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ON UNCERTAINTY PRINCIPLES IN THE FINITE DIMENSIONAL SETTING

SAIFALLAH GHOBBER AND PHILIPPE JAMING

ABSTRACT. The aim of this paper is to prove an uncertainty principle for the representation of a vector in two bases. Our result extends previously known “qualitative” uncertainty principles into more quantitative estimates. We then show how to transfer this result to the discrete version of the Short Time Fourier Transform.

1. INTRODUCTION

The aim of this paper is to deal with uncertainty principles in finite dimensional settings. Usually, an uncertainty principle says that a function and its Fourier transform can not be both well concentrated. Of course, one needs to give a precise meaning to “well concentrated” and we refer to [15, 12] for numerous versions of the uncertainty principle for the Fourier transform in various settings. Our aim here is to present results of that flavour for unitary operators on $\mathbb{C}^d$ and then to apply those results to the discrete short-time Fourier transform.

Before presenting our results, let us first introduce some notation. Let d be an integer and ℓ_d^2 be $\mathbb{C}^d$ equipped with its standard norm denoted $\|a\|_{\ell^2}$ or simply $\|a\|_2$ and the associated scalar product $\langle \cdot, \cdot \rangle$. More generally, for $0 < p < +\infty$, the ℓ^p -“norm” is defined by $\|a\|_{\ell^p} = \left(\sum_{j=0}^{d-1} |a_j|^p \right)^{1/p}$. For a set $E \subset \{0, \dots, d-1\}$ we will write E^c for its complementary, $|E|$ for the number of its elements. Further, for $a = (a_0, \dots, a_{d-1}) \in \ell_d^2$, we denote $\|a\|_{\ell^2(E)} = \left(\sum_{j \in E} |a_j|^2 \right)^{1/2}$. Finally, the support of a is defined as $\text{supp } a = \{j : a_j \neq 0\}$ and we set $\|a\|_{\ell^0} = |\text{supp } a|$.

Our aim here is to deal with finite dimensional analogues of the uncertainty principle where concentration is measured in the following sense:

Definition.

Let $T : \ell_d^2 \rightarrow \ell_d^2$ be a linear operator, $S, \Sigma \subset \{0, \dots, d-1\}$. Then (S, Σ) is said to be a

- weak annihilating pair (for T) if $\text{supp } a \subset S$ and $\text{supp } Ta \subset \Sigma$ implies that $a = 0$;
- strong annihilating pair (for T) if there exists a constant $C(S, \Sigma)$ such that for every $a \in \ell_n^2$

$$(1.1) \quad \|a\|_{\ell^2} \leq C(S, \Sigma) (\|a\|_{\ell^2(S^c)} + \|Ta\|_{\ell^2(\Sigma^c)}).$$

Of course, any strong annihilating pair is also a weak one. The corresponding notion for the Fourier transform has been extensively studied, and we refer to [15, 12] for more references.

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The advantage of the second notion over the first one is that it states that if the coordinates of a outside S and those of Ta outside Σ are small, then a itself is small.

It follows from a standard compactness argument (that we reproduce after Formula (2.5) below) that, in a finite dimensional setting, both notions are equivalent. However, this argument does not give any information on $C(S, \Sigma)$. It is our aim here to modify an argument from [10] to obtain quantitative information on this constant in terms of S and Σ . We will restrict our attention to the case where T is invertible. Such an operator can be seen as a change of basis and we are thus looking for uncertainty principles of the following form: “A finite dimensional vector can not have coordinates concentrated in two different bases”. Such uncertainty principles have been known for some time. The first occurrence in the case of T being the discrete Fourier transform seems to be [22] and was rediscovered in [10]. Further results in that case can be found in [25, 23]. For more general changes of basis, we refer to [9, 11, 13].

Before going on with the results of this paper, let us mention its close connection with the *Uniform Uncertainty Principle* introduced by E. Candès and T. Tao in their seminal papers on compressed sensing [4, 5, 6, 7]:

Definition.

Let $T : \ell_d^2 \rightarrow \ell_d^2$ be a unitary operator. Let $s \leq d$ be an integer and $\Omega \subset \{0, \dots, d-1\}$. Then (T, Ω, s) is said to have the Uniform Uncertainty Principle (also called the Restricted Isometry Property) if there exists $\delta_s \in (0, 1)$ such that, for every $S \subset \{0, \dots, d-1\}$ with $|S| = s$ and for every $a \in \ell_d^2$ with $\text{supp } a \subset S$

$$(1.2) \quad (1 - \delta_s) \|a\|_2^2 \leq \|Ta\|_{\ell^2(\Omega)}^2.$$

We will call δ_s the Restricted Isometry Constant of (T, Ω, s) .

Note that, as T is unitary,

$$(1.3) \quad \|Ta\|_{\ell^2(\Omega)}^2 = \|Ta\|_2^2 - \|Ta\|_{\ell^2(\Omega^c)}^2 \leq \|a\|_2^2 \leq (1 + \delta_s) \|a\|_2^2.$$

This inequality is sometimes included in the definition of the Uniform Uncertainty Principle and needed if T is not unitary.

The purpose of this property was to show that, one may recover a from the knowledge of Ta , under the restriction of a to be sufficiently *sparse*, that is $|\text{supp } a|$ to be sufficiently small. Moreover, a may be reconstructed by ℓ^1 -minimization. Let us mention how a recent result due to E. Candès [3] adapts in our case:

Theorem 1.1 (Candès [3]).

$T : \ell_d^2 \rightarrow \ell_d^2$ be a unitary operator, let $s \leq \frac{d-1}{2}$ be an integer and $\Omega \subset \{0, \dots, d-1\}$ and $\varepsilon > 0$. Assume that $(T, \Omega, 2s)$ satisfies the Uniform Uncertainty Principle with restricted isometry constant δ_{2s} satisfying $\delta_{2s} < \sqrt{2} - 1 = (\sqrt{2} + 1)^{-1}$.

Then, for every $a \in \ell_d^2$ with $|\text{supp } a| \leq s$, for every $e \in \ell_d^2$ with $\|e\|_2 < \varepsilon$, the solution $\underline{a}$ of

$$\min\{\|\tilde{a}\|_{\ell^1} : \tilde{a} \in \mathbb{C}^d, \|T\tilde{a} - (Ta + e)\|_2 \leq \varepsilon\}$$

satisfies

$$\|\underline{a} - a\|_2 \leq 2 \frac{1 + (\sqrt{2} - 1)\delta_{2s}}{1 - (\sqrt{2} + 1)\delta_{2s}} s^{-1/2} \|a - a_s^*\|_2 + \frac{4}{1 - (\sqrt{2} + 1)\delta_{2s}} \varepsilon$$

where a_s^* is a vector that minimizes $\|a - a_s\|_2$ among all vectors a_s such that $|\text{supp } a_s| \leq s$.

In particular, if a has support of size s (thus $a = a_s^*$) and $\varepsilon = 0$, then $\underline{a} = a$.

In [3] the operator T is *real*, that is, it maps $\mathbb{R}^d \rightarrow \mathbb{R}^d$. In order to adapt the result to complex operators, a stronger condition on δ_s needs to be imposed. On the other hand, we can take $\delta_s = 0$ in (1.3), which allows for some gain, so that we are back to the same condition as in [3]. This follows from straight forward adaptations of the proofs in [3].

Let us now mention how the Uniform Uncertainty Principle (UUP) is linked to the notion of annihilating pairs. If (T, Ω, s) has the UUP with constant δ_s then, for every S of cardinality s , (S, Ω^c) is an annihilating pair. More precisely, a standard computation (see (2.7) where we reproduce the simple argument) shows that

$$\|a\|_2 \leq \left(1 + \frac{1}{\sqrt{1 - \delta_s}}\right) (\|a\|_{\ell^2(S^c)} + \|Ta\|_{\ell^2(\Omega)}) \leq \frac{2}{\sqrt{1 - \delta_s}} (\|a\|_{\ell^2(S^c)} + \|Ta\|_{\ell^2(\Omega)}).$$

Conversely, assume that Σ is such that, for every S such that $|S| = s$, (S, Σ) is a strong annihilating pair for T . Let $C(\Sigma) = \sup_{|S|=s} C(S, \Sigma)$, then (T, Σ^c, s) satisfies the Uniform Uncertainty Principle with $\delta_s = 1 - C(\Sigma)^{-1}$.

Finally, we will apply our results to the discrete short-time Fourier transform. Let us describe these results in a slightly simplified setting. Let d be an integer, $f, g \in \ell_d^2$, the short-time Fourier transform of f with window g is defined by

$$V_g f(j, k) = \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} f(\ell) \overline{g(\ell - j)} e^{2i\pi k\ell/d}.$$

We refer to *e.g.* [17, 16, 21] for various applications of the discrete short-time Fourier transform in signal processing. Our aim is to show that this transform satisfies an uncertainty principle. To do so, we adapt a method that was originally developed in [19, 20] and improved in [8, 14], that allows to transfer a result about strong annihilating pairs for the discrete Fourier transform into a similar result for its short-time version. A typical result will then be the following:

Theorem 1.2. *Let Σ be a subset of $\{0, \dots, d-1\}^2$ with $|\Sigma| < d$. then*

$$\|f\|_2 \|g\|_2 \leq \frac{2\sqrt{2}}{1 - |\Sigma|/d} \left(\sum_{(j,k) \notin \Sigma} |V_g f(j, k)|^2 \right)^{1/2}.$$

This article is organized as follows: in the next section, we prove results about strong annihilating pairs for a change of basis. The following section deals with applications to the short-time Fourier transform. We devote the last section to a short conclusion.

2. THE UNCERTAINTY PRINCIPLE FOR EXPANSIONS IN TWO BASES.

2.1. Further notations on Hilbert spaces.

Let $\Phi = \{\Phi_j\}_{j=0, \dots, d-1}$ be a basis of $\mathbb{C}^d$ that is *normalized i.e.* $\|\Phi_j\|_2 = 1$ for all j . If $a \in \mathbb{C}^d$, then we may write $a = \sum_{i=0}^{d-1} a_i \Phi_i$. We will denote by $\|a\|_{\ell^p(\Phi)} = \|(a_0, \dots, a_{d-1})\|_{\ell^p}$ and

$\text{supp}_\Phi a = \{i : a_i \neq 0\}$. We also define $\|a\|_{\ell^2(\Phi, E)}$ in the obvious way when E is a subset of $\{0, \dots, d-1\}$. When no confusion can arise, we simply write $\|a\|_{\ell^2(E)}$.

Next, we will denote by $\Phi^* = \{\Phi_j^*\}_{j=1, \dots, d}$ the dual basis of Φ , that is the basis defined by $\langle \Phi_j, \Phi_k^* \rangle = \delta_{j,k}$ where $\delta_{j,k}$ is the Kronecker symbol, $\delta_{j,k} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$. Every $a \in \mathbb{C}^d$ can then be written as

$$a = \sum_{j=0}^{d-1} \langle a, \Phi_j^* \rangle \Phi_j.$$

Moreover, there exist two positive numbers $\alpha(\Phi)$ and $\beta(\Phi)$, called the *lower* and *upper Riesz bounds* of Φ such that

$$(2.4) \quad \alpha(\Phi) \|a\|_2 \leq \left(\sum_{j=0}^{d-1} |\langle a, \Phi_j^* \rangle|^2 \right)^{1/2} \leq \beta(\Phi) \|a\|_2.$$

If Φ and Ψ are two normalized bases of $\mathbb{C}^d$, we will define their *coherence* by

$$M(\Phi, \Psi) = \max_{0 \leq j, k \leq d-1} |\langle \Phi_j, \Psi_k \rangle|.$$

Obviously $M(\Phi, \Psi) \leq 1$ and, if Φ and Ψ are orthonormal bases, then $M(\Phi, \Psi) \geq \frac{1}{\sqrt{d}}$. Let us recall that if $M(\Phi, \Psi) = \frac{1}{\sqrt{d}}$, then Φ and Ψ are said to be *unbiased*. A typical example of a pair of unbiased bases is the standard basis and the Fourier basis of $\mathbb{C}^d$, see Section 3.1.

Let us recall that the Hilbert-Schmidt norm of a linear operator is the ℓ_d^2 norm of its matrix in an orthonormal basis Φ :

$$\|U\|_{HS} = \left(\sum_{i,j=0}^{d-1} |\langle U\Phi_i, \Phi_j \rangle|^2 \right)^{\frac{1}{2}}.$$

As is well known, this definition does not depend on the orthonormal basis and it controls the norm of $U : \ell_d^2 \rightarrow \ell_d^2$:

$$\|U\|_{\ell_d^2 \rightarrow \ell_d^2} := \max_{a \in \mathbb{C}^d : \|a\|_2=1} \|Ua\|_2 \leq \|U\|_{HS}.$$

2.2. The strong version of Elad and Bruckstein's Uncertainty Principle.

Let us start by giving a simple proof of a result of Elad and Bruckstein [11].

Lemma 2.1.

Let Φ and Ψ be two normalized bases of $\mathbb{C}^d$. Then, for every $a \in \mathbb{C}^d \setminus \{0\}$,

$$\|a\|_{\ell^0(\Phi)} \|a\|_{\ell^0(\Psi)} \geq \frac{1}{\left(\min \left\{ \frac{\beta(\Phi)}{\alpha(\Psi)} M(\Phi, \Psi^*), \frac{\beta(\Psi)}{\alpha(\Phi)} M(\Phi^*, \Psi) \right\} \right)^2}.$$

In particular,

$$\|a\|_{\ell^0(\Phi)} + \|a\|_{\ell^0(\Psi)} \geq \frac{2}{\min \left\{ \frac{\beta(\Phi)}{\alpha(\Psi)} M(\Phi, \Psi^*), \frac{\beta(\Psi)}{\alpha(\Phi)} M(\Phi^*, \Psi) \right\}}.$$

Proof. The second statement immediately follows from the first one. The proof mimics the proof given in [25] for the Fourier basis. For $a \neq 0$ and $j = 0, \dots, d-1$,

$$\begin{aligned} |\langle a, \Psi_j^* \rangle| &= \left| \sum_{k=0}^{d-1} \langle a, \Phi_k^* \rangle \langle \Phi_k, \Psi_j^* \rangle \right| \leq \max_{j,k=0,\dots,d-1} |\langle \Phi_k, \Psi_j^* \rangle| \sum_{k=0}^{d-1} |\langle a, \Phi_k^* \rangle| \\ &\leq M(\Phi, \Psi^*) |\text{supp}_\Phi a|^{1/2} \left(\sum_{k=0}^{d-1} |\langle a, \Phi_k^* \rangle|^2 \right)^{1/2} \\ &\leq \beta(\Phi) M(\Phi, \Psi^*) \|a\|_{\ell^0(\Phi)}^{1/2} \|a\|_{\ell^2} \\ &\leq \frac{\beta(\Phi)}{\alpha(\Psi)} M(\Phi, \Psi^*) \|a\|_{\ell^0(\Phi)}^{1/2} \left(\sum_{k=0}^{d-1} |\langle a, \Psi_k^* \rangle|^2 \right)^{1/2} \\ &\leq \frac{\beta(\Phi)}{\alpha(\Psi)} M(\Phi, \Psi^*) \|a\|_{\ell^0(\Phi)}^{1/2} \|a\|_{\ell^0(\Psi)}^{1/2} \max_{k=0,\dots,d-1} |\langle a, \Psi_k^* \rangle|. \end{aligned}$$

It follows that

$$\|a\|_{\ell^0(\Phi)} \|a\|_{\ell^0(\Psi)} \geq \left(\frac{\beta(\Phi)}{\alpha(\Psi)} M(\Phi, \Psi^*) \right)^{-2}.$$

Exchanging the roles of Φ and Ψ , we obtain the result. $\square$

Let us just note that if Φ and Ψ are two unbiased orthonormal bases, then the result reads $|\text{supp}_\Phi a| |\text{supp}_\Psi a| \geq d$. An other formulation would be the following:

Let S and Σ are two subsets of $\{1, \dots, d\}$ with $|S||\Sigma| < d$. If $\text{supp}_\Phi a \subset S$ and $\text{supp}_\Psi a \subset \Sigma$ then $a = 0$.

In other words, if we denote by $\mathcal{F}$ the (unitary) operator that is defined by $\mathcal{F}\Phi_i = \Psi_i$, then (S, Σ) is an annihilating pair for $\mathcal{F}$. When Φ is the standard basis and Ψ the Fourier basis, this result stems back to Matolcsi-Szücs [22] and Donoho-Stark [10]. It follows that (S, Σ) is also a *strong annihilating pair*, i.e. there exists a constant $C = C(S, \Sigma, \Phi, \Psi)$ such that, for every $a \in \mathbb{K}^d$,

$$(2.5) \quad \|a\|_2 \leq C \left(\|a\|_{\ell^2(\Phi, S^c)} + \|a\|_{\ell^2(\Psi, \Sigma^c)} \right).$$

Indeed, by homogeneity it is enough to prove (2.5) when $\|a\|_2 = 1$. But the unit sphere $\mathbb{S}^d = \{a \in \mathbb{C}^d : \|a\|_2 = 1\}$ of $\mathbb{C}^d$ is compact and the map $a \rightarrow \|a\|_{\ell^2(\Phi, S^c)} + \|a\|_{\ell^2(\Psi, \Sigma^c)}$ is continuous, thus its minimum D over $\mathbb{S}^d$ is reached in some a_0 . If this minimum were 0, then $\text{supp}_\Phi a_0 \subset S$ and $\text{supp}_\Psi a_0 \subset \Sigma$ thus $a_0 = 0$ which would contradict $\|a_0\|_2 = 1$. Thus $D > 0$ and (2.5) is satisfied with $C = D^{-1}$. However, this does not allow to obtain an estimate on the constant C . We will overcome this in the next theorem.

Theorem 2.2.

Let d be an integer. Let Φ and Ψ be two orthonormal bases of $\mathbb{C}^d$ and S, Σ be two subsets of $\{0, \dots, d-1\}$. Assume that $|S||\Sigma| < \frac{1}{M(\Phi, \Psi)^2}$. Then, for every $a \in \mathbb{C}^d$,

$$\|a\|_2 \leq \left(1 + \frac{1}{1 - M(\Phi, \Psi)(|S||\Sigma|)^{1/2}}\right) \left(\|a\|_{\ell^2(\Phi, S^c)} + \|a\|_{\ell^2(\Psi, \Sigma^c)}\right).$$

Remark :

For comparison with the previous lemma, recall that as Φ, Ψ are orthonormal they are equal to their dual bases.

Proof. Let U be the change of basis from Ψ to Φ , that is the linear operator defined by $U\Psi_i = \Phi_i$. We will still denote by U its matrix in the basis Φ_i , so that $U = [U_{i,j}]_{1 \leq i,j \leq d}$ is given by $U_{i,j} = \langle \Phi_j, \Psi_i \rangle$. As U is unitary, $U^*\Phi_i = \Psi_i$.

For a set $E \subset \{1, \dots, d\}$ let P_E be the projection $P_E a = \sum_{j \in E} \langle a, \Phi_j \rangle \Phi_j$. A direct computation then shows that $\|a\|_{\ell^2(\Phi, S^c)} = \|P_{S^c} a\|_2$ while

$$\begin{aligned} \|a\|_{\ell^2(\Psi, E)} &= \left(\sum_{j \in E} |\langle a, \Psi_j \rangle|^2 \right)^{1/2} = \left(\sum_{j \in E} |\langle a, U^* \Phi_j \rangle|^2 \right)^{1/2} \\ &= \left(\sum_{j \in E} |\langle Ua, \Phi_j \rangle|^2 \right)^{1/2} = \left(\sum_{j=1}^d |\langle P_E Ua, \Phi_j \rangle|^2 \right)^{1/2} \\ &= \|P_E Ua\|_2. \end{aligned}$$

Assume first that $a \in \mathbb{C}^d$ is such that $\text{supp}_\Phi a \subset S$. Then

$$\|P_\Sigma Ua\|_2 = \|P_\Sigma U P_S a\|_2 \leq \|P_\Sigma U P_S\|_{\ell^2 \rightarrow \ell^2} \|a\|_{\ell^2(\Phi, S)}.$$

It follows that

$$\begin{aligned} \|a\|_{\ell^2(\Psi, \Sigma^c)} &= \|P_{\Sigma^c} Ua\|_2 \geq \|Ua\|_2 - \|P_\Sigma Ua\|_2 = \|a\|_2 - \|P_\Sigma U P_S\|_{\ell^2 \rightarrow \ell^2} \|a\|_{\ell^2(\Phi, S)} \\ (2.6) \quad &= \left(1 - \|P_\Sigma U P_S\|_{\ell^2 \rightarrow \ell^2}\right) \|a\|_{\ell^2(\Phi, S)}. \end{aligned}$$

The last equality comes from the assumption $\text{supp}_\Phi x \subset S$ which implies $\|a\|_2 = \|a\|_{\ell^2(\Phi, S)}$.

Note that, if we are able to prove that $\|P_\Sigma U P_S\|_{\ell^2 \rightarrow \ell^2} < 1$, then this inequality implies that (S, Σ) is an annihilating pair. The following computation allows to estimate the constant $C(S, \Sigma)$ appearing in the definition of a strong annihilating pair: write $D = \left(1 - \|P_\Sigma U P_S\|_{\ell^2 \rightarrow \ell^2}\right)^{-1}$ then, for $a \in \mathbb{C}^d$,

$$\begin{aligned} \|a\|_2 &= \|P_S a\|_2 + \|P_{S^c} a\|_2 \leq D \|P_{\Sigma^c} U P_S a\|_2 + \|P_{S^c} a\|_2 \\ &= D \|P_{\Sigma^c} U(a - P_{S^c} a)\|_2 + \|P_{S^c} a\|_2 \\ (2.7) \quad &\leq D \|P_{\Sigma^c} Ua\|_2 + D \|U P_{S^c} a\|_2 + \|P_{S^c} a\|_2 \end{aligned}$$

since $\|P_{\Sigma^c} x\|_2 \leq \|x\|_2$ for every $x \in \mathbb{C}^d$. Now, as U is unitary, we get

$$\|a\|_2 \leq D \|P_\Sigma Ua\|_2 + (1 + D) \|P_{S^c} a\|_2$$

which immediately gives an estimate of the desired form with

$$C(S, \Sigma, \Phi, \Psi) = 1 + \left(1 - \|P_\Sigma U P_S\|_{\ell^2 \rightarrow \ell^2}\right)^{-1}.$$

It remains to give an upper bound on $\|P_\Sigma U P_S\|_{\ell^2 \rightarrow \ell^2}$:

$$\begin{aligned} \|P_\Sigma U P_S\|_{\ell^2 \rightarrow \ell^2} &\leq \|P_\Sigma U P_S\|_{HS} = \left(\sum_{i=1}^d \sum_{j=1}^d |\langle \Phi_i, P_\Sigma U P_S \Phi_j \rangle|^2 \right)^{1/2} \\ &= \left(\sum_{i \in \Sigma} \sum_{j \in S} |\langle \Phi_i, U \Phi_j \rangle|^2 \right)^{1/2} \\ (2.8) \quad &\leq M(\Phi, \Psi) (|S| |\Sigma|)^{1/2} \end{aligned}$$

which completes the proof of the theorem. $\square$

Remark :

Let $\Phi = \{\Phi_j\}_{j=0, \dots, d-1}$ and $\Psi = \{\Psi_j\}_{j=0, \dots, d-1}$ be two orthonormal bases of $\mathbb{C}^d$ and let U be the operator defined by $U\Psi_j = \Phi_j$ for $j = 0, \dots, d-1$.

It follows from (2.8) and (2.6) that, if $s|\Sigma| < \frac{1}{M(\Phi, \Psi)^2}$ then (U, s, Σ^c) satisfy the uniform uncertainty principle with restricted isometry constant $\delta_s \leq M(\Phi, \Psi)^2 s |\Sigma|$. As a consequence, Theorem 1.1 applies as soon as $s|\Sigma| \leq \frac{\sqrt{2}-1}{2} M(\Phi, \Psi)^{-2}$. Writing $\Omega = \Sigma^c$, we may deduce from this that, if $s \leq \frac{\sqrt{2}-1}{2(d-|\Omega|)M(\Phi, \Psi)^2}$, then any a such that $\|a\|_{\ell^0(\Phi)} \leq s$ can be recovered as the unique solution of

$$\min\{\|\tilde{a}\|_{\ell^1} : \langle \tilde{a}, \Psi_j \rangle = \langle a, \Psi_j \rangle \text{ for every } j \in \Omega\}.$$

Earlier results in that spirit may be found in [9, 11, 13].

Definition.

Let $C > 0$ and $\alpha > 1/2$. We will say that $a \in \mathbb{C}^d$ is (C, α) -compressible in the basis Φ if the j -th biggest coefficient $|\langle a, \Phi \rangle|^*(j)$ of a in the basis Φ satisfies $|\langle a, \Phi \rangle|^*(j) \leq \sqrt{2\alpha - 1} \frac{C}{j^\alpha} \|a\|$.

In order to illustrate our main theorem, let us show that a vector can not be too compressible in two different bases. We will restrict to a simple enough case, the proof being easy to adapt to more general settings:

Corollary 2.3.

Let Φ and Ψ be two unbiased orthonormal bases of $\mathbb{C}^d$. Let $d \geq 9$, $C > 0$ and $\alpha > 1/2$ be such that $C < \frac{(\sqrt{d}-3)^{\alpha-\frac{1}{2}}}{4\sqrt{d}}$. Then the only vector a that is (C, α) -compressible in both bases is 0.

Proof. Let $a \neq 0$ and assume that a is (C, α) -compressible in both bases. Without loss of generality, we may assume that $\|a\|_2 = 1$.

Let $\sigma = \sigma_\Phi$ be a permutation such that $(|\langle a, \Phi_{\sigma(j)} \rangle|)_{1 \leq j \leq d}$ is non-increasing. For $k = 0, \dots, d-1$ define $S_k = \{\sigma_\Phi(0), \dots, \sigma_\Phi(k)\}$, the set of the $k+1$ biggest coefficients of a in the basis Φ . Then

$$\begin{aligned} \|a\|_{\ell^2(\Phi, S_k^c)}^2 &= \sum_{j \notin S_k} |\langle a, \Phi_j \rangle|^2 = \sum_{j=k+1}^{d-1} |\langle a, \Phi_{\sigma(j)} \rangle|^2 \\ &\leq (2\alpha - 1)C^2 \sum_{j=k+1}^{d-1} j^{-2\alpha} \leq (2\alpha - 1)C^2 \int_k^{+\infty} \frac{dx}{x^{2\alpha}} = \frac{C^2}{k^{2\alpha-1}}. \end{aligned}$$

It follows that $\|a\|_{\ell^2(\Phi, S_k^c)} \leq \frac{C}{k^{\alpha-\frac{1}{2}}}$. In a similar way, we get $\|a\|_{\ell^2(\Psi, \Sigma_k^c)} \leq \frac{C}{k^{\alpha-\frac{1}{2}}}$ where Σ_k is the set of the $k+1$ biggest coefficients of a in the basis Ψ .

Let us now apply Theorem 2.2 with $S = S_k$ and $\Sigma = \Sigma_k$. Then, as long as $k+1 < \sqrt{d}$, $1 \leq \frac{2}{1 - \frac{k+1}{\sqrt{d}}} \times \frac{2C}{k^{\alpha-\frac{1}{2}}}$. In other words, $C \geq \frac{1}{4} \left(1 - \frac{k+1}{\sqrt{d}}\right) k^{\alpha-\frac{1}{2}}$.

Assume now that $d \geq 9$ and chose $k = [\sqrt{d}] - 2$ (where $[x]$ is the largest integer less than x) so that $k < \sqrt{d} - 1$. It follows that

$$C \geq \frac{1}{4} \left(1 - \frac{[\sqrt{d}] - 1}{\sqrt{d}}\right) ([\sqrt{d}] - 2)^{\alpha-\frac{1}{2}} \geq \frac{(\sqrt{d} - 3)^{\alpha-\frac{1}{2}}}{4\sqrt{d}}$$

which completes the proof. $\square$

Remark :

— This corollary may be seen as a discrete analogue of Hardy's Uncertainty Principle which states that an $L^2(\mathbb{R})$ function and its Fourier transform can not both decrease too fast (see [15, 12]).

— The above proof also works if the bases are not unbiased, in which case the condition on C has to be replaced by

$$C < \frac{M(\Phi, \Psi)}{4} \left(\frac{1}{M(\Phi, \Psi)} - 3 \right)^{\alpha-1/2}.$$

— Let Φ be an orthonormal basis of $\mathbb{C}^d$ and $a \in \mathbb{C}^d$ with $\|a\| = 1$ and $0 \leq p < 2$. From Bienaymé-Tchebichev, we get that, for $\lambda \geq 0$,

$$|\{j : |\langle a, \Phi_j \rangle| \geq \lambda\}| \leq \|a\|_{\ell^p(\Phi)}^p \lambda^{-p}.$$

Applying this to $\lambda = |\langle a, \Phi \rangle|^*(k)$ we get

$$k \leq |\{j : |\langle a, \Phi_j \rangle| \geq |\langle a, \Phi \rangle|^*(k)\}| \leq \|a\|_{\ell^p(\Phi)}^p (|\langle a, \Phi \rangle|^*(k))^{-p}$$

thus

$$|\langle a, \Phi \rangle|^*(k) \leq \frac{\|a\|_{\ell^p(\Phi)}}{k^{1/p}} = \sqrt{\frac{2}{p} - 1} \left(\sqrt{\frac{p}{p-2}} \|a\|_{\ell^p(\Phi)} \right) k^{-1/p}.$$

It follows that a is $\left(\sqrt{\frac{p}{2-p}} \|a\|_{\ell^p(\Phi)}, \frac{1}{p} \right)$ -compressible in Φ .

This shows that a vector can not have coefficients in two bases with too small ℓ^p -norm.

2.3. Results on annihilating pairs using probability techniques.

So far, we have only used deterministic techniques, which lead to rather weak results. In this section, we will recall some results that may be obtained using probability methods.

First, let us describe a model of random subsets of cardinality k . Let $k \leq d$ be an integer and let $\delta_0, \dots, \delta_{d-1}$ be d independent random variables take the value 1 with probability k/n and 0 with probability $1 - k/n$. We then define the *random subset of cardinality k* , $\Omega \subset \{0, \dots, d-1\}$ by $\Omega = \{i : \delta_i = 1\}$. The term “of cardinality k ” is justified by the fact that the average cardinality of Ω is k (which is immediate once one write $\mathbf{1}_\Omega = \sum_{j=0}^{d-1} \delta_j \mathbf{1}_j$). Moreover, one has the following standard estimate (see e.g. [1, Theorems A.1.12 and A.1.13] or [18]):

$$\mathbb{P} \left[|\Omega - k| \geq \frac{k}{2} \right] \leq 2e^{-k/10}.$$

Then Rudelson-Vershynin [24], (improving a result of Candès-Tao) proved the following theorem:

Theorem 2.4 (Rudelson-Vershynin [24]).

There exist two absolute constants C, c such that the following holds: let $\Phi = \{\Phi_0, \dots, \Phi_{d-1}\}$ and $\Psi = \{\Psi_0, \dots, \Psi_{d-1}\}$ be two unbiased orthonormal bases of $\mathbb{C}^d$ and let $T : \ell_d^2 \rightarrow \ell_d^2$ be defined by $T\psi_j = \Phi_j$ for $j = 0, \dots, d-1$.

Let $0 < \eta < 1$, $t > 1$ be real numbers and $s \leq d$ be an integer. Let k be an integer such that

$$(2.9) \quad k \simeq (Cts \log d) \log(Cts \log d) \log^2 s.$$

Then, with probability at least $1 - 7e^{-c(1-\eta)t}$, a random set Ω of cardinality k satisfies

$$k - \sqrt{tk} \leq |\Omega| \leq k + \sqrt{tk}$$

and (T, Ω, s) satisfies the Uniform Uncertainty Principle with Restricted Isometry Constant $\delta_s \leq 1 - \eta$. In particular, for any $S \subset \{0, \dots, d\}$ with $|S| \leq s$, for every $a \in \ell_d^2$,

$$(2.10) \quad \|a\|_{\ell^2} \leq \frac{2}{\sqrt{\eta}} (\|a\|_{\ell^2(\Phi, S^c)} + \|a\|_{\ell^2(\Psi, \Omega)}).$$

The parameter η is not present in their statement, but it can be obtained by straightforward modification of their proof.

Taking $s = \frac{d}{\log^5 d}$, $t = \frac{\log d}{2C}$ we obtain $k \simeq d/2$. Thus, with probability $\geq 1 - 5d^{-\kappa(1-\eta)}$ (κ some universal constant) Ω has cardinal $|\Omega| = d/2 + O(d^{1/2} \log^{1/2} d)$ and every set S with cardinal $|S| \leq \frac{d}{\log^5 d}$ and Ω^c form a strong annihilating pair in the sense of (2.10).

An other question that one may ask is the following. Given a set Σ , does there exist a “large” set S such that (S, Σ) is an annihilating pair? In order to answer this question, let us recall that Bourgain-Tzafriri [2] proved the following:

Theorem 2.5 (Bourgain-Tzafriri, [2]).

If $T : \ell_n^2 \rightarrow \ell_n^2$ is such that $\|Te_i\|_2 = 1$ for $i = 0, \dots, n-1$ (where the e_i ’s stand for the standard basis of ℓ_n^2), then there exists a set $\sigma \subset \{0, \dots, n-1\}$ with $|\sigma| \geq \frac{n}{240\|T\|_{\ell^2 \rightarrow \ell^2}^2}$ such that, for every $a = (a_j)_{j=0, \dots, n-1}$ with support in σ such that $\|Ta\| \geq \frac{1}{12}\|a\|$.

The values of the numerical constants where given in [18].

We may apply this theorem in the following way: consider two mutually unbiased orthonormal bases $\Phi = \{\phi_j\}$ and $\Psi = \{\psi_j\}$ of ℓ_d^2 and let $S, \Omega \subset \{0, \dots, d-1\}$ be two sets with $|S| = |\Omega| = n$ and enumerate them: $S = \{j_0, \dots, j_{n-1}\}$ and $\Omega = \{\omega_0, \dots, \omega_{n-1}\}$. Let T be the operator defined by $T\phi_{j_k} = \sqrt{\frac{d}{n}}\psi_{\omega_k}$ for $k = 0, \dots, n-1$. Then T satisfies the hypothesis of Bourgain-Tzafriri's Theorem and $\|T\|^2 \leq \frac{d}{n}$. Thus there exists $\sigma \subset S$ with $|\sigma| \geq n^2/240d$ such that, for every $a \in \ell_n^2$ with support in σ ,

$$\|a\|_{\ell^2(\Psi, \Omega)} \geq \frac{1}{12} \sqrt{\frac{n}{d}} \|a\|_{\ell^2(\sigma)}.$$

From which we immediately deduce the following (where $\Sigma = \Omega^c$):

Proposition 2.6.

Let Φ and Ψ be two mutually unbiased bases of ℓ_d^2 and let $S, \Sigma \subset \{0, \dots, d-1\}$ be two sets with $|S| + |\Sigma| = d$. Then there exists $\sigma \subset S$ such that $|\sigma| \geq \frac{(d - |\Sigma|)^2}{240d}$ and, for every $a \in \ell_d^2$,

$$(2.11) \quad \|a\|_{\ell_2} \leq \frac{13}{\sqrt{1 - |\Sigma|/d}} (\|a\|_{\ell^2(\Phi, \sigma^c)} + \|a\|_{\ell^2(\Psi, \Sigma^c)}).$$

Of course, this proposition only makes sense when $|\Sigma| \leq d - \sqrt{240d}$ otherwise there is no guarantee to have $\sigma \neq \emptyset$. We may thus rewrite (2.11) as

$$\|a\|_{\ell_2} \leq 4d^{1/4} (\|a\|_{\ell^2(\Phi, \sigma^c)} + \|a\|_{\ell^2(\Psi, \Sigma^c)}).$$

3. THE UNCERTAINTY PRINCIPLE FOR THE DISCRETE SHORT-TIME FOURIER TRANSFORM

3.1. Finite Abelian groups.

In this section, we recall some notations on the Fourier transform on finite Abelian groups. Results stated here may be found in [26] and (with slightly modified notations) in [21].

Throughout the remaining of this paper, we will denote by G a finite Abelian group for which the group law will be denoted additively. The identity element of G is denoted by 0. The dual group of characters $\hat{G}$ of G is the set of homomorphisms $\xi \in \hat{G}$ which map G into the multiplicative group $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. The set $\hat{G}$ is an Abelian group under pointwise multiplication and, as is customary, we shall write this commutative group operation additively. Note that G is isomorphic to $\hat{\hat{G}}$, in particular $|G| = |\hat{G}|$. Further, Pontryagin duality implies that $\hat{\hat{G}}$ can be canonically identified with G , a fact which is emphasized by writing $\langle \xi, x \rangle = \xi(x)$. Note that, as group operations are written additively,

$$\langle -\xi, x \rangle = \langle \xi, -x \rangle = \overline{\langle \xi, x \rangle}.$$

The Fourier transform $\mathcal{F}_G f = \hat{f} \in \mathbb{C}^{\hat{G}}$ of $f \in \mathbb{C}^G$ is given by

$$\hat{f}(\xi) = \frac{1}{|G|^{1/2}} \sum_{x \in G} f(x) \overline{\langle \xi, x \rangle}, \quad \xi \in \hat{G}.$$

The transform is unitary: $\|\hat{f}\|_2 = \|f\|_2$ and the inversion formula for the Fourier transform allows us to reconstruct the original function from its Fourier transform. Namely, for $f \in \hat{G}$ we have

$$f(x) = \mathcal{F}_{\hat{G}}[\bar{\hat{f}}](x) = \frac{1}{|G|^{1/2}} \sum_{\xi \in \hat{G}} \hat{f}(\xi) \langle \xi, x \rangle, \quad x \in G.$$

Moreover, as the normalized characters $\{|G|^{-1/2} \xi\}_{\xi \in \hat{G}}$ form an orthonormal basis of $\mathbb{C}^G$ that is unbiased with the standard basis we can reformulate Theorem 2.2 as follows:

Strong Uncertainty Principle on Finite Abelian Groups.

Let G be a finite Abelian group and let $S \subset G$ and $\Sigma \subset \hat{G}$ be such that $|S||\Sigma| < |G|$. Then, for every $f \in \mathbb{C}^G$,

$$(3.12) \quad \|f\|_2 \leq \frac{2}{1 - (|S||\Sigma|/|G|)^{1/2}} \left[\left(\sum_{x \notin S} |f(x)|^2 \right)^{1/2} + \left(\sum_{\xi \notin \Sigma} |\hat{f}(\xi)|^2 \right)^{1/2} \right].$$

The corresponding “weak” Uncertainty Principle appeared for the first time in [22] and has been re-discovered several times, including [10]. Note also that one may improve this results by choosing the sets S, Σ randomly ([28] for the cyclic groups) or when better information on the group is taken into account ([23, 25] for weak versions of this theorem, a proper estimate of the constants appearing in the corresponding strong versions remaining open).

For any $x \in G$, we define the translation operator T_x as the unitary operator on $\mathbb{C}^G$ given by $T_x f(y) = f(y - x)$, $y \in G$. Similarly, we define the modulation operator M_ξ for $\xi \in \hat{G}$ as the unitary operator defined by $M_\xi f = f \cdot \xi$, where here and in the following $f \cdot g$ denotes the pointwise product of $f, g \in \mathbb{C}^G$. Since $\widehat{M_\xi f} = T_\xi \hat{f}$, we refer to M_ξ also as a frequency shift operator. Note also that $\widehat{T_x f} = M_{-x} \hat{f}$.

We denote by $\pi(\lambda) = M_\xi T_x$, $\lambda = (x, \xi) \in G \times \hat{G}$ the *time-frequency shift operators*. Note that these are unitary operators. The *short-time Fourier transformation* $V_g^G : \mathbb{C}^G \rightarrow \mathbb{C}^{G \times \hat{G}}$ with respect to the window $g \in \mathbb{C}^G \setminus \{0\}$ is given by

$$V_g^G f(x, \xi) = \frac{1}{|G|^{1/2}} \langle f, \pi(x, \xi)g \rangle = \frac{1}{|G|^{1/2}} \sum_{y \in G} f(y) \overline{g(y - x)} \langle \xi, y \rangle = \mathcal{F}_G[f \cdot \overline{T_x g}](\xi)$$

where $f \in \mathbb{C}^G$. The inversion formula for the short-time Fourier transform is

$$f(y) = \frac{1}{|G|^{1/2} \|g\|_2^2} \sum_{(x, \xi) \in G \times \hat{G}} V_g^G f(x, \xi) g(y - x) \langle \xi, y \rangle.$$

Further, $\|V_g\|_2 = \|f\|_2 \|g\|_2$, in particular $V_g f = 0$ if and only if either $f = 0$ or $g = 0$.

Finally, let us note that a simple computation shows that

$$(3.13) \quad V_{\pi(b, v)g}^G \pi(a, u) f(x, \xi) = \langle u - v - \xi, a \rangle \langle v, x \rangle V_g^G f(x - a + b, \xi - u + v).$$

3.2. The symmetry lemma.

Let us first note that the short-time Fourier transform on $\hat{G}$ is defined by

$$V_\gamma^{\hat{G}} \varphi(\xi, x) = \frac{1}{|\hat{G}|^{1/2}} \langle \varphi, M_x T_\xi \gamma \rangle = \frac{1}{|\hat{G}|^{1/2}} \sum_{\eta \in \hat{G}} \varphi(\eta) \overline{\gamma(\eta - \xi)} \overline{\eta(x)}.$$

This is linked to V^G in the following way:

$$\begin{aligned} V_g^G f(x, \xi) &= \frac{1}{|G|^{1/2}} \langle f, \pi(x, \xi) g \rangle = \frac{1}{|G|^{1/2}} \langle \mathcal{F}^G f, \mathcal{F}^G [M_\xi T_x g] \rangle \\ &= \frac{1}{|\hat{G}|^{1/2}} \langle \mathcal{F}^G f, T_\xi M_{-x} \mathcal{F}^G g \rangle = \frac{\langle \xi, x \rangle}{|\hat{G}|^{1/2}} \langle \mathcal{F}^G f, M_{-x} T_\xi \mathcal{F}^G g \rangle, \end{aligned}$$

so that

$$(3.14) \quad V_g^G f(x, \xi) = \overline{\langle \xi, x \rangle} V_{\hat{g}}^{\hat{G}} \hat{f}(\xi, -x).$$

Lemma 3.1. *Let $f, g, h, k \in \mathbb{C}^G$. Then, for every $u \in G$ and every $\eta \in \hat{G}$,*

$$\mathcal{F}_{G \times \hat{G}} [V_g^G f \overline{V_h^G k}](\eta, u) = V_k^G f(-u, \eta) \overline{V_h^G g(-u, \eta)}.$$

Proof. First note that

$$\begin{aligned} \mathcal{F}_{G \times \hat{G}} [V_g^G f \overline{V_h^G k}](\eta, u) &= \frac{1}{|G|^{1/2} |\hat{G}|^{1/2}} \sum_{x \in G} \sum_{\xi \in \hat{G}} V_g^G f(x, \xi) \overline{V_h^G k(x, \xi)} \overline{\langle \eta, x \rangle} \overline{\langle \xi, u \rangle} \\ &= \frac{1}{|G|^{1/2} |\hat{G}|^{1/2}} \sum_{x \in G} \sum_{\xi \in \hat{G}} V_g^G f(x, \xi) \overline{V_h^{\hat{G}} \hat{k}(\xi, -x)} \overline{\langle \eta, x \rangle} \overline{\langle \xi, u - x \rangle} \end{aligned}$$

with (3.14). Using the definition of the short-time Fourier transform, this is further equal to

$$\begin{aligned} (3.15) \quad & \frac{1}{|\hat{G}| |G|} \sum_{x \in G} \sum_{\xi \in \hat{G}} \sum_{y \in G} \sum_{\zeta \in \hat{G}} f(y) \overline{g(y - x)} \overline{\langle \xi, y \rangle} \overline{\hat{k}(\zeta)} \hat{h}(\zeta - \xi) \langle \zeta, -x \rangle \overline{\langle \eta, x \rangle} \overline{\langle \xi, u + x \rangle} \\ &= \frac{1}{|\hat{G}| |G|} \sum_{x \in G} \sum_{\xi \in \hat{G}} \sum_{y \in G} \sum_{\zeta \in \hat{G}} f(y) \overline{g(y - x)} \overline{\hat{k}(\zeta)} \hat{h}(\zeta - \xi) \overline{\langle \eta + \zeta, x \rangle} \overline{\langle \xi, u - x + y \rangle} \end{aligned}$$

We will now invert the orders of summation. First

$$\begin{aligned} \frac{1}{|\hat{G}|^{1/2}} \sum_{\xi \in \hat{G}} \hat{h}(\zeta - \xi) \overline{\langle \xi, u - x + y \rangle} &= \frac{1}{|\hat{G}|^{1/2}} \sum_{\xi \in \hat{G}} \hat{h}(\xi + \zeta) \langle \xi, u - x + y \rangle \\ &= \frac{1}{|\hat{G}|^{1/2}} \sum_{\xi \in \hat{G}} \widehat{M_{-\zeta} h}(\xi) \langle \xi, u - x + y \rangle = [M_{-\zeta} h](u - x + y) \\ &= \langle \zeta, x \rangle \langle -\zeta, u + y \rangle h(u - x + y). \end{aligned}$$

Then

$$\begin{aligned}
& \frac{1}{|\hat{G}|^{1/2}|G|^{1/2}} \sum_{x \in G} \overline{g(y-x)} \langle \zeta + \eta, x \rangle \sum_{\xi \in \hat{G}} \hat{h}(\zeta - \xi) \overline{\langle \xi, u - x + y \rangle} \\
&= \frac{\overline{\langle \zeta, u + y \rangle}}{|G|^{1/2}} \sum_{x \in G} \overline{g(y-x)} \langle \eta, x \rangle h(u - x + y) \\
&= \frac{\overline{\langle \zeta, u + y \rangle}}{|G|^{1/2}} \sum_{z \in G} \overline{g(z)} h(z + u) \langle \eta, z \rangle \overline{\langle \eta, y \rangle} \\
&= \overline{\langle \zeta, u \rangle} \overline{\langle \eta + \zeta, y \rangle} \overline{V_h^G g(-u, \eta)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \frac{1}{|\hat{G}||G|^{1/2}} \sum_{\zeta \in \hat{G}} \overline{\hat{k}(\zeta)} \sum_{x \in G} \overline{g(y-x)} \langle \zeta + \eta, x \rangle \sum_{\xi \in \hat{G}} \hat{h}(\zeta - \xi) \overline{\langle \xi, u - x + y \rangle} \\
&= \frac{1}{|\hat{G}|^{1/2}} \sum_{\zeta \in \hat{G}} \overline{\hat{k}(\zeta)} \overline{\langle \zeta, u \rangle} \overline{\langle \eta + \zeta, y \rangle} \overline{V_h^G g(-u, \eta)} \\
&= \frac{\overline{\langle \eta, y \rangle} \overline{V_h^G g(-u, \eta)}}{|\hat{G}|^{1/2}} \sum_{\zeta \in \hat{G}} \overline{\hat{k}(\zeta)} \overline{\langle \zeta, y + u \rangle} \\
&= \overline{\langle \eta, y \rangle} \overline{V_h^G g(-u, \eta)} \overline{\hat{k}(y + u)}.
\end{aligned}$$

Finally, it remains to take the sum in the y -variable in 3.15 to obtain

$$\mathcal{F}_{G \times \hat{G}}[V_g^G f \overline{V_h^G k}](\eta, u) = V_k^G f(-u, \eta) \overline{V_h^G g(-u, \eta)}.$$

as announced. $\square$

Remark : The short-time Fourier transform can be defined on any locally Abelian group G and its dual $\hat{G}$ as

$$V_g f(x, \xi) = \int_G f(t) \overline{g(t-x)} \langle \xi, t \rangle d\nu(t), \quad (x, \xi) \in G \times \hat{G}$$

where $d\nu_G$ is the Haar measure on G . All results in this section go through to this more general context, the inversions of order of integrations being easily justified. In the case $G = \mathbb{R}^d$, this lemma was given independently in [19, 20].

3.3. The Uncertainty Principle for the short-time Fourier transform.

We will conclude this section with the following theorem that allows to transfer results about strong annihilating pairs in $G \times \hat{G}$ to Uncertainty Principles for the short-time Fourier transform.

Theorem 3.2. *Let $\Sigma \subset G \times \hat{G}$ and $\tilde{\Sigma} = \{(\xi, -x) : (x, \xi) \in \Sigma\} \subset \widehat{G \times \hat{G}} = \hat{G} \times G$. Assume that $(\Sigma, \tilde{\Sigma})$ is strong annihilating pair in $G \times \hat{G}$, i.e. that there is a constant $C(\Sigma)$ such that,*

for every $F \in \mathbb{C}^{G \times \hat{G}}$,

$$\|F\|_2^2 \leq C(\Sigma) \left(\sum_{(x,\xi) \notin \Sigma} |F(x,\xi)|^2 + \sum_{(x,\xi) \notin \tilde{\Sigma}} |\mathcal{F}_{G \times \hat{G}} F(\xi, x)|^2 \right)$$

then, for every $f, g \in \mathbb{C}^G$,

$$\|f\|_2^2 \|g\|_2^2 \leq 2C(\Sigma) \sum_{(x,\xi) \notin \Sigma} |V_g^G f(x, \xi)|^2.$$

Proof. We will adapt the proof in the case $G = \mathbb{R}^d$ given in [8] to our situation. Let us fix $f, g \in \mathbb{C}^G$.

We will only use Lemma 3.1 in a simple form: for $a \in G, \eta \in \hat{G}$ define the function $F_{a,\eta}$ on $G \times \hat{G}$ by

$$F_{a,\eta}(x, \xi) = \overline{\langle \xi - \eta, a \rangle} V_g^G f(x - a, \xi - \eta) \overline{V_f^G g(x, \xi)}.$$

Note that $F_{a,\eta}(x, \xi) = V_g^G \pi(a, \eta) f \overline{V_{\pi(a,\eta)f}^G g}$ so that then $\mathcal{F}_{G \times \hat{G}} F_{a,\eta}(\xi, x) = F_{a,\eta}(-x, \xi)$.

It follows that

$$\begin{aligned} \|F_{a,\eta}\|_2^2 &\leq C(\Sigma) \left(\sum_{(x,\xi) \notin \Sigma} |F_{a,\eta}(x, \xi)|^2 + \sum_{(x,\xi) \notin \tilde{\Sigma}} |F_{a,\eta}(-x, \xi)|^2 \right) \\ &= 2C(\Sigma) \sum_{(x,\xi) \notin \Sigma} |V_g^G f(x - a, \xi - \eta)|^2 |V_f^G g(x, \xi)|^2. \end{aligned}$$

Finally, summing this inequality over $(a, \eta) \in G \times \hat{G}$ gives

$$\|f\|_2^4 \|g\|_2^4 \leq 2C(\Sigma) \|f\|_2^2 \|g\|_2^2 \sum_{(x,\xi) \notin \Sigma} |V_f^G g(x, \xi)|^2$$

which completes the proof. $\square$

Combining this result with (3.12) we immediately get the following:

Corollary 3.3. *Let $\Sigma \subset G \times \hat{G}$ with $|\Sigma| < |G|$. Then, for every $f, g \in \mathbb{C}^G$,*

$$\|f\|_2^2 \|g\|_2^2 \leq \frac{8}{(1 - |\Sigma|/|G|)^2} \sum_{(x,\xi) \notin \Sigma} |V_f^G g(x, \xi)|^2.$$

The corresponding weak annihilating property for Σ was obtained by F. Krahmer, G. E. Pfander and P. Rashkov [21].

Finally, using the fact that two random events A and B that each occur with probability $\geq 1 - \alpha$, jointly occur with probability $\geq 1 - 2\alpha$, we deduce the following from Theorem 2.4:

Theorem 3.4.

There exist two absolute constants C, c such that the following holds: Let $0 < \eta < 1$, $t > 1$ be real numbers and $s \leq d$ be an integer. Let k be an integer such that

$$k \simeq (Cts \log d) \log(Cts \log d) \log^2 s.$$

Then, with probability at least $1 - 14e^{-c(1-\eta)t}$, a random set Ω of cardinality k satisfies

$$k - \sqrt{tk} \leq |\Omega| \leq k + \sqrt{tk}$$

and, for any $S \subset \{0, \dots, d\}$ with $|S| \leq s$, for every $f, g \in \ell_d^2$,

$$(3.16) \quad \|f\|_{\ell^2} \|g\|_{\ell^2} \leq \frac{2\sqrt{2}}{\sqrt{\eta}} \left(\sum_{x \notin S, \xi \in \Omega} |V_f^G g(x, \xi)|^2 \right)^{1/2}.$$

It would therefore be nice to have a “quadratic” analogue of Candès’s Theorem 1.1 in order to obtain reconstruction of f and g from lacunary data $\{V_f^G g(x, \xi), x \in S^c, \xi \in \Omega\}$.

4. CONCLUSION

In this paper, we have shown how to obtain quantitative uncertainty principles for the representation of a vector in two different bases. These estimates are stated in terms of annihilating pairs and both extend and simplify previously known qualitative results. We then apply our main theorem to the discrete short time Fourier transform, following the path of corresponding results in the continuous setting.

REFERENCES

- [1] N. ALON & J. H. SPENCER, *The probabilist method. 2nd Edition*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
- [2] J. BOURGAIN & L. TZAFRIRI, *Invertibility of “large” submatrices and applications to the geometry of Banach spaces and Harmonic Analysis*. Israel J. Math. **57** (1987), 137–224.
- [3] E. CANDÈS, *The Restricted Isometry Property and Its Implications for Compressed Sensing*. C. R. Acad. Sci. Paris Sér. I Math. **346** (2008), 589–592.
- [4] E. CANDÈS & T. TAO, *Decoding by linear programming*. IEEE Trans. Inform. Theory **51** (2005) 4203–4215.
- [5] E. CANDÈS & T. TAO, *Near-optimal signal recovery from random projections: universal encoding strategies*. IEEE Trans. Inform. Theory **52** (2006), 5406–5425.
- [6] E. CANDÈS, J. ROMBERG & T. TAO, *Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information*. IEEE Trans. Inform. Theory, **52** (2006) 489–509.
- [7] E. CANDÈS & J. ROMBERG, *Quantitative robust uncertainty principles and optimally sparse decompositions*. Found. of Comput. Math. **6** (2006), 227–254.
- [8] B. DEMANGE, *Uncertainty principles for the ambiguity function*. J. London Math. Soc. (2) **72** (2005), 717–730.
- [9] D. L. DONOHO & X. HUO, *Uncertainty principles and ideal atomic decomposition*. IEEE Trans. Inform. Theory **47** (2001), 2845–2862.
- [10] D. L. DONOHO & P. B. STARK, *Uncertainty principles and signal recovery*. SIAM J. Appl. Math. **49** (1989) 906–931.
- [11] M. ELAD & A. M. BRUCKSTEIN, *A Generalized Uncertainty Principle and Sparse Representation in Pairs of Bases*. IEEE Trans. Inform. Theory **48** (2002), 2558–2567.
- [12] G. B. FOLLAND & A. SITARAM, *The uncertainty principle — a mathematical survey*. J. Fourier Anal. Appl. **3** (1997), 207–238.
- [13] R. GRIBONVAL & M. NIELSEN, *Sparse representations in unions of base*. IEEE Trans. Inform. Theory **49** (2003) 3320–3325.
- [14] K. GRÖCHENIG & G. ZIMMERMANN, *Hardy’s theorem and the short-time Fourier transform of Schwartz functions*. J. London Math. Soc. (2) **63** (2001), 205–214.
- [15] V. HAVIN & B. JÖRCKE, *The uncertainty principle in harmonic analysis*. Springer-Verlag, Berlin, 1994.

- [16] M. HERMAN & T. STROHMER, *High Resolution Radar via Compressed Sensing*. IEEE Trans. Signal Processing, to appear.
- [17] S. D. HOWARD, A. R. CALDERBANK & W. MORAN, *The Finite Heisenberg-Weyl Groups in Radar and Communication*. EURASIP Journal on Applied Signal Processing **2006** (2006), Article ID 85685, 12 pages.
- [18] PH. JAMING, *Inversibilit Restreinte, Problème de Kadison-Singer et Applications l'Analyse Harmonique d'après J. Bourgain et L. Tzafriri- (sous la direction de M. Deschamps)*. Publications Mathématiques d'Orsay **94-24** (1994), 71–154.
- [19] PH. JAMING, *Principe d'incertitude qualitatif et reconstruction de phase pour la transformée de Wigner*. C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), 249–254.
- [20] A. J. E. M. JANSSEN, *Proof of a conjecture on the supports of Wigner distributions*. J. Fourier Anal. Appl. **4** (1998), 723–726.
- [21] F. KRAHMER, G. E. PFANDER & P. RASHKOV, *Uncertainty in timefrequency representations on finite Abelian groups and applications*. Appl. Compt. Harm. Anal. **25** (2008) 209–225.
- [22] T. MATOLCSI & J. SZUCS, *Intersection des mesures spectrales conjuguées*. C.R. Acad. Sci. Sr. I Math. **277** (1973), 841–843.
- [23] R. MESHULAM, *An uncertainty inequality for finite abelian groups*. European J. Combin. **27** (2006), 63–67.
- [24] M. RUDELSON & R. VERSHYNIN, *On sparse reconstruction from Fourier and Gaussian measurements*. Comm. Pure and Appl. Math. **61** (2008), 1025–1045.
- [25] T. TAO, *An uncertainty principle for cyclic groups of prime order*. Math. Res. Letters **12** (2005), 121–127.
- [26] A. TERRAS, *Fourier analysis on finite groups and application*. London Mathematical Society Student Texts, **43**. Cambridge University Press, Cambridge, 1999.
- [27] J. A. TROPP, *The random paving property for uniformly bounded matrices*. Studia Math. **185** (2008), 67–82.
- [28] J. A. TROPP, *On the linear independence of spikes and sines*. J. Fourier Anal. Appl, to appear, available at arXiv:0709.0517

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