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A NOTE ON THE CHAS-SULLIVAN PRODUCT

FRANÇOIS LAUDENBACH

ABSTRACT. We give a finite dimensional approach of the Chas-Sullivan product on the free loop space of a manifold, orientable or not.

1. INTRODUCTION

Let M be an n -dimensional manifold with empty boundary; it is not required to be compact nor orientable. Denote $LM = C^\infty(S^1, M)$ be its free loop space. In a famous paper [3], M. Chas and D. Sullivan, at least in the case when M is orientable, constructed a natural graded algebra structure on the homology $H_*(LM; \mathbb{Z})$, more precisely a product

$$H_i(LM; \mathbb{Z}) \otimes H_j(LM; \mathbb{Z}) \rightarrow H_{i+j-n}(LM; \mathbb{Z}),$$

in the same *intersection* spirit as the usual intersection product on $H_*(M; \mathbb{Z})$. A slightly different approach has been considered by R. Cohen and J. Jones in [6]. Recently, in [8], Y. Félix and J.-C. Thomas have put the *String Topology* into a broad homotopy theoretical setting. On the contrary, in the present note we propose a finite dimensional approach, very close to the spirit of [3], based on transversality arguments. We also treat the case of a non-orientable manifold using local coefficients instead of \mathbb{Z} .

We do not think of LM as a topological space but as a simplicial set (except in section 4). A k -simplex in LM is a smooth map $\Sigma : \Delta^k \times S^1 \rightarrow M$ where Δ^k is the standard k -simplex. Let 0 be the base point of $S^1 = \mathbb{R}/\mathbb{Z}$. The evaluation map $ev_0 : LM \rightarrow M$ is simplicial: $ev_0(\Sigma) = \sigma$ where $\sigma : \Delta^k \rightarrow M$ is the k -simplex of M defined by

$$\sigma(t) = \Sigma(t, 0).$$

It is easy to form a chain complex based on the simplices of LM and a bi-complex based on *bi-simplices* (pairs of simplices). In order to take into account the non-orientability, we limit ourselves to *small* simplices and bi-simplices (see 2.1). For defining an *intersection product* we consider *transverse* bi-complexes. By Thom's transversality theorem, they generate a sub-bicomplex which has the same homology. As we shall see, smallness and transverseness will only refer to the image of simplices and bisimplices by the evaluation map. The intersection product will be not far from being defined on the chain level: the intersection of two transverse cycles produces a singular smooth manifold in LM , which becomes a simplicial cycle once it is triangulated. According to Whitehead, such a triangulation is unique up to subdivision and

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isotopy. Therefore the product is well defined at the homology level.

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2. SIMPLICES AND BI-SIMPLICES AT THE MANIFOLD LEVEL

2.1. The manifold M under consideration is equipped with an atlas \mathcal{A} of charts. A *small* k -simplex is a smooth map

$$\sigma : \Delta^k \rightarrow M$$

whose image is contained in \mathcal{A} . For each simplex σ some particular chart $U(\sigma) \in \mathcal{A}$ is chosen once for all containing its image. A *small k -chain* is a linear combination $\xi = \sum n_i \sigma_i$ with coefficient $n_i \in \mathbb{Z}$ of finitely many small simplices σ_i . The *orientation twisted boundary* is defined by the following formula:

$$\partial\sigma = \sum_{i=0}^k \varepsilon(-1)^i F_i\sigma,$$

where $F_i\sigma$ is the i^{th} face of σ and ε is the sign of the Jacobian of change of coordinates from $U(F_i\sigma)$ to $U(\sigma)$ calculated at any point of the image of $F_i\sigma$. The small chains with this boundary form a chain complex whose homology is $H_*(M; \mathbb{Z}_{or})$, the homology with integral coefficients twisted by the orientation.

Given a k -simplex σ , a homotopy $\sigma^t : \Delta^k \rightarrow M$, $t \in [0, 1]$, with $\sigma^0 = \sigma$ induces a homotopy $(F\sigma)^t$ of each face $F\sigma$. So the following definition makes sense.

Definition 2.2. *Given a chain $\xi = \sum n_i \sigma_i$, a boundary preserving homotopy of ξ is a one parameter family ξ^t , $t \in [0, 1]$, $\xi^t = \sum n_i \sigma_i^t$, where $t \mapsto \sigma_i^t$ is a homotopy of σ_i into M ($\sigma_i^0 = \sigma_i$), and such that*

$$\partial\xi^t = (\partial\xi)^t.$$

For instance, if ξ is a cycle then ξ^t is a cycle for every $t \in [0, 1]$. When we consider such a homotopy of a small chain, we shall limit ourselves to the case when the homotopy is small, that is: each summand σ_i^t takes place in $U(\sigma_i)$. The following homotopy extension lemma is easy to prove.

Lemma 2.3. *Let τ be any simplex which is a face of one of the summands of ξ . Any (small) homotopy of τ extends as a (small) boundary preserving homotopy of ξ . The same statement holds for a family of homotopies with an extension depending continuously on the parameter.*

Definition 2.4. *A (p, q) -bi-simplex of $M \times M$ is a pair (u, v) where u is a p -simplex and v is a q -simplex of M . It is said to be small when both factors are small simplices of M .*

It is convenient to denote it $u \times v$ like the map $u \times v : \Delta^p \times \Delta^q \rightarrow M \times M, (x, y) \mapsto (u(x), v(y))$. The (small) bi-simplices generate a bi-complex $C_{**}(M \times M)$ whose elements are bi-chains, with two boundary operators twisted by the orientation,

$$\partial_1(u \times v) = (\partial u) \times v, \quad \partial_2(u \times v) = u \times \partial v,$$

and a total boundary operator

$$D(u \times v) = \partial u \times v + (-1)^p u \times \partial v.$$

Definition 2.5. *A (small) bi-simplex $u \times v$ is said to be transverse when the map $u \times v$ is transversal to the diagonal Δ_M and when all its faces (they are bi-simplices) are also transversal to the diagonal.*

In the sequel, all transverse bi-simplices will be small, without saying it. The advantage is that $W = (u \times v)^{-1}(\Delta_M)$ is a proper orientable submanifold of codimension n (with corners) in $\Delta^p \times \Delta^q$. Moreover, if we use the charts $U(u)$ and $U(v)$ whose product contains the image of $u \times v$, W receives a canonical orientation.

Notice that, when $u \times v$ is a transverse bi-simplex, a small homotopy of the factors keep them transverse. The transverse bi-simplices generate a sub-bicomplex $C_{**}^{tr}(M \times M)$ of $C_{**}(M \times M)$. A bi-chain is said to be transverse when each of its bi-simplices is transverse.

Lemma 2.6. *Let $\xi \times \eta$ be a bi-chain. There exists an arbitrarily C^∞ -small boundary preserving homotopy ξ^t of $\xi^0 = \xi$ such that $\xi^1 \times \eta$ is transverse. Moreover, when $\partial \xi \times \eta$ is transverse, the homotopy can be chosen stationary on $\partial \xi \times \eta$. The same statement holds for the second factor.*

Considering the composition of loops we have in mind, it is very important to make this approximation among the bi-chains.

Proof. We first consider the case when $\xi \times \eta = u \times v$ is a bi-simplex. The conclusion follows from Thom's transversality theorem, not in its elementary form but in the form known as the transversality with constraint ([10]). Indeed, only the first factor is moved for guaranteeing that bi-simplices remain bi-simplices during the deformation. We argue as follows. Let S be a small n -ball in the vector space containing the chart $U(u)$, so small that, for any $s \in S$, $u^s := u + s$ is still contained in $U(u)$. We introduce the family $(u^s \times v)$ parametrized by S . It is transversal to Δ_M as well as its restriction to any face of $\Delta^p \times \Delta^q$. Therefore, according to Sard's lemma (used *à la Thom*), for almost every $s \in S$, $u^s \times v$ is a transverse bi-simplex.

For the general case, it is useful to observe that, in the above argument, the space S of parameters can be chosen arbitrarily small. When considering a bi-chain $\xi \times \eta$, its k -skeleton is the collection of all the k -faces of the bi-simplices appearing in $\xi \times \eta$. Arguing recursively, we may assume that all bi-simplices in the k -skeleton are transverse. Let $(u \times v)$ be a $(k+1)$ -bi-simplex, which we endow with a family S of translations in $U(u)$. According to lemma 2.3, the translation by any $s \in S$ extends as a boundary preserving homotopy of ξ (ending at ξ_s) and it can be chosen small enough so that each bi-simplex of the k -skeleton remains transverse. Therefore, for almost $s \in S$, ξ_s is transverse along its k -skeleton and $u \times v$. Repeating this process, we make successively all $(k+1)$ -bi-simplices transverse.

For the relative version, we notice that, if Fu is a face of u such that $Fu \times v$ is transverse, then $u \times v$ is automatically transversal to Δ_M on a neighborhood of the domain of $Fu \times v$. In that case we can moderate the translation by s so that it becomes stationary along $Fu \times v$. The general relative version follows easily. \diamond

Let $[\xi]$ and $[\eta]$ be two classes of respective degree p and q in $H_*(M; \mathbb{Z}_{or})$. According to lemma 2.6, they can be represented by small cycles so that the bi-cycle $\xi \times \eta$ is transverse. This bi-cycle is unique up to *transverse homology*; more precisely we claim the following uniqueness lemma:

Lemma 2.7. *Let $\xi \times \eta$ and $\xi' \times \eta'$ be two transverse bi-cycles in the same bi-homology class ($[\xi] = [\xi']$ and $[\eta] = [\eta']$). There exists a transverse bi-chain Ω whose total boundary is $D\Omega = \xi' \times \eta' - \xi \times \eta$.*

Proof. There exists a cycle ξ'' , homotopic to ξ , such that $\xi'' \times \eta$ and $\xi'' \times \eta'$ are both transverse ($\xi \times \eta$ is already transverse and this property is preserved by approximation of ξ ; according to lemma 2.6, such an approximation exists making $\xi'' \times \eta'$ transverse). If ω is a $(p+1)$ -chain such that $\partial\omega = \xi - \xi''$, there is a boundary preserving homotopy $\omega^t, t \in [0, 1]$, of ω relative to its boundary so that $\omega^1 \times \eta$ is transverse. By the same argument there is a $(p+1)$ -chain ω' with $\partial\omega' = \eta - \eta'$, such that $\xi'' \times \omega'$ is transverse. Eventually, there exists ω'' with $\partial\omega'' = \xi' - \xi''$ such that $\omega'' \times \eta'$ is transverse. By concatenating the three transverse homologies $\omega^1 \times \eta$, $\xi'' \times \omega'$ and $\omega'' \times \eta'$ we get a transverse homology joining both given bi-cycles. \diamond

2.8. Intersection of cycles.

We are now ready for defining the intersection of cycles. To begin with, we consider a transverse bi-chain $\xi \times \eta$ of degree (p, q) , which reads as a sum of transverse bi-simplices

$$\xi \times \eta = \sum n_{ij} u_i \times v_j.$$

Let W_{ij} be the preimage of Δ_M by $u_i \times v_j$. As already said, W_{ij} is an oriented manifold with corners of codimension n . If $F(u_i \times v_j)$ is a face of the bi-simplex, FW_{ij} denotes the corresponding face of W_{ij} .

According to Whitehead [11], W_{ij} can be smoothly triangulated by a PL -triangulation T_{ij} . Moreover if some faces have been already triangulated, by using the relative version of Whitehead's theorem, T_{ij} can be chosen so that the triangulated faces are subcomplexes. If two bi-simplices have common faces $F(u_i \times v_j) = F(u_k \times v_\ell)$, then we have a canonical diffeomorphism $FW_{ij} \rightarrow FW_{k\ell}$ which we think of as an identification. The triangulations of these faces are chosen accordingly. We consider the chain of $\Delta_M \cong M$

$$\xi \cdot \eta := (-1)^{n(n-q)} \sum n_{ij}(u_i \times v_j)|(W_{ij}, T_{ij})$$

which is called the *intersection product*. Of course, as a chain it depends on the chosen triangulations. But, since the latter are unique up to subdividing and boundary preserving isotopy (that is, smooth isotopy of each simplex in W_{ij} keeping the triangulation property), the ambiguity is not severe. The sign, which we call the *Dold sign*, will be commented later.

Lemma 2.9. *When $\xi \times \eta$ is a transverse bi-cycle, the intersection product $\xi \cdot \eta$ is a cycle of degree $p + q - n$ (with orientation twisted coefficients). If $\xi \times \eta$ is changed by a transverse*

homology (in the sense of lemma 2.7), $\xi \cdot \eta$ is changed by a homology. Finally, $[\xi] \cdot [\eta]$ is well-defined in $H_{p+q-n}(M; \mathbb{Z}_{or})$.

Note that a change of triangulation of the W_{ij} 's may be thought of as a special case of a change by a transverse homology.

Proof. As W_{ij} is an oriented proper submanifold, (W_{ij}, T_{ij}) is a relative cycle in $\Delta^p \times \Delta^q$. Thus, the total boundary of $(u_i \times v_j)|(W_{ij}, T_{ij})$ is

$$((\partial u_i) \times v_j)|(W_{ij}, T_{ij}) + (-1)^p(u_i \times (\partial v_j))|(W_{ij}, T_{ij}).$$

By summing on ij we get the boundary of $\xi \cdot \eta$. As ξ and η are cycles, each hyperface in the latter sum appears twice with opposite sign. The rest of the statement is easy to prove. \diamond

Remark 2.10. In his book [7] (Chap. VIII, §13.3), A. Dold explains that the chosen sign makes the intersection product on homology and the (unsigned) cup-product on cohomology fit together via the Poincaré duality. An other advantage of this sign is the following. Set $\mathbb{H}_*(M; \mathbb{Z}_{or}) = H_{*+n}(M; \mathbb{Z}_{or})$. This regraded homology endowed with the above intersection product becomes a commutative ring in the graduate sense.

3. SIMPLICES AND BI-SIMPLICES AT THE FREE LOOP SPACE LEVEL

3.1. We recall the evaluation map $ev_0 : LM \rightarrow M$. A simplex $\Sigma : \Delta^k \times S^1 \rightarrow M$ is said to be *small* when $\sigma = ev_0(\Sigma)$ is so. The i^{th} -face of Σ is obtained by restricting Σ to $F_i \Delta^k \times S^1$. We have $F_i(ev_0(\Sigma)) = ev_0(F_i \Sigma)$. The *orientation twisted boundary* of Σ is

$$\partial \Sigma = \sum_{i=0}^k \varepsilon(-1)^i F_i \Sigma,$$

where ε is the sign of the Jacobian of change of coordinates from $U(ev_0(F_i \Sigma))$ to $U(ev_0(\Sigma))$. The small chains with this boundary form a sub-complex of $C_*(LM)$, the singular chain complex of LM , whose homology is $H_*(LM; \mathbb{Z}_{or})$. Indeed, this sub-complex is obtained from $C_*(LM)$ by two operations which induce homotopy equivalences: subdivision and smoothing. The notion of boundary preserving homotopy is similar to the one given in 2.1.

Lemma 3.2. *The evaluation map has the lifting property for boundary preserving homotopy of chain with any initial chain. Moreover, if the lifting of the homotopy is given along some faces this partial lifting can be extended to a global lifting.*

Proof. It is clear for $\Delta^k \times S^1 \times [0, 1]$ retracts onto $\Delta^k \times S^1 \times \{0\} \cup \Delta^k \times \{0\} \times [0, 1] \cup F \times S^1 \times [0, 1]$, where F is any union of faces in Δ^k . \diamond

A (p, q) -bi-simplex $u \times v$ in $LM \times LM$ is a map

$$\begin{aligned} \Delta^p \times \Delta^q \times S^1 &\rightarrow M \times M, \\ (x, y, \theta) &\mapsto (u(x, \theta), v(y, \theta)). \end{aligned}$$

It is said to be transverse when $(ev_0(u), ev_0(v))$ is a transverse bi-simplex of $M \times M$. In that case, the preimage of the diagonal Δ_M yields a submanifold with corners $W \subset \Delta^p \times \Delta^q$. For

each $(x, y) \in W$, the loops $u(x, -)$ and $v(y, -)$ are composable, as they have common base points $u(x, 0) = v(y, 0)$. Therefore, taking a triangulation of W and the Dold sign as in 2.8, we get a $(p + q - n)$ -chain of composable loops, which we call the *loop intersection product*:

$$u * v := (-1)^{n(n-q)}(u \times v)|W \times S^1.$$

Performing the composition (in some order, u before v) yields a $(p + q - n)$ -chain in LM , called the *Chas-Sullivan product* or *loop product* $u \cdot_{CS} v$. Of course, it depends on the choice of the triangulation of W . This operation extends linearly to the transverse bi-chains. When performing it on a bi-cycle, the result is a cycle in LM whose homology class in $H_{p+q-n}(LM; \mathbb{Z}_{or})$ is well-defined. Strictly speaking the system of coefficients is $ev_0^*(\mathbb{Z}_{or})$, which we write \mathbb{Z}_{or} for making short and we are doing the same each time a loop space is in question. More precisely, we have the following proposition.

Proposition 3.3. *Let $[\xi] \in H_p(LM; \mathbb{Z}_{or})$ and $[\eta] \in H_q(LM; \mathbb{Z}_{or})$. These classes can be represented by cycles in LM such that $\xi \times \eta$ is a transverse bi-cycle. The class of $\xi \cdot_{CS} \eta$ is uniquely defined in $H_{p+q-n}(LM; \mathbb{Z}_{or})$.*

Proof. Starting with arbitrary representatives of the given homology classes in LM , lemma 2.6 is applied: it produces a homotopy of their images by the evaluation map making them a transverse bi-cycle in $M \times M$. The lifting homotopy property (lemma 3.2) allows one to make $\xi \times \eta$ a transverse bi-cycle in $LM \times LM$. For this representative, the loop product $\xi \cdot_{CS} \eta$ is well defined. If another representative $\xi' \times \eta'$ is used, one can prove that $\xi' \times \eta'$ and $\xi \times \eta$ are joined by a transverse homology, that is a transverse bi-chain in $LM \times LM$. It is nothing but a loop version of the uniqueness lemma 2.7 and it can be deduced from the latter by applying the lifting homotopy property. As a consequence the homology class of $\xi \cdot_{CS} \eta$ is well defined. \diamond

Remark 3.4. Of course the composition of smooth loops produces a piecewise smooth loop only. There are two ways for correcting this default. One consists of doing a smoothing (boundary preserving) homotopy. The other consists of using $LM = C^0(S^1, M)$ equipped with a mixed simplicial structures: a k -simplex will be a continuous map $u : \Delta^k \times S^1 \rightarrow M$ whose restriction to $\Delta^k \times 0$ is smooth.

In the sequel we use the following simplified notation: $\xi \cdot \eta := \xi \cdot_{CS} \eta$, which is defined when the bi-cycle $\xi \times \eta$ is transverse, and $[\xi] \cdot [\eta] := [\xi \cdot_{CS} \eta]$ which is well defined. Actually, there is a 1-parameter family of compositions $- \cdot_s -$, $s \in [0, 1]$, defined as follows. Two loops u and v are said to be s -composable when $u(s) = v(0)$; in that case the composed loop $u \cdot_s v$ is made of $u|_{[0, 1-s]} * v * u|_{[1-s, 1]}$. When $s = 0$, it is the usual composition and when $s = 1$ we have $u \cdot_1 v = v \cdot u$ (notice that if two loops are 0-composable, they are also 1-composable).

Proposition 3.5. *At the homology level, the loop product is commutative up to sign. Precisely, if ξ and η are respectively a p -cycle and a q -cycle of LM , then $[\xi] \cdot [\eta] = (-1)^{(p-n)(q-n)}[\eta] \cdot [\xi]$. The loop product is also associative.*

If the regrading $\mathbb{H}_* = H_{*+n}$ is applied, then $\mathbb{H}_*(LM, \mathbb{Z}_{or})$ when endowed with the loop product becomes a graded commutative and associative ring.

Proof. We assume that the bi-cycle $\xi \times \eta$ is transverse. We first prove:

$$[\xi] \cdot [\eta] = [\xi \cdot_1 \eta].$$

Writing $\xi \times \eta = \sum n_{ij} u_i \times v_j$, one can make a boundary preserving homotopy of ξ so that, for every ij , the following map

$$(x, y, s) \in \Delta^p \times \Delta^q \times [0, 1] \mapsto (u_i(x, s), v_j(y, 0), s) \in M \times M \times [0, 1]$$

be transverse to $\Delta_M \times [0, 1]$. This transversality yields us a $(p + q - n + 1)$ -chain ω in LM whose boundary is $\partial\omega = \xi \cdot_1 \eta - \xi \cdot \eta$.

Thus we are reduced to prove:

$$[\xi \cdot_1 \eta] = (-1)^{(p-n)(q-n)} [\eta] \cdot [\xi].$$

On both sides of this equality the composition is the same. The only difference comes from the orientation of the manifold W_{ij} associated to each transverse bi-simplex $u_i \times v_j$ appearing in $\xi \times \eta$. The permutation of both factors in this bi-simplex induces a change $(-1)^n$ in the co-orientation of the diagonal Δ_M and a change $(-1)^{pq}$ due to the order of the factors Δ^p and Δ^q in the source. Moreover, the Dold sign $(-1)^{n(n-q)}$ is changed to $(-1)^{n(n-p)}$. All together, the sign rule follows.

If three chains ξ, η, ζ (of respective degrees p, q, r) are given in LM , $\xi \times \eta \times \zeta$ is easily defined to be transverse. If they are cycles and if the triple is transverse, the triple composition $[\xi] \cdot [\eta] \cdot [\zeta]$ is well defined in $H_{p+q+r-2n}(LM; \mathbb{Z}_{or})$. Moreover, in that situation the following facts are easily checked:

- $\xi \times \eta$ is transverse;
- $(\xi * \eta) \times \zeta$ is transverse (where $*$ stands for the loop intersection product);
- $(\xi \cdot \eta) \cdot \zeta$ coincides with $\xi \cdot \eta \cdot \zeta$ up to a canonical reparametrization of the circle.

The last item yields the associativity in homology once it is observed that the same is true for other bracketing. \diamond

4. A MULTIPLICATIVE SPECTRAL SEQUENCE

In this section, using our definition of the loop product, we discuss multiplicative properties which have been stated and proved by Mark Goresky and Nancy Hingston in [9], §12 (up to the coefficient system). The setting is the one that R. Bott first considered in his seminal paper [1], where he studied the standard n -sphere, and in the next paper [2]. We summarize his results as follows. Let M be an n -dimensional closed Riemannian manifold whose all primitive geodesics are simple loops with the same length ($= 1$, say). Denote Λ the space of loops parametrized proportionally to arc length. The class of regularity is not very important here. For instance take piecewise smooth loops; a good class is the Sobolev class $H^1(S^1, M)$. Let $\ell^2 : \Lambda \rightarrow \mathbb{R}$ be the squared length. Bott proved that it is a nondegenerate function (now called a Morse-Bott function) and he calculated the index of the critical points (that is, the closed geodesics). For $p \in \mathbb{N}$, let Λ_p be the subspace of loops of length $\leq p$ and Σ_p be the space of geodesics of length p . A geodesic in Σ_p is just a primitive geodesic which is traversed p times. As a manifold

$\Sigma_0 \cong M$ and, for $p \geq 1$, $\Sigma_p \cong UM$ where UM stands for the unitary tangent space to M . Let α_p be the index of the Hessian of ℓ^2 at any point of Σ_p ; obviously $\alpha_0 = 0$. Bott proved the iteration formula:

$$\alpha_p = p\alpha_1 + (p-1)(n-1).$$

Moreover he calculated (with \mathbb{Z}_2 coefficients) the spectral sequence derived from the filtration $\Lambda_0 \subset \Lambda_1 \subset \dots$ of Λ . We are going to consider the same spectral sequence, up to some regrading.

Definition 4.1. A spectral sequence $\{E_{p,q}^r, d^r\}_{r \geq 1}$ is said to be multiplicative when it is endowed with a product $E_{p,q}^r \otimes E_{p',q'}^r \rightarrow E_{p+p',q+q'}^r$ such that:

1) d^r is a derivation in the graded sense:

$$d^r(x \cdot y) = d^r(x) \cdot y + (-1)^{|x|} x \cdot d^r(y),$$

where $|\cdot|$ stands for the total degree $|x| = p + q$ when $x \in E_{p,q}^r$;

2) the product on $E_{*,*}^{r+1} \cong H_*(E_{*,*}^r, d^r)$ is induced by the one of $E_{*,*}^r$ on the ideal $\ker d^r$.

Following Chataur-Le Borgne in [4], we set

$$\mathbb{E}_{p,q}^1 = E_{p,q+n}^1 = H_{p+q+n}(\Lambda_p, \Lambda_{p-1}; ev_0^*(\mathbb{Z}_{or})).$$

The differential d^1 is the connecting homomorphism in the long exact sequence of the triple $(\Lambda_p, \Lambda_{p-1}, \Lambda_{p-2})$. More generally, $d^r : E_{*,*}^r \rightarrow E_{*,*}^r$ is defined by the usual algorithm associated to the filtration $\dots \Lambda_{p-1} \subset \Lambda_p \subset \Lambda_{p+1} \subset \dots \Lambda$ and the induced filtration on our chain sub-complex (see [5], Chap. 2). Notice that page 0 exists in this spectral sequence, but there is no multiplicative structure on it.

Proposition 4.2. The loop product endows $\{\mathbb{E}_{*,*}^r\}_{r \geq 1}$ with a multiplicative structure.

Proof. Here the system of coefficients $ev_0^*(\mathbb{Z}_{or})$ is understood. We first look at the wanted properties on page $r = 1$. When ξ is a relative i -cycle of the pair $(\Lambda_p, \Lambda_{p-1})$, η is a relative j -cycle of the pair $(\Lambda_{p'}, \Lambda_{p'-1})$ and $\xi \times \eta$ is transverse, the loop intersection product $\xi * \eta$ is a $(i + j - n)$ -chain of composable loops. By performing the composition we get a chain c of loops of length $\leq p + p'$ since the length of the composed loop is just the sum of the lengths of the entries. The boundary of c is a cycle in $\Lambda_{p+p'-1}$. The class of $[\xi] \cdot [\eta]$ is well defined in $H_{i+j-n}(\Lambda_{p+p'}, \Lambda_{p+p'-1})$. After regrading, this product is well adapted to the bi-degrees. There remains to check that d^1 is a derivation.

We look first at a transverse bi-simplex $u \times v$ of bi-degree (i, j) in $\Lambda \times \Lambda$. Let $W \subset \Delta^i \times \Delta^j$ be the preimage of the diagonal Δ_M by $ev_0(u) \times ev_0(v)$. Its boundary is made of two parts:

- $\partial_1 W := W \cap (\partial \Delta^i \times \Delta^j)$
- $\partial_2 W := W \cap (\Delta^i \times \partial \Delta^j)$

Each part gives rise to an $(i + j + n - 1)$ -chain in Λ which is a part of $\partial(u \cdot v)$. As in the proof of lemma 2.9 the second chain is endowed with the sign $(-1)^i$ according to the formula for the total boundary which is written after definition 2.4. Taking the Dold sign into account yields:

$$\partial(u \cdot v) = (\partial u) \cdot v + (-1)^{n-i} u \cdot (\partial v).$$

By summing such a formula over all bi-chains forming $\xi \times \eta$, we get:

$$\partial(\xi \cdot \eta) = (\partial \xi) \cdot \eta + (-1)^{n-i} \xi \cdot (\partial \eta).$$

After regrading, it becomes a derivation formula. When ξ and η are relative cycles as above, $d^1([\xi] \cdot [\eta]) = (d^1[\xi]) \cdot [\eta] + (-1)^{|\xi|}[\xi] \cdot d^1[\eta]$. Hence property 1 from definition 4.1 for $r = 1$.

The product on $\mathbb{E}_{*,*}^2$ is defined by taking ξ , a relative cycle of the pair $(\Lambda_p, \Lambda_{p-2})$, and η , a relative cycle in the pair $(\Lambda_{p'}, \Lambda_{p'-2})$, where $\xi \times \eta$ is transverse. Thus the chain of composed loops is a relative cycle of the pair $(\Lambda_{p+p'}, \Lambda_{p+p'-2})$. It can be checked that elements in $\ker d^1$ are representable by such cycles. Hence property 2 from definition 4.1 follows for $r = 1$.

It can be argued in the same way for every $r \geq 1$ once one remembers the definition of $E_{p,*}^r$ associated to the filtration of Λ . The product on $\mathbb{E}_{*,*}^r$ is defined by taking ξ , a relative cycle of the pair $(\Lambda_p, \Lambda_{p-r})$, and η , a relative cycle of the pair $(\Lambda_{p'}, \Lambda_{p'-r})$, where $\xi \times \eta$ is transverse. The boundary operator d^r is induced by ∂ and is not affected by the value of r . After regrading d^r , like d^1 , is a derivation and the product which is induced on its homology is the one of $E_{*,*}^{r+1}$. \diamond

4.3. The Thom isomorphism.

There is a Morse-Bott version of the famous Morse lemma. It yields a "normal" form for the Morse-Bott function near a critical manifold. We are going to apply it to the function $\ell^2 : \Lambda \rightarrow \mathbb{R}$ near the critical manifold Σ_p whose index is α_p . If one feels uncomfortable by applying this lemma in infinite dimension, one can take a finite dimensional approximation of Λ near Σ_p by considering the space of geodesics polygons (still parametrized proportionally to arc length) with Np edges of equal length, where $1/N$ is less than the injectivity radius. Let E_p^- be the vector bundle of rank α_p over Σ_p generated by the eigenvectors of the Hessian of ℓ^2 (with respect to some Riemannian metric on Λ) whose eigenvalues are negative; it is a sub-bundle of $T\Lambda|_{\Sigma_p}$. With notation borrowed from [9], let $\Sigma_p^- = \exp(E_p^-)$ denote the unstable manifold of Σ_p with respect to the gradient of ℓ^2 ; we are mainly interested in its germ along the critical manifold. Finally Λ_p^- denotes the open set of loops whose length is less than p . We have $\Sigma_p^- \setminus \Sigma_p \subset \Lambda_p^-$ (as a consequence of Taylor expansion).

The *Morse-Bott lemma* states:

- (1) Λ_p^- retracts by deformation onto Λ_{p-1} ;
- (2) Λ_p^- retracts by deformation onto $\Lambda_{p-1} \cup \Sigma_p^-$.

As a consequence, the inclusions of pairs induce the following isomorphisms:

$$H_*(\Lambda_p, \Lambda_{p-1}) \cong H_*(\Lambda_p, \Lambda_p^-) \cong H_*(\Sigma_p^-, \Sigma_p^- \setminus \Sigma_p).$$

Here the system of coefficients is $ev_0^*(\mathbb{Z}_{or})$. The orientation of the fibre bundle E_p^- being twisted as $ev_0^*(\mathbb{Z})$ (see [9] Prop. 12.2), we are ready to apply the Thom isomorphism:

$$h_p : H_*(\Sigma_p; \mathbb{Z}) \cong H_{*+\alpha_p}(\Sigma_p^-, \Sigma_p^- \setminus \Sigma_p; ev_0^*(\mathbb{Z}_{or})),$$

which can also be read, when $p > 0$,

$$h_p : H_*(UM; \mathbb{Z}) \cong H_{*+\alpha_p}(\Lambda_p, \Lambda_{p-1}; ev_0^*(\mathbb{Z}_{or})).$$

Notice also that the Gysin morphism

$$H_*(M; \mathbb{Z}_{or}) \rightarrow H_{*+n-1}(UM; \mathbb{Z})$$

makes $H_*(UM; \mathbb{Z})$ a $H_*(M; \mathbb{Z}_{or})$ -module for the intersection product. In the next proposition the coefficients are not written; they are meant as we just specified them.

Proposition 4.4. (Goresky-Hingston, [9] Theorem 12.5). *The Thom isomorphisms carry the intersection product of $H_*(UM)$ into the loop product of $\bigoplus_{p>0} H_{*+\alpha_p}(\Lambda_p, \Lambda_{p-1})$. Moreover, they carry the $H_*(M)$ -module structure of $H_*(UM)$ to the $H_*(\Lambda_0)$ -module structure of $H_*(\Lambda_p, \Lambda_{p-1})$.*

The proof below mainly follows the same line as [9]. But it is based on the notion of loop product that we have introduced in the previous section.

Proof. There are several steps.

A) One can factorize the intersection product int of $H_*(UM)$ in the following way:

$$\begin{array}{ccc} H_i(UM) \otimes H_j(UM) & \xrightarrow{int_0} & H_{i+j-n}(UM \times_M UM) \\ \text{\scriptsize } int \downarrow & & \downarrow \text{\scriptsize } int_1 \\ H_{i+j-(2n-1)}(UM) & \xrightarrow{=} & H_{i+j-(2n-1)}(UM \times_{UM} UM) \end{array}$$

Indeed, starting with a transverse bi-cycle $\xi \times \eta$ we may first intersect it with the fibered product $UM \times_M UM \subset UM \times UM$ yielding an $(i+j-n)$ -cycle ζ (this induces the morphism int_0), which, in turn, is transverse to the diagonal $\Delta_{UM} = UM \times_{UM} UM$ (this induces the morphism int_1).

B) Since $ev_0|_{\Sigma_p}$ is a smooth submersion, the fibered product $\Sigma_{p,p'} := \Sigma_p \times_M \Sigma_{p'}$ is a smooth manifold. A point of it is a pair of closed geodesics of respective lengths p and p' which are composable as loops. In general the composed loop is not a geodesic.

Similarly, since Σ_p^- is tangent to E_p^- , a fibre bundle over Σ_p , then $ev_0|_{\Sigma_p^-}$ is also a submersion onto M near Σ_p . Thus $\Sigma_{p,p'}^- := \Sigma_p^- \times_M \Sigma_{p'}^-$ is smooth near $\Sigma_{p,p'}$. The tangent space to $\Sigma_{p,p'}^-$ along $\Sigma_{p,p'}$ is the restriction of $E_p^- \times E_{p'}^-$, a vector bundle of rank $\alpha_p + \alpha_{p'}$. So we have a Thom isomorphism:

$$h_{p,p'} : H_*(\Sigma_{p,p'}) \cong H_{*+\alpha_p+\alpha_{p'}}(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \setminus \Sigma_{p,p'}).$$

At the chain level of small simplices, the Thom isomorphism is generated by the following Thom “extension”: take a small simplex in the base of a disk bundle (which hence is trivial over the considered small simplex) and cross it with the fiber. If $\xi \times \eta$ is a bi-cycle in $\Sigma_p \times \Sigma_{p'}$ transverse to $\Sigma_{p,p'}$, then the Thom extension $\tilde{\xi} \times \tilde{\eta}$ is transverse to $\Sigma_{p,p'}^-$ and its intersection with $\Sigma_{p,p'}^-$ is the Thom extension of the intersection cycle in the base $\Sigma_{p,p'}$. This proves that the Thom isomorphism carries the loop intersection product $H_i(\Sigma_p) \otimes H_j(\Sigma_{p'}) \rightarrow H_{i+j-n}(\Sigma_{p,p'})$ to the suitable relative version of the loop intersection product $H_{i+\alpha_p}(\Sigma_p^-) \otimes H_{j+\alpha_{p'}}(\Sigma_{p'}^-) \rightarrow H_{i+j+\alpha_p+\alpha_{p'}-n}(\Sigma_{p,p'}^-)$. Note that, after identification, the first morphism is nothing but int_0 from A).

C) We observe that $\Sigma_{p+p'}$ lifts (by a section of the composition map) as a submanifold of codimension $n-1$ in $\Sigma_{p,p'}$. Indeed, any smooth geodesic of length $p+p'$ splits uniquely into two geodesics of respective lengths p and p' . Conversely, $(\gamma, \gamma') \in \Sigma_{p,p'}$ belongs to this lifting

of $\Sigma_{p+p'}$ if and only if the initial velocities $\dot{\gamma}(0)$ and $\dot{\gamma}'(0)$ are positively proportional, which is a condition of codimension $n - 1$. Hence the composition is a map of pair:

$$comp : (\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \setminus \Sigma_{p+p'}) \rightarrow (\Lambda_{p+p'}, \Lambda_{p+p'} \setminus \Sigma_{p+p'}).$$

The normal bundle to $\Sigma_{p+p'}$ in $\Sigma_{p,p'}^-$, denoted by ν , is the direct sum $E_p^- \oplus E_{p'}^- \oplus \nu_{p,p'}$ where $\nu_{p,p'}$ denotes the normal bundle to $\Sigma_{p+p'}$ in $\Sigma_{p,p'}$. The rank of ν is $\alpha_p + \alpha_{p'} + n - 1$. According to Bott's iteration formula, it equals $\alpha_{p+p'}$, which is the rank of $E_{p+p'}^-$.

If ξ is a relative cycle in $(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \setminus \Sigma_{p+p'})$ transverse to $\Sigma_{p+p'}$, its trace in the pair $(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \setminus \Sigma_{p+p'})$ is the Thom extension of its intersection with $\Sigma_{p+p'}$. In other words we have the following commutative diagram:

$$\begin{array}{ccc} H_k(\Sigma_{p,p'}) & \xrightarrow{int_1} & H_{k-n+1}(\Sigma_{p+p'}) \\ \downarrow & & \downarrow \\ H_{k+\alpha_p+\alpha_{p'}}(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \setminus \Sigma_{p+p'}) & \longrightarrow & H_{k-n+1+\alpha_{p+p'}}(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \setminus \Sigma_{p+p'}), \end{array}$$

where the vertical arrows are the respective Thom isomorphisms. The proof of proposition 4.4 is not yet achieved because we are still at the level of composable loops and the composition has not yet been performed.

D) Some difficulty comes here from the fact that the composition map $comp : \Sigma_{p,p'}^- \rightarrow \Lambda_{p+p'}$ is singular along $\Sigma_{p+p'}$ in the direction of $\nu_{p,p'}$ since $comp$ maps $\Sigma_{p,p'}$ into the critical level set of ℓ whose value is $p + p'$. We are going to construct:

- a linear embedding $\varphi : \nu \rightarrow T\Lambda|_{\Sigma_{p+p'}}$, over the identity of $\Sigma_{p+p'}$, such that $Hess(\ell^2) \circ \varphi$ be negative definite;
- a homotopy from $comp$ to $exp \circ \varphi \circ exp^{-1}$ among the maps of pair

$$(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \setminus \Sigma_{p+p'}) \rightarrow (\Lambda_{p+p'}, \Lambda_{p+p'} \setminus \Sigma_{p+p'}).$$

If such items exist, $comp_* : H_*(\Sigma_{p,p'}^-, \Sigma_{p,p'}^- \setminus \Sigma_{p+p'})$ commutes with the respective Thom isomorphisms, as it is true for φ_* , and the proof of proposition 4.4 is finished.

We choose φ to be the identity of the factor $E_p^- \oplus E_{p'}^-$. This makes sense as $(E_p^-)_\gamma$ (resp. $(E_{p'}^-)_{\gamma'}$) embeds canonically into $(T\Lambda_{p+p'})_{\gamma*\gamma'}$ (thought of as the set of vector fields tangent to M along $\gamma * \gamma'$): it is a set of tangent vector fields which are vanishing along γ (resp. along γ').

Let $(\gamma, \gamma') \in \Sigma_{p+p'} \subset \Sigma_{p,p'}$ and X be a vector in $\nu_{(\gamma, \gamma')}$. This vector indicates an infinitesimal deformation of (γ, γ') among the pairs of composable closed smooth geodesics separating the directions of their initial velocities. Precisely, there is a one-parameter family (γ_u, γ'_u) , $u \in [0, \varepsilon)$, of pairs of closed geodesics such that $\gamma_u(0) = \gamma(0) = \gamma'(0) = \gamma'_u(0)$ and $\frac{d}{du}(\dot{\gamma}_u(0) - \dot{\gamma}'_u(0))|_{u=0} = X$ (up to a positive scalar); here we identify $\nu_{(\gamma, \gamma')}$ with the fibre $UM_{\gamma(0)}$. Instead of taking this deformation which leaves the length unchanged, we consider the following shortening deformation made of geodesic triangles T_u : the first edge in T_u is $\gamma_u(t)$, $t \in [0, 1 - \varepsilon]$, the second edge joins geodesically the point $\gamma_u(1 - \varepsilon)$ to the nearby point $\gamma'_u(\varepsilon)$ and the third edge is $\gamma'_u(t)$, $t \in [\varepsilon, 1]$ (the triangle is parametrized proportionally to arc length). We define $\varphi(X)$ to be the infinitesimal generator of this family. By estimating $\ell(T_u)$ it is easily seen that $Hess(\ell^2)(\varphi(X)) < 0$. Moreover $\varphi(X)$ is independent of $E_p^- \oplus E_{p'}^-$ since, as

a tangent vector field to M along the composed loop $\gamma * \gamma'$, it does not vanish at the junction point $\gamma(1) = \gamma'(0)$. This proves item a). For getting item b) it is sufficient to make ϵ tend to 0. \diamond

By taking the different regradings into account, the Thom isomorphism yields when $p > 0$:

$$\mathbb{E}_{p,q}^1 \cong H_{p+q+n-\alpha_p}(\Sigma_p, \mathbb{Z}) \cong H_{p+q+n-\alpha_p}(UM, \mathbb{Z}) \cong \mathbb{H}_{p+q+n-\alpha_p-(2n-1)}(UM, \mathbb{Z}),$$

where $\mathbb{H}_*(UM; \mathbb{Z})$ denotes the regraded intersection ring of the $(2n-1)$ -dimensional manifold UM . Similarly, after regrading the Gysin morphism, $H_*(M; \mathbb{Z}_{or}) \rightarrow H_{*+n-1}(UM; \mathbb{Z})$, becomes a morphism of degree 0, $\mathbb{H}_*(M; \mathbb{Z}_{or}) \rightarrow \mathbb{H}_*(UM; \mathbb{Z})$, making $\mathbb{H}_*(UM; \mathbb{Z})$ a $\mathbb{H}_*(M; \mathbb{Z}_{or})$ -module for the intersection product.

The multiplicative structure on $\mathbb{E}_{*,*}^1$ can be interpreted in terms of $\mathbb{H}_*(UM)$. The following statement is due to Chataur-Le Borgne in [4] (up to the coefficient system).

Proposition 4.5.

1) *There is an isomorphism of bigraded rings:*

$$\mathbb{E}_{*,*}^1 \cong \mathbb{H}_*(M; \mathbb{Z}_{or}) \oplus \mathbb{H}_*(UM; \mathbb{Z})[T]_{\geq 1}.$$

Here $\mathbb{E}_{*,*}^1$ is endowed with the bigraded ring structure yielded by proposition 4.2. The intersection rings $\mathbb{H}_*(M)$ and $\mathbb{H}_*(UM)$ have bi-degree $(0, *)$. The new variable T has bi-degree $(1, \alpha_1 + n - 2)$ and appears only at a positive power. Regarding $\mathbb{H}_*(UM)$ as a $\mathbb{H}_*(M)$ -module, the right hand side has a well-defined ring structure.

2) *The differential d^1 on page 1 vanishes at every place.*

3) *The page ∞ inherits the same isomorphism of bigraded rings as page 1:*

$$\mathbb{E}_{*,*}^\infty \cong \mathbb{H}_*(M; \mathbb{Z}_{or}) \oplus \mathbb{H}_*(UM; \mathbb{Z})[T]_{\geq 1}.$$

Proof. 1) We read the first page of the spectral sequence, $\mathbb{E}_{*,*}^1$, via the Thom isomorphism taking proposition 4.4 into account. For instance, we have (without writing the coefficients):

$$\mathbb{H}_0(UM) = H_{2n-1}(UM) = H_{2n-1+\alpha_1}(UM) \cong E_{1,2n-2+\alpha_1}^1 = \mathbb{E}_{1,n-2+\alpha_1}^1$$

and T is nothing but the image of the unity $\mu \in \mathbb{H}_0(UM)$ (that is, the fundamental class of UM) through the Thom isomorphism. Once the desired ring isomorphism is specified on T it extends globally using the multiplicative property of the Thom isomorphism.

2) As $\mathbb{H}_*(\Lambda_0)$ is a direct factor in $\mathbb{H}_*(\Lambda)$, $d^1 : \mathbb{E}_{1,*}^1 \rightarrow \mathbb{E}_{0,*}^1$ has to vanish. In particular, $d^1(T) = 0$. As d^1 is a derivation (prop. 4.2), it vanishes everywhere.

3) As a consequence, page 2 of the spectral sequence is isomorphic to page 1 as a bi-graded ring. Therefore the differential d^2 vanishes for the same reason as d^1 . Going recursively on the successive pages yields the conclusion. \diamond

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