



HAL
open science

On a mixed problem in Diophantine approximation

Yann Bugeaud, Bernard de Mathan

► **To cite this version:**

Yann Bugeaud, Bernard de Mathan. On a mixed problem in Diophantine approximation. *Acta Arithmetica*, 2009, à paraître. hal-00367593

HAL Id: hal-00367593

<https://hal.science/hal-00367593>

Submitted on 11 Mar 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On a mixed problem in Diophantine approximation

Yann BUGEAUD & Bernard de MATHAN

Abstract. *Let d be a positive integer. Let p be a prime number. Let α be a real algebraic number of degree $d+1$. We establish that there exist a positive constant c and infinitely many algebraic numbers ξ of degree d such that $|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_p, 1\} < cH(\xi)^{-d-1} (\log 3H(\xi))^{-1/d}$. Here, $H(\xi)$ and $\text{Norm}(\xi)$ denote the naïve height of ξ and its norm, respectively. This extends an earlier result of de Mathan and Teulié that deals with the case $d = 1$.*

1. Introduction

In analogy with the Littlewood conjecture, de Mathan and Teulié [7] proposed recently a ‘mixed Littlewood conjecture’. For any prime number p , the usual p -adic absolute value $|\cdot|_p$ is normalized in such a way that $|p|_p = p^{-1}$. We denote by $\|\cdot\|$ the distance to the nearest integer.

De Mathan–Teulié conjecture. *For every real number α and every prime number p , we have*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0. \quad (1.1)$$

Obviously, the above conjecture holds if α is rational or has unbounded partial quotients in its continued fraction expansion. Thus, it only remains to consider the case when α is an element of the set \mathbf{Bad}_1 of badly approximable real numbers, where

$$\mathbf{Bad}_1 = \{\alpha \in \mathbf{R} : \inf_{q \geq 1} q \cdot \|q\alpha\| > 0\}.$$

De Mathan and Teulié [7] proved that (1.1) holds for every quadratic real number α (recall that such a number is in \mathbf{Bad}_1) but, despite several recent results [4, 3], the general conjecture is still unsolved.

2000 *Mathematics Subject Classification* : 11J04; 11J61, 11J68 .

If we rewrite (1.1) under the form

$$\inf_{a,q \geq 1, \gcd(a,q)=1} q^2 \cdot \left| \alpha - \frac{a}{q} \right| \cdot |q|_p = 0,$$

then we have $|q|_p = \min\{|\text{Norm}(q/a)|_p, 1\}$. Hence, upon replacing α by $1/\alpha$, the de Mathan–Teulié conjecture can be reformulated as follows: For every irrational real number α , for every prime number p and every positive real number ε , there exists a non-zero rational number ξ satisfying

$$|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_p, 1\} < \varepsilon H(\xi)^{-2}.$$

Throughout this paper, the height $H(P)$ of an integer polynomial $P(X)$ is the maximal of the absolute values of its coefficients. The height $H(\xi)$ of an algebraic number ξ is the height of its minimal defining polynomial over the rational integers $a_0 + a_1X + \dots + a_dX^d$, and the norm of ξ , denoted by $\text{Norm}(\xi)$, is the rational number $(-1)^d a_0/a_d$.

This reformulation suggests us to ask the following question.

Problem 1. *Let d be a positive integer. Let α be a real number that is not algebraic of degree less than or equal to d . For every prime number p and every positive real number ε , does there exist a non-zero real algebraic number ξ of degree at most d satisfying*

$$|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_p, 1\} < \varepsilon H(\xi)^{-d-1}?$$

The answer to Problem 1 is clearly positive, unless (perhaps) when α is an element of the set \mathbf{Bad}_d of real numbers that are badly approximable by algebraic numbers of degree at most d , where

$$\mathbf{Bad}_d = \{\alpha \in \mathbf{R} : \text{There exists } c > 0 \text{ such that } |\alpha - \xi| > cH(\xi)^{-d-1}, \\ \text{for all algebraic numbers } \xi \text{ of degree at most } d\}.$$

For $d \geq 1$, the set \mathbf{Bad}_d contains the set of algebraic numbers of degree $d+1$, but it remains an open problem to decide whether this inclusion is strict for $d \geq 2$; see the monograph [2] for more information. The purpose of the present note is to give a positive answer to Problem 1 for every positive integer d and every real algebraic number α of degree $d+1$. This extends the result from [7] which deals with the case $d = 1$.

2. Results

Throughout this paper, for a prime number p , a number field \mathbf{K} , and a non-Archimedean place v on \mathbf{K} lying above p , we normalize the absolute value $|\cdot|_v$ in such a way that $|\cdot|_v$ and $|\cdot|_p$ coincide on \mathbf{Q} .

Our main result includes a positive answer to Problem 1 when α is a real algebraic number of degree $d+1$.

Theorem 1. *Let d be a positive integer. Let α be a real algebraic number of degree $d+1$ and denote by r the unit rank of $\mathbf{Q}(\alpha)$. Let p be a prime number. There exist positive constants c_1, c_2, c_3 and infinitely many real algebraic numbers ξ of degree d such that*

$$|\alpha - \xi| < c_1 H(\xi)^{-d-1}, \quad (2.1)$$

$$|\xi|_v < c_2 (\log 3H(\xi))^{-1/(rd)}, \quad (2.2)$$

for every absolute value $|\cdot|_v$ on $\mathbf{Q}(\xi)$ above the prime p , and

$$|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_p, 1\} < c_3 H(\xi)^{-d-1} (\log 3H(\xi))^{-1/r}. \quad (2.3)$$

Theorem 1 extends Théorème 2.1 of [7] that is only concerned with the case $d = 1$.

Under the assumptions of Theorem 1, Wirsing [10] established that there are infinitely many real algebraic numbers ξ satisfying (2.1).

The proof of Theorem 1 is very much inspired by a paper of Peck [8] on simultaneous rational approximation to real algebraic numbers. Roughly speaking, we use a method dual to Peck's to construct integer polynomials $P(X)$ that take small values at α , and we need an extra argument to ensure that our polynomials have a root ξ very close to α .

De Mathan [6] used the theory of linear forms in non-Archimedean logarithms to prove that Theorem 1 for $d = 1$ is best possible, in the sense that the absolute value of the exponent of $(\log 3H(\xi))$ in (2.2) cannot be too large. Next theorem extends this result to all values of d .

Theorem 2. *Let p be a prime number, d a positive integer and α a real algebraic number of degree $d+1$. Let λ be a positive real number. There exists a positive real number $\kappa = \kappa(\lambda)$ such that for every non-zero real algebraic number ξ of degree d satisfying*

$$|\alpha - \xi| \leq \lambda H(\xi)^{-d-1} \quad (2.4)$$

we have

$$|\xi|_v \geq (\log 3H(\xi))^{-\kappa}$$

for at least one absolute value $|\cdot|_v$ on $\mathbf{Q}(\xi)$ above the prime p .

As in [6], the proof of Theorem 2 rests on the theory of linear forms in non-Archimedean logarithms.

Let d be a positive integer. We recall that it follows from the p -adic version of the Schmidt Subspace Theorem that for every algebraic number α of degree $d+1$ and for every positive real number ε , there are only finitely many non-zero integer polynomials $P(X) = a_0 + a_1X + \dots + a_dX^d$ of degree at most d , with $a_0 \neq 0$, that satisfy

$$|P(\alpha)| \cdot |a_0|_p < H(P)^{-d-\varepsilon}.$$

Let ξ be a real algebraic number of degree at most d , and denote by $P(X) = a_0 + a_1X + \dots + a_dX^d$ its minimal defining polynomial over \mathbf{Z} . Then,

$$\min\{|\text{Norm}(\xi)|_p, 1\} \geq |a_0|_p$$

and there exists a constant $c(\alpha)$, depending only on α , such that

$$|P(\alpha)| \leq c(\alpha) H(\xi) \cdot |\xi - \alpha|.$$

Let ε be a positive real number. Applying the above statement deduced from the p -adic version of the Schmidt Subspace Theorem to these polynomials $P(X)$, we deduce that

$$|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_p, 1\} \geq H(P)^{-d-1-\varepsilon}$$

holds if $H(P)$ is sufficiently large. This implies that if ξ satisfies (2.4) and if $H(\xi)$ is sufficiently large, then we have

$$|\text{Norm}(\xi)|_p \geq H(\xi)^{-\varepsilon},$$

accordingly

$$\max_{v|p} |\xi|_v \geq H(\xi)^{-\varepsilon/d}.$$

The result of Theorem 2 is more precise, however we cannot obtain a good lower bound for $|\text{Norm}(\xi)|_p$.

We conclude this section by pointing out that Einsiedler and Kleinbock [4] showed that a slight modification of the de Mathan–Teulié conjecture easily follows from a theorem of Furstenberg [5, 1].

Theorem EK. *Let p_1 and p_2 be distinct prime numbers. Then*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_{p_1} \cdot |q|_{p_2} = 0$$

holds for every real number α .

In view of Theorem EK, we formulate the following question, presumably easier to solve than Problem 1.

Problem 2. *Let d be a positive integer. Let α be a real number that is not algebraic of degree less than or equal to d . For every distinct prime numbers p_1, p_2 and every positive real number ε , does there exist a non-zero real algebraic number ξ of degree at most d satisfying*

$$|\alpha - \xi| \cdot \min\{|\text{Norm}(\xi)|_{p_1}, 1\} \cdot \min\{|\text{Norm}(\xi)|_{p_2}, 1\} < \varepsilon H(\xi)^{-d-1}?$$

Theorem EK gives a positive answer to Problem 2 when $d = 1$.

The sequel of the paper is organized as follows. We gather several auxiliary results in Section 3, and Theorems 1 and 2 are established in Sections 4 and 5, respectively.

In the next sections, we fix a real algebraic number field \mathbf{K} of degree $d + 1$. The notation $A \ll B$ means, unless specific indications, that the implicit constant depends on \mathbf{K} . Furthermore, we write $A \asymp B$ if we have simultaneously $A \ll B$ and $B \ll A$.

3. Auxiliary lemmas

Let \mathbf{K} be a real algebraic number field of degree $d + 1$. Let \mathcal{O} denote its ring of integers, and let $\alpha_0 = 1, \alpha_1, \dots, \alpha_d$ be a basis of \mathbf{K} . Let D be a positive integer satisfying

$$D(\mathbf{Z} + \alpha_1\mathbf{Z} + \dots + \alpha_d\mathbf{Z}) \subset \mathcal{O} \subset \frac{1}{D}(\mathbf{Z} + \alpha_1\mathbf{Z} + \dots + \alpha_d\mathbf{Z})$$

and the corresponding inequalities for the dual basis β_0, \dots, β_d defined by

$$\mathrm{Tr}(\alpha_i\beta_j) = \delta_{i,j},$$

where Tr is the trace and $\delta_{i,j}$ is the Kronecker symbol.

We denote by $\sigma_0 = \mathrm{Id}, \dots, \sigma_d$, the complex embeddings of \mathbf{K} , numbered in such a way that $\sigma_0, \dots, \sigma_{r_1-1}$ are real, $\sigma_{r_1}, \dots, \sigma_d$ are imaginary and $\sigma_{r_1+r_2+j} = \overline{\sigma_{r_1+j}}$ for $0 \leq j < r_2$. Set also $r = r_1 + r_2 - 1$, and let $\varepsilon_1, \dots, \varepsilon_r$ be multiplicatively independent units in \mathbf{K} .

Lemma 1. *Let η be a unit in \mathcal{O} such that $-1 < \eta < 1$ and define the real number N by $|\eta| = N^{-1}$. The conditions*

$$|\sigma_j(\eta)| \asymp N^{1/d}, \quad 0 < j \leq d, \quad (3.1)$$

and

$$|\sigma_i(\eta)| \asymp |\sigma_j(\eta)|, \quad 0 < i < j \leq d, \quad (3.2)$$

are equivalent. Let $\gamma \neq 0$ be in \mathbf{K} and let Δ be a positive integer such that $\Delta\gamma \in \mathcal{O}$. If η satisfies (3.1) or (3.2), write

$$\gamma\eta = a_0 + \dots + a_d\alpha_d,$$

with a_0, \dots, a_d in \mathbf{Q} . We have $D\Delta a_k \in \mathbf{Z}$ for $k = 0, \dots, d$ and

$$\max_{k=0, \dots, d} |a_k| \asymp N^{1/d},$$

where the implicit constants depend on γ .

Proof. Since η is a unit, we have

$$\prod_{0 \leq j \leq d} \sigma_j(\eta) = \pm 1.$$

and (3.1) and (3.2) are clearly equivalent. The formula

$$a_k = \mathrm{Tr}(\gamma\eta\beta_k) = \gamma\eta\beta_k + \sum_{j=1}^d \sigma_j(\eta)\sigma_j(\gamma\beta_k)$$

implies that if η satisfies (3.1), then

$$|a_k| \ll N^{1/d}, \quad 0 \leq k \leq d.$$

Combined with

$$\sigma_1(\gamma)\sigma_1(\eta) = a_0 + \dots + a_d\sigma_1(\alpha_d),$$

this shows that

$$N^{1/d} \asymp |\sigma_1(\eta)| \ll \max_{k=0,\dots,d} |a_k|.$$

The proof of the lemma is complete. \square

Let α be a real algebraic number of degree $d + 1$. We keep the above notation with the field $\mathbf{K} = \mathbf{Q}(\alpha)$ and the basis $1, \alpha, \dots, \alpha^d$ of \mathbf{K} over \mathbf{Q} , and we display an immediate consequence of Lemma 1.

Corollary 1. *Let η be a unit in \mathcal{O} such that $-1 < \eta < 1$ and set $N = |\eta|^{-1}$. Then*

$$D\Delta\gamma\eta = P(\alpha),$$

where $P(X)$ is an integral polynomial of degree at most d satisfying

$$H(P) \asymp N^{1/d}, \quad |P(\alpha)| \asymp N^{-1},$$

and thus

$$|P(\alpha)| \asymp H(P)^{-d}.$$

Denote by τ_j , $j = 0, \dots, d$ the embeddings of \mathbf{K} into \mathbf{C}_p . Recall that the absolute value $|\cdot|_p$ on \mathbf{Q} has a unique extension to \mathbf{C}_p , that we also denote by $|\cdot|_p$. In Lemmata 2 to 4 below we work in \mathbf{C}_p . Let $P(X)$ be an irreducible integer polynomial of degree $n \geq 1$. Let ξ be a complex root of $P(X)$ and ξ_1, \dots, ξ_n be the roots of $P(X)$ in \mathbf{C}_p . We point out that the sets

$$\{|\xi|_v : v \text{ is above } p \text{ on } \mathbf{Q}(\xi)\}$$

and

$$\{|\xi_i|_p : 1 \leq i \leq n\}$$

coincide, since all the absolute values $|\cdot|_v$ and $|\cdot|_p$ coincide on \mathbf{Q} .

Keeping the notation of Lemma 1, we have the following auxiliary result.

Lemma 2. *Assume that $\gamma = \alpha_d$. Then*

$$|a_k|_p \ll \max_{0 \leq j \leq d} |\tau_j(\eta) - 1|_p, \quad 0 \leq k < d,$$

and

$$|a_d - 1|_p \ll \max_{0 \leq j \leq d} |\tau_j(\eta) - 1|_p.$$

Proof. Since

$$\mathrm{Tr}(\alpha_d\beta_k) = 0, \quad \text{for } k = 0, \dots, d-1,$$

we get

$$a_k = \mathrm{Tr}(\gamma\eta\beta_k) = \mathrm{Tr}(\alpha_d(\eta - 1)\beta_k) = \sum_{j=0}^d (\tau_j(\eta) - 1)\tau_j(\alpha_d\beta_k),$$

and deduce that

$$|a_k|_p \ll \max_{0 \leq j \leq d} |\tau_j(\eta) - 1|_p, \quad 0 \leq k < d.$$

It follows from

$$\mathrm{Tr}(\alpha_d \beta_d) = 1$$

that

$$a_d = 1 + \mathrm{Tr}(\alpha_d \beta_d (\eta - 1)) = 1 + \sum_{j=0}^d (\tau_j(\eta) - 1) \tau_j(\alpha_d \beta_d),$$

and we derive that

$$|a_d - 1|_p \ll \max_{0 \leq j \leq d} |\tau_j(\eta) - 1|_p.$$

This concludes the proof. \square

Lemma 3. *Let $0 < \delta < 1$. There exist arbitrarily large positive real numbers H and units η satisfying $\eta = H^{-d}$,*

$$\left| \frac{\sigma_j(\eta)}{\sigma_1(\eta)} - 1 \right| \leq \delta, \quad 2 \leq j \leq d, \quad (3.3)$$

and

$$|\tau_j(\eta) - 1|_p \ll (\log H)^{-1/r}, \quad 0 \leq j \leq d.$$

Proof. By taking suitable powers of the units $\varepsilon_1, \dots, \varepsilon_r$, we can assume that they are all positive, as well as their real conjugates, and that $|\tau_j(\varepsilon_i) - 1|_p < p^{-1/(p-1)}$ for $i = 1, \dots, r$ and $j = 0, \dots, d$. This is possible since $|\tau_j(\varepsilon_i)|_p = 1$ for $i = 1, \dots, r$ and $j = 0, \dots, d$. This allows us to consider the p -adic logarithms of each $\tau_j(\varepsilon_i)$. Our aim is to construct a suitable unit η of the form

$$\eta = \varepsilon_1^{\mu_1 p^s} \dots \varepsilon_r^{\mu_r p^s},$$

where $\mu_i \in \mathbf{Z}$. The conditions for (3.3) are then

$$p^s \left| \mu_1 \log \frac{|\sigma_j(\varepsilon_1)|}{|\sigma_1(\varepsilon_1)|} + \dots + \mu_r \log \frac{|\sigma_j(\varepsilon_r)|}{|\sigma_r(\varepsilon_r)|} \right| \leq C_1, \quad 2 \leq j \leq r,$$

where $C_1 = C_1(\delta) > 0$ is a constant, and

$$\left\| \frac{p^s}{2\pi} (\mu_1 \arg \sigma_j(\varepsilon_1) + \dots + \mu_r \arg \sigma_j(\varepsilon_r)) \right\| \leq C_2, \quad r_1 \leq j \leq r,$$

with $C_2 = C_2(\delta) > 0$. Set

$$Y_j = p^s \left(\mu_1 \log \frac{|\sigma_1(\varepsilon_1)|}{|\sigma_j(\varepsilon_1)|} + \dots + \mu_r \log \frac{|\sigma_1(\varepsilon_r)|}{|\sigma_j(\varepsilon_r)|} \right), \quad 2 \leq j \leq r,$$

and

$$Z_k = \frac{p^s}{2\pi} (\mu_1 \arg \sigma_k(\varepsilon_1) + \dots + \mu_r \arg \sigma_k(\varepsilon_r)) \in \mathbf{R}/\mathbf{Z}, \quad r_1 \leq k \leq r.$$

Taking $0 \leq \mu_i < M$, we have M^r points $(\mu_i)_{1 \leq i \leq r}$. The $(Y_j, Z_k)_{2 \leq j \leq r, r_1 \leq k \leq r}$ are in the product of intervals I_j , $2 \leq j \leq r$, of lengths $O(Mp^s)$ and of r_2 factors identical to \mathbf{R}/\mathbf{Z} . This set can be covered by $C_3(Mp^s)^{r-1}$ sets of diameter at most $\max\{C_1, C_2\}$, where C_3 is a constant depending on δ . By Dirichlet's *Schubfachprinzip*, choosing M such that

$$C_3(Mp^s)^{r-1} < M^r,$$

which can be done with

$$M \asymp p^{(r-1)s},$$

we get that there is $(\mu_1, \dots, \mu_r) \in \mathbf{Z}^r \setminus \{0\}$ such that

$$\max_{1 \leq i \leq r} |\mu_i| \ll M,$$

$$|Y_j| \leq C_1, \quad 2 \leq j \leq r,$$

and

$$\|Z_k\| \leq C_2, \quad r_1 \leq k \leq r.$$

Set then

$$\eta = (\varepsilon_1^{\mu_1} \dots \varepsilon_r^{\mu_r})^{p^s}$$

in such a way that $0 < \eta < 1$ (if needed, just consider $1/\eta$). This choice implies that

$$|\tau_i(\eta) - 1|_p = |\log_p \tau_i(\eta)|_p \leq p^{-s}, \quad 0 \leq i \leq d,$$

and

$$|\log \eta| \ll p^s M \ll p^{rs},$$

and the lemma is proved. □

Lemma 4. *Let $P(X) \in \mathbf{C}_p[X]$ be a polynomial of degree d , and write*

$$P(X) = a_0 + \dots + a_d X^d.$$

Let ξ_i ($1 \leq i \leq d$) be the roots of $P(X)$ in \mathbf{C}_p . Let c be a real number satisfying $0 \leq c \leq 1$. If

$$|\xi_i|_p \leq c, \quad 1 \leq i \leq d,$$

we get

$$|a_k|_p \leq c|a_d|_p, \quad 0 \leq k < d. \tag{3.4}$$

Conversely, if (3.4) holds, then we have

$$|\xi_i|_p \leq c^{1/d}, \quad 1 \leq i \leq d.$$

Proof. Since

$$P(X) = a_d \prod_{1 \leq i \leq d} (X - \xi_i),$$

if $|\xi_i|_p \leq c \leq 1$ for $i = 1, \dots, d$, then we have

$$|a_k|_p \leq c|a_d|_p, \quad \text{for } k = 0, \dots, d-1.$$

Conversely, if

$$|a_k|_p \leq c|a_d|_p, \quad 0 \leq k < d,$$

and if $\xi \in \mathbf{C}_p$ is such that

$$a_d \xi^d + \dots + a_0 = 0,$$

then, there exists k with $0 \leq k < d$ and

$$|a_k \xi^k|_p \geq |a_d \xi^d|_p,$$

thus,

$$|\xi|_p^d \leq |\xi|_p^{d-k} \leq c.$$

This completes the proof of the lemma. \square

We conclude this section with two lemmas used in the proof of Theorem 2. The first of them was proved by Peck [8].

Lemma 5. *There exists a sequence $(\eta_m)_{m \geq 1}$ of positive units in \mathcal{O} such that*

$$\eta_m \asymp e^{-dm}$$

and

$$|\sigma_j(\eta_m)| \asymp e^m, \quad 1 \leq j \leq d.$$

Proof. Let us search the unit η_m under the form

$$\eta_m = \varepsilon_1^{\mu_1} \dots \varepsilon_r^{\mu_r},$$

with $\mu_i \in \mathbf{Z}$. We construct *real* numbers ν_1, \dots, ν_r such that

$$\nu_1 \log \varepsilon_1 + \dots + \nu_r \log \varepsilon_r = -dm \tag{3.5}$$

and

$$\nu_1 \log |\sigma_j(\varepsilon_1)| + \dots + \nu_r \log |\sigma_j(\varepsilon_r)| = m, \quad 1 \leq j \leq d. \tag{3.6}$$

Taking into account that, by complex conjugation, the equations (3.6) corresponding to an index j with $r_1 \leq j < r_1 + r_2$ and to the index $j + r_2$ are identical, and that the sum of (3.5) and equations (3.6) is zero, we simply have to deal with a Cramer system, since the matrix $(\sigma_j(\varepsilon_i))_{1 \leq j \leq r, 1 \leq i \leq r}$ is regular. We solve this system and then replace every ν_i by a rational integer μ_i such that $|\mu_i - \nu_i| \leq 1/2$. \square

Lemma 6. *Let λ' be a positive real number. Let $(\eta_m)_{m \geq 1}$ be a sequence of positive units as in Lemma 5. There exists a finite set $\Gamma = \Gamma(\lambda')$ of non-zero elements of \mathbf{K} such that for all integer polynomial $P(X)$ of degree at most d that satisfies*

$$|P(\alpha)| \leq \lambda' H(P)^{-d}, \tag{3.7}$$

there exist a positive integer m and γ in Γ for which

$$P(\alpha) = \gamma\eta_m.$$

Proof. Below, all the constants implicit in \ll depend on \mathbf{K} and on λ' . Let m be a positive integer such that

$$H(P) \asymp e^m,$$

and set

$$\gamma = P(\alpha)\eta_m^{-1}.$$

Since $D\alpha^k$ is an algebraic integer for $k = 0, \dots, d$, the algebraic number $D\gamma$ is an algebraic integer, and, by (3.7),

$$|\gamma| \ll 1.$$

Furthermore, for $j = 1, \dots, d$, we have

$$|\sigma_j(\gamma)| = |P(\sigma_j(\alpha))| \cdot |\sigma_j(\eta_m^{-1})| \ll H(P)e^{-m} \ll 1.$$

The algebraic integers $D\gamma \in \mathcal{O}$ and all their complex conjugates being bounded, they form a finite set. \square

4. Proof of Theorem 1

Let δ be in $(0, 1)$ to be selected later. Apply Lemma 3 with this δ to get a unit η and apply Lemma 1 with this unit and with $\gamma = \alpha^d$. Since $D^2\alpha^d\eta \in \mathbf{Z} + \dots + \alpha^d\mathbf{Z}$, we get

$$D^2\eta\alpha^d = a_0 + a_1\alpha + \dots + a_d\alpha^d = P(\alpha),$$

where, by Corollary 1, $P(X)$ is an integer polynomial of degree d and

$$|P(\alpha)| \asymp H(P)^{-d} \asymp H^{-d}.$$

By Lemma 2, each coefficient of $P(X)$ has its p -adic absolute value $\ll (\log 3H(P))^{-1/r}$, except the leading coefficient, whose p -adic absolute value equals $|D|_p^2$.

We then infer from Lemma 4 that all the roots of $P(X)$ in \mathbf{C}_p have their p -adic absolute value $\ll (\log 3H(P))^{-1/(dr)}$. This proves (2.2).

It now remains for us to guarantee that $P(X)$ has a root very close to α . To this end, we proceed to check that

$$|P'(\alpha)| \gg H(P).$$

Since

$$P'(\alpha) = a_1 + \dots + da_d\alpha^{d-1},$$

we get

$$P'(\alpha) = D^2 (\text{Tr}(\eta\alpha^d\beta_1) + 2\alpha\text{Tr}(\eta\alpha^d\beta_2) + \dots + d\alpha^{d-1}\text{Tr}(\eta\alpha^d\beta_d)),$$

hence,

$$P'(\alpha) = D^2 \sum_{i=0}^d \sum_{k=1}^d k \alpha^{k-1} \sigma_i(\eta \alpha^d \beta_k).$$

Let us write

$$P'(\alpha) = D^2 \sum_{i=0}^d A_i \sigma_i(\eta)$$

with

$$A_i = \sigma_i(\alpha^d) \sum_{k=1}^d k \alpha^{k-1} \sigma_i(\beta_k), \quad i = 0, \dots, d.$$

Observe first that

$$\sum_{i=1}^d A_i \neq 0.$$

Indeed, if this is not the case, then, working with the unit $\eta = 1$, that is, with $P(X) = D^2 X^d$ and $P'(\alpha) = dD^2 \alpha^{d-1}$, we get

$$d\alpha^{d-1} = A_0 = \alpha^d \sum_{k=1}^d k \alpha^{k-1} \beta_k,$$

hence,

$$d = \sum_{k=1}^d k \alpha^k \beta_k.$$

Taking the trace, and recalling that $\text{Tr}(\alpha^k \beta_k) = 1$, we get

$$d(d+1) = \sum_{k=1}^d k,$$

a contradiction.

Write

$$P'(\alpha) = D^2 \sum_{i=1}^d A_i \sigma_i(\eta) + O(H^{-d}) = D^2 \sigma_1(\eta) \sum_{i=1}^d A_i + B$$

with

$$|B| \leq D^2 \sum_{2 \leq i \leq d} |A_i| \cdot |\sigma_1(\eta)| \cdot \left| \frac{\sigma_i(\eta)}{\sigma_1(\eta)} - 1 \right| + O(H^{-d}).$$

Selecting now δ such that

$$\delta \sum_{2 \leq i \leq d} |A_i| \leq \frac{1}{3} \left| \sum_{i=1}^d A_i \right|,$$

we infer from Lemma 3 that

$$|P'(\alpha)| \geq \frac{1}{2}D^2 \left| \sigma_1(\eta) \sum_{i=1}^d A_i \right|,$$

when H is sufficiently large. This gives

$$|P'(\alpha)| \gg |\sigma_1(\eta)| \gg H.$$

Consequently, $P(X)$ has a root ξ such that

$$|\alpha - \xi| \ll H(P)^{-d-1} \ll H(\xi)^{-d-1}.$$

Classical arguments (see at the end of the proof of Theorem 2.11 in [2]) show that ξ must be real and of degree d if H is sufficiently large. This proves (2.1). Inequality (2.3) follows from (2.1) and (2.2) together with the fact that ξ is of degree d . This completes the proof of the theorem.

5. Proof of Theorem 2

The constants implicit in \ll and \gg below depend on \mathbf{K} , p and λ . There exists a positive real number λ' , depending on λ and on d , such that the minimal defining polynomial $P(X)$ of any real number ξ of sufficiently large height and for which (2.4) holds is of degree d and satisfies

$$|P(\alpha)| \leq \lambda' H(P)^{-d}.$$

Let $(\eta_m)_{m \geq 1}$ be as in Lemma 5. By Lemma 6, it is sufficient to prove Theorem 2 for the integer polynomials $P(X)$ as above such that

$$P(\alpha) = \gamma \eta_m = a_0 + a_1 \alpha + \dots + a_d \alpha^d.$$

Let ξ_i be the roots of $P(X)$ in \mathbf{C}_p and set

$$u := \max_{1 \leq i \leq d} |\xi_i|_p.$$

Assume that $u \leq 1$. It follows from Lemma 4 that

$$|a_k|_p \leq u |a_d|_p, \quad 0 \leq k < d.$$

Dividing $P(X)$ by $p^s = |a_d|_p^{-1}$ if necessary, we can assume that $|a_d|_p = 1$, and we obtain that

$$|a_k|_p \leq u, \quad 0 \leq k < d.$$

For $j = 1, \dots, d$, we then have

$$\gamma\eta_m\alpha^{-d} - \tau_j(\gamma\eta_m\alpha^{-d}) = \sum_{k=0}^{d-1} a_k(\alpha^{k-d} - \tau_j(\alpha^{k-d})),$$

hence,

$$|\gamma\eta_m\alpha^{-d} - \tau_j(\gamma\eta_m\alpha^{-d})|_p \ll u.$$

Since $|\eta_m|_p = 1$, we get that

$$\left| \frac{\tau_j(\eta_m)}{\eta_m} \frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} - 1 \right|_p \ll u.$$

Upon writing

$$\eta_m = \varepsilon_1^{\mu_{1,m}} \dots \varepsilon_r^{\mu_{r,m}},$$

we have thus

$$u \gg \left| \left(\frac{\tau_j(\varepsilon_1)}{\varepsilon_1} \right)^{-\mu_{1,m}} \dots \left(\frac{\tau_j(\varepsilon_r)}{\varepsilon_r} \right)^{-\mu_{r,m}} \frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} - 1 \right|_p$$

If

$$\frac{\tau_j(\eta_m)}{\eta_m} = \frac{\gamma\tau_j(\alpha^d)}{\tau_j(\gamma)\alpha^d}$$

holds for $j = 1, \dots, d$, the number

$$\gamma\eta_m\alpha^{-d}$$

is equal to all its conjugates, hence is rational, and we have

$$P(\alpha) = b\alpha^d$$

with $b \in \mathbf{Q}$, hence $P(X) = bX^d$, a contradiction. For every m , there thus exists an index j such that $1 \leq j \leq d$ and

$$\left(\frac{\tau_j(\varepsilon_1)}{\varepsilon_1} \right)^{-\mu_{1,m}} \dots \left(\frac{\tau_j(\varepsilon_r)}{\varepsilon_r} \right)^{-\mu_{r,m}} \frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} \neq 1.$$

Consequently, by the theory of linear forms in non-Archimedean logarithms (see e.g., Kunrui Yu [9]), there exists a positive constant κ such that

$$u \gg \left(\max_{1 \leq i \leq r} |\mu_{i,m}| \right)^{-\kappa}.$$

Since $\eta_m \asymp H(P)^{-d}$ and

$$|\log \eta_m| \asymp \max_{1 \leq i \leq r} |\mu_{i,m}|,$$

the matrix $(\log |\sigma_j(\varepsilon_i)|)_{1 \leq i \leq r, 1 \leq j \leq r}$ being regular, we conclude that

$$u \gg (\log 3H(\xi))^{-\kappa}.$$

This completes the proof of the theorem. □

References

- [1] M. D. Boshernitzan, *Elementary proof of Furstenberg's Diophantine result*, Proc. Amer. Math. Soc. 122 (1994), 67–70.
- [2] Y. Bugeaud, *Approximation by algebraic numbers*, Cambridge Tracts in Mathematics 160, Cambridge, 2004.
- [3] Y. Bugeaud, M. Drmota, and B. de Mathan, *On a mixed Littlewood conjecture in Diophantine approximation*, Acta Arith. 128 (2007), 107–124.
- [4] M. Einsiedler and D. Kleinbock, *Measure rigidity and p -adic Littlewood-type problems*, Compositio Math. 143 (2007), 689–702.
- [5] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory 1 (1967), 1–49.
- [6] B. de Mathan, *On a mixed Littlewood conjecture for quadratic numbers*, J. Théor. Nombres Bordeaux 17 (2005), 207–215.
- [7] B. de Mathan et O. Teulié, *Problèmes diophantiens simultanés*, Monatsh. Math. 143 (2004), 229–245.
- [8] L. G. Peck, *Simultaneous rational approximations to algebraic numbers*, Bull. Amer. Math. Soc. 67 (1961), 197–201.
- [9] Kunrui Yu, *p -adic logarithmic forms and group varieties II*, Acta Arith. 89 (1999), 337–378.
- [10] E. Wirsing, *Approximation mit algebraischen Zahlen beschränkter Grades*, J. reine angew. Math. 206 (1961), 67–77.

Yann Bugeaud
Université Louis Pasteur
Mathématiques
7, rue René Descartes
67084 STRASBOURG Cedex (France)
bugeaud@math.u-strasbg.fr

Bernard de Mathan
Université Bordeaux I
Institut de Mathématiques
351, cours de la Libération
33405 TALENCE Cedex (France)
demathan@math.u-bordeaux1.fr