notice that for every realization of the renewal process  $\tau$ , the  $U_{\tau_{n-1}+1}^{\tau_n}$ 's are independent under the law  $\mu$ , so that

$$\frac{1}{N} \mathbb{E}_\mu(\log Z_{N, \beta, h, x}^c) \tag{16}$$

$$\leq \frac{1}{N} \log \mathbb{E} \left( e^{h \sum_{n=1}^{\iota_N} 1_{n \in \tau}} e^{\beta \sum_{n=1}^{\iota_N} d_{\tau_n - \tau_{n-1}}} \prod_{n=1}^{\iota_N} \mathbb{E}_\mu(e^{\beta U_{\tau_{n-1}+1}^{\tau_n}}) 1_{N \in \tau} \right) \tag{17}$$

and since  $U_{\tau_{n-1}+1}^{\tau_n} \leq \omega_{\tau_n} + d_{\tau_n - \tau_{n-1}}$

$$\begin{aligned} \frac{1}{N} \mathbb{E}_\mu(\log Z_{N,\beta,h,x}^c) &\leq \frac{1}{N} \log \mathbb{E} \left( e^{h \sum_{n=1}^N 1_{n \in \tau}} e^{\sum_{n=1}^N \Lambda(\beta) + 2\beta d_{\tau_n - \tau_{n-1}}} 1_{N \in \tau} \right) \\ &\leq \frac{1}{N} \log Z_{N,0,h+\Lambda(\beta)+2\beta \sup_{i \geq 1} \{d_i\}}^c \end{aligned}$$

because  $\sum_{n=1}^N d_{\tau_n - \tau_{n-1}} \leq \iota_N d_1$ . By taking the limit as  $N \rightarrow \infty$ , we obtain

$$F(\beta, h) \leq F(0, h + \Lambda(\beta) + 2\beta d_1)$$

and Theorem 8.1 follows.  $\square$

It would be tempting to use this for the case of correlated gaussian disorder. Indeed, any gaussian process  $\omega$  with correlation function  $\rho = (\rho_n)_{n \geq 0}$  can be written as

$$\omega_n = H((\varepsilon_{n-i})_{i \geq 0}) := \sum_{i \geq 0} a_i \varepsilon_{n-i}$$

where the  $\varepsilon_n$ 's are i.i.d standard gaussian r.v and the coefficients  $(a_i)_{i \geq 0}$  are given by the spectral distribution of  $\omega$ . Unfortunately, the  $d_i$ 's are not finite in that case (except when  $a_i = 0$ ).

We will now give an example where  $d_1$  (or rather an upper bound) can be computed. The law of  $\zeta_0$  is  $\mathcal{N}(0, 1)$  and we define the function  $s$  such that  $s(x) = 1$  if  $x \geq 0$  and  $s(x) = -1$  otherwise. Let  $\omega$  be defined by

$$\omega_n := \sum_{k \geq 0} 2^{-(k+1)} s(\zeta_{n-k}) = H(\zeta_n, \zeta_{n-1}, \dots, \zeta_0, \zeta_{-1}, \dots)$$

which is well defined since  $\sum_{k \geq 0} 2^{-(k+1)} \mathbb{E}(|s(\zeta_{n-k})|) = 1$ . In fact, the law of  $\omega_0$  is explicit. It is the uniform law on the segment  $[-1, 1]$ , so  $\Lambda$  is defined on  $\mathbb{R}$ , and if  $\beta > 0$

$$\Lambda(\beta) = \log \left( \frac{e^\beta - e^{-\beta}}{2\beta} \right)$$

Moreover, the  $d_n$ 's are summable. Indeed, we have that for all  $u \in \mathbb{R}^{\mathbb{Z}}$

$$\begin{aligned} &|H(\zeta_n, \zeta_{n-1}, \dots, \zeta_0, \zeta_{-1}, \dots) - H(\zeta_n, \zeta_{n-1}, \dots, \zeta_1, u_0, u_{-1}, \dots)| \\ &= \left| \sum_{k \geq n} 2^{-(k+1)} (s(\zeta_{n-k}) - s(u_{n-k})) \right| \\ &\leq \sum_{k \geq n} 2^{-(k+1)} |s(\zeta_{n-k}) - s(u_{n-k})| \\ &\leq \sum_{k \geq n} 2^{-k} \\ &= 2^{-(n-1)} \end{aligned}$$



which implies that  $d_n \leq 2^{-(n-1)}$  and in particular,  $d_1 \leq 1$  so Theorem 8.1 gives

$$h_c(\beta) \geq h_c(0) - \log \left( \frac{e^\beta - e^{-\beta}}{2\beta} \right) - 2\beta$$

## References

- [1] ASMUSSEN, S.(2003). *Applied Probabilities and Queues*. Applications of Mathematics **51**, Springer-Verlag, New York.
- [2] BRYC, W., DEMBO, A.(1996). *Large deviations and strong mixing*. Ann. IHP, B, **32**, p.549-569.
- [3] CHEN, X. Y., BAO L. J., MO, J. Y., WANG, Y.(2003). *Characterizing Long-range Correlation Properties in Nucleotide Sequences*. Chinese Chemical Letters Vol. 14, n°5, p. 503-504
- [4] CORNFELD, I.P, FOMIN, S.V, SINAI, YA.G.(1982). *Ergodic Theory*. Springer-Verlag
- [5] COSSETTE, H, MARCEAU, E, MAUME-DESCHAMPS, V.(2008). *Adjustment coefficient for risk processes in some dependent contexts*. arXiv:0901.0182v1 [math.ST]
- [6] DEDECKER, J, DOUKHAN, P, LANG, G, LEÓN, JR, LOUICHI, S, PRIEUR, C.(2007). *Weak Dependence: With Examples and Applications*.Lect. Notes Stat. **190**.
- [7] ELLIS, R.S.(1985). *Entropy, Large Deviations, and Statistical Mechanics*.A Series of Comprehensive Studies in Mathematics 271, Springer-Verlag
- [8] GIACOMIN, G. (2007). *Random Polymer Models*. Imperial College Press
- [9] GIACOMIN, G.(2008). *Renewal sequences, disordered potentials, and pinning phenomena*. arXiv:0807.4285v1 [math-ph]
- [10] HAMMERSLEY, J.M.(1962). *Generalization of the fundamental theorem on subadditive functions*.Math. proc. Philos. Soc., **58**, 235-238
- [11] PENG C.-K., BULDYREV, S. V., GOLDBERGER, A. L., HAVLIN, S., SCIORTINO, F., SIMONS, M., STANLEY, H E.(1992). *Long-Range Correlations in Nucleotide Sequences*.Nature, **356**, pp. 168-170.

- [12] STEELE, J.M (1989). *Kingman's subadditive ergodic theorem*. Ann. Inst. Henri Poincaré, Vol. 25, n°1, p. 93-98