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# Lifshitz Tails for Generalized Alloy Type Random Schrödinger Operators

Frédéric KLOPP\* and Shu NAKAMURA†

## Abstract

We study Lifshitz tails for random Schrödinger operators where the random potential is alloy type in the sense that the single site potentials are independent, identically distributed, but they may have various function forms. We suppose the single site potentials are distributed in a finite set of functions, and we show that under suitable symmetry conditions, they have Lifshitz tail at the bottom of the spectrum except for special cases. When the single site potential is symmetric with respect to all the axes, we give a necessary and sufficient condition for the existence of Lifshitz tails. As an application, we show that certain random displacement models have Lifshitz singularity at the bottom of the spectrum, and also complete the study of continuous Anderson type models undertaken in [10].

## 1 Introduction

Consider the continuous alloy type (or Anderson) random Schrödinger operator:

$$(1.1) \quad H_\omega = -\Delta + V_0 + V_\omega \text{ where } V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma V(x - \gamma)$$

on  $\mathbb{R}^d$ ,  $d \geq 1$ , where

- $V_0$  is a periodic potential;
- $V$  is a compactly supported single site potential;
- $(\omega_\gamma)_{\gamma \in \mathbb{Z}^d}$  are independent identically distributed random coupling constants.

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Let  $\Sigma$  be the almost sure spectrum of  $H_\omega$  and  $E_- = \inf \Sigma$ . When  $V$  has a fixed sign, it is well known that the  $E_- = \inf(\sigma(-\Delta + V_{\bar{b}}))$  if  $V \leq 0$  and  $E_- = \inf(\sigma(-\Delta + V_{\bar{a}}))$  if  $V \geq 0$ . Here,  $\bar{x}$  is the constant vector  $\bar{x} = (x)_{\gamma \in \mathbb{Z}^d}$ . Moreover, for  $E$  a real energy, one defines the *integrated density of states* by

$$(1.2) \quad N(E) = \lim_{L \rightarrow +\infty} \frac{\#\{\text{eigenvalues of } H_{\omega,L}^N \leq E\}}{L^d}$$

where

$$(1.3) \quad H_{\omega,L}^N = -\Delta + V_0 + V_\omega \quad \text{on } L^2(C_L(0))$$

with Neumann boundary conditions, where  $C_L(0)$  is defined by (1.4). It is well-known that  $N(E)$  exists and is non-random, i.e.,  $N(E)$  is independent of  $\omega$ , almost surely; it has been the object of a lot of studies.

In particular, it is well known that the integrated density of states of the Hamiltonian admits a Lifshitz tail near  $E_-$ , i.e.,

$$\lim_{E \rightarrow E_-^+} \frac{\log |\log N(E)|}{\log(E - E_-)} < 0.$$

Actually, the limit can often be computed and in many cases is equal to  $-d/2$ ; we refer to [3, 7, 8, 11, 13, 14, 15] for extensive reviews and more precise statements.

In the present paper, we mainly consider a generalized Bernoulli alloy type model that we define below: we allow the single site potential to have various function forms (with a discrete distribution). We give a necessary and sufficient condition to have Lifshitz tail under a symmetry assumption on the single site potentials. The results we obtain are then applied to the random displacement models studied recently by Baker, Loss and Stolz ([1, 2]), and also to complete the study of the occurrence of Lifshitz tails for alloy type models initiated in [10].

## 1.1 The model

Let us now describe our model. We let  $d \geq 1$  and we study operators on  $\mathcal{H} = L^2(\mathbb{R}^d)$ . We denote

$$(1.4) \quad C_\ell(x) = \{y \in \mathbb{R}^d \mid 0 \leq y_j - x_j \leq \ell, j = 1, \dots, d\}$$

be the cube with the size  $\ell > 0$  and  $x$  as a corner. Let  $V_0 \in C^0(\mathbb{R}^d)$  be a background potential, which is periodic with respect to  $\mathbb{Z}^d$ .

Let  $v_k \in C_c^0(C_1(0))$ ,  $k = 1, \dots, M$ , be single site potentials where  $M \in \mathbb{N}$ . We consider the random Schrödinger operator:

$$H_\omega = -\Delta + V_0 + V_\omega \quad \text{on } \mathcal{H} = L^2(\mathbb{R}^d),$$

where

$$V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} v_{\omega(\gamma)}(x - \gamma)$$

is the random potential and  $\{\omega(\gamma) \mid \gamma \in \mathbb{Z}^d\}$  are independent identically distributed (i.i.d.) random variables with values in  $\{1, \dots, M\}$ .

To fix ideas, let us assume

$$(1.5) \quad \inf \sigma(H_\omega) = 0, \quad \text{a.s. } \omega$$

which can always be achieved by shifting  $V_0$  by a constant.

We denote

$$H_k^N = -\Delta + V_0 + v_k \quad \text{on } L^2(C_1(0))$$

with Neumann boundary conditions on the boundary  $\partial C_1(0)$ .

Define

**Assumption A.** (1)  $V_0$  is symmetric about the plane  $\{x \mid x_d = 1/2\}$ .  
(2) There exists  $m \in \{1, \dots, M\}$  such that

$$\inf \sigma(H_k^N) = 0 \quad \text{for } k = 1, \dots, m,$$

and

$$\inf \sigma(H_k^N) > 0 \quad \text{for } k > m.$$

(3) Moreover, for  $k = 1, \dots, m$ ,  $v_k(x)$  is symmetric about  $\{x_d = 1/2\}$ .

*Remark 1.1.* Note that in this assumption, we only require symmetry with respect to a single coordinate hyperplane that we chose to be the  $d$ -th one.

If one assumes that  $V_0$  and the  $(v_k)_{1 \leq k \leq M}$  are reflection symmetric with respect to all the coordinate planes (see e.g. [1, 2, 10]), the standard characterization of the almost sure spectrum (see e.g. [11, 7]) and lower bounding  $H_\omega$  by the direct sum of its Neumann restrictions to the cubes  $(C_1(\gamma))_{\gamma \in \mathbb{Z}^d}$  show that, as a consequence of (1.5), one obtains

- for all  $m \in \{1, \dots, M\}$ ,  $\inf \sigma(H_k^N) \geq 0$ ;
- there exists  $m \in \{1, \dots, M\}$  such that  $\inf \sigma(H_m^N) = 0$ .

## 1.2 The results

We study the Lifshitz singularity for the integrated density of states (IDS) at the zero energy. Recall that the IDS is defined by (1.2)

We first consider a relatively easy case:

**Theorem 1.2.** *Suppose Assumption A with  $m < M$ . Then*

$$(1.6) \quad \limsup_{E \rightarrow +0} \frac{\log |\log N(E)|}{\log E} \leq -\frac{1}{2}.$$

We expect (1.6) holds with  $-d/2$  in the right hand side, which is known to be optimal (see e.g Theorem 0.2 and Section 2.2 in [10]).

If  $m = M$ , then we need further classification of the potential functions. We denote the standard basis of  $\mathbb{R}^d$  by

$$\mathbf{e}_j = (\delta_{ji})_{i=1}^d \in \mathbb{R}^d, \quad j = 1, \dots, d,$$

and we define an operator  $H_{k\ell(j)}^N$  on  $L^2(U_j)$  as

$$(1.7) \quad U_j = C_1(0) \cup C_1(\mathbf{e}_j), \quad j = 1, \dots, d.$$

We set

$$(1.8) \quad H_{k\ell(j)}^N = \begin{cases} -\Delta + V_0(x) + v_k(x) & \text{on } C_1(0) \\ -\Delta + V_0(x) + v_\ell(x - \mathbf{e}_j) & \text{on } C_1(\mathbf{e}_j) \end{cases}$$

with Neumann boundary conditions on  $\partial U_j$ , where  $k, \ell \in \{1, \dots, m\}$  and  $j \in \{1, \dots, d\}$ . We define

$$(1.9) \quad v_j \underset{j}{\sim} v_\ell \quad \stackrel{def}{\iff} \quad \inf \sigma(H_{k\ell(j)}^N) = 0.$$

Namely,  $v_k \underset{j}{\sim} v_\ell$  implies the coupling of two local Hamiltonians  $H_k^N$  and  $H_\ell^N$  does not increase the ground state energy. We note that  $v_k \not\underset{j}{\sim} v_\ell$  generically for  $k \neq \ell$ .

**Theorem 1.3.** *Suppose Assumption A with  $m = M$ . Suppose moreover that  $v_k \not\underset{d}{\sim} v_\ell$  for some  $k \neq \ell$ . Then (1.6) holds, i.e.,  $H_\omega$  has Lifshitz singularities at the zero energy.*

In order to obtain a more precise result on the existence and the absence of Lifshitz singularities, we make a stronger symmetry assumption on the potentials.

**Assumption B.** In addition to satisfying Assumption A,  $V_0$  and  $v_k$  are symmetric about  $\{x \mid x_j = 1/2\}$  for all  $j = 1, \dots, d$ , and  $k = 1, \dots, m = M$ .

**Theorem 1.4.** *Suppose Assumption B. Then*

- (i) *If  $v_k \not\underset{j}{\sim} v_\ell$  for some  $j$  and  $k \neq \ell$ , then (1.6) holds.*
- (ii) *If  $v_k \underset{j}{\sim} v_\ell$  for all  $j$  and  $k, \ell$ , then the van Hove property holds, namely, there exists  $C > 0$  such that*

$$(1.10) \quad \frac{1}{C} E^{d/2} \leq N(E) \leq C E^{d/2}.$$

In (1.10), the asymptotic is new only for  $E$  small; for  $E$  large, it is a consequence of Weyl's law. The example in Section 3 of [10] is a special case of (ii) of Theorem 1.4.

In a previous paper [10], we used the concavity of the ground state energy with respect to the random parameters, and also used an operator theoretical trick to reduce the problem to monotonous perturbation case. These methods are not available under the assumptions of the present paper. Instead, we employ a quadratic inequality similar to the Poincaré inequality, and take advantage of the positivity of certain Dirichlet-to-Neumann operators to obtain a lower bound of the ground state energy for Schrödinger operators on a strip. This estimate is quasi one dimensional, and this is why we obtain Lifshitz tail estimate with the exponent corresponding to one dimensional case. We do believe that this method can be refined to obtain the optimal exponent, though we have not been successful so far.

This paper is organized as follows. We discuss the eigenvalue estimate on a strip in Section 2 and prove our main theorems in Section 3. We discuss an application to random displacement models in Section 4, and an application to the model studied in [10] in Section 5.

Throughout this paper, we use the following notations:  $\mathbb{P}(\cdot)$  denotes the probability measure for the random potential, and  $\mathbb{E}(\cdot)$  denotes the expectation;  $\mathcal{D}(A)$  denotes the definition domain of an operator  $A$ ;  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $L^2$ -spaces;  $\partial\Omega$  denotes the boundary of a domain  $\Omega$ ; and  $\#\Lambda$  denotes the cardinality of a set  $\Lambda$ .

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## 2 Lower bounds on the ground state energy

Throughout this section, we suppose  $v_1, \dots, v_m$  satisfy Assumption A. Let  $a > 0$ ,

$$\Omega_0 = [0, 1]^{d-1} \times [-a, 0] \subset \mathbb{R}^d,$$

and let  $W_0 \in C^0(\Omega_0)$  be a real-valued function on  $\Omega_0$ . We set

$$P_0^N = -\Delta + W_0 \quad \text{on } L^2(\Omega_0)$$

with Neumann boundary conditions. Let  $L \in \mathbb{N}$ ,

$$\Omega_1 = [0, 1]^{d-1} \times [0, L]$$

and let  $W_1 \in C^0(\Omega_1)$  such that

$$W_1 = V_0 + v_{k(\ell)}(x - \ell \mathbf{e}_d) \quad \text{if } x \in C_1(\ell \mathbf{e}_d), \ell = 0, \dots, L-1,$$

where  $\{k(\ell)\}_{\ell=0}^{L-1}$  is a sequence with values in  $\{1, \dots, m\}$ . We then set

$$\Omega = \Omega_0 \cup \Omega_1, \quad W(x) = \begin{cases} W_0(x) & \text{if } x \in \Omega_0 \\ W_1(x) & \text{if } x \in \Omega_1 \end{cases}$$

and set

$$P^N = -\Delta + W \quad \text{on } L^2(\Omega)$$

with Neumann boundary conditions. Then, the main result of this section is as follows.

**Theorem 2.1.** *Suppose  $\inf \sigma(P_0^N) > 0$ , and suppose  $v_{k(\ell)} \underset{d}{\sim} v_{k(\ell')}$  for  $\ell, \ell' \in \{0, \dots, L-1\}$ . Then, there exists  $C > 0$  such that  $C$  is independent of  $L$  and of the sequence  $\{k(\ell)\}$ , and such that*

$$\inf \sigma(P^N) \geq \frac{1}{CL^2}.$$

In the following, we suppose  $v_k \underset{d}{\sim} v_\ell$  for all  $k, \ell$  for simplicity (and without loss of generality). We prove Theorem 2.1 by a series of lemmas. First, we show a variant of the classical Poincaré inequality. Let  $\Gamma$  be the trace operator from  $H^1(\Omega_1)$  to  $L^2(S)$  with  $S = [0, 1]^{d-1} \times \{0\}$ , i.e.,

$$\Gamma\varphi(x') = \varphi(x', 0) \quad \text{for } x' \in [0, 1]^{d-1}, \varphi \in C^0(\Omega_1),$$

and  $\Gamma$  extends to a bounded operator from  $H^1(\Omega_1)$  to  $L^2(S)$ .

**Lemma 2.2.** *Let  $\varphi \in H^1(\Omega_1)$ . Then*

$$\frac{2}{L} \|\Gamma\varphi\|_{L^2(S)}^2 + \|\nabla\varphi\|_{L^2(\Omega_1)}^2 \geq \frac{1}{L^2} \|\varphi\|_{L^2(\Omega_1)}^2.$$

*Proof.* It suffices to show the estimate for  $\varphi \in C^1(\Omega_1)$ . Since

$$\varphi(x', t) = \varphi(x', 0) + \int_0^t \partial_{x_d} \varphi(x', s) ds, \quad x' \in [0, 1]^{d-1}, t \in [0, L],$$

we have

$$\begin{aligned} |\varphi(x', t)| &\leq |\varphi(x', 0)| + \int_0^t |\partial_{x_d} \varphi(x', s)| ds \\ &\leq |\varphi(x', 0)| + \sqrt{t} \left( \int_0^L |\nabla \varphi(x', s)|^2 ds \right)^{1/2} \end{aligned}$$

by the Cauchy-Schwarz inequality. This implies

$$\begin{aligned} \|\varphi\|_{L^2(\Omega_1)}^2 &\leq \int \int_0^L \left\{ |\varphi(x', 0)| + \sqrt{t} \left( \int_0^L |\nabla \varphi(x', s)|^2 ds \right)^{1/2} \right\}^2 dt dx' \\ &\leq 2 \int \int_0^L |\varphi(x', 0)|^2 ds dx' + 2 \int_0^L t dt \times \|\nabla \varphi\|_{L^2(\Omega_1)}^2 \\ &= 2L \|\Gamma\varphi\|_{L^2(S)}^2 + L^2 \|\nabla \varphi\|_{L^2(\Omega_1)}^2 \end{aligned}$$

and the claim follows.  $\square$

For  $k \in \{1, \dots, M\}$ , we set

$$q_k(\varphi, \psi) = \int_{C_1(0)} (\nabla\varphi \cdot \nabla\bar{\psi} + v_k\varphi\bar{\psi}) dx, \quad \varphi, \psi \in H^1(C_1(0)),$$

which is the quadratic form corresponding to  $H_k^N$ . Let  $\Psi_k$  be the positive ground state for  $H_k^N$ , which is unique up to a constant. Since  $\inf \sigma(H_k^N) = 0$ , we expect  $\varphi/\Psi_k$  is close to a constant if  $q_k(\varphi, \varphi)$  is close to 0, and this observation is justified by the following lemma.

**Lemma 2.3.** *There exists  $c_1 > 0$  such that*

$$\|\nabla(\varphi/\Psi_k)\|_{L^2(C_1(0))}^2 \leq c_1 q_k(\varphi, \varphi), \quad \varphi \in H^1(C_1(0)), k = 1, \dots, m.$$

*Proof.* This is a consequence of the so-called *ground state transform*. It suffices to show the inequality when  $\varphi \in C^1(C_1(0))$ . We set  $f = \varphi/\Psi_k$ . Then we have

$$\begin{aligned} q_k(\varphi, \varphi) &= \langle \nabla(f\Psi_k), \nabla(f\Psi_k) \rangle + \langle v_k f\Psi_k, f\Psi_k \rangle \\ &= \|\Psi_k(\nabla f)\|^2 + \langle \Psi_k \nabla f, f \nabla \Psi_k \rangle + \langle f \nabla \Psi_k, \Psi_k \nabla f \rangle \\ &\quad + \langle f \nabla \Psi_k, f \nabla \Psi_k \rangle + \langle v_k f \Psi_k, f \Psi_k \rangle \\ &= \|\Psi_k(\nabla f)\|^2 + \langle \nabla(|f|^2 \Psi_k), \nabla \Psi_k \rangle + \langle v_k |f|^2 \Psi_k, \Psi_k \rangle \\ &= \|\Psi_k(\nabla f)\|^2 + q_k(|f|^2 \Psi_k, \Psi_k). \end{aligned}$$

Since  $q_k(|f|^2 \Psi_k, \Psi_k) = \langle (H_k^N)^{1/2} |f|^2 \Psi_k, (H_k^N)^{1/2} \Psi_k \rangle = 0$ , we have

$$q_k(\varphi, \varphi) = \|\Psi_k(\nabla f)\|^2 \geq (\inf |\Psi_k|)^2 \|\nabla f\|^2,$$

and we may choose  $c_1 = (\min_k \inf |\Psi_k|)^{-2}$ .  $\square$

**Lemma 2.4.** *Suppose  $v_k \sim_d v_\ell$ . Then, there exists  $\mu_1, \mu_2 > 0$  such that*

$$\mu_1 \Psi_k(x', 0) = \mu_2 \Psi_\ell(x', 0), \quad \text{for } x' \in [0, 1]^{d-1}.$$

*Proof.* Consider  $H_{k\ell(d)}^N$  in  $U_d$  (see (1.7) and (1.8) in Section 1), and let  $\Phi \in L^2(U_d)$  be the positive ground state of  $H_{k\ell(j)}^N$ . We set

$$\varphi_1 = \Phi|_{C_1(0)}, \quad \varphi_2(\cdot) = \Phi(\cdot + \mathbf{e}_d)|_{C_1(0)}.$$

Then  $\varphi_1, \varphi_2$  are positive and  $q_k(\varphi_1, \varphi_1) = q_\ell(\varphi_2, \varphi_2) = 0$ . By the variational principle and the uniqueness of the ground states, we learn

$$\varphi_1 = \mu_1 \Psi_k, \quad \varphi_2 = \mu_2 \Psi_\ell$$

with some  $\mu_1, \mu_2 > 0$ . By Assumption A,  $\Psi_k$  and  $\Psi_\ell$  are symmetric about  $\{x_d = 1/2\}$ , and hence

$$\mu_1 \Psi_k(x', 0) = \mu_1 \Psi_k(x', 1) = \varphi_1(x', 1) = \varphi_2(x', 0) = \mu_2 \Psi_\ell(x', 0)$$

for  $x' \in [0, 1]^{d-1}$ , where we have used the continuity of  $\Phi$  on  $\{x_d = 1\}$ .  $\square$



Now, let  $\Omega_1$  and  $W_1$  be as in the beginning of Section 2, and define

$$P_1^N = -\Delta + W_1 \quad \text{on } L^2(\Omega_1)$$

with Neumann boundary conditions. We set

$$Q_1(\varphi, \psi) = \int_{\Omega_1} (\nabla \varphi \cdot \nabla \bar{\psi} + W_1 \varphi \bar{\psi}) dx = \langle (P_1^N)^{1/2} \varphi, (P_1^N)^{1/2} \psi \rangle$$

for  $\varphi, \psi \in H^1(\Omega_1) = \mathcal{D}((P_1^N)^{1/2})$ . Then, we have

**Lemma 2.5.** *There exists  $c_2 > 0$  such that  $c_2$  is independent of  $L$  and of the sequence  $\{k(\ell)\}$ , and*

$$\frac{1}{L} \|\Gamma \varphi\|_{L^2(S)}^2 + Q_1(\varphi, \varphi) \geq \frac{1}{c_2 L^2} \|\varphi\|_{L^2(\Omega_1)}^2$$

for  $\varphi \in H^1(\Omega_1)$ .

*Proof.* By Lemma 2.4, there exist  $\mu_1, \dots, \mu_m > 0$  such that

$$\mu_1 \Psi_1(x', 0) = \mu_2 \Psi_2(x', 0) = \dots = \mu_m \Psi_m(x', 0).$$

We set

$$\Psi(x) = \mu_{k(\ell)} \Psi_{k(\ell)}(x - \ell \mathbf{e}_d) \quad \text{if } \ell \leq x_d \leq \ell + 1,$$

and then  $\Psi \in H^1(\Omega_1)$  by the above observation. Moreover,  $\Psi$  is the ground state for  $P_1^N$ , unique up to a constant. We apply Lemma 2.1 to  $\varphi/\Psi$ , and we have

$$\begin{aligned} \frac{1}{L^2} \|\varphi\|_{L^2(\Omega_1)}^2 &\leq \frac{1}{L^2} (\sup \Psi)^2 \|\varphi/\Psi\|_{L^2(\Omega_1)}^2 \\ &\leq \frac{(\sup \Psi)^2}{L} \|\Gamma(\varphi/\Psi)\|_{L^2(S)}^2 + (\sup \Psi)^2 \|\nabla(\varphi/\Psi)\|_{L^2(\Omega_1)}^2 \\ &\leq \left( \frac{\sup \Psi}{\inf \Psi} \right)^2 \frac{1}{L} \|\Gamma \varphi\|_{L^2(S)}^2 + c_1 (\sup \Psi)^2 Q_1(\varphi, \varphi), \end{aligned}$$

where we have used Lemma 2.3 in the last inequality. The claim follows immediately.  $\square$

We next consider  $P_0 = -\Delta + W_0$  on  $L^2(\Omega_0)$  and its Dirichlet-to-Neumann operator. As in Theorem 2.1, we suppose

$$\alpha = \inf \sigma(P_0^N) > 0.$$

We set

$$P'_0 = -\Delta + W_0 \quad \text{on } L^2(\Omega_0) \quad \text{with } \mathcal{D}((P'_0)^{1/2}) = \{\varphi \in H^1(\Omega_0) \mid \Gamma \varphi = 0\},$$

where  $\Gamma$  is the trace operator from  $H^1(\Omega_1)$  to  $L^2(S)$ .  $P'_0$  defines a self-adjoint operator, and each  $\varphi \in \mathcal{D}(P'_0)$  satisfies Dirichlet boundary conditions on  $S$  and Neumann boundary conditions on  $\partial\Omega_0 \setminus S$ . Let  $\lambda < \alpha$ . By a standard argument of the theory of elliptic boundary value problems (see, e.g., Folland [4]), for any  $g \in H^{3/2}(S)$ , there exists a unique  $\psi \in H^2(\Omega_0)$  such that

$$(2.1) \quad (-\Delta + W_0 - \lambda)\psi = 0, \quad \Gamma\psi = g$$

and that satisfies Neumann boundary conditions on  $\partial\Omega_0 \setminus S$ . Then, the map

$$T(\lambda) : g \mapsto \Gamma(\partial_\nu\psi) \in H^{1/2}(S)$$

defines a bounded linear map from  $H^{3/2}(S)$  to  $H^{1/2}(S)$ , where  $\partial_\nu = \partial/\partial x_d$  is the outer normal derivative on  $S$ . We consider  $T(\lambda)$  as an operator on  $L^2(S)$ , and it is called the *Dirichlet-to-Neumann operator*.

**Lemma 2.6.**  *$T(\lambda)$  is a symmetric operator. Moreover, if  $\lambda_0 < \alpha$  then  $T(\lambda) \geq \varepsilon$  for  $0 \leq \lambda \leq \lambda_0$  with some  $\varepsilon > 0$ .*

*Proof.* Let  $\varphi, \psi \in H^2(\Omega_0)$  such that  $\Gamma\varphi = f$ ,  $\Gamma\psi = g$ , and

$$(-\Delta + W_0 - \lambda)\varphi = (-\Delta + W_0 - \lambda)\psi = 0$$

with Neumann boundary conditions on  $\partial\Omega_0 \setminus S$ . By Green's formula we have

$$\begin{aligned} 0 &= \langle (-\Delta + W_0 - \lambda)\varphi, \psi \rangle - \langle \varphi, (-\Delta + W_0 - \lambda)\psi \rangle \\ &= - \int_S \partial_\nu\varphi \cdot \bar{\psi} + \int_S \varphi \cdot \partial_\nu\bar{\psi} = -\langle T(\lambda)f, g \rangle_{L^2(S)} + \langle f, T(\lambda)g \rangle_{L^2(S)}, \end{aligned}$$

and hence  $T(\lambda)$  is symmetric. Similarly, we have

$$\begin{aligned} 0 &= \langle (-\Delta + W_0 - \lambda)\varphi, \varphi \rangle \\ &= - \int_S \partial_\nu\varphi \cdot \bar{\varphi} + \int_{\Omega_0} |\nabla\varphi|^2 + \int_{\Omega_0} (W_0 - \lambda)|\varphi|^2 \\ &= -\langle T(\lambda)f, f \rangle + Q_0(\varphi, \varphi) - \lambda\|\varphi\|^2, \end{aligned}$$

where  $Q_0(\varphi, \varphi) = \int_{\Omega_0} (|\nabla\varphi|^2 + W_0|\varphi|^2) dx$ . Hence, we learn

$$\langle T(\lambda)f, f \rangle = Q_2(\varphi, \varphi) - \lambda\|\varphi\|^2 \geq Q_0(\varphi, \varphi) - \lambda_0\|\varphi\|^2.$$

The form in the right hand side is equivalent to  $\|\varphi\|_{H^1(\Omega_0)}^2$  since  $\lambda_0 < \alpha$ . Hence, it is bounded from below by  $\varepsilon\|f\|_{L^2(S)}^2$  with some  $\varepsilon > 0$  by virtue of the boundedness of the trace operator from  $H^1(\Omega_0)$  to  $L^2(S)$ .  $\square$

We note that  $T(\lambda)$  extends to a self-adjoint operator on  $L^2(S)$  by the Friedrichs extension, though we do not use the fact in this paper.

*Proof of Theorem 2.1.* Let  $\varphi$  be the ground state of  $P^N$  on  $\Omega$  with the ground state energy  $\lambda \geq 0$ . If  $\lambda \geq \lambda_0 > 0$  with some fixed  $\lambda_0$  (independently of  $L$ ), then the statement is obvious, and hence we may assume  $0 \leq \lambda \leq \lambda_0 < \alpha = \inf \sigma(P_0^N)$  without loss of generality.

Let  $f = \Gamma\varphi \in H^{3/2}(S)$ . Since  $\varphi$  satisfies Neumann boundary conditions on  $\partial\Omega_0 \setminus S$ , we learn  $\partial_\nu \varphi|_S = T(\lambda)\varphi$ . On the other hand, by Green's formula, we have

$$\begin{aligned} \int_{\Omega_1} P^N \varphi \cdot \bar{\varphi} &= \int_S \partial_n \varphi \cdot \bar{\varphi} + \int_{\Omega_1} |\nabla \varphi|^2 + W_1 |\varphi|^2 \\ &= \langle T(\lambda)f, f \rangle_{L^2(S)} + Q_1(\varphi, \varphi) \\ &\geq \varepsilon \|f\|_{L^2(S)}^2 + Q_1(\varphi, \varphi) \end{aligned}$$

by Lemma 2.6. Now, we apply Lemma 2.5 to learn that the right hand side is bounded from below by  $(1/c_2 L^2) \|\varphi\|_{L^2(\Omega_1)}^2$ . Since  $P^N \varphi = \lambda \varphi$  and  $\|\varphi\|_{L^2(\Omega_1)} \neq 0$ , this implies  $\lambda \geq 1/c_2 L^2$  for sufficiently large  $L$ .  $\square$

### 3 Proof of main theorems

Here, we mainly discuss the proof of Theorems 1.2 and 1.3, and we prove Theorem 1.4 at the end of the section. We thus suppose Assumption A with either  $m < M$  or that there exists  $k, k'$  such that  $v_k \not\stackrel{d}{\sim} v_{k'}$ .

For notational simplicity, we assume the reflections of  $v_k$  at  $\{x_d = 1/2\}$  are included in the possible set of potentials  $\{v_k\}$ . This does not change the conditions on  $\{v_1, \dots, v_m\}$ , but we might need to add the reflections of  $\{v_{m+1}, \dots, v_M\}$ . This does not affect the following arguments.

We write

$$\Lambda = \{p \in \mathbb{Z}^{d-1} \mid 0 \leq p_j \leq L-1, j = 1, \dots, d-1\}$$

and, for  $p \in \Lambda$ , we set

$$\Sigma_p = \bigcup_{k=0}^{L-1} C_1((p, k))$$

so that  $C_L(0)$  is decomposed (see Fig. 1) as

$$C_L(0) = \bigcup_{p \in \Lambda} \Sigma_p$$

which is a disjoint union except for the boundaries of the strips.

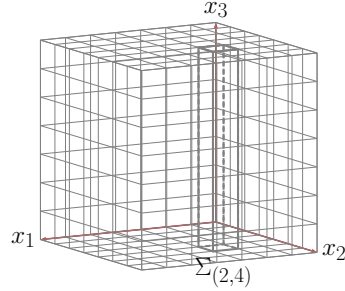


Figure 1: Chopping the cube into strips

For a given  $V_\omega$  and  $p \in \Lambda$ , we consider the restriction of  $H_\omega$  to  $\Sigma_p$ , i.e.,

$$\tilde{H}_p^N = \Delta + V_0 + \sum_{\ell=0}^{L-1} v_{\omega((p,\ell))}(x - (p,\ell)) \quad \text{on } L^2(\Sigma_p)$$

with Neumann boundary conditions on  $\partial\Sigma_p$ . By the standard Neumann bracketing, we learn

$$H_{\omega,L}^N \geq \bigoplus_{p \in \Lambda} \tilde{H}_p^N \quad \text{on } L^2(C_L(0)) \cong \bigoplus_{p \in \Lambda} L^2(\Sigma_p),$$

and hence, in particular,

$$(3.1) \quad \inf \sigma(H_{\omega,L}^N) \geq \min_{p \in \Lambda} \inf \sigma(\tilde{H}_p^N).$$

Under our assumptions, one of the following holds for each  $p \in \Lambda$ :

(a)<sub>p</sub>:  $\omega((p,\ell)) > m$  for some  $\ell$ , or  $v_{\omega((p,\ell))} \not\underset{d}{\sim} v_{\omega((p,\ell'))}$  for some  $\ell, \ell' \in \{0, \dots, L-1\}$ .

(b)<sub>p</sub>: For all  $\ell, \ell' \in \{0, \dots, L-1\}$ ,  $\omega((p,\ell)) \leq m$  and  $v_{\omega((p,\ell))} \underset{d}{\sim} v_{\omega((p,\ell'))}$ .

We note that the probability of (b)<sub>p</sub> to occur is less than  $\mu^{-L}$  with some  $\mu < 1$  independent of  $L$ . Since  $\{\omega(\gamma)\}$  are independent, we have

$$(3.2) \quad \mathbb{P}((b)_p \text{ holds for some } p \in \Lambda) \leq L^d \mu^{-L},$$

which is small if  $L$  is large. For the moment, then, we suppose (a)<sub>p</sub> holds for all  $p \in \Lambda$ .

We denote  $V^p(x)$  be the potential function of  $\tilde{H}_p^N$  on  $\Sigma_p$ . Let

$$\hat{\Sigma}_p = (p + [0, 1]^{d-1}) \times (\mathbb{R}/(2L\mathbb{Z}))$$

and set  $\hat{V}^p(x) = V^p(x', |x_d|)$  for  $x = (x', x_d) \in (p + [0, 1]^{d-1}) \times [-L, L] \cong \hat{\Sigma}_p$ , i.e.,  $\hat{V}^p$  is the extension of  $V^p$  by the reflection at  $\{x_d = 0\}$ . We note  $\hat{V}^p$  is continuous on  $\hat{\Sigma}_p$ . We now consider

$$\hat{H}_p^N = \Delta + \hat{V}^p \quad \text{on } L^2(\hat{\Sigma}_p)$$

with Neumann boundary conditions. It is easy to see

$$(3.3) \quad \inf \sigma(\tilde{H}_p^N) \geq \inf \sigma(\hat{H}_p^N).$$

In fact, if  $\Phi$  is the ground state of  $\tilde{H}_p^N$ , then we extend  $\Phi$  by reflection to obtain  $\hat{\Phi} \in H^1(\hat{\Sigma}_p)$  and we have

$$\frac{\langle \hat{H}_p^N \hat{\Phi}, \hat{\Phi} \rangle}{\|\hat{\Phi}\|^2} = \frac{\langle \tilde{H}_p^N \Phi, \Phi \rangle}{\|\Phi\|^2} = \inf \sigma(\tilde{H}_p^N)$$

and the claim (3.3) follows by the variational principle.

Since we assume  $(a)_p$ ,  $\Sigma_p$  can be decomposed to subsegments  $\Sigma_p = \bigcup_{j=1}^K \Xi_j$  such that each  $\Xi_j$  satisfies the following conditions: We write

$$\Xi_j = \bigcup_{\ell=0}^{\nu} C_1(p, \kappa + \ell), \quad \kappa \in \mathbb{Z}, \quad 0 \leq \nu < L,$$

and

$$\hat{V}^p(x) = v_{\beta(\ell)}(x - (p, \ell)) \quad \text{for } x \in C_1(p, \kappa + \ell), \quad \ell \in \{0, \dots, \nu\}$$

with  $\beta(\ell) \in \{1, \dots, M\}$ . Then either one of the following holds

- (i)  $\beta(0) \in \{m+1, \dots, M\}$ ;  $\beta(\ell) \in \{1, \dots, m\}$  for  $\ell \geq 1$ ; and  $v_{\beta(\ell)} \underset{d}{\sim} v_{\beta(\ell')}$  for  $\ell, \ell' \in \{1, \dots, \nu\}$ .
- (ii)  $\beta(\ell) \in \{1, \dots, m\}$  for all  $\ell$ ;  $v_{\beta(0)} \not\underset{d}{\sim} v_{\beta(1)}$ ; and  $v_{\beta(\ell)} \underset{d}{\sim} v_{\beta(\ell')}$  for  $\ell, \ell' \in \{2, \dots, \nu\}$ .

The proof of this claim is an easy combinatorics, though somewhat lengthy to write down using symbols. We omit the details.

We again decompose  $\hat{H}_p^N$ . We denote the restriction of  $\hat{H}_p^N$  to  $\Xi_j$  by  $P_j$  on  $L^2(\Xi_j)$  with Neumann boundary conditions. Then, again by Neumann bracketing, we learn

$$\hat{H}_p^N \geq \bigoplus_{j=1}^{\kappa} P_j \quad \text{on } L^2(\hat{\Sigma}_p) \cong \bigoplus_{j=1}^{\kappa} L^2(\Xi_j),$$

and in particular,

$$(3.4) \quad \inf \sigma(\hat{H}_p^N) \geq \min_j \inf \sigma(P_j).$$

Now if (i) holds for  $\Xi_j$ , then we set  $a = 1$  and use Theorem 2.1 for  $P_j$ . Since  $\inf \sigma(H_{\beta(0)}^N) > 0$  by Assumption A and  $\nu \leq L$ , we learn

$$\inf \sigma(P_j) \geq \frac{1}{C(\nu-1)^2} \geq \frac{1}{C(L-1)^2}.$$

If (ii) holds for  $\Xi_j$ , then we set  $a = 2$  and use Theorem 2.1 for  $P_j$ . Since  $v_{\beta(0)} \not\underset{d}{\sim} v_{\beta(1)}$ , we have  $\inf \sigma(H_{\beta(0)\beta(1)(d)}^N) > 0$ . Thus we have

$$\inf \sigma(P_j) \geq \frac{1}{C(\nu-2)^2} \geq \frac{1}{C(L-2)^2}.$$

Combining these with (3.1), (3.3) and (3.4), we conclude

$$(3.5) \quad \inf \sigma(H_{\omega, L}^N) \geq \frac{c_3}{L^2}$$

with some  $c_3 > 0$ , provided  $(a)_p$  holds for all  $p \in \Lambda$ .

*Proof of Theorems 1.2 and 1.3.* For  $E > 0$ , we set

$$\sqrt{\frac{c_3}{E}} < L \leq \sqrt{\frac{c_3}{E}} + 1$$

so that, by virtue of (3.5),

$$\inf \sigma(H_{\omega,L}^N) > E$$

provided Condition  $(a)_p$  holds for all  $p \in \Lambda$ . As noted in (3.2), the probability of the events that  $(b)_p$  holds for some  $p \in \Lambda$  is bounded by

$$\mathbb{P}((b)_p \text{ for some } p \in \Lambda) \leq L^d \mu^{-L} \leq c_4 E^{-d/2} e^{-c_5 E^{-1/2}}$$

with some  $c_4, c_5 > 0$ . On the other hand, since the potential  $V_0 + V_\omega$  is uniformly bounded, we have

$$\#\{\text{eigenvalues of } H_{\omega,L}^N \leq \alpha\} \leq c_6 L^d$$

for any  $\omega$  with some  $c_6 > 0$ . Thus we have

$$\begin{aligned} L^{-d} \mathbb{E}(\#\{\text{e.v. of } H_{\omega,L}^N \leq E\}) &\leq L^{-d} (c_6 L^d) \mathbb{P}((b)_p \text{ for some } p \in \Lambda) \\ &\leq c_4 c_6 E^{-d/2} e^{-c_5 E^{-1/2}} \leq c_7 e^{-(c_5 - \varepsilon) E^{-1/2}} \end{aligned}$$

for  $0 < \varepsilon < c_5$  with some  $c_7 > 0$ . By the Neumann bracketing again, we have

$$N(E) \leq L^{-d} \mathbb{E}(\#\{\text{e.v. of } H_{\omega,L}^N \leq E\}) \leq c_7 e^{-(c_5 - \varepsilon) E^{-1/2}}$$

and Theorems 1.2 and 1.3 follow immediately from this estimate.  $\square$

In fact, we have proved

$$\liminf_{E \rightarrow +0} \frac{|\log N(E)|}{E^{-1/2}} > 0,$$

and this statement is slightly stronger than (1.6).

*Proof of Theorem 1.4.* (i) This statement is an immediate consequence of Assumption B and Theorem 1.3. We just replace the  $x_d$ -axis by the  $x_j$ -axis where  $v_k \not\sim_j v_\ell$  for some  $k, \ell$ .

(ii) We use the ground state transform as in the proof of Lemmas 2.3–2.5. Under our conditions, there exist  $\mu_1, \dots, \mu_m > 0$  such that

$$\mu_1 \Psi_1(x) = \mu_2 \Psi_2(x) = \dots = \mu_m \Psi_m(x) \quad \text{for } x \in \partial C_1(0).$$

For given  $H_{\omega,L}^N$ , we set

$$\Phi(x) = \mu_k \Psi_k(x) \quad \text{if } x \in C_1(\gamma) \text{ with } \omega(\gamma) = k.$$

Then it is easy to see that  $\Phi$  is the positive ground state of  $H_{\omega,L}^N$  with the energy 0. Let  $Q(\cdot, \cdot)$  be the quadratic form corresponding to  $H_{\omega,L}^N$ . For  $\varphi \in H^1(C_L(0))$ , we set  $f = \varphi/\Phi$ . As in the proof of Lemma 2.3, we have

$$Q(\varphi, \varphi) = \|\Phi(\nabla f)\|^2$$

and hence

$$(\inf \Phi)^2 \|\nabla f\|^2 \leq Q(\varphi, \varphi) \leq (\sup \Phi)^2 \|\nabla f\|^2.$$

This implies

$$K^{-2} \frac{\|\nabla f\|^2}{\|f\|^2} \leq \frac{Q(\varphi, \varphi)}{\|\varphi\|^2} \leq K^2 \frac{\|\nabla f\|^2}{\|f\|^2}$$

where  $K = \max_k \sup(\mu_k \Psi_k) / \min_k \inf(\mu_k \Psi_k)$ . By the min-max principle, we learn

$$\begin{aligned} K^{-2} \#\{\text{e.v. of } (-\Delta)_L^N \leq E\} &\leq \#\{\text{e.v. of } H_{\omega,L}^N \leq E\} \\ &\leq K^2 \#\{\text{e.v. of } (-\Delta)_L^N \leq E\} \end{aligned}$$

where  $(-\Delta)_L^N$  is the Laplacian on  $C_L(0)$  with Neumann boundary conditions. Taking the limit  $L \rightarrow +\infty$ , we have

$$(3.6) \quad K^{-2} c_d E^{d/2} \leq N(E) \leq K^2 c_d E^{d/2},$$

where  $c_d$  is the volume of the unit ball in  $\mathbb{R}^d$ . This completes the proof of Theorem 1.4.  $\square$

## 4 Application to random displacement models

We now consider a model recently studied by Baker, Loss and Stolz in [1, 2]. Combining their results with Theorem 1.2, we show that this model exhibits Lifshitz singularities at the ground state energy.

We consider a random Schrödinger operator of the form:

$$H_\omega = -\Delta + V_\omega \quad \text{on } L^2(\mathbb{R}^d)$$

where

$$V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} q(x - \gamma - \omega(\gamma))$$

with i.i.d. random variables  $\{\omega(\gamma) \mid \gamma \in \mathbb{Z}^d\}$  which take values in  $C_1(0)$ .

**Assumption C.** (1) There exists  $\delta \in (0, 1/2)$  such that  $\omega(\gamma)$  takes values in a finite set

$$\Theta \subset \{x \in \mathbb{R}^d \mid \delta \leq x_j \leq 1 - \delta, \forall j \in \{1, \dots, d\}\}.$$

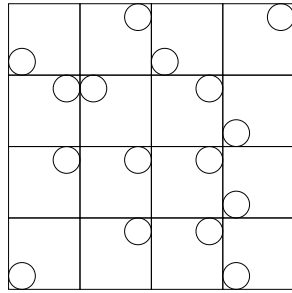
Moreover

$$\Theta \supset \Delta = \{x \in \mathbb{R}^d \mid x_j = \delta \text{ or } 1 - \delta, \forall j \in \{1, \dots, d\}\}$$

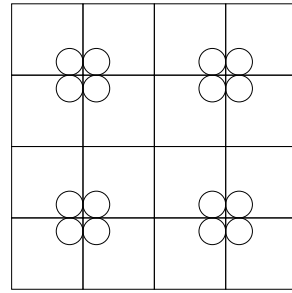
and  $\mathbb{P}(\omega(\gamma) = x) > 0$  for  $x \in \Delta$ .

(2)  $q \in C_0(\mathbb{R}^d)$  and it is supported in  $\{x \mid |x_j| \leq \delta, j \in \{1, \dots, d\}\}$ . Moreover,  $q$  is symmetric about  $\{x \mid x_j = 0\}, j = 1, \dots, d$ .

(3) Let  $H_q^N = -\Delta + q$  on  $L^2(\{|x| \leq 1\})$  with Neumann boundary conditions, and let  $\phi$  be the ground state. Then,  $\phi$  is not a constant outside  $\text{Supp } q$ . Note that this is relevant only if the ground state energy is 0.



(a) A typical random configuration

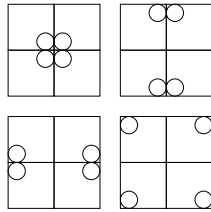


(b) The minimizing configuration

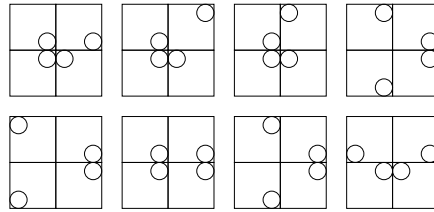
Figure 2: An example in two dimensions

Let  $H_{1,\beta}^N = -\Delta + q(x - \beta)$  on  $L^2(C_1(0))$  with Neumann boundary conditions, where  $\beta \in \Theta$ . Baker, Loss and Stolz [1] showed that  $\inf \sigma(H_{1,\beta}^N)$  takes its minimum (with respect to  $\beta$ ) if and only if  $\beta \in \Delta$ .

In particular, they showed that for  $H_{\omega,2\ell}^N$  the Neumann restriction of  $H_\omega$  to  $C_{2\ell}(0)$  the minimal value of the ground state energy was obtained for clustered configuration (see Fig 2).



(a) The minimal  $2 \times 2$  configurations



(b) Other  $2 \times 2$  configurations

Figure 3:  $2 \times 2$  configurations in two dimensions

We cannot directly apply our result to this model, since  $q(x - \beta)$  is not symmetric for  $\beta \in \Delta$ . However, they also showed that if we consider the operator  $H_\omega$  restricted to  $C_2(0)$  and if  $d \geq 2$ , then the minimum is attained by



$2^d$  symmetric configurations, which are equivalent to each other by translations (see [2] and Fig. 3). Thus, we can apply our results by considering  $H_\omega$  as a  $2\mathbb{Z}^d$ -ergodic random Schrödinger operators, i.e., by considering  $C_2(0)$  as the unit cell. Then, this model satisfies Assumption A with  $M = (\#\Theta)^{2^d}$  and  $m = 2^d$ .

**Theorem 4.1.** *Let  $d \geq 2$ , and suppose Assumption C for some  $\delta \in (0, 1/2)$ . Then, (1.6) holds at the bottom of the spectrum of  $H_\omega$ , a.s.*

We note that if  $d = 1$ , this result does not hold, and the IDS may have logarithmic singularity at the bottom of the spectrum ([2]). In view of our results, such singularities can occur for the lack of symmetry of the minimizing configurations.

## 5 The alloy type model studied in [10]

In a previous paper on Lifshitz tails for sign indefinite alloy type random Schrödinger operators [10], we studied the model (1.1) for a single site potential  $V$  satisfying the reflection symmetry Assumption B.

We now recall some of the results of that work. Let the support of the random variables  $(\omega_\gamma)_\gamma$  be contained in  $[a, b]$  and assume both  $a$  and  $b$  belong to the essential support of the random variables.

Consider now the operator  $H_\lambda^N = -\Delta + \lambda V$  with Neumann boundary conditions on the cube  $C_1(0) = [0, 1]^d$ . Its spectrum is discrete, and we let  $E_-(\lambda)$  be its ground state energy. It is a simple eigenvalue and  $\lambda \mapsto E_-(\lambda)$  is a real analytic concave function defined on  $\mathbb{R}$ . Let  $E_-$  be the infimum of the almost sure spectrum of  $H_\omega$  then

**Proposition 5.1** ([10]). *Under Assumption B,*

$$E_- = \inf(E_-(a), E_-(b)).$$

As for Lifshitz tails, we proved

**Theorem 5.1** ([10]). *Suppose Assumption B is satisfied. Assume moreover that*

$$(5.1) \quad E_-(a) \neq E_-(b).$$

*Then*

$$\limsup_{E \rightarrow E_-^+} \frac{\log |\log N(E)|}{\log(E - E_-)} \leq -\frac{d}{2} - \alpha_+$$

*where we have set  $c = a$  if  $E_-(a) < E_-(b)$  and  $c = b$  if  $E_-(a) > E_-(b)$ , and*

$$\alpha_+ = -\frac{1}{2} \liminf_{\varepsilon \rightarrow 0} \frac{\log |\log \mathbb{P}(\{|c - \omega_0| \leq \varepsilon\})|}{\log \varepsilon} \geq 0.$$

The technique developed in [10] did not allow us to treat the case  $E_-(a) = E_-(b)$ . Clearly, if the random variables  $(\omega_\gamma)_\gamma$  are non trivial and Bernoulli distributed, i.e., if  $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) = 1$  and  $\mathbb{P}(\omega_0 = a) > 0$ ,  $\mathbb{P}(\omega_0 = b) > 0$ , Theorem 1.4 tells us that the Lifshitz tails hold if and only if  $aV \not\sim_j bV$  for some  $j \in \{1, \dots, d\}$  (see (1.9)). So we are just left with the case when the random variables  $(\omega_\gamma)_\gamma$  are not Bernoulli distributed.

We prove

**Theorem 5.2.** *Suppose assumption B is satisfied and that*

$$(5.2) \quad E_-(a) = E_-(b).$$

*Assume moreover that the i.i.d. random variables  $(\omega_\gamma)_\gamma$  are not Bernoulli distributed i.e.  $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) < 1$ .*

*Then*

$$(5.3) \quad \limsup_{E \rightarrow E_-^+} \frac{\log |\log N(E)|}{\log(E - E_-)} \leq -\frac{1}{2}.$$

So we show that Lifshitz tails also hold in this case. As already noted we believe that (5.4) is not optimal and that  $-1/2$  should be replaced by  $-d/2$ . Moreover, depending on the tail of the distributions of the random variables  $(\omega_\gamma)_\gamma$  near  $a$  and  $b$ , the lim sup in (5.4) should be a limit, the inequality should become an equality, the exponent  $-1/2$  should be replaced by  $-d/2$  plus a possibly vanishing constant (see Section 0 of [10] for the case  $E_-(a) \neq E_-(b)$ ).

Combining Theorems 5.1 and 5.2 with the Wegner estimates obtained in [9, 6] and the multiscale analysis as developed in [5], we learn

**Theorem 5.3.** *Assume Assumption B. Assume, moreover, that the common distribution of the random variables admits an absolutely continuous density. Then, the bottom edge of the spectrum of  $H_\omega$  exhibits complete localization in the sense of [5].*

This result improves upon Theorem 0.3 of [10].

## 5.1 The proof of Theorem 5.2

Recall that  $H_{\omega,L}^N$  is defined in (1.3). It is well known that, at  $E$ , a continuity point of  $N(E)$ , the sequence

$$N_L^N(E) = \mathbb{E} \left( \frac{\#\{\text{eigenvalues of } H_{\omega,L}^N \leq E\}}{L^d} \right)$$

is decreasing and converges to  $N(E)$  (see e.g. [11, 7]). As

$$(5.4) \quad N_L^N(E) \leq C \mathbb{P}(\{\inf \sigma(H_{\omega,L}^N) \leq E\})$$

it is sufficient to prove an upper bound for  $\mathbb{P}(\{\inf \sigma(H_{\omega,L}^N) \leq E\})$  for a well chosen value of  $L$ .

Define  $E_{-,L}(\omega) = \inf \sigma(H_{\omega,L}^N)$ . It only depends on  $(\omega_\gamma)_{\gamma \in Z_L}$ , where

$$Z_L = \{\gamma \in \mathbb{Z}^d \mid 0 \leq \gamma_j < L, j = 1, \dots, d\}.$$

One has

**Lemma 5.1.** *The function  $\omega \mapsto E_{-,L}(\omega)$  is real analytic and strictly concave on  $[a, b]^{Z_L}$ .*

*Proof.* Though this is certainly a well known result, for the sake of completeness, we give the proof. The ground state being simple,  $\omega \mapsto E_{-,L}(\omega)$  is real analytic in  $\omega$ .

As  $H_\omega$  depends affinely on  $\omega$ , by the variational characterization of the ground state energy,  $E_{-,L}(\omega)$  is the infimum of a family of affine functions of  $\omega$ . So it is concave.

The strict concavity is obtained using perturbation theory. Let  $\varphi_L(\omega)$  be the unique normalized positive ground state associated to  $E_{-,L}(\omega)$  and  $H_{\omega,L}^N$ . The ground state energy being simple, this ground state is a real analytic function of  $\omega$ ; differentiating once the eigenvalue equation and the normalization condition of the ground state, as the ground state is normalized and real, one obtains

$$(5.5) \quad (H_{\omega,L}^N - E_{-,L}(\omega))\partial_{\omega_\gamma}\varphi_L(\omega) = (\partial_{\omega_\gamma}E_{-,L}(\omega) - V(\cdot - \gamma))\varphi_L(\omega)$$

and

$$(5.6) \quad \langle \partial_{\omega_\gamma}\varphi_L(\omega), \varphi_L(\omega) \rangle = 0.$$

A second differentiation yields

$$\begin{aligned} (H_{\omega,L}^N - E_{-,L}(\omega))\partial_{\omega_\gamma\omega_\beta}^2\varphi_L(\omega) &= \partial_{\omega_\gamma\omega_\beta}^2E_{-,L}(\omega)\varphi_L(\omega) \\ &\quad + (\partial_{\omega_\gamma}E_{-,L}(\omega) - V(\cdot - \gamma))\partial_{\omega_\beta}\varphi_L(\omega) \\ &\quad + (\partial_{\omega_\beta}E_{-,L}(\omega) - V(\cdot - \beta))\partial_{\omega_\gamma}\varphi_L(\omega). \end{aligned}$$

Hence, using (5.5) and (5.6), we compute

$$\begin{aligned} \partial_{\omega_\gamma\omega_\beta}^2E_{-,L}(\omega) &= -\langle V(\cdot - \gamma)\partial_{\omega_\beta}\varphi_L(\omega), \varphi_L(\omega) \rangle - \langle V(\cdot - \beta)\partial_{\omega_\gamma}\varphi_L(\omega), \varphi_L(\omega) \rangle \\ &= -2\text{Re}(\langle (H_{\omega,L}^N - E_{-,L}(\omega))^{-1}\psi_\beta, \psi_\gamma \rangle) \end{aligned}$$

where

- $\psi_\gamma = \Pi V(\cdot - \gamma)\varphi_L(\omega)$
- $\Pi$  is the orthogonal projector on the orthogonal to  $\varphi_L(\omega)$ .

Hence, for  $(a_\gamma)_\gamma$  complex numbers,

$$\sum_{\gamma, \beta} \partial_{\omega_\gamma \omega_\beta}^2 E_{-,L}(\omega) a_\gamma \bar{a}_\beta = -2\text{Re} \left( \langle (H_{\omega,L}^N - E_{-,L}(\omega))^{-1} \Pi u_a, \Pi u_a \rangle \right)$$

where  $u_a = (\sum_\gamma a_\gamma V(\cdot - \gamma)) \varphi_L(\omega)$ . Note that, as  $V$  is not trivial, the assumption  $E_-(a) = E_-(b)$  implies that  $V$  changes sign, i.e., there exists  $x_+ \neq x_-$  such that  $V(x_-) \cdot V(x_+) < 0$ . Now, the vector  $\Pi u_a$  vanishes if and only if  $u_a$  is colinear to  $\varphi_L(\omega)$  which cannot happen as  $V$  is not constant and  $\varphi_L(\omega)$  does not vanish on open sets by the unique continuation principle. On the other hand,  $E_{-,L}(\omega)$  being a simple eigenvalue associated to  $\varphi_L(\omega)$ ,  $\Pi(H_{\omega,L}^N - E_{-,L}(\omega))^{-1} \Pi \geq c \Pi$  for some  $c > 0$ . So the Hessian of  $\omega \mapsto E_{-,L}(\omega)$  is positive definite. This completes the proof of Lemma 5.1.  $\square$

We now turn to the proof of Theorem 5.2. As the random variables are not Bernoulli distributed, i.e.,  $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) < 1$ , we can fix  $\varepsilon > 0$  sufficiently small such that  $\mathbb{P}(\omega_0 \in [a, a + \varepsilon]) + \mathbb{P}(\omega_0 \in (b - \varepsilon, b]) < 1$ . By strict concavity of  $E_-(\lambda)$ , one has  $E_-(a) < E_-(a + \varepsilon)$  and  $E_-(b) < E_-(b - \varepsilon)$ . In Section 2, we have proved

**Lemma 5.2.** *Assume  $E_-(a) = E_-(b)$ . There exists  $C > 0$  such, for all  $L \geq 0$ , if  $\omega \in \{a, b, a + \varepsilon, b - \varepsilon\}^{Z^L}$  is such that*

**(P)** *for all  $p \in \Lambda$ , there exists  $\ell \in \{0, \dots, L - 1\}$  such that*

$$\omega_{(p,\ell)} \in \{a + \varepsilon, b - \varepsilon\}$$

then

$$(5.7) \quad E_{-,L}(\omega) \geq E_-(a) + \frac{1}{CL^2}.$$

To complete the proof of Theorem 5.2, we first extend lemma 5.2 using the concavity of the ground state energy to

**Lemma 5.3.** *Assume  $E_-(a) = E_-(b)$ . There exists  $C > 0$  such, for all  $L \geq 0$ , if  $\omega \in \Omega_L$  is such that*

**(P')** *for all  $p \in \Lambda$ , there exists  $\ell \in \{0, \dots, L - 1\}$  such that*

$$\omega_{(p,\ell)} \in [a + \varepsilon, b - \varepsilon]$$

then (5.7) holds (with the same constant as in Lemma 5.2).

Let us postpone the proof of this result to complete that of Theorem 5.2. Pick  $E > E_-(a) = E_-(b)$ . We use (5.4) and pick  $L = c(E - E_-(a))^{1/2}$ . Pick  $c > 0$  sufficiently small that  $Cc^2 < 1$ . Then, Lemma 5.2 tells us that,

if  $\omega \in [a, b]^{Z_L}$  satisfies (P'), then  $E_-(\omega) > E$ . So, the set  $\Omega_L(E) := \{\omega \in \Omega_L; E_-(\omega) > E\}$  satisfies

$$\Omega_L \setminus \Omega_L(E) \subset \{\omega \in \Omega_L; \exists p \in \Lambda, \text{ s.t. } \forall \ell, \omega_{(p,\ell)} \in [a, a + \varepsilon) \cup (b - \varepsilon, b]\}.$$

Hence,

$$\begin{aligned} \mathbb{P}(\Omega_L \setminus \Omega_L(E)) &\leq \sum_{p \in \Lambda} \mathbb{P}(\{\omega_{(p,\ell)} \in [a, a + \varepsilon) \cup (b - \varepsilon, b] \text{ for } \forall \ell\}) \\ &= L^{d-1} (\mathbb{P}(\omega_0 \in [a, a + \varepsilon)) + \mathbb{P}(\omega_0 \in (b - \varepsilon, b]))^L \end{aligned}$$

This yields the announced exponential decay and completes the proof of Theorem 5.2.  $\square$

*Proof of Lemma 5.3.* We will proceed in two steps. First, we prove that, if  $\omega$  satisfies (P') and all its coordinates that are not in  $[a + \varepsilon, b - \varepsilon]$  are either equal to  $a$  or to  $b$ , then (5.7) holds (with the same constant as in Lemma 5.2). This comes from the concavity of the ground state and the fact that any such point is a convex combination of points satisfying (P). Indeed, take such a point  $\omega$  and let  $\Gamma(\omega)$  be the set of coordinates such that  $\omega_\gamma \in [a + \varepsilon, b - \varepsilon]$ . Define  $K(\omega) = \{a + \varepsilon, b - \varepsilon\}^{\Gamma(\omega)}$ . Then, there exists a convex combination  $(\mu_\eta)_{\eta \in K(\omega)}$  such that

$$(\omega_\gamma)_{\gamma \in \Gamma(\omega)} = \sum_{\eta \in K(\omega)} \mu_\eta \eta, \quad \sum_{\eta \in K(\omega)} \mu_\eta = 1, \quad \mu_\eta \geq 0.$$

Hence,

$$\omega = \sum_{\eta \in K(\omega)} \mu_\eta \tilde{\eta} \text{ where } (\tilde{\eta})_\gamma = \begin{cases} \eta_\gamma & \text{if } \gamma \in \Gamma(\omega), \\ \omega_\gamma & \text{if } \gamma \notin \Gamma(\omega). \end{cases}$$

That  $\omega$  satisfies (5.7) then follows from the concavity of  $\omega \mapsto E_{-,L}(\omega)$ , that is Lemma 5.1, and from Lemma 5.2.

To complete the proof of Lemma 5.3, it suffices to show that a point  $\omega$  satisfying (P') can be written a convex combination of points of the type defined above. This is done as above. Indeed, pick  $\omega$  satisfying (P'). Define  $L(\omega) = \{a, b\}^{(Z_L \setminus \Gamma(\omega))}$ . Then, there exists a convex combination  $(\mu_\eta)_{\eta \in L(\omega)}$  such that

$$(\omega_\gamma)_{\gamma \in (Z_L \setminus \Gamma(\omega))} = \sum_{\eta \in L(\omega)} \mu_\eta \eta, \quad \sum_{\eta \in L(\omega)} \mu_\eta = 1, \quad \mu_\eta \geq 0.$$

Hence,

$$\omega = \sum_{\eta \in L(\omega)} \mu_\eta \tilde{\eta} \text{ where } (\tilde{\eta})_\gamma = \begin{cases} \eta_\gamma & \text{if } \gamma \notin \Gamma(\omega), \\ \omega_\gamma & \text{if } \gamma \in \Gamma(\omega). \end{cases}$$

That  $\omega$  satisfies (5.7) then follows from the concavity of  $\omega \mapsto E_{-,L}(\omega)$  and from the first step. This completes the proof of Lemma 5.3.  $\square$

## References

- [1] J. Baker, M. Loss, and G. Stolz: Minimizing the ground state energy of an electron in a randomly deformed lattice. *Commun. Math. Phys.* **283**, (2):397–415, 2008.
- [2] J. Baker, M. Loss, and G. Stolz: Low energy properties of the random displacement models. Preprint, <http://arxiv.org/abs/0808.0670>.
- [3] R. Carmona and J. Lacroix: *Spectral Theory of Random Schrödinger Operators*. Birkhäuser, Boston, 1990.
- [4] G. B. Folland: *Introduction to Partial Differential Equations*. (2nd Ed.) Princeton Univ. Press, New Jersey, 1995.
- [5] F. Germinet and A. Klein. Bootstrap multiscale analysis and localization in random media. *Commun. Math. Phys.*, **222**, (2):415–448, 2001.
- [6] P. Hislop and F. Klopp. The integrated density of states for some random operators with nonsign definite potentials. *J. Func. Anal.*, **195**, 12–47, 2002.
- [7] W. Kirsch: Random Schrödinger operators. In A. Jensen H. Holden (Eds.): *Schrödinger Operators*, (Proceedings, Sønderborg, Denmark 1988.) *Lecture Notes in Physics* **345**, Berlin, 1989. Springer-Verlag.
- [8] W. Kirsch: Random Schrödinger operators and the density of states. In *Stochastic aspects of classical and quantum systems*, *Lecture Notes in Mathematics* **1109**, 68–102. Springer, Berlin, 1985.
- [9] F. Klopp. Localization for some continuous random Schrödinger operators. *Commun. Math. Phys.*, **167**, 553–570, 1995.
- [10] F. Klopp, S. Nakamura: Spectral extrema and Lifshitz tails for non monotonous alloy type models. *Commun. Math. Phys.* **287**, 1133–1143, 2009.
- [11] L. Pastur and A. Figotin. *Spectra of random and almost-periodic operators*, *Grundlehren der Mathematischen Wissenschaften* **297**. Springer-Verlag, Berlin, 1992.
- [12] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press, New York, 1978.
- [13] P. Stollmann. *Caught by disorder : bound states in random media*, *Progress in Mathematical Physics* **20**. Birkhäuser, Boston, MA, 2001.

- [14] I. Veselić. Integrated density of states and Wegner estimates for random Schrödinger operators. In *Spectral theory of Schrödinger operators, Contemp. Math.* **340**, 97–183. American Math. Soc., Providence, RI, 2004.
- [15] I. Veselić. *Existence and regularity properties of the integrated density of states of random Schrödinger operators. Lecture Notes in Mathematics* **1917**. Springer-Verlag, Berlin, 2008.