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► **To cite this version:**

Yoshinori Morimoto, Chao-Jiang Xu. Ultra-analytic effect of Cauchy problem for a class of kinetic equations. *Journal of Differential Equations*, 2009, 247, pp.597-617. hal-00368263

**HAL Id: hal-00368263**

**<https://hal.science/hal-00368263>**

Submitted on 17 Mar 2009

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# Ultra-analytic effect of Cauchy problem for a class of kinetic equations

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## Abstract

The smoothing effect of the Cauchy problem for a class of kinetic equations is studied. We firstly consider the spatially homogeneous nonlinear Landau equation with Maxwellian molecules and inhomogeneous linear Fokker-Planck equation to show the ultra-analytic effects of the Cauchy problem. Those smoothing effect results are optimal and similar to heat equation. In the second part, we study a model of spatially inhomogeneous linear Landau equation with Maxwellian molecules, and show the analytic effect of the Cauchy problem.

Key words: Landau equation, Fokker-Planck equation, ultra-analytic effect of Cauchy problem.

AMS Classification: 35A05, 35B65, 35D10, 35H20, 76P05, 84C40

## 1. Introduction

It is well known that the Cauchy problem of heat equation possesses the ultra-analytic effect phenomenon, namely, if  $u(t, x)$  is the solution of the following Cauchy problem :

$$\begin{cases} \partial_t u - \Delta_x u = 0, & x \in \mathbb{R}^d; \quad t > 0 \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^d), \end{cases}$$

then under the uniqueness hypothesis, the solution  $u(t, \cdot) = e^{t\Delta_x} u_0$  is an ultra-analytic function for any  $t > 0$ . We give now the definition of function spaces  $\mathcal{A}^s(\Omega)$  where  $\Omega$  is an open subset of  $\mathbb{R}^d$ .

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*Preprint submitted to Elsevier*

*March 17, 2009*

**Definition 1.1.** For  $0 < s < +\infty$ , we say that  $f \in \mathcal{A}^s(\Omega)$ , if  $f \in C^\infty(\Omega)$ , and there exists  $C > 0, N_0 > 0$  such that

$$\|\partial^\alpha f\|_{L^2(\Omega)} \leq C^{|\alpha|+1}(\alpha!)^s, \quad \forall \alpha \in \mathbb{N}^d, |\alpha| \geq N_0.$$

If the boundary of  $\Omega$  is smooth, by using Sobolev embedding theorem, we have the same type estimate with  $L^2$  norm replaced by any  $L^p$  norm for  $2 < p \leq +\infty$ . On the whole space  $\Omega = \mathbb{R}^d$ , it is also equivalent to

$$e^{c_0(-\Delta)^{\frac{1}{2s}}}(\partial^{\beta_0} f) \in L^2(\mathbb{R}^d)$$

for some  $c_0 > 0$  and  $\beta_0 \in \mathbb{N}^d$ , where  $e^{c_0(-\Delta)^{\frac{1}{2s}}}$  is the Fourier multiplier defined by

$$e^{c_0(-\Delta)^{\frac{1}{2s}}} u(x) = \mathcal{F}^{-1}\left(e^{c_0|\xi|^{\frac{1}{s}}}\hat{u}(\xi)\right).$$

If  $s = 1$ , it is usual analytic function. If  $s > 1$ , it is Gevrey class function. For  $0 < s < 1$ , it is called ultra-analytic function. Notice that all polynomial functions are ultra-analytic for any  $s > 0$ .

It is obvious that if  $u_0 \in L^2(\mathbb{R}^d)$  then, for any  $t > 0$  and any  $k \in \mathbb{N}$ , we have  $u(t, \cdot) = e^{-t(-\Delta_x)^k} u_0 \in \mathcal{A}^{\frac{1}{2k}}(\mathbb{R}^d)$ , namely, there exists  $C > 0$  such that for any  $m \in \mathbb{N}$ ,

$$\begin{aligned} \|(t^m \partial_x^{2km})u(t, \cdot)\|_{L^2(\mathbb{R}^d)} &\leq C^k m \|t(-\Delta_x)^k u(t, \cdot)\|_{L^2(\mathbb{R}^d)} \\ &\leq \|u_0\|_{L^2(\mathbb{R}^d)} C^k m m! \leq \tilde{C}^{2km+1} ((2km)!)^{\frac{1}{2k}}, \end{aligned}$$

where  $\partial_x^{2km} = \sum_{|\alpha|=2km, \alpha \in \mathbb{N}^d} \partial_x^\alpha$ . We say that the diffusion operators  $(-\Delta_x)^k$  possess the ultra-analytic effect property if  $k > 1/2$ , the analytic effect property if  $k = 1/2$  and the Gevrey effect property if  $0 < k < 1/2$ .

We study the Cauchy problem for spatially homogeneous Landau equation

$$\begin{cases} f_t = Q(f, f) \equiv \nabla_v(\bar{a}(f) \cdot \nabla_v f - \bar{b}(f)f), & v \in \mathbb{R}^d, t > 0, \\ f|_{t=0} = f_0, \end{cases} \quad (1.1)$$

where  $\bar{a}(f) = (\bar{a}_{ij}(f))$  and  $\bar{b}(f) = (\bar{b}_1(f), \dots, \bar{b}_d(f))$  are defined as follows (convolution is w. r. t. the variable  $v \in \mathbb{R}^d$ )

$$\bar{a}_{ij}(f) = a_{ij} \star f, \quad \bar{b}_j(f) = \sum_{i=1}^d (\partial_{v_i} a_{ij}) \star f, \quad i, j = 1, \dots, d,$$

with

$$a_{ij}(v) = \left( \delta_{ij} - \frac{v_i v_j}{|v|^2} \right) |v|^{\gamma+2}, \quad \gamma \in [-3, 1].$$

We consider hereafter only the Maxwellian molecule case which corresponds to  $\gamma = 0$ . We introduce also the notation, for  $l \in \mathbb{R}$ ,  $L_l^p(\mathbb{R}^d) = \{f; (1 + |v|^2)^{l/2} f \in L^p(\mathbb{R}^d)\}$  is the weighted function space.

We prove the following ultra-analytic effect results for the nonlinear Cauchy problem (1.1).

**Theorem 1.1.** *Let  $f_0 \in L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$  and  $0 < T \leq +\infty$ . If  $f(t, x) > 0$  and  $f \in L^\infty([0, T]; L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d))$  is a weak solution of the Cauchy problem (1.1), then for any  $0 < t < T$ , we have*

$$f(t, \cdot) \in \mathcal{A}^{1/2}(\mathbb{R}^d),$$

and moreover, for any  $0 < T_0 < T$ , there exists  $c_0 > 0$  such that for any  $0 < t \leq T_0$

$$\|e^{-c_0 t \Delta_v} f(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq e^{\frac{d}{2}t} \|f_0\|_{L^2(\mathbb{R}^d)}. \quad (1.2)$$

In [17], they proved the Gevrey regularity effect of the Cauchy problem for linear spatially homogeneous non-cut-off Boltzmann equation. By a careful revision for the proof of Theorem 1.2 of [17], one can also prove that the solution of the Cauchy problem (1.10) in [17] belongs to  $\mathcal{A}^{\frac{1}{2\alpha}}(\mathbb{R}^d)$  for any  $t > 0$ , where  $0 < \alpha < 1$  is the order of singularity of collision kernel of Boltzmann operator. Hence, if  $\alpha \geq 1/2$ , there is also the ultra-analytic effect phenomenon. Now the above Theorem 1.1 shows that, for Landau equation, the ultra-analytic effect phenomenon holds in nonlinear case, which is an optimal regularity result.

The ultra-analytic effect property is also true for the Cauchy problem of the following generalized Kolmogorov operators

$$\begin{cases} \partial_t u + v \cdot \nabla_x u + (-\Delta_v)^\alpha u = 0, & (x, v) \in \mathbb{R}^{2d}; \quad t > 0 \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^{2d}), \end{cases}$$

where  $0 < \alpha < \infty$ , and the classical Kolmogorov operators is corresponding to  $\alpha = 1$ . By Fourier transformation, the explicit solution of the above Cauchy problem is given by

$$\hat{u}(t, \eta, \xi) = e^{-\int_0^t |\xi + s\eta|^{2\alpha} ds} \hat{u}_0(\eta, \xi + t\eta).$$

Since there exists  $c_\alpha > 0$  (see Lemma 3.1 below) such that

$$c_\alpha (t|\xi|^{2\alpha} + t^{2\alpha+1}|\eta|^{2\alpha}) \leq \int_0^t |\xi + s\eta|^{2\alpha} ds, \quad (1.3)$$

we have

$$e^{c_\alpha(t(-\Delta_v)^\alpha + t^{2\alpha+1}(-\Delta_x)^\alpha)} u(t, \cdot, \cdot) \in L^2(\mathbb{R}^{2d}),$$

i. e.  $u(t, \cdot, \cdot) \in \mathcal{A}^{1/(2\alpha)}(\mathbb{R}^{2d})$  for any  $t > 0$ .

Notice that this ultra-analytic (if  $\alpha > 1/2$ ) effect phenomenon is similar to heat equations of  $(x, v)$  variables. That is, this means  $v \cdot \nabla_x + (-\Delta_v)^\alpha$  is equivalent to  $(-\Delta_x)^\alpha + (-\Delta_v)^\alpha$  by time evolution in ‘‘some sense’’, though the equation is only transport for  $x$  variable.

We consider now a more complicate equation, the Cauchy problem for linear Fokker-Planck equation :

$$\begin{cases} f_t + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + v f), & (x, v) \in \mathbb{R}^{2d}, \quad t > 0; \\ f|_{t=0} = f_0. \end{cases} \quad (1.4)$$

This equation is a natural generalization of classical Kolmogorov equation, and a simplified model of inhomogeneous Landau equation (see [20, 21]). The local property of

this equation is the same as classical Kolmogorov equation since the add terms  $\nabla_v \cdot (vf)$  is a first order term, but for the studies of kinetic equation,  $v$  is velocity variable, and hence it is in whole space  $\mathbb{R}_v^d$ . Then there occurs additional difficulty for analysis of this equation.

The definition of weak solution in the function space  $L^\infty([0, T[; L^2(\mathbb{R}_{x,v}^{2d}) \cap L_1^1(\mathbb{R}_{x,v}^{2d}))$  for the Cauchy problem is standard in the distribution sense, where for  $1 \leq p < +\infty, l \in \mathbb{R}$

$$L_l^p(\mathbb{R}_{x,v}^{2d}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^{2d}); (1 + |v|^2)^{l/2} f \in L^p(\mathbb{R}_{x,v}^{2d}) \right\}.$$

The existence of weak solution is similar to full Landau equation (see [1, 13]). We get also the following ultra-analytic effect result.

**Theorem 1.2.** *Let  $f_0 \in L^2(\mathbb{R}_{x,v}^{2d}) \cap L_1^1(\mathbb{R}_{x,v}^{2d}), 0 < T \leq +\infty$ . Assume that  $f \in L^\infty([0, T[; L^2(\mathbb{R}_{x,v}^{2d}) \cap L_1^1(\mathbb{R}_{x,v}^{2d}))$  is a weak solution of the Cauchy problem (1.4). Then, for any  $0 < t < T$ , we have*

$$f(t, \cdot, \cdot) \in \mathcal{A}^{1/2}(\mathbb{R}^{2d}).$$

Furthermore, for any  $0 < T_0 < T$  there exists  $c_0 > 0$  such that for any  $0 < t \leq T_0$ , we have

$$\left\| e^{-c_0(t\Delta_v + t^3\Delta_x)} f(t, \cdot, \cdot) \right\|_{L^2(\mathbb{R}^{2d})} \leq e^{\frac{d}{2}t} \|f_0\|_{L^2(\mathbb{R}^{2d})}. \quad (1.5)$$

**Remark 1.1.** *The ultra-analyticity results of the above two theorems are optimal for the smoothness properties of solutions. From these results, we obtain a good understanding for the hypoellipticity of kinetic equations (see [11, 14]), and also the relationship, established by Villani [19] and Desvillettes-Villani [10], between the nonlinear Landau equation (with Maxwellian molecules) and the linear Fokker-Planck equation.*

We consider now the spatially inhomogeneous Landau equation

$$\begin{cases} f_t + v \cdot \nabla_x f = Q(f, f), & (x, v) \in \mathbb{R}^{2d}, \quad t > 0; \\ f|_{t=0} = f_0(x, v). \end{cases} \quad (1.6)$$

The problem is now much more complicate since the solution  $f$  is the function of  $(t, x, v)$  variables. We consider it here only in the linearized framework around the normalized Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}},$$

which is the equilibrium state because  $Q(\mu, \mu) = 0$ . Setting  $f = \mu + g$ , we consider the diffusion part of linear Landau collision operators,

$$Q(\mu, g) = \nabla_v \left( \bar{a}(\mu) \cdot \nabla_v g - \bar{b}(\mu)g \right)$$

where

$$\begin{aligned} \bar{a}_{ij}(\mu) &= a_{ij} \star \mu = \delta_{ij}(|v|^2 + 1) - v_i v_j, \\ \bar{b}_j(\mu) &= \sum_{i=1}^d (\partial_{v_i} a_{ij}) \star \mu = -v_j, \quad i, j = 1, \dots, d. \end{aligned}$$

In particular, it follows that

$$\sum_{ij=1}^d \bar{a}_{ij}(\mu) \xi_i \xi_j \geq |\xi|^2, \quad \text{for all } (v, \xi) \in \mathbb{R}^{2d}. \quad (1.7)$$

We then consider the following Cauchy problem

$$\begin{cases} g_t + v \cdot \nabla_x g = \nabla_v \cdot (\bar{a}(\mu) \cdot \nabla_v g - \bar{b}(\mu) g), & (x, v) \in \mathbb{R}^{2d}, \quad t > 0; \\ g|_{t=0} = g_0. \end{cases} \quad (1.8)$$

We can also look this equation as a linear model of spatially inhomogeneous Landau equation, which is much more complicate than linear Fokker-Planck equation (1.4), since the coefficients of diffusion part are now variables. The existence and  $C^\infty$  regularity of weak solution for the Cauchy problem have been considered in [1]. We prove now the following;

**Theorem 1.3.** *Let  $g_0 \in L^2(\mathbb{R}_{x,v}^{2d}) \cap L_2^1(\mathbb{R}_{x,v}^{2d})$ ,  $0 < T \leq +\infty$ . Assume that  $g \in L^\infty(]0, T[; L^2(\mathbb{R}_{x,v}^{2d}) \cap L_2^1(\mathbb{R}_{x,v}^{2d}))$  is a weak solution of the Cauchy problem (1.8). Then, for any  $0 < t < T$ , we have*

$$g(t, \cdot, \cdot) \in \mathcal{A}^1(\mathbb{R}^{2d}).$$

Furthermore, for any  $0 < T_0 < T$  there exist  $C, c > 0$  such that for any  $0 < t \leq T_0$ , we have

$$\left\| e^{c(t(-\Delta_v)^{1/2} + t^2(-\Delta_x)^{1/2})} g(t, \cdot, \cdot) \right\|_{L^2(\mathbb{R}^{2d})} \leq e^{Ct} \|g_0\|_{L^2(\mathbb{R}^{2d})}. \quad (1.9)$$

In this theorem, we only consider the analytic effect result for the Cauchy problem (1.8), neglecting the symmetric term  $Q(g, \mu)$  in the linearized operators of Landau collision operator (cf.,(1.15) of [1]) because of the technical difficulty, see the remark in the end of section 4.

There have been many results about the regularity of solutions for Boltzmann equation without angular cut-off and Landau equation, see [1, 2, 3, 6, 7, 9, 12, 15, 16] for the  $C^\infty$  smoothness results, and [4, 5, 8, 18, 17] for Gevrey regularity results for Boltzmann equation and Landau equation in both cases : the spatially homogeneous and inhomogeneous cases. As for the analytic and Gevrey regularities, we remark that the propagation of Gevrey regularities of solutions is investigated in [5] for full nonlinear spatially homogeneous Landau equations, including non-Maxwellian molecule case, and the local Gevrey regularity for all variables  $t, x, v$  is considered in [4] for some semi-linear Fokker-Planck equations. Comparing those results, the ultra-analyticity for  $x, v$  variables showed in Theorem 1.1 is strong although the Maxwellian molecule case is only treated. As a related result for spatially homogeneous Boltzmann equation in the Maxwellian molecule case, we refer [8], where the propagation of Gevrey and ultra-analytic regularity is studied uniformly in time variable  $t$ . Throughout the present paper, we focus the smoothing effect of the Cauchy problem, and the uniform smoothness estimate near to  $t = 0$ . Concerning further details of the analytic and Gevrey regularities of solutions for Landau equations and Boltzmann equation without angular cut-off, we refer the introduction of [5] and references therein.

## 2. Spatially homogeneous Landau equations

We consider the Cauchy problem (1.1) and prove Theorem 1.1 in this section. We refer to the works of C. Villani [19, 20] for the essential properties of homogeneous Landau equations. We suppose the existence of weak solution  $f(t, v) > 0$  in  $L^\infty(]0, T[; L^1_2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))$ . The conservation of mass, momentum and energy reads,

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t, v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv \equiv 0.$$

Without loss of generality, we can suppose that

$$\begin{aligned} \int_{\mathbb{R}^d} f(t, v) dv &= 1, & \text{unit mass} \\ \int_{\mathbb{R}^d} f(t, v) v_j dv &= 0, \quad j = 1, \dots, d; & \text{zero mean velocity} \\ \int_{\mathbb{R}^d} f(t, v) |v|^2 dv &= T_0, & \text{unit temperature} \\ \int_{\mathbb{R}^d} f(t, v) v_j v_k dv &= T_j \delta_{jk}, & \sum_j^d T_j = T_0 \\ T_j = \int_{\mathbb{R}^d} f(t, v) v_j^2 dv &> 0, \quad j = 1, \dots, d; & \text{directional temperatures.} \end{aligned}$$

Then we have,

$$\bar{a}_{jk}(f) = \delta_{jk}(|v|^2 + T_0 - T_j) - v_j v_k; \quad (2.1)$$

$$\bar{b}_j(f) = -v_j; \quad (2.2)$$

$$\sum_{j,k}^d \bar{a}_{jk}(f) \xi_j \xi_k \geq C_1 |\xi|^2, \quad \forall (v, \xi) \in \mathbb{R}^{2d}. \quad (2.3)$$

where  $C_1 = \min_{1 \leq j \leq d} \{T_0 - T_j\} > 0$ .

Now for  $N > \frac{d}{4} + 1$  and  $0 < \delta < 1/N$ ,  $c_0 > 0$ ,  $t > 0$ , set

$$G_\delta(t, |\xi|) = \frac{e^{c_0 t |\xi|^2}}{(1 + \delta e^{c_0 t |\xi|^2}) (1 + \delta c_0 t |\xi|^2)^N}.$$

Since  $G_\delta(t, \cdot) \in L^\infty(\mathbb{R}^d)$ , we can use it as Fourier multiplier, denoted by

$$G_\delta(t, D_v) f(t, v) = \mathcal{F}^{-1} \left( G_\delta(t, |\xi|) \hat{f}(t, \xi) \right).$$

Then, for any  $t > 0$ ,

$$G_\delta(t) = G_\delta(t, D_v) : L^2(\mathbb{R}^d) \rightarrow H^{2N}(\mathbb{R}^d) \subset C_b^2(\mathbb{R}^d).$$

The object of this section is to prove the uniform bound (with respect to  $\delta > 0$ ) of

$$\|G_\delta(t, D_v) f(t, \cdot)\|_{L^2(\mathbb{R}^d)}.$$

Since  $f(t, \cdot) \in L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$  is a weak solution, we can take

$$G_\delta(t)^2 f(t, \cdot) = G_\delta(t, D_v)^2 f(t, \cdot) \in H^{2N}(\mathbb{R}^d),$$

as test function in the equation of (1.1), whence we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|G_\delta(t)f(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 + \sum_{j,k=1}^d \int_{\mathbb{R}^d} \bar{a}_{jk}(f) \left( \partial_{v_j} G_\delta(t)f(t, v) \right) \\
& \quad \times \overline{\left( \partial_{v_k} G_\delta(t)f(t, v) \right)} dv \\
= & \frac{1}{2} \left( \left( \partial_t G_\delta(t) \right) f, G_\delta(t)f \right)_{L^2(\mathbb{R}^d)} + \sum_{j=1}^d \int_{\mathbb{R}^d} \left( \partial_{v_j} (v_j f(t, v)) \right) \overline{G_\delta(t)^2 f(t, v)} dv \\
& + \sum_{j,k=1}^d \int_{\mathbb{R}^d} \left\{ \bar{a}_{jk}(f) \left( G_\delta(t) \partial_{v_j} f(t, v) \right) - G_\delta(t) \left( \bar{a}_{jk}(f) \partial_{v_j} f(t, v) \right) \right\} \\
& \quad \times \overline{\left( \partial_{v_k} G_\delta(t)f(t, v) \right)} dv.
\end{aligned}$$

To estimate the terms in the above equality, we prove the following two propositions.

**Proposition 2.1.** *We have*

$$\begin{aligned}
C_1 \|\nabla_v G_\delta(t)f(t)\|_{L^2(\mathbb{R}^d)}^2 & \leq \sum_{j,k=1}^d \int_{\mathbb{R}^d} \bar{a}_{jk}(f) \left( \partial_{v_j} G_\delta(t, D_v)f(t, v) \right) \\
& \quad \times \overline{\left( \partial_{v_k} G_\delta(t, D_v)f(t, v) \right)} dv. \tag{2.4}
\end{aligned}$$

$$\left| \left( \left( \partial_t G_\delta(t) \right) f, G_\delta(t)f \right)_{L^2} \right| \leq c_0 \|\nabla_v G_\delta(t)f(t)\|_{L^2}^2. \tag{2.5}$$

$$\begin{aligned}
\operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} \left( \partial_{v_j} (v_j f(t, v)) \right) \overline{G_\delta(t)^2 f(t, v)} dv \\
\leq \frac{d}{2} \|G_\delta(t)f(t)\|_{L^2}^2 + 2c_0 t \|\nabla_v G_\delta(t)f(t)\|_{L^2}^2. \tag{2.6}
\end{aligned}$$

**Proof :** The estimate (2.4) is exactly the elliptic condition (2.3). By using the Fourier transformation, (2.5) is deduced from the following calculus

$$\partial_t G_\delta(t, |\xi|) = c_0 |\xi|^2 G_\delta(t, |\xi|) \left( \frac{1}{1 + \delta e^{c_0 t} |\xi|^2} - \frac{N\delta}{1 + \delta c_0 t |\xi|^2} \right) = c_0 |\xi|^2 G_\delta(t, |\xi|) J_{N,\delta}$$

where

$$|J_{N,\delta}| = \left| \frac{1}{1 + \delta e^{c_0 t} |\xi|^2} - \frac{N\delta}{1 + \delta c_0 t |\xi|^2} \right| \leq 1.$$

To treat (2.6), we use

$$\partial_{\xi_j} G_\delta(t, |\xi|) = 2c_0 t \xi_j G_\delta(t, |\xi|) J_{N,\delta}. \tag{2.7}$$



Then, we have

$$\begin{aligned}
& \operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} \left( \partial_{v_j} (v_j f(t, v)) \right) \overline{G_\delta(t, D_v)^2 f(t, v)} dv \\
&= -\operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} v_j G_\delta(t, D_v) f(t, v) \overline{\left( \partial_{v_j} G_\delta(t, D_v) f(t, v) \right)} dv \\
&\quad - \operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} \left( [G_\delta(t, D_v), v_j] f(t, v) \right) \overline{\left( \partial_{v_j} G_\delta(t, D_v) f(t, v) \right)} dv \\
&= \frac{d}{2} \|G_\delta(t) f(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 - \operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^d} \left( [G_\delta(t, D_v), v_j] f(t, v) \right) \\
&\quad \times \overline{\left( \partial_{v_j} G_\delta(t, D_v) f(t, v) \right)} dv.
\end{aligned}$$

Using Fourier transformation and (2.7), we have that for  $t > 0$ ,

$$\begin{aligned}
& - \sum_{j=1}^d \int_{\mathbb{R}^3} \left( [G_\delta(t, D_v), v_j] f(t, v) \right) \overline{\left( \partial_{v_j} G_\delta(t, D_v) f(t, v) \right)} dv \\
&= - \sum_{j=1}^d \int_{\mathbb{R}^d} \left( G_\delta(t, D_v) v_j f(t, v) - v_j G_\delta(t, D_v) f(t, v) \right) \overline{\left( \partial_{v_j} G_\delta(t, D_v) f(t, v) \right)} dv \\
&= \sum_{j=1}^d \int_{\mathbb{R}^d} \left\{ i \partial_{\xi_j} (G_\delta(t, |\xi|) \hat{f}(t, \xi)) - G_\delta(t, |\xi|) (i \partial_{\xi_j} \hat{f}(t, \xi)) \right\} G_\delta(t, |\xi|) \overline{i \xi_j \hat{f}(t, \xi)} d\xi \\
&= \sum_{j=1}^d \int_{\mathbb{R}^3} \left( \partial_{\xi_j} G_\delta(t, |\xi|) \right) \hat{f}(t, \xi) \xi_j G_\delta(t, |\xi|) \overline{\hat{f}(t, \xi)} d\xi \\
&= 2c_0 t \int_{\mathbb{R}^d} |\xi|^2 |G_\delta(t, |\xi|) \hat{f}(t, \xi)|^2 J_{N, \delta} d\xi \leq 2c_0 t \int_{\mathbb{R}^d} |\xi|^2 |G_\delta(t, |\xi|) \hat{f}(t, \xi)|^2 d\xi,
\end{aligned}$$

which give (2.6). The proof of Proposition 2.1 is now complete.

For the commutator term, the special structure of the operator implies

**Proposition 2.2.**

$$\begin{aligned}
& \sum_{j,k=1}^d \int_{\mathbb{R}^d} \left\{ \bar{a}_{jk}(f) \left( G_\delta(t, D_v) \partial_{v_j} f(t, v) \right) - G_\delta(t, D_v) \left( \bar{a}_{jk}(f) \partial_{v_j} f(t, v) \right) \right\} \\
&\quad \times \overline{\left( \partial_{v_k} G_\delta(t, D_v) f(t, v) \right)} dv = 0.
\end{aligned}$$

**Proof :** We introduce now polar coordinates on  $\mathbb{R}_\xi^d$  by setting  $r = |\xi|$  and  $\omega = \xi/|\xi| \in \mathbb{S}^{d-1}$ . Note that  $\partial/\partial \xi_j = \omega_j \partial/\partial r + r^{-1} \Omega_j$  where  $\Omega_j$  is a vector field on  $\mathbb{S}^{d-1}$ , and (see

[14], Proposition 14.7.1)

$$\sum_{j=1}^d \omega_j \Omega_j = 0, \quad \sum_{j=1}^d \Omega_j \omega_j = d - 1. \quad (2.8)$$

By using Fourier transformation, we have

$$\begin{aligned} & - \sum_{j,k=1}^d \int_{\mathbb{R}^d} \left\{ \bar{a}_{jk}(f) \left( G_\delta(t, D_v) \partial_{v_j} f(t, v) \right) - G_\delta(t, D_v) \left( \bar{a}_{jk}(f) \partial_{v_j} f(t, v) \right) \right\} \\ & \quad \times \overline{\left( \partial_{v_k} G_\delta(t, D_v) f(t, v) \right)} dv \\ & = \int_{\mathbb{R}^d} \left\{ \sum_{j,k=1}^d \xi_k \left[ \left( \delta_{jk} \Delta_\xi - \partial_{\xi_k} \partial_{\xi_j} \right), G_\delta(t, |\xi|) \right] \xi_j \hat{f}(t, \xi) \right\} \times G_\delta(t, |\xi|) \overline{\hat{f}(t, \xi)} d\xi. \end{aligned}$$

Noting, in polar coordinates on  $\mathbb{R}_\xi^d$ ,

$$\Delta_\xi = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \sum_{j=1}^d \Omega_j^2,$$

we have, denoting by  $\tilde{G}(r^2) = G_\delta(t, r)$ ,

$$\begin{aligned} & \sum_{j,k=1}^d \omega_k \left[ \left( \delta_{jk} \left\{ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} \right\} \right. \right. \\ & \quad \left. \left. - \left\{ (\omega_k \partial / \partial r + r^{-1} \Omega_k) (\omega_j \partial / \partial r + r^{-1} \Omega_j) \right\} \right), \tilde{G}(r^2) \right] \omega_j \\ & = \left[ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}, \tilde{G}(r^2) \right] \\ & \quad - \left[ \left( \sum_{k=1}^d (\omega_k^2 \partial / \partial r + r^{-1} \omega_k \Omega_k) \sum_{j=1}^d (\omega_j^2 \partial / \partial r + r^{-1} \Omega_j \omega_j) \right), \tilde{G}(r^2) \right] \\ & = \left[ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}, \tilde{G}(r^2) \right] - \left[ \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \frac{d-1}{r}, \tilde{G}(r^2) \right] = 0, \end{aligned}$$

where we have used (2.8). Then we finish the proof of Proposition 2.2.

**Remark 2.1.** *In the above proof of Proposition 2.2, we have used the polar coordinates in the dual variable of  $v$ , which is essentially related to a form of the Landau operator with Maxwellian molecules. We notice that the same relation (in  $v$  variable) was described by Villani [19] and Desvillettes-Villani [10].*

**End of proof of Theorem 1.1 :**

From Propositions 2.1 and 2.2, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|G_\delta(t) f(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 + (C_1 - \frac{1}{2} c_0 - 2c_0 t) \|\nabla_v G_\delta(t) f(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 \\ & \leq \frac{d}{2} \|G_\delta(t) f(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

For any  $0 < T_0 < T$ , choose  $c_0$  small enough such that  $C_1 - \frac{1}{2}c_0 - 2c_0T_0 \geq 0$ . Then we get

$$\frac{d}{dt} \|G_\delta(t)f(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq \frac{d}{2} \|G_\delta(t)f(t, \cdot)\|_{L^2(\mathbb{R}^d)}. \quad (2.9)$$

Integrating the inequality (2.9) on  $]0, t[$ , we obtain

$$\|G_\delta(t)f(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq e^{\frac{d}{2}t} \|f_0\|_{L^2(\mathbb{R}^d)}. \quad (2.10)$$

Take limit  $\delta \rightarrow 0$  in (2.10). Then we get

$$\|e^{-c_0 t \Delta_v} f(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq e^{\frac{d}{2}t} \|f_0\|_{L^2(\mathbb{R}^d)} \quad (2.11)$$

for any  $0 < t \leq T_0$ . We have now proved  $f(t, \cdot) \in \mathcal{A}^{1/2}(\mathbb{R}^d)$  and Theorem 1.1.

### 3. Linear Fokker-Planck equations

In the paper [19], there is an exact solution for spatially homogeneous linear Fokker-Planck equation. In the inhomogeneous case we can also obtain an exact solution of the Cauchy problem (1.4). Denote by

$$\hat{f}(t, \eta, \xi) = \mathcal{F}_{x,v}(f(t, x, v)),$$

the partial Fourier transformation of  $f$  with respect to  $(x, v)$  variable. Then, by Fourier transformation for  $(x, v)$  variables, the linear Fokker-Planck equation (1.4) becomes

$$\begin{cases} \frac{\partial}{\partial t} \hat{f}(t, \eta, \xi) - \eta \cdot \nabla_\xi \hat{f}(t, \eta, \xi) + \xi \cdot \nabla_\eta \hat{f}(t, \eta, \xi) = -|\xi|^2 \hat{f}(t, \eta, \xi); \\ \hat{f}|_{t=0} = \mathcal{F}(f_0)(\eta, \xi). \end{cases}$$

Therefore we obtain the exact solution

$$\hat{f}(t, \xi, \eta) = \hat{f}(0, \xi e^{-t} + \eta(1 - e^{-t}), \eta) \exp\left(-\int_0^t |\xi e^{\tau-t} + \eta(1 - e^{\tau-t})|^2 d\tau\right).$$

Note that

$$\begin{aligned} & \int_0^t |\xi e^{-\tau} + \eta(1 - e^{-\tau})|^2 d\tau \\ &= \frac{1 - e^{-2t}}{2} |\xi|^2 + (1 - e^{-t})^2 \xi \cdot \eta + \left(t - \frac{3 + e^{-2t}}{2} + 2e^{-t}\right) |\eta|^2 \\ &= \left(X - \frac{X^2}{2}\right) |\xi|^2 + X^2 \xi \cdot \eta + \left(-\log(1 - X) - X - \frac{X^2}{2}\right) |\eta|^2, \end{aligned}$$

where  $X = 1 - e^{-t} \sim t$ . We have for  $0 < K < 2/3$

$$\int_0^t |\xi e^{-\tau} + \eta(1 - e^{-\tau})|^2 d\tau \geq X(1 - 1/(2K) - X/2) |\xi|^2 + (1/3 - K/2) X^3 |\eta|^2.$$

Hence for  $t \sim X < 2 - 1/K$ , we get

$$f(t, \cdot, \cdot) \in \mathcal{A}^{1/2}(\mathbb{R}^{2d}),$$

so that the ultra-analytic effect holds for any  $t > 0$  by means of the semi-group property. But we cannot get the uniform estimate (1.5).

We present now the proof of (1.5) which implies the ultra-analytic effect, by commutator estimates similarly as for homogeneous Landau equation. Set

$$w(t, \eta, \xi) = \hat{f}(t, \eta, \xi - t\eta).$$

Then the Cauchy problem (1.4) is equivalent to

$$\begin{cases} \frac{\partial}{\partial t} w(t, \eta, \xi) = -|\xi - t\eta|^2 w(t, \eta, \xi) - (\xi - t\eta) \cdot \nabla_{\xi} w(t, \eta, \xi); \\ w|_{t=0} = \mathcal{F}(f_0)(\eta, \xi). \end{cases} \quad (3.1)$$

Since we need to study the function  $\int_0^t |\xi - s\eta|^2 ds$ , we prove the following estimate.

**Lemma 3.1.** *For any  $\alpha > 0$ , there exists a constant  $c_{\alpha} > 0$  such that*

$$\int_0^t |\xi - s\eta|^{\alpha} ds \geq c_{\alpha} (t|\xi|^{\alpha} + t^{\alpha+1}|\eta|^{\alpha}). \quad (3.2)$$

**Remark 3.1.** *If  $\alpha = 2$ , we can get the above estimate by direct calculation. The following simple proof is due to Seiji Ukai.*

**Proof :** Setting  $s = t\tau$  and  $\tilde{\eta} = t\eta$ , we see that the estimate is equivalent to

$$\int_0^1 |\xi - \tau\tilde{\eta}|^{\alpha} d\tau \geq c_{\alpha} (|\xi|^{\alpha} + |\tilde{\eta}|^{\alpha}).$$

Since this is trivial when  $\tilde{\eta} = 0$ , we may assume  $\tilde{\eta} \neq 0$ . If  $|\xi| < |\tilde{\eta}|$  then

$$\begin{aligned} \int_0^1 |\xi - \tau\tilde{\eta}|^{\alpha} d\tau &\geq |\tilde{\eta}|^{\alpha} \int_0^1 \left| \tau - \frac{|\xi|}{|\tilde{\eta}|} \right|^{\alpha} d\tau \\ &= |\tilde{\eta}|^{\alpha} \left\{ \int_0^{|\xi|/|\tilde{\eta}|} \left( \frac{|\xi|}{|\tilde{\eta}|} - \tau \right)^{\alpha} d\tau + \int_{|\xi|/|\tilde{\eta}|}^1 \left( \tau - \frac{|\xi|}{|\tilde{\eta}|} \right)^{\alpha} d\tau \right\} \\ &\geq \frac{|\tilde{\eta}|^{\alpha}}{\alpha+1} \min_{0 \leq \theta \leq 1} (\theta^{\alpha+1} + (1-\theta)^{\alpha+1}) = \frac{|\tilde{\eta}|^{\alpha}}{2^{\alpha}(\alpha+1)} \\ &\geq \frac{1}{2^{\alpha+1}(\alpha+1)} (|\xi|^{\alpha} + |\tilde{\eta}|^{\alpha}). \end{aligned}$$

If  $|\xi| \geq |\tilde{\eta}|$  then

$$\begin{aligned} \int_0^1 |\xi - \tau\tilde{\eta}|^{\alpha} d\tau &\geq |\xi|^{\alpha} \int_0^1 \left( 1 - \tau \frac{|\tilde{\eta}|}{|\xi|} \right)^{\alpha} d\tau \geq |\xi|^{\alpha} \int_0^1 (1 - \tau)^{\alpha} d\tau \\ &= \frac{|\xi|^{\alpha}}{\alpha+1} \geq \frac{1}{2(\alpha+1)} (|\xi|^{\alpha} + |\tilde{\eta}|^{\alpha}). \end{aligned}$$

Hence we obtain (3.2).

Set now

$$\phi(t, \eta, \xi) = c_0 \left( \int_0^t |\xi - s\eta|^2 ds - \frac{c_2}{2} t^3 |\eta|^2 \right),$$

where  $c_0 > 0$  is a small constant to choose later, and  $c_2$  is the constant in (3.2) with  $\alpha = 2$ . Then (3.2) implies

$$\phi(t, \eta, \xi) \geq c_0 \frac{c_2}{2} (t|\xi|^2 + t^3|\eta|^2). \quad (3.3)$$

Let  $N = (2d + 1)/4$ . For  $0 < \delta < 1/4N^2$  and  $t > 0$ , set

$$G_\delta = G_\delta(t, \eta, \xi) = \frac{e^{\phi(t, \eta, \xi)}}{(1 + \delta e^{\phi(t, \eta, \xi)}) (1 + \delta(|\eta|^2 + |\xi|^2))^N}. \quad (3.4)$$

Since  $G_\delta(t, \cdot \cdot) \in L^\infty(\mathbb{R}^{2d})$ , we can use it as Fourier multiplier, denoted by

$$(G_\delta(t, D_x, D_v)u)(t, x, v) = \mathcal{F}_{\eta, \xi}^{-1} \left( G_\delta(t, \eta, \xi) \hat{u}(t, \eta, \xi) \right).$$

**Lemma 3.2.** *Assume that  $f(t, \cdot) \in L^2(\mathbb{R}_{x, v}^{2d}) \cap L_1^1(\mathbb{R}_{x, v}^{2d})$  for any  $t \in ]0, T[$ . Then  $\nabla_\xi w(t, \eta, \xi) \in L^\infty(\mathbb{R}_{\eta, \xi}^{2d})$ , and*

$$|\xi - t\eta| G_\delta(t, \eta, \xi)^2 \bar{w}(t, \eta, \xi), |\eta| G_\delta(t, \eta, \xi)^2 \bar{w}(t, \eta, \xi), \nabla_\xi \left( G_\delta(t, \eta, \xi)^2 \bar{w}(t, \eta, \xi) \right) \quad (3.5)$$

belong to  $L^2(\mathbb{R}_{\eta, \xi}^{2d})$  for any  $t \in ]0, T[$ .

**Proof :** Since  $\partial_{\xi_j} w = -i\mathcal{F}(v_j f)$ , it follows from  $f \in L_1^1(\mathbb{R}_{x, v}^{2d})$  that  $\nabla_\xi w(t, \eta, \xi) \in L^\infty(\mathbb{R}_{\eta, \xi}^{2d})$ . Noting

$$|\xi - t\eta| G_\delta(t, \eta, \xi)^2, |\eta| G_\delta(t, \eta, \xi)^2 \in L^\infty(\mathbb{R}_{\eta, \xi}^{2d}),$$

we see that the first two terms of (3.5) are obvious. To check the last term in (3.5), note

$$\begin{aligned} \partial_{\xi_j} G_\delta(t, \eta, \xi) &= 2c_0 t \left( \xi_j - \frac{t}{2} \eta_j \right) G_\delta(t, \eta, \xi) \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} \\ &\quad - \frac{2N\delta \xi_j}{(1 + \delta(|\eta|^2 + |\xi|^2))} G_\delta(t, \eta, \xi). \end{aligned} \quad (3.6)$$

Then, we have

$$\begin{aligned} \nabla_\xi \left( G_\delta(t, \eta, \xi)^2 \bar{w}(t, \eta, \xi) \right) &= G_\delta(t, \eta, \xi)^2 \nabla_\xi \bar{w}(t, \eta, \xi) + \nabla_\xi \left( G_\delta(t, \eta, \xi)^2 \right) \bar{w}(t, \eta, \xi) \\ &= G_\delta(t, \eta, \xi)^2 \nabla_\xi \bar{w}(t, \eta, \xi) + 4c_0 t \left( \xi - \frac{t}{2} \eta \right) \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} G_\delta(t, \eta, \xi)^2 \bar{w}(t, \eta, \xi) \\ &\quad - \frac{4N\delta \xi}{(1 + \delta(|\eta|^2 + |\xi|^2))} G_\delta(t, \eta, \xi)^2 \bar{w}(t, \eta, \xi). \end{aligned}$$

Since  $G_\delta(t, \eta, \xi)^2 \in L^2(\mathbb{R}_{x, v}^{2d})$  we have

$$G_\delta(t, \eta, \xi)^2 \nabla_\xi \bar{w}(t, \eta, \xi) \in L^2(\mathbb{R}^{2d}).$$

Using

$$\left| \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} \right| \leq 1, \quad \left| \frac{2N\delta \xi}{(1 + \delta(|\eta|^2 + |\xi|^2))} \right| \leq 1,$$

and

$$\begin{aligned} & \left| \left( \xi - \frac{t}{2}\eta \right) G_\delta(t, \eta, \xi)^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} \bar{w}(t, \eta, \xi) \right| \leq \left| \xi - \frac{t}{2}\eta \right| G_\delta(t, \eta, \xi)^2 |\bar{w}(t, \eta, \xi)| \\ & \leq |\xi - t\eta| G_\delta(t, \eta, \xi)^2 |\bar{w}(t, \eta, \xi)| + \frac{t}{2} |\eta| G_\delta(t, \eta, \xi)^2 |\bar{w}(t, \eta, \xi)| \in L^2(\mathbb{R}^{2d}). \end{aligned}$$

We have proved Lemma 3.2

We take now  $G_\delta(t, \eta, \xi)^2 \bar{w}(t, \eta, \xi)$  as test function in the equation of (3.1). Then we have

$$\begin{aligned} & \frac{d}{dt} \|G_\delta(t, \cdot, \cdot) w(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})}^2 + 2 \int_{\mathbb{R}^{2d}} |(\xi - t\eta) G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi \\ & = 2 \sum_{j=1}^d \int_{\mathbb{R}^{2d}} w(t, \eta, \xi) \overline{\left( \partial_{\xi_j} (\xi_j - t\eta_j) G_\delta(t, \eta, \xi)^2 w(t, \eta, \xi) \right)} d\eta d\xi \\ & \quad + \left( \left( \partial_t G_\delta(t, \cdot, \cdot) \right) w(t, \cdot, \cdot), G_\delta(t, \cdot, \cdot) w(t, \cdot, \cdot) \right)_{L^2(\mathbb{R}^{2d})}. \end{aligned} \quad (3.7)$$

We prove now the following;

**Proposition 3.1.** *We have*

$$\begin{aligned} & \left( \left( \partial_t G_\delta(t, \cdot, \cdot) \right) w, G_\delta(t, \cdot, \cdot) w \right)_{L^2(\mathbb{R}^{2d})} \\ & = c_0 \int_{\mathbb{R}^{2d}} |(\xi - t\eta) G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi \\ & \quad - \frac{3}{2} c_0 c_2 t^2 \int_{\mathbb{R}^{2d}} |\eta|^2 |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi. \end{aligned} \quad (3.8)$$

$$\begin{aligned} & \operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^{2d}} w(t, \eta, \xi) \overline{\partial_{\xi_j} \left( (\xi_j - t\eta_j) G_\delta(t, \eta, \xi)^2 w(t, \eta, \xi) \right)} d\eta d\xi \\ & \leq \left( 2c_0 t + \frac{c_0 t^2}{3c_2} + c_0 \right) \int_{\mathbb{R}^{2d}} |(\xi - t\eta) G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi \\ & \quad + \frac{d + 2N^2 \delta / c_0}{2} \|G_\delta(t, \cdot, \cdot) w(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})}^2 \\ & \quad + \frac{3}{4} c_0 c_2 t^2 \int_{\mathbb{R}^{2d}} |\eta|^2 |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi. \end{aligned} \quad (3.9)$$

**Proof of Proposition 3.1 :** The estimate (3.8) is deduced from

$$\partial_t G_\delta(t, \eta, \xi) = c_0 (|\xi - t\eta|^2 - \frac{3}{2} c_2 t^2 |\eta|^2) G_\delta(t, \eta, \xi) \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})}.$$

Since it follows from (3.6) that

$$\begin{aligned}
\mathcal{I} &= \operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^{2d}} w(t, \eta, \xi) \overline{\partial_{\xi_j} \left( (\xi_j - t\eta_j) G_\delta(t, \eta, \xi)^2 w(t, \eta, \xi) \right)} d\eta d\xi \\
&= \operatorname{Re} 2c_0 t \sum_{j=1}^d \int_{\mathbb{R}^{2d}} (\xi_j - t\eta_j) \left( \xi_j - \frac{t}{2}\eta_j \right) |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi \\
&\quad - \operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^{2d}} \frac{2N\delta\xi_j(\xi_j - t\eta_j)}{(1 + \delta(|\eta|^2 + |\xi|^2))} |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi \\
&\quad - \operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^{2d}} (\xi_j - t\eta_j) \left( \partial_{\xi_j} G_\delta(t, \eta, \xi) w(t, \eta, \xi) \right) \overline{G_\delta(t, \eta, \xi) w(t, \eta, \xi)} d\eta d\xi,
\end{aligned}$$

we get

$$\begin{aligned}
\mathcal{I} &= 2c_0 t \sum_{j=1}^d \int_{\mathbb{R}^{2d}} (\xi_j - t\eta_j) \left( \xi_j - \frac{t}{2}\eta_j \right) |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi \\
&\quad - \sum_{j=1}^d \int_{\mathbb{R}^{2d}} \frac{2N\delta\xi_j(\xi_j - t\eta_j)}{(1 + \delta(|\eta|^2 + |\xi|^2))} |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi \\
&\quad \quad \quad + \frac{d}{2} \|G_\delta(t, \cdot, \cdot) w(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})}^2 \\
&= 2c_0 t \int_{\mathbb{R}^{2d}} |\xi - t\eta|^2 |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi \\
&\quad + c_0 t^2 \int_{\mathbb{R}^{2d}} (\xi - t\eta) \cdot \eta |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi \\
&\quad - \sum_{j=1}^d \int_{\mathbb{R}^{2d}} \frac{2N\delta\xi_j(\xi_j - t\eta_j)}{(1 + \delta(|\eta|^2 + |\xi|^2))} |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi \\
&\quad \quad \quad + \frac{d}{2} \|G_\delta(t, \cdot, \cdot) w(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})}^2.
\end{aligned}$$

For the last term, noting

$$\sum_{j=1}^d \frac{2N\delta\xi_j(\xi_j - t\eta_j)}{(1 + \delta(|\eta|^2 + |\xi|^2))} \leq \frac{(N^2/c_0)\delta^2|\xi|^2 + c_0|\xi - t\eta|^2}{(1 + \delta(|\eta|^2 + |\xi|^2))} \leq N^2\delta/c_0 + c_0|\xi - t\eta|^2$$

we finally obtain

$$\begin{aligned}
\mathcal{I} &\leq \left( 2c_0 t + \frac{c_0 t^2}{3c_2} + c_0 \right) \int_{\mathbb{R}^{2d}} |(\xi - t\eta) G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 d\eta d\xi \\
&\quad + \frac{d + 2N^2\delta/c_0}{2} \|G_\delta(t, \cdot, \cdot) w(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})}^2 \\
&\quad + \frac{3}{4} c_0 c_2 t^2 \int_{\mathbb{R}^{2d}} |\eta|^2 |G_\delta(t, \eta, \xi) w(t, \eta, \xi)|^2 \frac{1}{(1 + \delta e^{\phi(t, \eta, \xi)})} d\eta d\xi.
\end{aligned}$$

Thus we have proved Proposition 3.1.

**End of proof of Theorem 1.2 :**

Now the equation (3.7), the estimate (3.8) and (3.9) deduce

$$\begin{aligned} & \frac{d}{dt} \|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})}^2 \\ & + \left(2 - 3c_0 - 4c_0t - \frac{2c_0t^2}{3c_2}\right) \int_{\mathbb{R}^{2d}} |(\xi - t\eta)G_\delta(t, \eta, \xi)w(t, \eta, \xi)|^2 d\eta d\xi \\ & \leq (d + 2N^2\delta/c_0) \|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})}^2. \end{aligned}$$

Then for any  $0 < T_0 < T$  choose  $c_0 > 0$  (depends on  $T_0$ ) small enough such that

$$2 - 3c_0 - 4c_0T_0 - \frac{2c_0T_0^2}{3c_2} \geq 0,$$

then for any  $0 < t \leq T_0$ ,

$$\frac{d}{dt} \|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})} \leq \frac{d + 2N^2\delta/c_0}{2} \|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})},$$

which gives

$$\|G_\delta(t, \cdot, \cdot)w(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})} \leq e^{\frac{d+2N^2\delta/c_0}{2}t} \|f_0\|_{L^2(\mathbb{R}^{2d})}.$$

Take  $\delta \rightarrow 0$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} e^{c_0 \int_0^t |\xi - s\eta|^2 ds - c_1 t^3 |\eta|^2} |\hat{f}(t, \eta, \xi - t\eta)|^2 d\eta d\xi \\ & = \int_{\mathbb{R}^{2d}} e^{c_0 \int_0^t |\xi + (t-s)\eta|^2 ds - c_1 t^3 |\eta|^2} |\hat{f}(t, \eta, \xi)|^2 d\eta d\xi \leq e^{dt} \|f_0\|_{L^2(\mathbb{R}^{2d})}^2. \end{aligned}$$

By using (3.3), we get finally

$$\|e^{-\tilde{c}_0(t\Delta_v + t^3\Delta_x)} f(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^{2d})} \leq e^{\frac{d}{2}t} \|f_0\|_{L^2(\mathbb{R}^{2d})}$$

for any  $0 < t \leq T_0$ , where  $\tilde{c}_0 = \frac{c_0 c_2}{2} > 0$ . This is the desired estimate (1.5), which implies

$$f(t, \cdot, \cdot) \in \mathcal{A}^{1/2}(\mathbb{R}^{2d}).$$

We have thus proved Theorem 1.2.

#### 4. Linear model of inhomogeneous Landau equations

We prove now the Theorem 1.3 in this section. By the change of variables  $(t, x, v) \rightarrow (t, x + vt, v)$ , the Cauchy problem (1.8) is reduced to

$$\begin{cases} f_t = (\nabla_v - t\nabla_x) \left( \bar{a}(\mu) \cdot (\nabla_v - t\nabla_x) f - \bar{b}(\mu) f \right), \\ f|_{t=0} = g_0(x, v), \end{cases} \quad (4.1)$$



where  $f(t, x, v) = g(t, x + vt, v)$ . Recall that

$$\begin{aligned}\bar{a}_{ij}(\mu) &= a_{ij} \star \mu = \delta_{ij}(|v|^2 + 1) - v_i v_j ; \\ \bar{b}_j(\mu) &= \sum_{i=1}^d (\partial_{v_i} a_{ij}) \star \mu = -v_j ; \quad i, j = 1, \dots, d,\end{aligned}$$

and

$$\sum_{ij=1}^d \bar{a}_{ij}(\mu) \xi_i \xi_j \geq |\xi|^2, \quad \text{for all } (v, \xi) \in \mathbb{R}^{2d}.$$

In view of this Cauchy problem , we set

$$\Psi(t, \eta, \xi) = c_0 \int_0^t |\xi - s\eta| ds,$$

for a sufficiently small  $c_0 > 0$  which will be chosen later on. Then we can use the (3.2) with  $\alpha = 1$  to estimate  $\Psi$ . Set

$$F_\delta(t, \eta, \xi) = \frac{e^\Psi}{(1 + \delta e^\Psi)(1 + \delta \Psi)^N}$$

for  $N = d + 1, 0 < \delta \leq \frac{1}{N}$ . If  $A$  is a first order differential operator of  $(t, \eta, \xi)$  variables then we have

$$AF_\delta = \left( \frac{1}{1 + \delta e^\Psi} - \frac{N\delta}{1 + \delta \Psi} \right) (A\Psi)F_\delta, \quad (4.2)$$

and

$$\left| \frac{1}{1 + \delta e^\Psi} - \frac{N\delta}{1 + \delta \Psi} \right| \leq 1.$$

Taking

$$F_\delta(t, D_x, D_v)^2 f = F_\delta(t)^2 f \in H^{2N}(\mathbb{R}^{2d})$$

as a test function in the weak solution formula of (4.1), we have

$$\begin{aligned}& \frac{1}{2} \frac{d}{dt} \|F_\delta(t)f\|_{L^2(\mathbb{R}^{2d})}^2 + \left( \bar{a}(\mu) \left( (\nabla_v - t\nabla_x) F_\delta(t)f \right), \left( (\nabla_v - t\nabla_x) F_\delta(t)f \right) \right)_{L^2(\mathbb{R}^{2d})} \\ &= - \sum_{j=1}^d \int_{\mathbb{R}^{2d}} v_j f \overline{\left( (\partial_{v_j} - t\partial_{x_j}) F_\delta(t)^2 f \right)} dx dv + \frac{1}{2} \left( (\partial_t F_\delta) f, F_\delta(t)f \right)_{L^2(\mathbb{R}^{2d})} \\ &+ \sum_{j,k=1}^d \int_{\mathbb{R}^{2d}} \left\{ \bar{a}_{jk}(\mu) \left( F_\delta(t) (\partial_{v_j} - t\partial_{x_j}) f - F_\delta(t) \left( \bar{a}_{jk}(\mu) (\partial_{v_j} - t\partial_{x_j}) f \right) \right) \right. \\ &\quad \left. \times \overline{\left( (\partial_{v_k} - t\partial_{x_k}) F_\delta(t)f \right)} \right\} dx dv.\end{aligned}$$

We prove now the following results.

**Proposition 4.1.** *We have*

$$\begin{aligned} & \|(\nabla_v - t\nabla_x)F_\delta(t)f\|_{L^2(\mathbb{R}^{2d})}^2 \\ & \leq \left( \bar{a}(\mu) \left( (\nabla_v - t\nabla_x)F_\delta(t)f, \left( (\nabla_v - t\nabla_x)F_\delta(t)f \right) \right) \right)_{L^2(\mathbb{R}^{2d})}. \end{aligned} \quad (4.3)$$

$$\left| \left( \left( \partial_t F_\delta(t) \right) f, F_\delta(t)f \right)_{L^2} \right| \leq c_0 \|(\nabla_v - t\nabla_x)F_\delta(t)f\|_{L^2} \|F_\delta(t)f\|_{L^2}. \quad (4.4)$$

$$\begin{aligned} -\operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^{2d}} v_j f \overline{\left( (\partial_{v_j} - t\partial_{x_j})F_\delta(t)^2 f \right)} & \leq \frac{d}{2} \|F_\delta(t)f\|_{L^2}^2 \\ & + c_0 t \|(\nabla_v - t\nabla_x)F_\delta f(t)\|_{L^2} \|F_\delta f(t)\|_{L^2}. \end{aligned} \quad (4.5)$$

**Proof :** The estimate (4.3) is a direct consequence of the elliptic condition (1.7). Using the Fourier transformation and noting (4.2), we see that (4.4) is derived from

$$\partial_t F_\delta(t, \eta, \xi) = \left( \frac{1}{1 + \delta e^\Psi} - \frac{N\delta}{1 + \delta\Psi} \right) (\partial_t \Psi) F_\delta, \quad \partial_t \Psi = c_0 |\xi - t\eta|.$$

For (4.5), we have firstly

$$-\operatorname{Re} \sum_{j=1}^d \int_{\mathbb{R}^6} v_j F_\delta(t)f \overline{\left( (\partial_{v_j} - t\partial_{x_j})F_\delta(t)f \right)} = \frac{d}{2} \|F_\delta(t)f\|_{L^2}^2.$$

For the commutators  $[v_j, F_\delta(t)]$ , using Fourier transformation, we have that for  $t > 0$  and  $\hat{f} = \hat{f}(t, \eta, \xi)$

$$\begin{aligned} & - \sum_{j=1}^d \int_{\mathbb{R}^{2d}} \left( [F_\delta(t, D_x, D_v), v_j] f(t, x, v) \right) \overline{\left( (\partial_{v_j} - t\partial_{x_j})F_\delta(t, D_x, D_v) f(t, x, v) \right)} dx dv \\ & = - \sum_{j=1}^d \int_{\mathbb{R}^{2d}} \left( F_\delta(t, D_x, D_v) v_j f(t) - v_j F_\delta(t, D_x, D_v) f(t) \right) \\ & \quad \times \overline{\left( (\partial_{v_j} - t\partial_{x_j})F_\delta(t, D_v) f(t) \right)} dx dv \\ & = \sum_{j=1}^3 \int_{\mathbb{R}^{2d}} \left\{ i\partial_{\xi_j} (F_\delta(t, \eta, \xi) \hat{f}(t)) - F_\delta(t, \eta, \xi) (i\partial_{\xi_j} \hat{f}(t)) \right\} F_\delta(t, \eta, \xi) \\ & \quad \times \overline{i(\xi_j - t\eta_j) \hat{f}(t) d\eta d\xi} \\ & = \sum_{j=1}^d \int_{\mathbb{R}^{2d}} \left( \partial_{\xi_j} F_\delta(t, \eta, \xi) \right) \hat{f}(t) (\xi_j - t\eta_j) \overline{F_\delta(t, \eta, \xi) \hat{f}(t) d\eta d\xi} \\ & \leq c_0 t \int_{\mathbb{R}^{2d}} |\xi - t\eta| |F_\delta(t, \eta, \xi) \hat{f}(t)|^2 d\eta d\xi \leq c_0 t \|(\nabla_v - t\nabla_x)F_\delta f(t)\|_{L^2} \|F_\delta f(t)\|_{L^2}, \end{aligned}$$

where, in view of (4.2), we have used the fact that

$$\left| \sum_{j=1}^d (\partial_{\xi_j} \Psi)(t, \eta, \xi) \times (\xi_j - t\eta_j) \right| \leq c_0 \int_0^1 \left| \sum_{j=1}^3 \frac{\xi_j - s\eta_j}{|\xi - s\eta|} (\xi_j - t\eta_j) \right| ds \leq c_0 t |\xi - t\eta|.$$

Thus (4.5) has been proved.

For the commutator terms, we have

**Proposition 4.2.** *There exists a constant  $C_1 > 0$  independent of  $\delta > 0$  such that*

$$\begin{aligned} & \left| \sum_{j,k=1}^d \int_{\mathbb{R}^{2d}} \left\{ \bar{a}_{jk}(\mu) \left( F_\delta(t) (\partial_{v_j} - t\partial_{x_j}) \right) f - F_\delta(t) \left( \bar{a}_{jk}(\mu) (\partial_{v_j} - t\partial_{x_j}) f \right) \right\} \right. \\ & \qquad \qquad \qquad \left. \times \overline{\left( (\partial_{v_k} - t\partial_{x_k}) F_\delta(t) f \right)} \right| \\ & \leq C_1 \left\{ (c_0 t)^2 \|(\nabla_v - t\nabla_x) F_\delta(t) f\|_{L^2}^2 + \|F_\delta(t) f\|_{L^2}^2 \right\}. \end{aligned} \quad (4.6)$$

**Proof :** In order to prove (4.6), we introduce the polar coordinates of  $\xi$  centered at  $t\eta$ , that is ,

$$r = |\xi - t\eta| \quad \text{and} \quad \omega = \frac{\xi - t\eta}{|\xi - t\eta|} \in \mathbb{S}^{d-1}.$$

Note again that  $\partial/\partial\xi_j = \omega_j \partial/\partial r + r^{-1} \Omega_j$  where  $\Omega_j$  is a vector field on  $\mathbb{S}^{d-1}$ . We have again

$$\sum_{j=1}^d \omega_j \Omega_j = 0, \quad \sum_{j=1}^d \Omega_j \omega_j = d - 1, \quad .$$

By means of Plancherel formula, we have

$$\begin{aligned} & \sum_{j,k=1}^d \int_{\mathbb{R}^{2d}} \left\{ \bar{a}_{jk}(\mu) \left( F_\delta(t) (\partial_{v_j} - t\partial_{x_j}) \right) - F_\delta(t) \left( \bar{a}_{jk}(\mu) (\partial_{v_j} - t\partial_{x_j}) f \right) \right\} \\ & \qquad \qquad \qquad \times \overline{\left( (\partial_{v_k} - t\partial_{x_k}) F_\delta(t) f \right)} \\ & = - \int_{\mathbb{R}^{2d}} \left\{ \sum_{j,k=1}^d (\xi_k - t\eta_k) \left[ \left( \delta_{jk} \Delta_\xi - \partial_{\xi_k} \partial_{\xi_j} \right), F_\delta(t, \eta, \xi) \right] (\xi_j - t\eta_j) \hat{f}(t) \right\} \\ & \qquad \qquad \qquad \times \overline{F_\delta(t, \eta, \xi) \hat{f}(t)} d\xi d\eta \\ & = J. \end{aligned}$$

Noting again

$$\Delta_\xi = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \sum_{l=1}^d \Omega_l^2,$$

we have with  $\tilde{F}_\delta(t, \eta, r, \omega) = F_\delta(t, \eta, r \cdot \omega + t\eta) = F_\delta(t, \eta, \xi)$

$$\begin{aligned} & - \sum_{j,k=1}^d \omega_k \left[ \left( \delta_{jk} \left\{ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \sum_{l=1}^d \Omega_l^2 \right\} \right. \right. \\ & \qquad \qquad \qquad \left. \left. - \left\{ \left( \omega_k \frac{\partial}{\partial r} + r^{-1} \Omega_k \right) \left( \omega_j \frac{\partial}{\partial r} + r^{-1} \Omega_j \right) \right\} \right), \tilde{F}_\delta \right] \omega_j \\ & = - \left[ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}, \tilde{F}_\delta \right] \end{aligned}$$

$$\begin{aligned}
& + \left[ \left( \sum_{k=1}^d (\omega_k^2 \frac{\partial}{\partial r} + r^{-1} \omega_k \Omega_k) \sum_{j=1}^d (\omega_j^2 \frac{\partial}{\partial r} + r^{-1} \Omega_j \omega_j) \right), \tilde{F}_\delta \right] \\
& - \frac{1}{r^2} \sum_{j=1}^d \omega_j \left[ \sum_{l=1}^d \Omega_l^2, \tilde{F}_\delta \right] \omega_j = A_1 + A_2 + A_3.
\end{aligned}$$

Note again that

$$A_1 + A_2 = - \left[ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}, \tilde{F}_\delta \right] + \left[ \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \frac{d-1}{r}, \tilde{F}_\delta \right] = 0.$$

On the other hand, we have in view of (4.2)

$$\begin{aligned}
A_3 &= -\frac{1}{r^2} \sum_{j,l=1}^d \omega_j \left( 2\Omega_l [\Omega_l, \tilde{F}_\delta] - [\Omega_l, [\Omega_l, \tilde{F}_\delta]] \right) \omega_j \\
&= -\frac{1}{r^2} \sum_{j,l=1}^d \omega_j \left( 2\Omega_l \left( \frac{(\Omega_l \Psi)}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_l \Psi)}{1 + \delta \Psi} \right) \tilde{F}_\delta \right. \\
&\quad \left. - \left( \left( \frac{(\Omega_l \Psi)}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_l \Psi)}{1 + \delta \Psi} \right)^2 + \left( \Omega_l \left( \frac{(\Omega_l \Psi)}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_l \Psi)}{1 + \delta \Psi} \right) \right) \right) \tilde{F}_\delta \right) \omega_j.
\end{aligned}$$

Putting  $w_j = \omega_j \tilde{F}_\delta$   $w$  with  $w(t, \eta, r, \omega) = \hat{f}(t, \eta, r \cdot \omega + t\eta)$ , we have

$$\begin{aligned}
J &= \operatorname{Re} J = \operatorname{Re} \int_{\mathbb{R}_\eta^d} \int_0^\infty \int_{S^{d-1}} r^2 (A_3 w) \overline{\tilde{F}_\delta w} r^{d-1} dr d\omega d\eta \\
&= - \sum_{j,l=1}^d \operatorname{Re} \int_{\mathbb{R}_\eta^d} \int_0^\infty \int_{S^{d-1}} \left\{ 2\Omega_l \left( \frac{(\Omega_l \Psi)}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_l \Psi)}{1 + \delta \Psi} \right) w_j \right\} \overline{w_j} r^{d-1} dr d\omega d\eta \\
&+ \sum_{j,l=1}^d \int_{\mathbb{R}_\eta^d} \int_0^\infty \int_{S^{d-1}} \left( \left( \frac{(\Omega_l \Psi)}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_l \Psi)}{1 + \delta \Psi} \right)^2 \right. \\
&\quad \left. + \left( \Omega_l \left( \frac{(\Omega_l \Psi)}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_l \Psi)}{1 + \delta \Psi} \right) \right) \right) |w_j|^2 r^{d-1} dr d\omega d\eta \\
&= J_1 + J_2.
\end{aligned}$$

Since  $\Omega_l^* = -\Omega_l + (d-1)\omega_l$ , the integration by parts gives

$$\begin{aligned}
J_1 &= - \sum_{j,l=1}^d \int_{\mathbb{R}_\eta^d} \int_0^\infty \int_{S^{d-1}} \left\{ \left( \Omega_l \left( \frac{(\Omega_l \Psi)}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_l \Psi)}{1 + \delta \Psi} \right) \right) \right. \\
&\quad \left. + (d-1)\omega_l \left( \frac{(\Omega_l \Psi)}{1 + \delta e^\Psi} - \frac{N\delta(\Omega_l \Psi)}{1 + \delta \Psi} \right) \right\} |w_j|^2 r^{d-1} dr d\omega d\eta.
\end{aligned}$$

Hence we obtain

$$J = \sum_{j,l=1}^d \int_{\mathbb{R}_\eta^d} \int_0^\infty \int_{S^{d-1}} \left\{ \left( \frac{1}{1 + \delta e^\Psi} - \frac{N\delta}{1 + \delta \Psi} \right)^2 (\Omega_l \Psi)^2 \right. \quad (4.7)$$

$$\begin{aligned}
& -(d-1)\omega_l \left( \frac{1}{1+\delta e^\Psi} - \frac{N\delta}{1+\delta\Psi} \right) (\Omega_l\Psi) \Big\} |w_j|^2 r^{d-1} dr d\omega d\eta \\
= & \int_{\mathbb{R}_\eta^d} \int_0^\infty \int_{S^{d-1}} \left\{ \left( \frac{1}{1+\delta e^\Psi} - \frac{N\delta}{1+\delta\Psi} \right)^2 \left( \sum_{l=1}^d (\Omega_l\Psi)^2 \right) \right. \\
& \left. -(d-1) \left( \frac{1}{1+\delta e^\Psi} - \frac{N\delta}{1+\delta\Psi} \right) \left( \sum_{l=1}^d \omega_l (\Omega_l\Psi) \right) \right\} |\tilde{F}_\delta w|^2 r^{d-1} dr d\omega d\eta.
\end{aligned}$$

Since there exists a constant  $C_d > 0$  such that

$$|\Omega_l\Psi| = c_0 r \left| \sum_{j=1}^d \int_0^t \frac{\xi_j - s\eta_j}{|\xi - s\eta|} ds (\Omega_l\omega_j) \right| \leq c_0 C_d t r, \quad (4.8)$$

we have

$$\begin{aligned}
|J| \leq C'_d \Big\{ & (c_0 t)^2 \int_{\mathbb{R}_\eta^d} \int_0^\infty \int_{S^{d-1}} r^2 |\tilde{F}_\delta w|^2 r^{d-1} dr d\omega d\eta \\
& + \int_{\mathbb{R}_\eta^d} \int_0^\infty \int_{S^{d-1}} |\tilde{F}_\delta w|^2 r^{d-1} dr d\omega d\eta \Big\},
\end{aligned}$$

which yields (4.6). The proof of Proposition 4.2 is now complete.

### End of proof of Theorem 1.3 :

From Propositions 4.1 and 4.2, there exist constants  $C_2, C_3 > 0$  independent of  $\delta > 0$  and  $t > 0$  such that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|(F_\delta f)(t)\|_{L^2(\mathbb{R}^{2d})}^2 + \left( \frac{1}{2} - (c_0 t)^2 C_2 \right) \|(\nabla_v - t\nabla_x)(F_\delta f)(t)\|_{L^2(\mathbb{R}^{2d})}^2 \\
\leq C_3 \|(F_\delta f)(t)\|_{L^2(\mathbb{R}^{2d})}^2.
\end{aligned}$$

So that if  $\frac{1}{2} - (c_0 t)^2 C_2 \geq 0$ , we have,

$$\frac{d}{dt} \|(F_\delta f)(t)\|_{L^2(\mathbb{R}^{2d})} \leq C_3 \|(F_\delta f)(t)\|_{L^2(\mathbb{R}^{2d})}. \quad (4.9)$$

Using the fact  $(F_\delta f)(0) = \frac{1}{1+\delta} g_0$ , we get

$$\|(F_\delta f)(t)\|_{L^2(\mathbb{R}^{2d})} \leq e^{C_3 t} \|g_0\|_{L^2(\mathbb{R}^{2d})}.$$

Take the limit  $\delta \rightarrow 0$ . Then we have

$$\int_{\mathbb{R}^{2d}} e^{2\Psi(t,\eta,\xi)} |\hat{f}(t,\eta,\xi)|^2 d\eta d\xi \leq e^{2C_3 t} \|g_0\|_{L^2(\mathbb{R}^{2d})}^2. \quad (4.10)$$

On the other hand, by Lemma 3.1, there exists a  $c_1 > 0$  such that

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} e^{2\Psi(t,\eta,\xi)} |\hat{f}(t,\eta,\xi)|^2 d\eta d\xi &= \int_{\mathbb{R}^{2d}} e^{2c_0 \int_0^t |\xi - s\eta| ds} |\hat{g}(t,\eta,\xi - t\eta)|^2 d\eta d\xi \\
&= \int_{\mathbb{R}^{2d}} e^{2c_0 \int_0^t |\xi + (t-s)\eta| ds} |\hat{g}(t,\eta,\xi)|^2 d\eta d\xi \\
&\geq \int_{\mathbb{R}^{2d}} e^{2c_0 c_1 (t|\xi| + t^2|\eta|)} |\hat{g}(t,\eta,\xi)|^2 d\eta d\xi.
\end{aligned}$$

Finally, for any  $0 < T_0 < T$ , choosing  $c_0 > 0$  small enough such that  $\frac{1}{2} - (c_0 T_0)^2 C_2 \geq 0$ , we have proved,

$$\int_{\mathbb{R}^{2d}} \left| e^{c_0 c_1 (t(-\Delta_v)^{1/2} + t^2(-\Delta_x)^{1/2})} g(t, x, v) \right|^2 dx dv \leq e^{2C_3 t} \|g_0\|_{L^2(\mathbb{R}^{2d})}^2 \quad \text{for any } 0 < t \leq T_0,$$

which completes the proof of Theorem 1.3 with  $C = 2C_3$  depending only on  $d$ .

**Remark 4.1.** *The formulas (4.7) and (4.8) show that we cannot get the ultra-analytic effect of order 1/2 as in Theorem 1.2. It is the same reason why we do not consider the symmetric term  $Q(g, \mu)$  in the equation (1.8) as in [1].*

**Acknowledgments:** Authors wish to express their hearty gratitude to Seiji Ukai who communicated Lemma 3.1. The research of the first author was supported by Grant-in-Aid for Scientific Research No.18540213, Japan Society of the Promotion of Science, and the second author would like to thank the support of Kyoto University for his visit there.

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