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The grazing collision limit of Kac caricature of Bose-Einstein particles

Thibaut Allemand* Giuseppe Toscani†

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Abstract

We discuss the grazing collision limit of certain kinetic models of Bose-Einstein particles obtained from a suitable modification of the one-dimensional Kac caricature of a Maxwellian gas without cut-off. We recover in the limit a nonlinear Fokker-Planck equation which presents many similarities with the one introduced by Kaniadakis and Quarati in [14]. In order to do so, we perform a study of the moments of the solution. Moreover, as is typical in Maxwell models, we make an essential use of the Fourier version of the equation.

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1 Introduction

The quantum dynamics of many body systems is often modelled by a nonlinear Boltzmann equation which exhibits a gas-particle-like collision behavior. The

*DMA, École Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05, France

†Dipartimento di matematica, Università di Pavia, Via Ferrata 1, 27100 Pavia, Italy

application of quantum assumptions to molecular encounters leads to some divergences from the classical kinetic theory [6] and despite their formal analogies the Boltzmann equation for classical and quantum kinetic theory present very different features. The interest in the quantum framework of the Boltzmann equation has increased noticeably in the recent years. Although the quantum Boltzmann equation, for a single specie of particles, is valid for a gas of fermions as well as for a gas of bosons, blow up of the solution in finite time can occur only in the bosonic case at low temperature. As a consequence the quantum Boltzmann equation for a gas of bosons represents the most challenging case both mathematically and numerically. In particular this equation has been successfully used for computing non-equilibrium situations where Bose-Einstein condensate occurs. From Chapman and Cowling [6] one can learn that the Boltzmann Bose-Einstein equation (BBE) is established by imposing that, when the mean distance between neighboring molecules is comparable with the size of the quantum wave fields with which molecules are surrounded, a state of congestion results. For a gas composed of Bose-Einstein identical particles, according to quantum theory, the presence of a like particle in the velocity-range dv increases the probability that a particle will enter that range; the presence of $f(v)dv$ particles per unit volume increases this probability in the ratio $1 + \delta f(v)$. This fundamental assumption yields the Boltzmann Bose-Einstein equation

$$\frac{\partial f}{\partial t} = Q_{QBE}(f)(t, v), \quad t \in \mathbb{R}_+, \quad v \in \mathbb{R}^3, \quad (1)$$

where

$$Q_{QBE}(f)(t, v) = \int_{\mathbb{R}^3 \times S^2} B(v - v_*, \omega) (f' f'_* (1 + \delta f)(1 + \delta f_*) - f f_* (1 + \delta f')(1 + \delta f'_*)) dv_* d\omega, \quad (2)$$

where as usual we denoted

$$f = f(v), \quad f_* = f(v_*), \quad f' = f(v'), \quad f'_* = f(v'_*),$$

and the pairs (v, v_*) (respectively (v', v'_*)) are the post- (respectively pre-) collision velocities in a elastic binary collision. In (2) $B(z, \omega)$ is the collision kernel which is a nonnegative Borel function of $|z|, |< z, \omega >|$ only

$$B(z, \omega) = B\left(|z|, \frac{|< z, \omega >|}{|z|}\right), \quad (z, \omega) \in \mathbb{R}^3 \times S^2. \quad (3)$$

The solutions $f(v, t)$ are velocity distribution functions (i.e., the density functions of particle number), $\delta = (h/m)^3/g$, h is the Planck's constant, m and g are the mass and the "statistical weight" of a particle (see [18] for details).

For a non relativistic particle, by setting $v(p) = p/m$, the collision operator Q_{QBE} can be rewritten in general form as follows [23, 24]

$$Q_{QBE}(f)(t, p) = \int_{\mathbb{R}^9} W(p, p_*, p', p'_*) (f' f'_* (1 + \delta f)(1 + \delta f_*) - f f_* (1 + \delta f')(1 + \delta f'_*)) dp_* dp' dp'_* \quad (4)$$

where W is a nonnegative measure called transition rate, which is of the form

$$W(p, p_*, p', p'_*) = \Omega(p, p_*, p', p'_*) \delta(p, +p_* - p' - p'_*) \delta(\mathcal{E}(p) + \mathcal{E}(p_*) - \mathcal{E}(p') - \mathcal{E}(p'_*)),$$

where δ represents the Dirac measure and $\mathcal{E}(p)$ is the energy of the particle. The quantity $W dp' dp'_*$ is the probability for the initial state (p, p_*) to scatter and become a final state of two particles whose momenta lie in a small region $dp' dp'_*$. The function Ω is directly related to the differential cross section (see (3)), a quantity that is intrinsic to the colliding particles and the kind of interaction between them. The collision operator (4) is simplified by assuming a boson distribution which only depends on the total energy $e = \mathcal{E}(p)$. In this last case $f = f(e, t)$ is the boson density in energy space.

Together with the Boltzmann description given by the collision operators (2)-(4), other kinetic models for Bose-Einstein particles have been introduced so far. In particular, a related model described by means of Fokker-Planck type non linear operators has been proposed by Kompaneets [16] to describe the evolution of the radiation distribution $f(x, t)$ in a homogeneous plasma when radiation interacts with matter via Compton scattering

$$Q_K(f)(t, \rho) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left[\rho^4 \left(\frac{\partial f}{\partial \rho} + f + f^2 \right) \right], \quad \rho \in (0, +\infty) \quad (5)$$

In that context the coordinate ρ represents a momentum coordinate, $\rho = |p|$. More precisely, an equation which includes (5) as a particular case is obtained in [16] as a leading term for the corresponding Boltzmann equation under the crucial assumption that the scattering cross section is of the classical Thomson type (see [11] for details).

The fundamental assumption which leads to the correction in the Boltzmann collision operator (2), namely the fact that the presence of $f(v)dv$ particles per unit volume increases the probability that a particle will enter the velocity range dv in the ratio $1 + \delta f(v)$, has been recently used by Kaniadakis and Quarati [14, 13] to propose a correction to the drift term of the Fokker-Planck equation in presence of quantum indistinguishable particles, bosons or fermions. In their model, the collision operator (2) is substituted by

$$Q_{FP}(f)(t, v) = \nabla \cdot [\nabla f + v f(1 + \delta f)]. \quad (6)$$

Maybe the most remarkable difference between the kinetic operators (2) and (6) is that, while the former is such that mass, momentum and energy are collision invariant, the latter does not admit the energy as collision invariant. This suggests that the operator (6) would not result directly through an asymptotic procedure from the Bose-Einstein collision operator (2), but instead from some linearized version, where only the mass is preserved under the collision mechanism.

For a mathematical analysis of the quantum Boltzmann equation in the space homogeneous isotropic case we refer to [9, 10, 11, 18, 19]. We remark that already the issue of giving mathematical sense to the collision operator is highly nontrivial (particularly if positive measure solutions are allowed, as required by a careful analysis of the equilibrium states). All the mathematical results, however, require very strong cut-off assumptions on the cross-section [11, 18].

Also, accurate numerical discretization of the quantum Boltzmann equation, which maintain the basic analytical and physical features of the continuous problem, namely, mass and energy conservation, entropy growth and equilibrium distributions have been introduced recently in [2, 20]. Related works [17, 21, 22] in which fast methods for Boltzmann equations were derived using different

techniques like multipole methods, multigrid methods and spectral methods, are relevant to quote.

At the Fokker-Planck level, the qualitative analysis of the Kompaneets equation described by the operator (5) has been exhaustively studied in [8], while the numerical simulation has been done by Buet and Cordier [3]. To our knowledge, the mathematical study of the Fokker-Planck equation (6) introduced by Kaniadakis and Quarati [14] has been done only very recently [5], where the one-dimensional version of (6) has been studied.

In the case of the quantum Boltzmann equation the asymptotic equivalence between the binary collision operators (2), (4) and the Fokker-Planck type operators (5) and (6) is unknown. This is not the case for the classical binary collisions in a elastic gas, where the asymptotic equivalence between the Boltzmann and the Fokker-Planck-Landau equations has been proven rigorously in a series of papers by Villani [26, 27] by means of the so-called *grazing collision* asymptotics.

The same asymptotic procedure, in the case of the one-dimensional Kac equation [12], showed the asymptotic equivalence between Kac collision operator and the linear Fokker-Planck operator [25]. The method of proof in [25] takes advantage from the relatively simple structure of Kac equation. Taking this into account, in order to establish the asymptotic connection between the Boltzmann equation for Bose Einstein particles and its Fokker-Planck description, we will introduce a one-dimensional kinetic model in the spirit of Kac caricature of a Maxwell gas with a singular kernel. Then we will study the grazing collision limit of the equation, which leads to a Fokker-Planck type equation in which the drift is of the form of equation (6), but the coefficient of the (linear) diffusion term depends on time through the density function. More precisely, the Fokker-Planck collision operator reads

$$Q_{FP}(f)(t, v) = A_t(f) \frac{\partial^2 f}{\partial v^2} + B_t(f) \frac{\partial}{\partial v} (vf(1 + \delta f)), \quad (7)$$

where

$$A_t(f) = \int_{\mathbb{R}} v^2 f(v, t) (1 + \delta f(v, t)) dv$$

and

$$B_t(f) = \int_{\mathbb{R}} f(v, t) dv.$$

The paper is organized as follows. In the next Section we will introduce the model, together with some simplifications (mollified model), then in Section 3 we formally show the convergence in the grazing collision limit. We then focus on a mollified model to get rigorous results: in Section 4 we prove existence of a weak solution, in Section 5 we focus on the moments of the solution and on some regularity properties. Finally, in Section 6, we will deal with the grazing collision limit.

2 The Kac caricature of a Bose Einstein gas

The simplest one-dimensional model which maintains almost all physical properties of the Boltzmann equation for a Bose-Einstein gas can be obtained by

generalizing Kac caricature of a Maxwell gas to Bose-Einstein particles. This one-dimensional model reads as follows

$$\begin{cases} \frac{\partial f}{\partial t} = Q_{QBE}(f)(t, v), & t \in \mathbb{R}_+, \quad v \in \mathbb{R}, \\ f(0, v) = f_0(v), \end{cases} \quad (8)$$

where

$$Q_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} (f' f'_*(1 + \delta f)(1 + \delta f_*) - f f_*(1 + \delta f')(1 + \delta f'_*)) dv_* d\theta. \quad (9)$$

For the sake of brevity, we used the usual notations

$$f \equiv f(t, v), \quad f' \equiv f(t, v'), \quad f_* \equiv f(t, v_*), \quad f'_* \equiv f(t, v'_*),$$

The initial datum f_0 is a nonnegative measurable function. The pre-collision velocities (v', v'_*) are defined by the Kac rotation rule [12], which is given by

$$\begin{cases} v' = v \cos \theta - v_* \sin \theta \\ v'_* = v \sin \theta + v_* \cos \theta. \end{cases} \quad (10)$$

Collisions (10) imply the conservation of the energy at each collision

$$v^2 + v_*^2 = v'^2 + v_*'^2. \quad (11)$$

Let us observe that the system (10) can be reversed so that we can write the post-collision velocities with respect to the pre-collision ones

$$\begin{cases} v = v' \cos \theta + v'_* \sin \theta \\ v_* = -v' \sin \theta + v'_* \cos \theta. \end{cases} \quad (12)$$

The parameter δ in (9) is a positive constant. The choice $\delta = 0$ would lead us back to standard Kac model, whereas δ negative would lead us to the Boltzmann-Fermi-Dirac equation, whose features are very different from that of the Boltzmann-Bose-Einstein equation.

The cross-section $\beta(\theta)$ is a function defined over $(-\frac{\pi}{2}, \frac{\pi}{2})$. In the original Kac equation [12], $\beta(\theta)$ is assumed constant, which implies that collisions spread out uniformly with respect to the angle θ . Following Desvillettes [7], we will here assume that the cross-section is suitable to concentrate collisions on the *grazing* ones (these collisions are those that are neglected when the cut-off assumption is made). This corresponds to satisfy one or more of the following hypotheses

H1 $\beta(\theta)$ is a nonnegative even function.

H2 $\beta(\theta)$ satisfies a non-cutoff assumption on the form

$$\beta(\theta) \sim \frac{1}{|\theta|^{1+\nu}} \quad \text{when } \theta \rightarrow 0, \quad (13)$$

with $1 < \nu < 2$. That is,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\sin \theta| d\theta = +\infty$$

whereas

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\sin \theta|^{\nu+\varepsilon} d\theta < +\infty$$

for all $\varepsilon > 0$.

H3 $\beta(\theta)$ is zero near $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, namely there exists a small constant $\varepsilon_0 > 0$ such that

$$\beta(\theta) = 0 \quad \forall \theta \in \left(-\frac{\pi}{2}, -\frac{\pi}{2} + \varepsilon_0\right] \cap \left[\frac{\pi}{2} - \varepsilon_0, \frac{\pi}{2}\right).$$

In the case in which the classical Kac equation is concerned, the asymptotic equivalence between the non cut-off Kac equation and the linear Fokker-Planck equation as collisions become grazing has been proven in [25]. Hence, the passage to grazing collisions in (9), would give us the correct Fokker-Planck type operator which leads the initial density towards the Bose-Einstein distribution.

Due to the symmetries of the kernel (9) and to the microscopic conservation of the energy (11), it can be easily shown, at least at a formal level, that the mass and the global energy of the solution are conserved

$$\int_{\mathbb{R}} f(t, v) dv = \int_{\mathbb{R}} f_0(v) dv,$$

and

$$\int_{\mathbb{R}} v^2 f(t, v) dv = \int_{\mathbb{R}} v^2 f_0(v) dv,$$

for all $t > 0$. Moreover, if

$$H(f) = \int_{\mathbb{R}} \left(\frac{1}{\delta} (1 + \delta f) \log(1 + \delta f) - f \log f \right) dv$$

denotes the Bose-Einstein entropy, the time derivative of $H(f)$ is given by

$$D(f) = \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} \Gamma \left(f f'_*(1 + \delta f)(1 + \delta f_*), \right. \\ \left. f f_*(1 + \delta f')(1 + \delta f'_*) \right) dv dv_* d\theta$$

with

$$\Gamma(a, b) = \begin{cases} (a - b) \log(a/b), & a > 0, \quad b > 0; \\ +\infty, & a > 0, \quad b = 0 \quad \text{or} \quad a = 0, \quad b > 0; \\ 0, & a = b = 0. \end{cases} \quad (14)$$

Then, since $D(f) \geq 0$ the solution $f(t)$ to equation (8) satisfies formally an H -theorem: $H(f(t))$ is monotonically increasing unless $f(t)$ coincides with the Bose-Einstein density f_{BE} , defined by the relationship

$$\frac{f_{BE}(v)}{1 + \delta f_{BE}(v)} = a e^{-bv^2}, \quad (15)$$

where a and b are positive constant chosen to satisfy the mass and energy conservation for f_{BE} .

It can be easily verified by direct inspection that the fourth order term in (9) cancels out from the collision integral, so that it can be rewritten as

$$Q_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} (f' f'_* (1 + \delta f + \delta f_*) - f f_* (1 + \delta f' + \delta f'_*)) dv_* d\theta. \quad (16)$$

In trying to give a rigorous signification to equation (8), several difficulties arise. In fact, our non-cutoff cross-section $\beta(\theta)$ does not allow us neither to use the same change of variable as in [18], nor to use the same weak formulation as in [7]. A sufficient condition to give a sense to the collision kernel would be that $f \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$. It would even be enough that such a condition hold for the quantum part, that is for the f involved in the terms of the form $1 + \delta f$.

To satisfy that condition, we introduce a new model where the quantum part is smoothed. Let ψ be a mollifier, that is

1. $\psi \in C_c^\infty(\mathbb{R})$
2. $\psi \geq 0$
3. $\int_{\mathbb{R}} \psi(v) dv = 1$.

Then, let

$$\tilde{f}(t, v) = \int_{\mathbb{R}} f(t, v - w) \psi(w) dw = f(t, \cdot) *_v \psi.$$

The function \tilde{f} is regular in the velocity variable, and relies uniformly in all the $L^p(\mathbb{R})$ spaces (for $1 \leq p \leq +\infty$) since the $L^1(\mathbb{R})$ norm of $f(t, \cdot)$ is constant. Moreover, $\tilde{f}(t, \cdot)$ is as close as we want to $f(t, \cdot)$ in all these norms, provided ψ is well chosen, so that our new model is nothing but an approximation of (8):

$$\begin{cases} \frac{\partial f}{\partial t} = \tilde{Q}_{QBE}(f), & t \in \mathbb{R}_+, v \in \mathbb{R} \\ f(0, v) = f_0(v) \end{cases} \quad (17)$$

with

$$\tilde{Q}_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} (f' f'_* (1 + \delta \tilde{f})(1 + \delta \tilde{f}_*) - f f_* (1 + \delta \tilde{f}')(1 + \delta \tilde{f}'_*)) dv_* d\theta. \quad (18)$$

This approximation still formally preserves mass and energy, while maintaining the same nonlinearity of the original collision operator. It has to be remarked, however, that both the validity of the H -theorem and the explicit form of the steady solution are lost. Other approximations can be introduced, which do not exhibit this problem. Among others, the operator

$$\tilde{Q}_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} \left(\frac{f'}{1 + \delta f'} \frac{f'_*}{1 + \delta f'_*} - \frac{f}{1 + \delta f} \frac{f_*}{1 + \delta f_*} \right) \cdot (1 + \delta \tilde{f})(1 + \delta \tilde{f}_*)(1 + \delta \tilde{f}')(1 + \delta \tilde{f}'_*) dv_* d\theta. \quad (19)$$

preserves mass and energy, satisfies the H -theorem and possesses the right steady state. Unlikely, the nonlinearity of (19) is difficult to handle for our purposes since its Fourier transform can not be written in a simple form.

Let us end this Section with a few notations. The functional spaces that will be used in the following, apart from the usual Lebesgue spaces, are the weighted Lebesgue spaces, defined, for $p > 0$, by the norm

$$\|g\|_{L_p^1(\mathbb{R})} = \int_{\mathbb{R}} (1 + |v|^p) |g(v)| dv.$$

We will also need some Sobolev spaces, defined for $0 < s < 1$ by the norm

$$\|g\|_{H^s(\mathbb{R})}^2 = \|g\|_{L^2}^2 + |g|_{H^s}^2$$

where

$$|g|_{H^s}^2 = \iint_{\mathbb{R}^2} \frac{|g(x+y) - g(y)|^2}{|y|^{1+2s}} dx dy.$$

Our convention for the Fourier transform is the following:

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(v) e^{-iv\xi} dv$$

and the inverse Fourier transform is given by

$$f(v) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{iv\xi} d\xi.$$

We will sometimes use the notations

$$m = \int_{\mathbb{R}} f_0(v) dv$$

and

$$e = \int_{\mathbb{R}} v^2 f_0(v) dv.$$

3 Formal results

In this section we will show how the grazing collision limit work on the Kac model for bosons (20)

$$\begin{cases} \frac{\partial f}{\partial t} = Q_{QBE}(f)(t, v), & t \in \mathbb{R}_+, \quad v \in \mathbb{R}, \\ f(0, v) = f_0(v), \end{cases} \quad (20)$$

where

$$Q_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} (f' f'_*(1 + \delta f)(1 + \delta f_*) - f f_*(1 + \delta f')(1 + \delta f'_*)) dv_* d\theta. \quad (21)$$

As we pointed out before, it is not known how to prove that solutions to this model exist; hence we will give only formal results, assuming that a solution to this equation exists and is regular enough.

3.1 H -theorem and regularity of the solution

Formally, the solutions to (20) satisfy the H -theorem as pointed out in the previous section:

Theorem 1. *Let f be a solution of the problem (20), with $f_0 \in L \log L(\mathbb{R})$, and assume that $H(f)$ and $D(f)$ are well defined. Then*

$$H(f(t, \cdot)) = H(f_0) + \int_0^t D(f(s, \cdot)) ds \quad \forall t > 0.$$

Consequently, the entropy H is increasing along the solution.

Using the H -theorem, we can give an *a priori* regularity estimate on the solution f to (20).

Theorem 2. *Let β satisfy the properties (H1), (H2) and (H3), and let $f(t, v)$ be a solution of the problem (20) with the initial datum $f_0 \in L^1_2 \cap L \log L$. Assume that f satisfies the H -theorem. Then we have*

$$\log(1 + \delta f) \in L^2_{loc}(\mathbb{R}_+; H^{\nu/2}(\mathbb{R})).$$

If in addition $f \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$, then

$$f \in L^2_{loc}(\mathbb{R}_+; H^{\nu/2}(\mathbb{R})).$$

Proof. We will do the computations as if the cross-section and the function f were smooth. Using the classical changes of variable $(v, v_*, \theta) \mapsto (v', v'_*, -\theta)$ and $(v, v_*) \mapsto (v_*, v)$ which have unit jacobian, we have:

$$\begin{aligned} D(f) &= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} \left(f' f'_*(1 + \delta f)(1 + \delta f_*) - f f_*(1 + \delta f')(1 + \delta f'_*) \right) \\ &\quad \times \log \frac{f' f'_*(1 + \delta f)(1 + \delta f_*)}{f f_*(1 + \delta f')(1 + \delta f'_*)} dv dv_* d\theta \\ &= -\frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} \left(f' f'_*(1 + \delta f)(1 + \delta f_*) - f f_*(1 + \delta f')(1 + \delta f'_*) \right) \\ &\quad \times \log \left(f f_*(1 + \delta f')(1 + \delta f'_*) \right) dv dv_* d\theta \\ &= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} \left(f' f'_*(1 + \delta f)(1 + \delta f_*) - f f_*(1 + \delta f')(1 + \delta f'_*) \right) \\ &\quad \times \log \left(f(1 + \delta f') \right) dv dv_* d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f f_*(1 + \delta f')(1 + \delta f'_*) \log \frac{f(1 + \delta f')}{f'(1 + \delta f)} dv dv_* d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) \left(f(1 + \delta f') \log \frac{f(1 + \delta f')}{f'(1 + \delta f)} \right. \\ &\quad \left. - f(1 + \delta f') + f'(1 + \delta f) \right) dv dv_* d\theta \\ &\quad + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) \left(f(1 + \delta f') - f'(1 + \delta f) \right) dv dv_* d\theta \\ &= I_1 + I_2. \end{aligned}$$

The term I_2 can be treated easily, because it can be written as

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*)(f - f') dv dv_* d\theta,$$

and the presence of $f - f'$ involves strong cancellations. In fact, the term I_2 verifies

$$I_2 = \|f\|_{L^1}^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left(1 - \frac{1}{\cos \theta}\right) d\theta. \quad (22)$$

To prove (22), consider that

$$\begin{aligned} I_2 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(f - f') dv dv_* d\theta + \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_* f'_*(f - f') dv dv_* d\theta \\ &= I_2^1 + I_2^2 \end{aligned}$$

Thanks to the change of variable $(v, v_*, \theta) \mapsto (v', v'_*, -\theta)$, we see that $I_2^2 = 0$. On the second part of I_2^1 , we use the change of variable $v \mapsto v'$ with v_* and θ fixed, which jacobian is

$$\frac{dv}{dv'} = \frac{1}{\cos \theta}.$$

Therefore

$$I_2^1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left(1 - \frac{1}{\cos \theta}\right) \int_{\mathbb{R}^2} f_* f dv dv_* d\theta$$

and from this (22) follows.

Using now the inequality

$$x \log \frac{x}{y} - x + y \geq (\sqrt{x} - \sqrt{y})^2, \quad \forall x, y > 0,$$

we obtain

$$I_1 \geq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) \left(\sqrt{f(1 + \delta f')} - \sqrt{f'(1 + \delta f)} \right)^2 dv dv_* d\theta.$$

Therefore

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) \left(\sqrt{f(1 + \delta f')} - \sqrt{f'(1 + \delta f)} \right)^2 dv dv_* d\theta \leq D(f) + c_1 \|f\|_{L^1}^2 \quad (23)$$

where

$$c_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left(\frac{1}{\cos \theta} - 1 \right) d\theta > 0.$$

Now, we write

$$\sqrt{f(1 + \delta f')} - \sqrt{f'(1 + \delta f)} = (\log(1 + \delta f') - \log(1 + \delta f)) \frac{\sqrt{f(1 + \delta f')} - \sqrt{f'(1 + \delta f)}}{\log(1 + \delta f') - \log(1 + \delta f)}.$$

For $0 < a < x$, let

$$\phi(x) = \frac{\sqrt{x(1 + \delta a)} - \sqrt{a(1 + \delta x)}}{\log(1 + \delta a) - \log(1 + \delta x)}.$$

Then, there exists some constant $c_2 > 0$ that does not depend on x or a such that

$$|\phi(x)| > c_2 \quad \forall x > a.$$

From this inequality we deduce that

$$c_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) \left(\log(1 + \delta f') - \log(1 + \delta f) \right)^2 \leq D(f) + c_1 \|f\|_{L^1}^2$$

It has been shown in [1] that if F is a real function such that $F(f) \in L^2(\mathbb{R})$ satisfying

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_* \left(F(f') - F(f) \right)^2 \leq D(f) + c_1 \|f\|_{L^1}^2,$$

then the following inequality holds:

$$\|F(f)\|_{H^{\nu/2}}^2 \leq D(f) + c_1 \|f\|_{L^1}^2.$$

Taking $F(f) = c_2 \log(1 + \delta f)$ the result follows. \square

3.2 Moments of the solution and the grazing collision limit

We can now make precise assumptions on the asymptotics of the grazing collisions, namely in letting the kernel β concentrate on the singularity $\theta = 0$. We will introduce a family of kernels $\{\beta_\varepsilon(|\theta|)\}_{\varepsilon>0}$ satisfying hypotheses (H1) and (H2), with

$$\forall \theta_0 > 0 \quad \sup_{\theta > \theta_0} \beta_\varepsilon(|\theta|) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad (24)$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^\pi \beta_\varepsilon(|\theta|) \theta^2 d\theta = 1 \quad (25)$$

This can be obtained in several ways, for example taking, for $0 < \mu < 1$

$$\beta_\varepsilon(|\theta|) = \frac{2(1-\mu)}{\varepsilon|\theta|^{2+\mu}} \quad 0 \leq |\theta| \leq \varepsilon^{1/(1-\mu)},$$

$$\beta_\varepsilon(|\theta|) = \frac{(1-\mu)\varepsilon}{|\theta|^{2+\mu}} \quad \text{elsewhere.}$$

Let f_ε be a solution of

$$\begin{cases} \frac{\partial f_\varepsilon}{\partial t} = Q_{QBE}^\varepsilon(f_\varepsilon)(t, v), & t \in \mathbb{R}_+, \quad v \in \mathbb{R}, \\ f_\varepsilon(0, v) = f_0(v), \end{cases} \quad (26)$$

where $\beta_\varepsilon(\theta)$ has replaced $\beta(\theta)$. The grazing collision limit is obtained when $\varepsilon \rightarrow 0$. Our strategy is first to take the Fourier transform in the velocity variable of equation (26), then to pass to the limit $\varepsilon \rightarrow 0$ in the Fourier formulation, and finally to recognize at the limit the Fourier formulation of a quantum Fokker-Planck equation.

Let us take the Fourier transform of (26):

$$\frac{\partial \hat{f}_\varepsilon(t, \xi)}{\partial t} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^2} \beta_\varepsilon(\theta) f_\varepsilon f_{\varepsilon,*} (1 + \delta f_\varepsilon(v') + \delta f_\varepsilon(v'_*)) (e^{-iv'\xi} - e^{-iv\xi}) dv dv_* d\theta.$$

We can split the integral on the right-hand side into three parts. The first part gives

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^2} \beta_\varepsilon(\theta) f_\varepsilon f_{\varepsilon,*} (e^{-iv'\xi} - e^{-iv\xi}) dv dv_* d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_\varepsilon(\theta) \left(\hat{f}_\varepsilon(\xi \cos \theta) \hat{f}_\varepsilon(\xi \sin \theta) - \hat{f}_\varepsilon(\xi) \hat{f}_\varepsilon(0) \right) d\theta. \end{aligned}$$

The second part can be evaluated using the inverse Fourier transform of the function f_ε which is supposed to rely in L^2

$$\begin{aligned} & \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^2} \beta_\varepsilon(\theta) f_\varepsilon f_{\varepsilon,*} f_\varepsilon(v') (e^{-iv'\xi} - e^{-iv\xi}) dv dv_* d\theta \\ &= \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_\varepsilon(\theta) \int_{\mathbb{R}^3} f_\varepsilon f_{\varepsilon,*} \hat{f}_\varepsilon(\eta) e^{i\eta(v \cos \theta - v_* \sin \theta)} \\ & \quad \cdot \left(e^{-i\xi(v \cos \theta - v_* \sin \theta)} - e^{-iv\xi} \right) d\eta dv dv_* d\theta \\ &= \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_\varepsilon(\theta) \int_{\mathbb{R}} \hat{f}_\varepsilon((\xi - \eta) \cos \theta) \hat{f}_\varepsilon((\xi - \eta) \sin \theta) \hat{f}_\varepsilon(\eta) d\eta d\theta \\ & \quad - \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_\varepsilon(\theta) \int_{\mathbb{R}} \hat{f}_\varepsilon(\xi - \eta \cos \theta) \hat{f}_\varepsilon(\eta \sin \theta) \hat{f}_\varepsilon(\eta) d\eta d\theta. \end{aligned}$$

The third term can be computed in the same way. At the end we get that the Fourier transform of f_ε satisfies

$$\begin{aligned} \frac{\partial \hat{f}_\varepsilon(t, \xi)}{\partial t} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_\varepsilon(\theta) \left(\hat{f}_\varepsilon(\xi \cos \theta) \hat{f}_\varepsilon(\xi \sin \theta) - \hat{f}_\varepsilon(\xi) \hat{f}_\varepsilon(0) \right) d\theta \\ & \quad + \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_\varepsilon(\theta) \int_{\mathbb{R}} \left[\hat{f}_\varepsilon((\xi - \eta) \cos \theta) \hat{f}_\varepsilon((\xi - \eta) \sin \theta) \right. \\ & \quad \quad \quad \left. - \hat{f}_\varepsilon(\xi - \eta \cos \theta) \hat{f}_\varepsilon(\eta \sin \theta) \right] \hat{f}_\varepsilon(\eta) d\eta d\theta \quad (27) \\ & \quad + \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_\varepsilon(\theta) \int_{\mathbb{R}} \left[\hat{f}_\varepsilon(\xi \cos \theta - \eta \sin \theta) \hat{f}_\varepsilon(-\xi \sin \theta - \eta \cos \theta) \right. \\ & \quad \quad \quad \left. - \hat{f}_\varepsilon(\xi - \eta \sin \theta) \hat{f}_\varepsilon(-\eta \cos \theta) \right] \hat{f}_\varepsilon(\eta) d\eta d\theta. \end{aligned}$$

To pass to the limit $\varepsilon \rightarrow 0$, we need some regularity on \hat{f}_ε , or equivalently, f_ε must have bounded moments. The following result will be proved at the end of Section 6.

Theorem 3. *Assume that the cross-section β satisfies (H1), (H2), with $1 < \nu < \sqrt{5} - 1$. Assume that there exists a solution f_ε to the problem (26). Assume*

that f_ε is regular, in the sense that $\|f_\varepsilon(t)\|_{L^\infty(\mathbb{R})}$ and $\|f_\varepsilon(t)\|_{H^{\frac{\nu}{2}}}$ are in $L^p(\mathbb{R}_+)$ for some p big enough; then there exists some constants $\lambda, T > 0$ independent of ε such that

$$\sup_{0 \leq t < T} \|f_\varepsilon(t)\|_{L^1_4} \leq \max \left\{ \lambda, \|f_0\|_{L^1_4} \right\}.$$

This result formally shows that the Fourier transform of f_ε (in the velocity variable) is four times derivable, with bounded derivatives. It is then possible to use Taylor expansions in the formulation (27) :

$$\begin{aligned} & \frac{\partial \hat{f}_\varepsilon(t, \xi)}{\partial t} \\ &= \frac{1}{\varepsilon^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left[\varepsilon \theta \xi \left(\hat{f}_\varepsilon(\xi) \hat{f}'_\varepsilon(0) \right. \right. \\ & \quad \left. \left. - \frac{\delta}{2\pi} \int_{\mathbb{R}} \left(\hat{f}_\varepsilon(\xi - \eta) \hat{f}_\varepsilon(\eta) \hat{f}'_\varepsilon(0) - \hat{f}'_\varepsilon(-\eta) \hat{f}_\varepsilon(\eta) \hat{f}_\varepsilon(\xi) \right) d\eta \right) \right. \\ & \quad \left. + \varepsilon^2 \frac{\theta^2}{2} \left(\xi^2 \hat{f}''_\varepsilon(0) \hat{f}_\varepsilon(\xi) - \xi \hat{f}'_\varepsilon(\xi) \hat{f}_\varepsilon(0) - \frac{\delta}{2\pi} \int_{\mathbb{R}} \left(\hat{f}'_\varepsilon(\xi - \eta) \hat{f}_\varepsilon(\eta) \hat{f}_\varepsilon(0) \right. \right. \right. \\ & \quad \left. \left. + ((\eta - \xi)^2 - \eta^2) \hat{f}_\varepsilon(\xi - \eta) \hat{f}_\varepsilon(\eta) \hat{f}''_\varepsilon(0) + \xi \hat{f}'_\varepsilon(\xi) \hat{f}_\varepsilon(\eta) \hat{f}_\varepsilon(-\eta) \right. \right. \\ & \quad \left. \left. + \xi^2 \hat{f}''_\varepsilon(-\eta) \hat{f}_\varepsilon(\eta) \hat{f}_\varepsilon(\xi) + 2\eta \xi \hat{f}'_\varepsilon(\xi) \hat{f}_\varepsilon(\eta) \hat{f}'_\varepsilon(-\eta) \right) d\eta \right] + \theta^2 O(\varepsilon^3) \Big] d\theta. \end{aligned} \quad (28)$$

Letting ε go to 0, we obtain the grazing collision limit, and we recognize in the limit the Fourier form of a quantum Fokker-Planck equation:

Theorem 4. *Let (f_ε) be a family of solutions to (26) where the cross-section β satisfies (H1), (H2), with $1 < \nu < \sqrt{5} - 1$. Assume that the solutions f_ε are regular enough ($L^\infty((0, T); L^1 \cap L^\infty(\mathbb{R}))$) should be enough thanks to Theorem 2). Then there exists a distribution f such that*

$$\hat{f}_\varepsilon \rightarrow \hat{f} \quad \text{when } \varepsilon \rightarrow 0$$

in $L^\infty((0, T); W^{4, \infty}(\mathbb{R}))$, and f is a solution to

$$\frac{\partial f}{\partial t} = \left(\int_{\mathbb{R}} f(v) dv \right) \frac{\partial}{\partial v} (vf(1 + \delta f)) + \left(\int_{\mathbb{R}} v^2 f(v) (1 + \delta f(v)) dv \right) \frac{\partial^2 f}{\partial v^2}. \quad (29)$$

The study of equation (29) should be close to what is done in [4, 5].

4 Existence theorems for the mollified model

To give rigorous results, we will from now on work on the regularized model

$$\begin{cases} \frac{\partial f}{\partial t} = \tilde{Q}_{QBE}(f), & t \in \mathbb{R}_+, v \in \mathbb{R} \\ f(0, v) = f_0(v) \end{cases} \quad (30)$$

with

$$\begin{aligned} \tilde{Q}_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} (f' f'_*(1 + \delta \tilde{f})(1 + \delta \tilde{f}_*) \\ - f f_*(1 + \delta \tilde{f}')(1 + \delta \tilde{f}'_*)) dv_* d\theta. \end{aligned} \quad (31)$$

The goal of this Section is to prove the existence of a solution to the problem (30). To start with, we first consider the case of a cross-section with cutoff.

Theorem 5. *Let $f_0 \in L^1_2(\mathbb{R})$ be a nonnegative function. Assume that the cross-section satisfies $\beta \in L^1(-\frac{\pi}{2}; \frac{\pi}{2})$ and (H1). Then there exists a unique solution $f \in L^\infty(\mathbb{R}_+; L^1_2(\mathbb{R}))$ to the problem (30), which is nonnegative, and preserves mass and energy.*

The proof is a consequence of the following theorem from Wild [15]:

Theorem 6. *Let E be a given Banach space and $P : E^N \rightarrow E$ (for $N \geq 2$) be a N -linear operator satisfying the inequality*

$$\|P(u_1, \dots, u_N)\| \leq C_p \|u_1\| \cdots \|u_N\| \quad \text{for all } u_i \in E.$$

Let

$$u(t) = \sum_{k=0}^{\infty} b_k e^{-t} (1 - e^{(1-N)t})^k u_k,$$

where

$$u_k = \sum_{i_1 + \dots + i_N = k-1} \frac{b_{i_1} \cdots b_{i_N}}{k(N-1)b_k} P(u_{i_1}, \dots, u_{i_N}) \quad \text{for } k \geq 1$$

and the b_k are the coefficients of the Taylor expansion of the function

$$v(x) - (1-x)^{\frac{1}{1-N}} = \sum_{k=0}^{\infty} b_k x^k.$$

Let

$$t_0 = \frac{1}{1-N} \log(1 + C_p^{-1} \|u_0\|^{1-N})$$

and

$$t_1 = \begin{cases} \frac{1}{1-N} \log(1 - C_p^{-1} \|u_0\|^{1-N}) & \text{if } 1 > C_p^{-1} \|u_0\|^{1-N} \\ \infty & \text{if } 1 \leq C_p^{-1} \|u_0\|^{1-N}. \end{cases}$$

Then $u(t)$ is uniformly convergent on compact subsets of (t_0, t_1) and is the solution of the equation

$$\frac{du}{dt} = -u + P(u, \dots, u),$$

with the initial condition

$$u(0) = u_0.$$

Proof of Theorem 5. For $f, g, h \in L_2^1(\mathbb{R})$, let

$$P(f, g, h) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} \left[f' g_* \left(\frac{\int_{\mathbb{R}} h}{\|f_0\|_{L^1}} + \delta \tilde{h} + \delta \tilde{h}'_* \right) - f g_* \left(\frac{\int_{\mathbb{R}} h}{\|f_0\|_{L^1}} + \delta \tilde{h}' + \delta \tilde{h}'_* \right) \right] dv_* d\theta + f \left(\int_{\mathbb{R}} g \right) \left(\frac{\int_{\mathbb{R}} h}{\|f_0\|_{L^1}} + 2\delta \|\psi\|_{L^\infty} \int_{\mathbb{R}} h \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) d\theta.$$

Let

$$K = \|f_0\|_{L^1} (1 + 2\delta \|\psi\|_{L^\infty} \|f_0\|_{L^1}) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) d\theta.$$

Let us consider the following problem

$$\begin{cases} \frac{\partial f}{\partial t} + Kf = P(f, f, f), & t \in \mathbb{R}_+, v \in \mathbb{R} \\ f(0, v) = f_0. \end{cases} \quad (32)$$

The operator $P : (L_2^1)^3 \rightarrow L_2^1$ is trilinear, and satisfies the inequality

$$\|P(f, g, h)\|_{L_2^1} \leq C_P \|f\|_{L_2^1} \|g\|_{L_2^1} \|h\|_{L_2^1} \quad \forall f, g, h \in L_2^1(\mathbb{R}),$$

with

$$C_P = \int \beta(\theta) d\theta \left(\frac{2}{\|f_0\|_{L^1}} + 4\delta \|\psi\|_{L^\infty} \right).$$

Assume for the moment that $K = 1$. Thanks to Theorem 6, there exists some $T > 0$ such that there exists a solution $f \in L^\infty(0, T; L_2^1(\mathbb{R}))$ to the problem (32). This solution can be written as a Wild sum, which reads

$$f(t) = \sum_{k=0}^{+\infty} b_k e^{-t} (1 - e^{-2t})^k f_k,$$

where

$$f_k = \sum_{i_1+i_2+i_3=k-1} \frac{b_{i_1} b_{i_2} b_{i_3}}{2k b_k} P(f_{i_1}, f_{i_2}, f_{i_3}) \quad \text{for } k \geq 1$$

and

$$f_{k=0} = f_0.$$

The numbers b_k are the coefficients of the Taylor expansion of

$$\frac{1}{\sqrt{1-x}} = \sum_{k=0}^{+\infty} b_k x^k.$$

One can easily see that all the b_k are positive. Moreover

$$0 \leq f, g, h \in L_2^1 \implies P(f, g, h) \geq 0.$$

Thus, the solution f of (32) is nonnegative. Moreover, owing to the definition of $P(\cdot, \cdot, \cdot)$, one can verify that this solution preserves the mass. From that we deduce that f is solution of (30) on $(0, T)$. But it preserves the mass, and it

relies in L_2^1 , so that it also preserves the energy. Since the time T depends only on $\|f_0\|_{L^1}, \|f_0\|_{L_2^1}, \|\psi\|_{L^\infty}, \|\beta\|_{L^1(-\frac{\pi}{2}, \frac{\pi}{2})}$, we can use the same arguments on the time intervals $(T, 2T), (2T, 3T)$, etc, to get a solution on $(0, +\infty)$. Finally, in the case $K \neq 1$, it is enough to rescale the time to get the right formulae, and to obtain the same conclusions. \square

The following theorem claims the existence of a solution of (30) in the non-cutoff case in some weak sense.

Theorem 7. *Let $f_0 \in L_2^1(\mathbb{R})$ be a nonnegative function. Let β satisfy the assumptions (H1) and (H2). Then, there exists a distribution g such that its Fourier transform $\hat{g} \in L^\infty(\mathbb{R}_+; C^0 \cap L^\infty(\mathbb{R}))$ is a solution of (33), the Fourier form of equation (30). Moreover, \hat{g} preserves the mass.*

Proof. We introduce, for $n \in \mathbb{N}_*$, the cross-section

$$\beta_n(\theta) = \beta(\theta) \wedge n = \min(\beta(\theta), n),$$

and we denote by f^n the solution of the problem (30) corresponding to the cross-section β_n . This solution exists thanks to Theorem 5. For all n and all $t > 0$, $f^n(t)$ relies in L^1 , so that we can define its Fourier transform. Moreover, $\tilde{Q}_{QBE}(f^n)$ also relies in L^1 . Hence, we can write the following equation:

$$\frac{\partial \hat{f}^n(t, \xi)}{\partial t} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^2} \beta_n(\theta) f^n f_*^n (1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v'_*)) (e^{-iv'\xi} - e^{-iv\xi}) dv dv_* d\theta.$$

As in Section 3 and since $\mathcal{F}(\tilde{f}^n)(\eta) = \hat{f}^n(\eta) \hat{\psi}(\eta)$, we get

$$\begin{aligned} \frac{\partial \hat{f}^n(t, \xi)}{\partial t} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \left(\hat{f}^n(\xi \cos \theta) \hat{f}^n(\xi \sin \theta) - \hat{f}^n(\xi) \hat{f}^n(0) \right) d\theta \\ &+ \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \int_{\mathbb{R}} \left[\hat{f}^n((\xi - \eta) \cos \theta) \hat{f}^n((\xi - \eta) \sin \theta) \right. \\ &\quad \left. - \hat{f}^n(\xi - \eta \cos \theta) \hat{f}^n(\eta \sin \theta) \right] \hat{f}^n(\eta) \hat{\psi}(\eta) d\eta d\theta \quad (33) \\ &+ \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \int_{\mathbb{R}} \left[\hat{f}^n(\xi \cos \theta - \eta \sin \theta) \hat{f}^n(-\xi \sin \theta - \eta \cos \theta) \right. \\ &\quad \left. - \hat{f}^n(\xi - \eta \sin \theta) \hat{f}^n(-\eta \cos \theta) \right] \hat{f}^n(\eta) \hat{\psi}(\eta) d\eta d\theta. \end{aligned}$$

Note that here the mass is the quantity $\hat{f}(0)$. Since the second moment of $f^n(v, t)$ is finite and conserved in time, its Fourier transform is two times differentiable, and satisfies

$$\sup_{t>0} \|\partial_{\xi\xi}^2 \hat{f}^n(t, \xi)\|_{L^\infty} \leq \int_{\mathbb{R}} v^2 f_0(v) dv.$$

Hence, we can use the Taylor formula at the order 2

$$h(\theta) = h(0) + \theta h'(0) + \theta^2 \int_0^1 (1-s) h''(s\theta) ds$$

on the functions $\theta \mapsto \hat{f}^n(\xi \cos \theta) \hat{f}^n(\xi \sin \theta)$, $\theta \mapsto \hat{f}^n((\xi - \eta) \cos \theta) \hat{f}^n((\xi - \eta) \sin \theta)$, $\theta \mapsto \hat{f}^n(\xi - \eta \cos \theta) \hat{f}^n(\eta \sin \theta)$, $\theta \mapsto \hat{f}^n(\xi \cos \theta - \eta \sin \theta) \hat{f}^n(-\xi \sin \theta - \eta \cos \theta)$ and $\theta \mapsto \hat{f}^n(\xi - \eta \sin \theta) \hat{f}^n(-\eta \cos \theta)$.

Using the notations $m = \int_{\mathbb{R}} f_0(v) dv$ and $e = \int_{\mathbb{R}} v^2 f_0(v) dv$, we get the following estimates

$$\begin{aligned} & \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \left(\hat{f}^n(\xi \cos \theta) \hat{f}^n(\xi \sin \theta) - \hat{f}^n(\xi) \hat{f}^n(0) \right) d\theta \right| \\ & \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^2 \beta(\theta) d\theta \left(4|\xi|^2 m e + 2|\xi| m^{3/2} e^{1/2} \right), \end{aligned} \quad (34)$$

$$\begin{aligned} & \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \left(\hat{f}^n((\xi - \eta) \cos \theta) \hat{f}^n((\xi - \eta) \sin \theta) \right. \right. \\ & \quad \left. \left. - \hat{f}^n(\xi - \eta \cos \theta) \hat{f}^n(\eta \sin \theta) \right) \mathcal{F}(\tilde{f}^n)(\eta) d\theta \right| \\ & \leq |\hat{\psi}(\eta)| m \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^2 \beta(\theta) d\theta \left(4(|\xi - \eta|^2 + |\eta|^2) m e + 2(|\xi - \eta| + |\eta|) m^{3/2} e^{1/2} \right) \end{aligned} \quad (35)$$

and

$$\begin{aligned} & \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \left(\hat{f}^n(\xi \cos \theta - \eta \sin \theta) \hat{f}^n(-\xi \sin \theta - \eta \cos \theta) \right. \right. \\ & \quad \left. \left. - \hat{f}^n(\xi - \eta \sin \theta) \hat{f}^n(-\eta \cos \theta) \right) \mathcal{F}(\tilde{f}^n)(\eta) d\theta \right| \\ & \leq |\hat{\psi}(\eta)| m \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^2 \beta(\theta) d\theta \left(4((|\xi| + |\eta|)^2 + |\eta|^2) m e + 2(|\xi| + 2|\eta|) m^{3/2} e^{1/2} \right) \end{aligned} \quad (36)$$

The right member of (34) is integrable in time on any interval $[t_1, t_2] \subset \mathbb{R}_+$, and the right-members of (35) and (36) are integrable in (t, η) on any $[t_1, t_2] \times \mathbb{R}$ with $0 < t_1 < t_2$, since $\psi \in C_c^\infty(\mathbb{R})$. Therefore, to pass to the limit in (33), it is enough for \hat{f}^n to converge pointwise on $\mathbb{R}_+ \times \mathbb{R}$. But inequalities (34), (35) and (36) ensure that for all compact set $\mathcal{K} \subset \mathbb{R}$, there exists a constant C depending only on \mathcal{K} , m , e , ψ , β , such that

$$|\hat{f}^n(t_1, \xi) - \hat{f}^n(t_2, \xi)| \leq C |t_1 - t_2| \quad \forall 0 < t_1 < t_2, \quad \forall \xi \in \mathcal{K}.$$

Then, thanks to Ascoli's theorem, there exists a function $\hat{g} \in L^\infty([t_1, t_2] \times \mathcal{K})$ such that, up to the extraction of a subsequence,

$$\|\hat{f}^n - \hat{g}\|_{L^\infty([t_1, t_2] \times \mathcal{K})} \xrightarrow{n \rightarrow +\infty} 0.$$

All this being true for every t_1, t_2, \mathcal{K} , we deduce that \hat{g} is well defined on $\mathbb{R}_+ \times \mathbb{R}$ and that $\hat{g} \in L^\infty(\mathbb{R}_+; L^\infty \cap C^0(\mathbb{R}))$. We can therefore pass to the limit in (33).

Finally, we obtained the existence of a function $\hat{g}(\xi, t)$ which satisfies (33) with the original cross-section β , and such that

$$\hat{g}(t, 0) = m \quad \forall t > 0.$$

□

5 Moments of the cutoff solutions, regularity of the non cutoff solution

The second step of our analysis is the study of the regularity of the Fourier transform of the solution obtained in the previous Section. This can be done by investigating the moments of the solution to the cut-off equation. Since it is enough for our needs, we will limit ourselves to the fourth moment. Let

$$A = \int_{-\pi/2}^{\pi/2} \beta(\theta) \cos \theta \sin^2 \theta d\theta$$

$$A_\eta = \int_{-\pi/2}^{\pi/2} \beta(\theta) \cos \theta |\sin \theta|^{1+\eta} d\theta, \quad \nu - 1 < \eta < 1,$$

$$A_* = \int_{-\pi/2}^{\pi/2} \beta(\theta) \cos^2 \theta \sin^2 \theta d\theta$$

and chose $n_\beta \in \mathbb{N}$ large enough such that, for all $n \geq n_\beta$,

$$\int_{-\pi/2}^{\pi/2} \beta_n(\theta) \cos^2 \theta \sin^2 \theta d\theta \geq \frac{A_*}{2}$$

with $\beta_n(\theta) = \min(\beta(\theta), n)$. Our result is the following

Theorem 8. *Let $\beta(\theta)$ satisfy the assumptions (H1) and (H2), and let $f^n(t, v)$ be the solution of the problem (30) with cross-section β_n and with nonnegative initial datum $f_0 \in L^1_4(\mathbb{R})$. Then there are explicit constants $a > 0$, $c > 0$, $C > 0$ such that we have*

$$\sup_{n \geq n_\beta} \|f^n(t)\|_{L^1_4} \leq (\|f_0\|_{L^1_4} + ct)e^{at}, \quad t \geq 0, \quad (37)$$

$$\sup_{n \geq n_\beta} \sup_{t \in \mathbb{R}_+} \|f^n(t)\|_{L^1_4} \leq \max\{C, \|f_0\|_{L^1_4}\}. \quad (38)$$

Moreover, a, c in (37) depend only on $\|f_0\|_{L^1}$, $\|f_0\|_{L^1_2}$, δ , ψ and A , whereas C in (38) depends only on $\|f_0\|_{L^1}$, $\|f_0\|_{L^1_2}$, δ , ψ and A , A_* , A_η and η .

Proof. Let us take $\phi(v) = v^4$ as test function. Thanks to the symmetries of the kernel we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} v^4 f^n(v, t) dv \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \int_{\mathbb{R}^2} (v'^4 - v^4 + v_*'^4 - v_*^4) f^n f_*^n (1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v_*')) dv dv_* d\theta. \end{aligned}$$

From the collision rule (10) it follows that

$$\begin{aligned}
& (v')^4 + (v'_*)^4 - v^4 - v_*^4 \\
&= (v^4 + v_*^4)(\cos^4 \theta + \sin^4 \theta - 1) + 12v^2 v_*^2 \sin^2 \theta \cos^2 \theta \\
&\quad + 4vv_* \cos \theta \sin \theta (v^2(\sin^2 \theta - \cos^2 \theta) + v_*^2(\cos^2 \theta - \sin^2 \theta)) \\
&= -2 \cos^2 \theta \sin^2 \theta (v^4 + v_*^4) + 12v^2 v_*^2 \sin^2 \theta \cos^2 \theta \\
&\quad + 4 \cos \theta \sin \theta \cos(2\theta) vv_*(v_*^2 - v^2).
\end{aligned}$$

Consequently

$$\begin{aligned}
& \frac{d}{dt} \|f^n(t)\|_{L^1_4} \\
&= - \int_{-\pi/2}^{\pi/2} \beta_n(\theta) \cos^2 \theta \sin^2 \theta \iint_{\mathbb{R}^2} (v^4 + v_*^4) f^n f_*^n \left(1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v'_*)\right) dv dv_* d\theta \\
&+ 6 \int_{-\pi/2}^{\pi/2} \beta_n(\theta) \cos^2 \theta \sin^2 \theta \iint_{\mathbb{R}^2} v^2 v_*^2 f^n f_*^n \left(1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v'_*)\right) dv dv_* d\theta \\
&+ 2 \int_{-\pi/2}^{\pi/2} \beta_n(\theta) \cos \theta \sin \theta \cos(2\theta) \iint_{\mathbb{R}^2} vv_* f^n f_*^n \left(1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v'_*)\right) dv dv_* d\theta \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

In what follows we always assume $n \geq n_\beta$. For the first and second terms I_1, I_2 we use

$$\frac{1}{2} A_* \leq \int_{-\pi/2}^{\pi/2} \beta_n(\theta) \cos^2 \theta \sin^2 \theta d\theta \leq A_*$$

and

$$\|\tilde{f}^n(t)\|_{L^\infty} \leq m \|\psi\|_{L^\infty}$$

to get

$$\begin{aligned}
I_1 &\leq - \left(\frac{1}{2} \int_{-\pi/2}^{\pi/2} \beta(\theta) \cos^2 \theta \sin^2 \theta d\theta \right) 2m \int_{\mathbb{R}} |v|^4 f^n(t, v) dv \\
&= m^2 A_* - mA_* \|f^n(t)\|_{L^1_4},
\end{aligned} \tag{39}$$

and

$$I_2 \leq 6(1 + 2\delta m \|\psi\|_{L^\infty}) A_* e^2. \tag{40}$$

For the third term I_3 we compute by change of variable $(v, v_*, \theta) \mapsto (v, v_*, -\theta)$

$$\begin{aligned}
I_3 &= \delta \int_{-\pi/2}^{\pi/2} \beta_n(\theta) \cos \theta \sin \theta \cos(2\theta) \\
&\quad \times \iint_{\mathbb{R}^2} vv_*(v_*^2 - v^2) f^n f_*^n \left(\tilde{f}^n(v') + \tilde{f}^n(v'_*) - \tilde{f}^n(\bar{v}') - \tilde{f}^n(\bar{v}'_*) \right) dv dv_* d\theta
\end{aligned}$$

where

$$\bar{v}' = v \cos \theta + v_* \sin \theta, \quad \bar{v}'_* = -v \sin \theta + v_* \cos \theta.$$

Recall

$$v' = v \cos \theta - v_* \sin \theta, \quad v'_* = v \sin \theta + v_* \cos \theta$$

and

$$\left\| \frac{\partial}{\partial v} \tilde{f}^n(\cdot, t) \right\|_{L^\infty} \leq m \|\psi'\|_{L^\infty}.$$

It follows that

$$\begin{aligned} |\tilde{f}^n(v') + \tilde{f}^n(v'_*) - \tilde{f}^n(\bar{v}') - \tilde{f}^n(\bar{v}'_*)| &\leq |\tilde{f}^n(v') - \tilde{f}^n(\bar{v}')| + |\tilde{f}^n(v'_*) - \tilde{f}^n(\bar{v}'_*)| \\ &\leq 2m \|\psi'\|_{L^\infty} (|v| + |v_*|) |\sin \theta|. \end{aligned} \quad (41)$$

The left hand side is also less than $2m \|\psi\|_{L^\infty}$ (since $f \geq 0$). Thus

$$|\tilde{f}^n(v') + \tilde{f}^n(v'_*) - \tilde{f}^n(\bar{v}') - \tilde{f}^n(\bar{v}'_*)| \leq 2m (\|\psi\|_{L^\infty})^{1-\eta} (\|\psi'\|_{L^\infty})^{1-\eta} (|v|^\eta + |v_*|^\eta) |\sin \theta|^\eta. \quad (42)$$

By (41) and (42) we obtain two estimates for I_3 :

$$\begin{aligned} I_3 &\leq 2m\delta \|\psi'\|_{L^\infty} A \int \int_{\mathbb{R}^2} |v| |v_*| (|v| + |v_*|) (v_*^2 + v^2) f^n f_*^n dv dv_* \\ &= K_{m,e} \delta \|\psi'\|_{L^\infty} A \|f^n(t)\|_{L^1_4}, \end{aligned} \quad (43)$$

and

$$\begin{aligned} I_3 &\leq 2m (\|\psi\|_{L^\infty})^{1-\eta} (\|\psi'\|_{L^\infty})^{1-\eta} \delta A_\eta \int \int_{\mathbb{R}^2} |v| |v_*| (|v|^\eta + |v_*|^\eta) (v_*^2 + v^2) f^n f_*^n dv dv_* \\ &\leq K_{m,e} (\|\psi\|_{L^\infty})^{1-\eta} (\|\psi'\|_{L^\infty})^{1-\eta} \delta A_\eta (\|f^n(t)\|_{L^1_4})^{\frac{3+\eta}{4}} := C_1 (\|f^n(t)\|_{L^1_4})^{\frac{3+\eta}{4}}. \end{aligned} \quad (44)$$

Here $K_{m,e} > 0$ depends only on m and e .

To prove the first estimate (37) we omit the negative term I_1 and use (40), (43) and notice that $A_* \leq A$ to get

$$\frac{d}{dt} \|f^n(t)\|_{L^1_4} \leq 6e^2 (1 + 2\delta m \|\psi\|_{L^\infty}) A + K_{m,e} \delta \|\psi'\|_{L^\infty} A \|f^n(t)\|_{L^1_4}, \quad t > 0.$$

This gives by Gronwall lemma

$$\sup_{n \geq n_\beta} \|f^n(t)\|_{L^1_4} \leq (\|f_0\|_{L^1_4} + ct) e^{at}, \quad t \geq 0$$

where

$$c = 6e^2 (1 + 2\delta m \|\psi\|_{L^\infty}) A, \quad a = K_{m,e} \delta \|\psi'\|_{L^\infty} A.$$

To prove the second estimate (38) we use (39), (40) and (44) to see that

$$\frac{d}{dt} \|f^n(t)\|_{L^1_4} \leq C_2 + C_1 (\|f^n(t)\|_{L^1_4})^{\frac{3+\eta}{4}} - mA_* \|f^n(t)\|_{L^1_4}, \quad t > 0.$$

Here and below C_1 , C_2 and C_3 depend only on $m, e, \psi, \delta, A, A_*, A_\eta$ and η . Applying the following inequality

$$Y^\alpha \leq (1/\varepsilon)^{\frac{\alpha}{1-\alpha}} + \varepsilon Y, \quad \alpha \in (0, 1), Y \geq 0, \varepsilon > 0$$

to

$$Y = \|f^n(t)\|_{L^1_4}, \quad \alpha = \frac{3+\eta}{4}, \quad \varepsilon = \frac{mA_*}{2C_1}$$

gives

$$C_1(\|f^n(t)\|_{L^1_4})^{\frac{3+\eta}{4}} \leq C_1 \left(\frac{2C_1}{mA_*} \right)^{\frac{3+\eta}{1-\eta}} + \frac{mA_*}{2} \|f^n(t)\|_{L^1_4}$$

and so

$$\frac{d}{dt} \|f^n(t)\|_{L^1_4} \leq C_3 - \frac{mA_*}{2} \|f^n(t)\|_{L^1_4}, \quad t > 0.$$

Therefore,

$$\sup_{n \geq n_\beta} \sup_{t \geq 0} \|f^n(t)\|_{L^1_4} \leq \max \left\{ \frac{2C_3}{mA_*}, \|f_0(t)\|_{L^1_4} \right\}.$$

□

Passing to the limit $n \rightarrow +\infty$ in inequality (38) we obtain

Theorem 9. *Let β satisfy (H1) and (H2), and let g be the weak solution of (30) defined in Theorem 7, with nonnegative initial data $f_0 \in L^1_4$. Then, $\hat{g}(t)$ is C^3 for all t and*

$$\sup_{t > 0} \{ \|\hat{g}(t)\|_{L^\infty} + \|\partial_\xi \hat{g}(t)\|_{L^\infty} + \|\partial_{\xi^2}^2 \hat{g}(t)\|_{L^\infty} + \|\partial_{\xi^3}^3 \hat{g}(t)\|_{L^\infty} + \|\partial_{\xi^4}^4 \hat{g}(t)\|_{L^\infty} \} < +\infty.$$

Moreover, \hat{g} conserves the energy, in the sense that $\partial_{\xi\xi}^2 \hat{g}(t, 0) = -e$ for all $t > 0$.

Proof. The conservation of the mass and inequality (38) imply that there exists a constant $C > 0$ which do not depend on n such that

$$\begin{cases} \|\partial_\xi \hat{f}^n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq C \\ \|\partial_{\xi^2}^2 \hat{f}^n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq C \\ \|\partial_{\xi^3}^3 \hat{f}^n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq C \\ \|\partial_{\xi^4}^4 \hat{f}^n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq C. \end{cases}$$

Since $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is the dual space of the Banach space $L^1(\mathbb{R}_+ \times \mathbb{R})$, the four sequences converge (up to the extraction of a subsequence) in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ weak-*; the limits can only be respectively $\partial_\xi \hat{g}$, $\partial_{\xi^2}^2 \hat{g}$, $\partial_{\xi^3}^3 \hat{g}$ and $\partial_{\xi^4}^4 \hat{g}$ (since the convergence in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ weak-* implies the convergence in the distributional sense). Moreover, we have the inequalities, for $1 \leq i \leq 4$,

$$\|\partial_{\xi^i}^i \hat{g}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq \liminf_{n \rightarrow +\infty} \|\partial_{\xi^i}^i \hat{f}^n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})}.$$

Finally, we have the embedding

$$W^{4,\infty}(\mathbb{R}) \hookrightarrow C^3(\mathbb{R}),$$

so that $\hat{g} \in L^\infty(\mathbb{R}_+; C^3(\mathbb{R}))$.

It remains to prove that the energy is conserved. Let us fix some time $t_0 > 0$. It is clear that for all integer n , we have

$$\|\partial_{\xi^2}^2 \hat{f}^n(t_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq e.$$

Therefore, up to the extraction of a subsequence, there exists a function $h \in L^\infty(\mathbb{R})$ such that

$$\partial_{\xi^2}^2 \hat{f}^n(t_0, \cdot) \rightharpoonup h \quad \text{weak} - * \quad L^\infty(\mathbb{R}).$$

But it is clear that $h = \partial_{\xi^2}^2 g(t_0, \cdot)$, since for all function $\phi \in C^\infty(\mathbb{R})$ with compact support,

$$\int_{\mathbb{R}} \hat{f}^n(t_0, \xi) \phi''(\xi) d\xi \rightarrow \int_{\mathbb{R}} \hat{g}(t_0, \xi) \phi''(\xi) d\xi,$$

or, after two integrations per part on both sides,

$$\int_{\mathbb{R}} \partial_{\xi^2}^2 \hat{f}^n(t_0, \xi) \phi(\xi) d\xi \rightarrow \int_{\mathbb{R}} \partial_{\xi^2}^2 \hat{g}(t_0, \xi) \phi(\xi) d\xi.$$

Let us define an approximation of the Dirac measure

$$\Phi_p(\xi) = \begin{cases} p & \text{if } -\frac{1}{2p} < \xi < \frac{1}{2p} \\ 0 & \text{otherwise} \end{cases}$$

We have

$$\begin{aligned} \left| \partial_{\xi^2}^2 \hat{f}^n(t_0, 0) - \partial_{\xi^2}^2 \hat{g}(t_0, 0) \right| &\leq \left| \partial_{\xi^2}^2 \hat{f}^n(t_0, 0) - \int_{\mathbb{R}} \Phi_p(\xi) \partial_{\xi^2}^2 \hat{f}^n(t_0, \xi) d\xi \right| \\ &+ \left| \int_{\mathbb{R}} \Phi_p(\xi) \left(\partial_{\xi^2}^2 \hat{f}^n(t_0, \xi) - \partial_{\xi^2}^2 \hat{g}(t_0, \xi) \right) d\xi \right| \\ &+ \left| \int_{\mathbb{R}} \Phi_p(\xi) \partial_{\xi^2}^2 \hat{g}(t_0, \xi) d\xi - \partial_{\xi^2}^2 \hat{g}(t_0, 0) \right|. \end{aligned}$$

The first and the third terms converge toward 0 when p converges to infinity, independently of n since

$$\|\partial_{\xi^3}^3 \hat{f}^n(t_0, \cdot)\|_{L^\infty} \leq C$$

with $C > 0$ independent of n . As for the second term, once p has been fixed, it converges to 0 since $\Phi_p \in L^1(\mathbb{R})$, and the result follows. \square

6 The grazing collision limit

We are now in a position to perform the grazing collision limit in equation (17). The mechanism is the same as in Section 3: we introduce a family of kernels $\{\beta_\varepsilon(|\theta|)\}_{\varepsilon>0}$ satisfying hypotheses (H1) and (H2), with

$$\forall \theta_0 > 0 \quad \sup_{\theta > \theta_0} \beta_\varepsilon(|\theta|) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad (45)$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^\pi \beta_\varepsilon(|\theta|) \theta^2 d\theta = 1 \quad (46)$$

Let g_ε be the weak solution of the problem (30) in the sense that it satisfies equation (33), where $\beta(\theta)$ has been replaced by $\beta_\varepsilon(\theta)$.

Theorem 10. *Let $\beta(\theta)$ satisfy assumptions (H1), (H2), and let $\beta_\varepsilon(\theta)$ satisfy (45) and (46). Let g_ε be the weak solution of the problem (30) where $\beta(\theta)$ has been replaced by $\beta_\varepsilon(\theta)$, with the nonnegative initial data f_0 satisfying $f_0 \in L^1_+(\mathbb{R})$.*

Then, for all $T > 0$, there exists a distribution g whose Fourier transform satisfy $\hat{g} \in L^\infty(0, T; W^{4, \infty}(\mathbb{R}))$, and such that, up to the extraction of a subsequence,

$$\|\hat{g}_\varepsilon - \hat{g}\|_{L^\infty([t_1, t_2] \times \mathcal{K})} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \forall 0 < t_1 < t_2 < T \text{ and } \mathcal{K} \subset \mathbb{R} \text{ compact,}$$

$$\partial_{\xi^i}^i \hat{g}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \partial_{\xi^i}^i \hat{g} \text{ in } L^\infty((0, T) \times \mathbb{R}) \text{ weak-}^*, \quad 1 \leq i \leq 4,$$

and such that \hat{g} is a solution of the Fourier form of the equation

$$\frac{\partial h}{\partial t} = \left(\int_{\mathbb{R}} h(v) dv \right) \frac{\partial}{\partial v} (vh(1 + \delta \tilde{h})) + \left(\int_{\mathbb{R}} v^2 h(v) (1 + \delta \tilde{h}(v)) dv \right) \frac{\partial^2 h}{\partial v^2}, \quad (47)$$

Remark 11. If $\delta = 0$, equation (47) reduces to the classical linear Fokker-Planck equation.

Proof. To pass to the limit, we need some regularity on the solution g_ε . However, Theorem 9 is not sufficient, since the bound on the fourth derivative of \hat{g}_ε depends on A_η which becomes an unbounded quantity when replacing β by β_ε and as $\varepsilon \rightarrow 0$. Therefore, we use inequality (37) to obtain a regularity result on \hat{g}_ε which is the same than Theorem 9, except that it works only on any time interval $(0, T)$. The gain is that the constants used remain bounded as $\varepsilon \rightarrow 0$ when replacing β by β_ε .

Let us fix some $T > 0$; \hat{g}_ε is four times differentiable for almost any time in $(0, T)$, and these derivatives are bounded uniformly in time and independently of ε , provided the initial datum f_0 relies in $L^1_4(\mathbb{R})$. We now act as if $\beta(\theta)$ was integrable, and then we will obtain the result by an approximation argument. A Taylor expansion in ε under the integral sign gives

$$\begin{aligned} & \frac{\partial \hat{g}_\varepsilon(t, \xi)}{\partial t} \\ &= \frac{1}{\varepsilon^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left[\varepsilon \theta \xi \left(\hat{g}_\varepsilon(\xi) \hat{g}'_\varepsilon(0) \right. \right. \\ & \quad \left. \left. - \frac{\delta}{2\pi} \int_{\mathbb{R}} \left(\hat{g}_\varepsilon(\xi - \eta) \hat{g}_\varepsilon(\eta) \hat{g}'_\varepsilon(0) \hat{\psi}(\eta) - \hat{g}'_\varepsilon(-\eta) \hat{g}_\varepsilon(\eta) \hat{g}_\varepsilon(\xi) \hat{\psi}(\eta) \right) d\eta \right) \right. \\ & \quad \left. + \varepsilon^2 \frac{\theta^2}{2} \left(\xi^2 \hat{g}''_\varepsilon(0) \hat{g}_\varepsilon(\xi) - \xi \hat{g}'_\varepsilon(\xi) \hat{g}_\varepsilon(0) - \frac{\delta}{2\pi} \int_{\mathbb{R}} \hat{\psi}(\eta) \left(\hat{g}'_\varepsilon(\xi - \eta) \hat{g}_\varepsilon(\eta) \hat{g}_\varepsilon(0) \right. \right. \right. \\ & \quad \left. \left. + ((\eta - \xi)^2 - \eta^2) \hat{g}_\varepsilon(\xi - \eta) \hat{g}_\varepsilon(\eta) \hat{g}''_\varepsilon(0) + \xi \hat{g}'_\varepsilon(\xi) \hat{g}_\varepsilon(\eta) \hat{g}_\varepsilon(-\eta) \right. \right. \\ & \quad \left. \left. + \xi^2 \hat{g}''_\varepsilon(-\eta) \hat{g}_\varepsilon(\eta) \hat{g}_\varepsilon(\xi) + 2\eta \xi \hat{g}'_\varepsilon(\xi) \hat{g}_\varepsilon(\eta) \hat{g}'_\varepsilon(-\eta) \right) d\eta \right] + \theta^2 O(\varepsilon^3) \Big] d\theta. \end{aligned} \quad (48)$$

Since $\beta(\theta)$ is an even function, the first-order terms vanish.

By Theorem 8 there exists a constant $\lambda_T > 0$ which do not depend on ε such that

$$\sup_{0 < t < T} \left\{ \sum_{i=0}^4 \|\partial_{\xi^i}^i \hat{g}_\varepsilon(t, \cdot)\|_{L^\infty} \right\} \leq \lambda_T.$$

Using equation (48), we see that the family $(\hat{g}_\varepsilon)_\varepsilon$ is equicontinuous, so that we can use Ascoli's theorem, which says that there exists a function $\hat{g} \in L^\infty((0, T) \times \mathbb{R})$ such that, up to the extraction of a subsequence,

$$\|\hat{g}_\varepsilon - \hat{g}\|_{L^\infty([t_1, t_2] \times \mathcal{K})} \xrightarrow{\varepsilon \rightarrow 0} 0$$

for all $0 < t_1 < t_2 < T$ and all compact set $\mathcal{K} \subset \mathbb{R}$. In addition, all the results of Theorem 9 are still valid for \hat{g} on $(0, T)$. Therefore, thanks to both the uniform convergence for \hat{g} and the convergence in $L^\infty((0, T) \times \mathbb{R})$ weak-* for its derivatives, we can pass to the limit in equation (48), and we get, using classical formulae on the Fourier transform and the conservation of mass and energy, that \hat{g} satisfies the equation which is the Fourier transform of equation (47). \square

Proof of Theorem 3. To study the grazing collision limit, we needed some regularity on the Fourier transform of the solution. This is equivalent to have a uniform bound on some higher moment of the solution. In proving Theorem 8, we used the regularity of the mollified part, more precisely the fact that this part was in C^1 . In fact it could be enough to use the $H^{\frac{\nu}{2}}$ regularity which follows from the H -theorem. Indeed, the terms that raise problems in the proof of Theorem 8 are \tilde{I}_1 and \tilde{I}_2 . Let us see how to treat the first one. We have

$$\begin{aligned} |\tilde{I}_1| &= \left| \int \beta(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f f_* (f(v') - f(\tilde{v}')) dv dv_* d\theta \right| \\ &= \left| \int \beta(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f(v) f(v_*) \left[v(1 - \cos \theta) \right]^\alpha \right. \\ &\quad \left. \frac{f(v') - f(\tilde{v}')}{[v(1 - \cos \theta)]^\alpha} dv dv_* d\theta \right| \\ &\leq \int \beta(\theta) |\sin \theta| |1 - \cos \theta|^\alpha \cos^3 \theta \left(\int_{\mathbb{R}^2} (v^{1+\alpha} v_* (v^2 - v_*^2))^p f(v)^p f(v_*)^p dv dv_* \right)^{1/p} \\ &\quad \cdot \left(\int_{\mathbb{R}^2} \frac{|f(v') - f(\tilde{v}')|^q}{|v(1 - \cos \theta)|^{\alpha q}} dv dv_* \right)^{1/q} d\theta. \end{aligned}$$

In order to deal with moments not exceeding the fourth one, we use Hölder's inequality, with p such that $(3 + \alpha)p = 4$. Consequently $q = \frac{4}{1-\alpha}$. Moreover, in order to recognize the semi-norm of f in the Sobolev space $H^{\nu/2}$ in the last term of the product, we need to set

$$\alpha q = 1 + \nu,$$

and thus

$$\alpha = \frac{1 + \nu}{5 + \nu}$$

(note that $0 < \alpha < 1$). With these constants, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^2} \frac{|f(v') - f(\tilde{v}')|^q}{|v(1 - \cos \theta)|^{\alpha q}} dv dv_* \right)^{1/q} &= \frac{1}{|1 - \cos \theta|^{1/q}} \left(\int_{\mathbb{R}^2} \frac{|f(v') - f(\tilde{v}')|^q}{|v(1 - \cos \theta)|^{\alpha q}} dv d(1 - \cos \theta) v_* \right)^{1/q} \\ &\leq \frac{2^{1-\frac{2}{q}}}{|1 - \cos \theta|^{\frac{1}{q}} |\sin \theta|^{\frac{1}{q}}} \|f(t)\|_{L^\infty}^{1-\frac{2}{q}} |f(t)|_{H^{\nu/2}}^{\frac{2}{q}}. \end{aligned}$$

Finally

$$|\tilde{I}_1| \leq C \int \beta(\theta) |\cos^3 \theta| |\sin \theta|^{1-\frac{1}{q}} |1 - \cos \theta|^{\alpha-\frac{1}{q}} d\theta \left(\int_{\mathbb{R}} v^4 f(t, v) dv \right)^{\frac{2}{p}} \|f(t)\|_{L^\infty} |f(t)|_{H^{\nu/2}}^{\frac{2}{q}}$$

The integral in θ is finite if and only if

$$2 \left(\alpha - \frac{1}{q} \right) - \nu - \frac{1}{q} > -1.$$

Recalling that

$$q = \frac{4}{1 - \alpha}$$

and that

$$\alpha = \frac{1 + \nu}{5 + \nu},$$

this request is equivalent to

$$\nu^2 + 2\nu - 4 < 0,$$

which is verified for $-1 - \sqrt{5} < \nu < \sqrt{5} - 1$. Combining this with the previous constraints on the parameter ν , we obtain that our bound of the fourth moment works for all ν verifying

$$1 < \nu < \sqrt{5} - 1.$$

The treatment of \tilde{I}_2 is quite more simple, since it requires only some changes of variable. Indeed, using the changes of variable $v \mapsto v - v_* \sin \theta$ and $\theta \mapsto -\theta$

$$\begin{aligned} \tilde{I}_2 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f(v) f(v_*) (f(\tilde{v}') - f(v)) dv dv_* d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f(v) f(v_*) f(\tilde{v}') dv dv_* d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta \cos^3 \theta \cdot \\ &\quad \cdot \int_{\mathbb{R}^2} (v + v_* \sin \theta) v_* ((v + v_* \sin \theta)^2 - v_*^2) f(v + v_* \sin \theta) f(v_*) f(v) dv dv_* d\theta \\ &= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta \cos^3 \theta \cdot \\ &\quad \cdot \int_{\mathbb{R}^2} (v - v_* \sin \theta) v_* ((v - v_* \sin \theta)^2 - v_*^2) f(\tilde{v}') f(v_*) f(v) dv dv_* d\theta \end{aligned}$$

This implies

$$\tilde{I}_2 \leq C(m, e) \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \theta d\theta \right) \|f(t)\|_{L^\infty} \left(e^2 + m^{\frac{1}{4}} \left(\int_{\mathbb{R}} v^4 f(t, v) dv \right)^{\frac{3}{4}} \right).$$

□

7 Conclusions

In this paper we investigated the asymptotic equivalence between the (mollified) Kac caricature of a Bose-Einstein gas and a nonlinear Fokker-Planck type equation in the so-called grazing collision limit. The limit equation differs from the analogous one present in the literature [13], since in our case the linear diffusion has a diffusivity which depends on the solution itself, in order to guarantee the conservation of energy. Our analysis refers to a mollified version of the equation, due to the difficulties of handle the third order nonlinearity present in the Bose-Einstein correction. A further inside on the true model, done in the first part of the paper, shows that a proof of the boundedness of the solution would be sufficient to avoid the presence of the mollifier.

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