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The grazing collision limit of Kac caricature of Bose-Einstein particles

Thibaut Allemand* Giuseppe Toscani†

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Abstract

We discuss the grazing collision limit of certain kinetic models of Bose-Einstein particles obtained from a suitable modification of the one-dimensional Kac caricature of a Maxwellian gas without cut-off. We recover in the limit a nonlinear Fokker-Planck equation which presents many similarities with the one introduced by Kaniadakis and Quarati in [13]. In order to do so, we perform a study of the moments of the solution. Moreover, as is typical in Maxwell models, we make an essential use of the Fourier version of the equation.

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1 Introduction

The quantum dynamics of many body systems is often modelled by a nonlinear Boltzmann equation which exhibits a gas-particle-like collision behavior. The

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application of quantum assumptions to molecular encounters leads to some divergences from the classical kinetic theory [5] and despite their formal analogies the Boltzmann equation for classical and quantum kinetic theory present very different features. The interest in the quantum framework of the Boltzmann equation has increased noticeably in the recent years. Although the quantum Boltzmann equation, for a single specie of particles, is valid for a gas of fermions as well as for a gas of bosons, blow up of the solution in finite time may occur only in the latter case. As a consequence the quantum Boltzmann equation for a gas of bosons represents the most challenging case both mathematically and numerically. In particular this equation has been successfully used for computing non-equilibrium situations where Bose-Einstein condensate occurs. From Chapman and Cowling [5] one can learn that the Boltzmann Bose-Einstein equation (BBE) is established by imposing that, when the mean distance between neighboring molecules is comparable with the size of the quantum wave fields with which molecules are surrounded, a state of congestion results. For a gas composed of Bose-Einstein identical particles, according to quantum theory, the presence of a like particle in the velocity-range dv increases the probability that a particle will enter that range; the presence of $f(v)dv$ particles per unit volume increases this probability in the ratio $1 + \delta f(v)$. This fundamental assumption yields the Boltzmann Bose-Einstein equation

$$\frac{\partial f}{\partial t} = Q_{QBE}(f)(t, v), \quad t \in \mathbb{R}_+, \quad v \in \mathbb{R}^3, \quad (1)$$

where

$$Q_{QBE}(f)(t, v) = \int_{\mathbb{R}^3 \times S^2} B(v - v_*, \omega) (f' f'_* (1 + \delta f)(1 + \delta f_*) - f f_* (1 + \delta f')(1 + \delta f'_*)) dv_* d\omega, \quad (2)$$

where as usual we denoted

$$f = f(v), \quad f_* = f(v_*), \quad f' = f(v'), \quad f'_* = f(v'_*),$$

and the pairs (v, v_*) (respectively (v', v'_*)) are the post- (respectively pre-) collision velocities in a elastic binary collision. In (2) $B(z, \omega)$ is the collision kernel which is a nonnegative Borel function of $|z|, |< z, \omega >|$ only

$$B(z, \omega) = B\left(|z|, \frac{|< z, \omega >|}{|z|}\right), \quad (z, \omega) \in \mathbb{R}^3 \times S^2. \quad (3)$$

The solutions $f(v, t)$ are velocity distribution functions (i.e., the density functions of particle number), $\delta = (h/m)^3/g$, h is the Planck's constant, m and g are the mass and the "statistical weight" of a particle (see [17] for details).

For a non relativistic particle, by setting $v(p) = p/m$, the collision operator Q_{QBE} can be rewritten in general form as follows [22, 23]

$$Q_{QBE}(f)(t, p) = \int_{\mathbb{R}^9} W(p, p_*, p', p'_*) (f' f'_* (1 + \delta f)(1 + \delta f_*) - f f_* (1 + \delta f')(1 + \delta f'_*)) dp_* dp' dp'_* \quad (4)$$

where W is a nonnegative measure called transition rate, which is of the form

$$W(p, p_*, p', p'_*) = \Omega(p, p_*, p', p'_*) \delta(p, +p_* - p' - p'_*) \delta(\mathcal{E}(p) + \mathcal{E}(p_*) - \mathcal{E}(p') - \mathcal{E}(p'_*)),$$

where δ represents the Dirac measure and $\mathcal{E}(p)$ is the energy of the particle. The quantity $W dp' dp'_*$ is the probability for the initial state (p, p_*) to scatter and become a final state of two particles whose momenta lie in a small region $dp' dp'_*$. The function Ω is directly related to the differential cross section (see (3)), a quantity that is intrinsic to the colliding particles and the kind of interaction between them. The collision operator (4) is simplified by assuming a boson distribution which only depends on the total energy $e = \mathcal{E}(p)$. In this last case $f = f(e, t)$ is the boson density in energy space.

Together with the Boltzmann description given by the collision operators (2)-(4), other kinetic models for Bose-Einstein particles have been introduced so far. In particular, a related model described by means of Fokker-Planck type non linear operators has been proposed by Kompaneets [15] to describe the evolution of the radiation distribution $f(x, t)$ in a homogeneous plasma when radiation interacts with matter via Compton scattering

$$Q_K(f)(t, \rho) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left[\rho^4 \left(\frac{\partial f}{\partial \rho} + f + f^2 \right) \right], \quad \rho \in (0, +\infty) \quad (5)$$

In that context the coordinate ρ represents a momentum coordinate, $\rho = |p|$. More precisely, an equation which includes (5) as a particular case is obtained in [15] as a leading term for the corresponding Boltzmann equation under the crucial assumption that the scattering cross section is of the classical Thomson type (see [10] for details).

The fundamental assumption which leads to the correction in the Boltzmann collision operator (2), namely the fact that the presence of $f(v)dv$ particles per unit volume increases the probability that a particle will enter the velocity range dv in the ratio $1 + \delta f(v)$, has been recently used by Kaniadakis and Quarati [13, 12] to propose a correction to the drift term of the Fokker-Planck equation in presence of quantum indistinguishable particles, bosons or fermions. In their model, the collision operator (2) is substituted by

$$Q_{FP}(f)(t, v) = \nabla \cdot [\nabla f + v f(1 + \delta f)]. \quad (6)$$

Maybe the most remarkable difference between the kinetic operators (2) and (6) is that, while the former is such that mass, momentum and energy are collision invariant, the latter does not admit the energy as collision invariant. This suggests that the operator (6) would not result directly through an asymptotic procedure from the Bose-Einstein collision operator (2), but instead from some linearized version, where only the mass is preserved under the collision mechanism.

For a mathematical analysis of the quantum Boltzmann equation in the space homogeneous isotropic case we refer to [8, 9, 10, 17, 18]. We remark that already the issue of giving mathematical sense to the collision operator is highly nontrivial (particularly if positive measure solutions are allowed, as required by a careful analysis of the equilibrium states). All the mathematical results, however, require very strong cut-off assumptions on the cross-section [10, 17].

Also, accurate numerical discretization of the quantum Boltzmann equation, which maintain the basic analytical and physical features of the continuous problem, namely, mass and energy conservation, entropy growth and equilibrium distributions have been introduced recently in [2, 19]. Related works [16, 20, 21] in which fast methods for Boltzmann equations were derived using different

techniques like multipole methods, multigrid methods and spectral methods, are relevant to quote.

At the Fokker-Planck level, the qualitative analysis of the Kompaneets equation described by the operator (5) has been exhaustively studied in [7], while the numerical simulation has been done by Buet and Cordier [3]. To our knowledge, the mathematical study of the Fokker-Planck equation (6) introduced by Kaniadakis and Quarati [13] has been done only very recently [4], where the one-dimensional version of (6) has been studied.

In the case of the quantum Boltzmann equation the asymptotic equivalence between the binary collision operators (2), (4) and the Fokker-Planck type operators (5) and (6) is unknown. This is not the case for the classical binary collisions in a elastic gas, where the asymptotic equivalence between the Boltzmann and the Fokker-Planck-Landau equations has been proven rigorously in a series of papers by Villani [25, 26] by means of the so-called *grazing collision* asymptotics.

The same asymptotic procedure, in the case of the one-dimensional Kac equation [11], showed the asymptotic equivalence between Kac collision operator and the linear Fokker-Planck operator [24]. The method of proof in [24] takes advantage from the relatively simple structure of Kac equation. Taking this into account, in order to establish the asymptotic connection between the Boltzmann equation for Bose Einstein particles and its Fokker-Planck description, we will introduce a one-dimensional kinetic model in the spirit of Kac caricature of a Maxwell gas with a singular kernel. Then we will study the grazing collision limit of the equation, which leads to a Fokker-Planck type equation in which the drift is of the form of equation (6), but the coefficient of the (linear) diffusion term depends on time through the density function. More precisely, the Fokker-Planck collision operator reads

$$Q_{FP}(f)(t, v) = A_t(f) \frac{\partial^2 f}{\partial v^2} + \frac{\partial}{\partial v}(vf(1 + \delta f)), \quad (7)$$

where

$$A_t(f) = \int_{\mathbb{R}} v^2 f(v, t)(1 + \delta f(v, t))dv. \quad (8)$$

The paper is organized as follows. In the next Section we will introduce the model, together with some simplifications (mollified model) that allow to prove in Section 3 existence of a weak solution. Then in Section 4, we will focus on the moments of the solution and on some regularity properties. In Section 5, we will deal with the grazing collision limit. The last part of the work, Section 6 will contain some results for the non mollified model.

2 The Kac caricature of a Bose Einstein gas

The simplest one-dimensional model which maintains almost all physical properties of the Boltzmann equation for a Bose-Einstein gas can be obtained by generalizing Kac caricature of a Maxwell gas to Bose-Einstein particles. This one-dimensional model reads as follows

$$\begin{cases} \frac{\partial f}{\partial t} = Q_{QBE}(f)(t, v), & t \in \mathbb{R}_+, \quad v \in \mathbb{R}, \\ f(0, v) = f_0(v), \end{cases} \quad (9)$$

where

$$Q_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} (f' f'_* (1 + \delta f)(1 + \delta f_*) - f f_* (1 + \delta f')(1 + \delta f'_*)) dv_* d\theta. \quad (10)$$

For the sake of brevity, we used the usual notations

$$f \equiv f(t, v), \quad f' \equiv f(t, v'), \quad f_* \equiv f(t, v_*), \quad f'_* \equiv f(t, v'_*),$$

The initial datum f_0 is a nonnegative measurable function. The pre-collision velocities (v', v'_*) are defined by the Kac rotation rule [11], which is given by

$$\begin{cases} v' = v \cos \theta - v_* \sin \theta \\ v'_* = v \sin \theta + v_* \cos \theta. \end{cases} \quad (11)$$

Collisions (11) imply the conservation of the energy

$$v^2 + v_*^2 = v'^2 + v_*'^2. \quad (12)$$

Let us observe that the system (11) can be reversed and that we can write the post-collision velocities with respect to the pre-collision ones

$$\begin{cases} v = v' \cos \theta + v'_* \sin \theta \\ v_* = -v' \sin \theta + v'_* \cos \theta. \end{cases} \quad (13)$$

The parameter δ in (10) is a positive constant. The choice $\delta = 0$ would lead us back to standard Kac model, whereas δ negative would lead us to the Boltzmann-Fermi-Dirac equation, which features are very different from that of the Boltzmann-Bose-Einstein equation.

The cross-section $\beta(\theta)$ is a function defined over $(-\frac{\pi}{2}, \frac{\pi}{2})$. In the original Kac equation [11], $\beta(\theta)$ is assumed constant, which implies that collisions spread out uniformly with respect to the angle θ . Following Desvillettes [6], we will here assume that the cross-section is suitable to concentrate collisions on the *grazing* ones (these collisions are those that are neglected when the cut-off assumption is made). This corresponds to satisfy one or more of the following hypotheses

H1: $\beta(\theta)$ is a nonnegative even function.

H2: $\beta(\theta)$ satisfies a non-cutoff assumption on the form

$$\beta(\theta) \sim \frac{1}{|\theta|^{1+\nu}} \quad \text{when } \theta \rightarrow 0, \quad (14)$$

with $1 < \nu < 2$. That is,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\sin \theta| d\theta = +\infty$$

whereas

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\sin \theta|^{\nu+\varepsilon} d\theta < +\infty$$

for all $\varepsilon > 0$.

H3: $\beta(\theta)$ is zero near $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, namely there exists an $\varepsilon_0 > 0$ such that

$$\beta(\theta) = 0 \quad \forall \theta \in \left(-\frac{\pi}{2}, -\frac{\pi}{2} + \varepsilon_0\right] \cap \left[\frac{\pi}{2} - \varepsilon_0, \frac{\pi}{2}\right).$$

In the case in which the classical Kac equation is concerned, the asymptotic equivalence between the non cut-off Kac equation and the linear Fokker-Planck equation as collisions become grazing has been proven in [24]. Hence, the passage to grazing collisions in (10), would give us the correct Fokker-Planck type operator which leads the initial density towards the Bose-Einstein distribution.

Due to the symmetries of the kernel (10) and to the microscopic conservation of the energy (12), it can be easily shown, at least at a formal level, that the mass and the global energy of the solution are conserved

$$\int_{\mathbb{R}} f(t, v) dv = \int_{\mathbb{R}} f_0(v) dv,$$

and

$$\int_{\mathbb{R}} v^2 f(t, v) dv = \int_{\mathbb{R}} v^2 f_0(v) dv,$$

for all $t > 0$. Moreover, if

$$H(f) = \int_{\mathbb{R}} \left(\frac{1}{\delta} (1 + \delta f) \log(1 + \delta f) - f \log f \right) dv$$

denotes the Bose-Einstein entropy, the time derivative of $H(f)$ is given by

$$D(f) = \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} \Gamma \left(f f'_*(1 + \delta f)(1 + \delta f_*), \right. \\ \left. f f_*(1 + \delta f')(1 + \delta f'_*) \right) dv dv_* d\theta$$

with

$$\Gamma(a, b) = \begin{cases} (a - b) \log(a/b), & a > 0, \quad b > 0; \\ +\infty, & a > 0, \quad b = 0 \quad \text{or} \quad a = 0, \quad b > 0; \\ 0, & a = b = 0. \end{cases} \quad (15)$$

Then, since $D(f) \geq 0$ the solution $f(t)$ to equation (9) satisfies formally an H -theorem. $H(f(t))$ is monotonically increasing unless $f(t)$ coincides with the Bose-Einstein density f_{BE} , defined by the relationship

$$\frac{f_{BE}(v)}{1 + \delta f_{BE}(v)} = a e^{-bv^2}, \quad (16)$$

where a and b are positive constant chosen to satisfy the mass and energy conservation for f_{BE} .

It can be easily verified by direct inspection that the fourth order term in (10) cancels out from the collision integral, so that it can be rewritten as

$$Q_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} (f' f'_*(1 + \delta f + \delta f_*) \\ - f f_*(1 + \delta f' + \delta f'_*)) dv_* d\theta. \quad (17)$$

In trying to give a rigorous signification to equation (9), several difficulties arise. In fact, our non-cutoff cross-section $\beta(\theta)$ does not allow us nor to use the same change of variable as in [17], neither to use the same weak formulation as in [6]. A sufficient condition to give a sense to the collision kernel would be that $f \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$. It would even be enough that such a condition hold for the quantum part, that is for the f involved in the terms of the form $1 + \delta f$. To satisfy that condition, we modify the quantum part by smoothing it. Let ψ be a mollifier, that is

1. $\psi \in C_c^\infty(\mathbb{R})$
2. $\psi \geq 0$
3. $\int_{\mathbb{R}} \psi(v) dv = 1$.

Then, let

$$\tilde{f}(t, v) = \int_{\mathbb{R}} f(t, v - w) \psi(w) dw = f(t, \cdot) *_v \psi.$$

The function \tilde{f} is regular in the velocity variable, and relies uniformly in all the $L^p(\mathbb{R})$ spaces (for $1 \leq p \leq +\infty$) since the $L^1(\mathbb{R})$ norm of $f(t, \cdot)$ is constant. Moreover, $\tilde{f}(t, \cdot)$ is as close as we want to $f(t, \cdot)$ in all these norms, provided ψ is well chosen, so that our new model is nothing but an approximation of (9):

$$\begin{cases} \frac{\partial f}{\partial t} = \tilde{Q}_{QBE}(f), & t \in \mathbb{R}_+, v \in \mathbb{R} \\ f(0, v) = f_0(v) \end{cases} \quad (18)$$

with

$$\begin{aligned} \tilde{Q}_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} & (f' f'_* (1 + \delta \tilde{f})(1 + \delta \tilde{f}_*) \\ & - f f_* (1 + \delta \tilde{f}')(1 + \delta \tilde{f}'_*)) dv_* d\theta. \end{aligned} \quad (19)$$

This approximation still formally preserves mass and energy, while maintaining the same nonlinearity of the original collision operator. It has to be remarked, however, that both the validity of the H -theorem and the explicit form of the steady solution are lost. Other approximations can be introduced, which do not exhibit this problem. Among others, the operator

$$\begin{aligned} \tilde{Q}_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} & \left(\frac{f'}{1 + \delta f'} \frac{f'_*}{1 + \delta f'_*} - \frac{f}{1 + \delta f} \frac{f_*}{1 + \delta f_*} \right) \\ & \cdot (1 + \delta \tilde{f})(1 + \delta \tilde{f}_*)(1 + \delta \tilde{f}')(1 + \delta \tilde{f}'_*) dv_* d\theta. \end{aligned} \quad (20)$$

preserves mass and energy, satisfies the H -theorem and possesses the right steady state. Unlikely, the nonlinearity of (20) is difficult to handle for our purposes.

The remaining of our work will concentrate on the study of (18). In Section 3, we will study the existence of a solution to this problem. Then in Section 4, we will focus on the moments of the solution and on some regularity. In Section

5, we will finally study the grazing collision limit. Our work will end in Section 6 with some partial result for the non mollified model (9).

Let us end this Section with a few notations. The functional spaces that will be used in the following, apart from the usual Lebesgue spaces, are the weighted Lebesgue spaces, defined, for $p > 0$, by the norm

$$\|g\|_{L^1_p(\mathbb{R})} = \int_{\mathbb{R}} (1 + |v|^p) |g(v)| dv.$$

We will also need some Sobolev spaces, defined for $0 < s < 1$ by the norm

$$\|g\|_{H^s(\mathbb{R})}^2 = \|g\|_{L^2}^2 + |g|_{H^s}^2$$

where

$$|g|_{H^s}^2 = \iint_{\mathbb{R}^2} \frac{|g(x+y) - g(y)|^2}{|y|^{1+2s}} dx dy.$$

Our convention for the Fourier transform is the following:

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(v) e^{-iv\xi} dv$$

and the inverse Fourier transform is given by

$$f(v) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{iv\xi} d\xi.$$

We will sometimes use the notations

$$m = \int_{\mathbb{R}} f_0(v) dv$$

and

$$e = \int_{\mathbb{R}} v^2 f_0(v) dv.$$

3 Existence theorems

The goal of this Section is to prove the existence of a solution to the problem (18). To start with, we first consider the case of a cross-section with some cutoff.

We have

Theorem 1. *Let $f_0 \in L^1_2(\mathbb{R})$ be a nonnegative function. Assume that the cross-section satisfies $\beta \in L^1(-\frac{\pi}{2}; \frac{\pi}{2})$ and (H1). Then there exists a unique solution $f \in L^\infty(\mathbb{R}_+; L^1_2(\mathbb{R}))$ to the problem (18), which is nonnegative, and preserves mass and energy.*

Proof. For $f, g, h \in L^1_2(\mathbb{R})$, let

$$\begin{aligned} P(f, g, h) = & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} \left[f' g'_* \left(\frac{\int_{\mathbb{R}} h}{\|f_0\|_{L^1}} + \delta \tilde{h} + \delta \tilde{h}'_* \right) - f g_* \left(\frac{\int_{\mathbb{R}} h}{\|f_0\|_{L^1}} \right. \right. \\ & \left. \left. + \delta \tilde{h}' + \delta \tilde{h}'_* \right) \right] dv_* d\theta + f \left(\int_{\mathbb{R}} g \right) \left(\frac{\int_{\mathbb{R}} h}{\|f_0\|_{L^1}} + 2\delta \|\psi\|_{L^\infty} \int_{\mathbb{R}} h \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) d\theta. \end{aligned}$$

Let

$$K = \|f_0\|_{L^1} (1 + 2\delta\|\psi\|_{L^\infty}\|f_0\|_{L^1}) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) d\theta.$$

Let us consider the following problem

$$\begin{cases} \frac{\partial f}{\partial t} + Kf = P(f, f, f), & t \in \mathbb{R}_+, v \in \mathbb{R} \\ f(0, v) = f_0. \end{cases} \quad (21)$$

The operator $P : (L_2^1)^3 \rightarrow L_2^1$ is trilinear, and satisfies the inequality

$$\|P(f, g, h)\|_{L_2^1} \leq C_P \|f\|_{L_2^1} \|g\|_{L_2^1} \|h\|_{L_2^1} \quad \forall f, g, h \in L_2^1(\mathbb{R}),$$

with

$$C_P = \int \beta(\theta) d\theta \left(\frac{2}{\|f_0\|_{L^1}} + 4\delta\|\psi\|_{L^\infty} \right).$$

Assume for the moment that $K = 1$. Thanks to a theorem of [14], there exists some $T > 0$ such that there exists a solution $f \in L^\infty(0, T; L_2^1(\mathbb{R}))$ to the problem (21). This solution can be written as a Wild sum, which reads

$$f(t) = \sum_{k=0}^{+\infty} b_k e^{-t} (1 - e^{-2t})^k f_k,$$

where

$$f_k = \sum_{i_1+i_2+i_3=k-1} \frac{b_{i_1} b_{i_2} b_{i_3}}{2k b_k} P(f_{i_1}, f_{i_2}, f_{i_3}) \quad \text{for } k \geq 1$$

and

$$f_{k=0} = f_0.$$

The numbers b_k are the coefficients of the Taylor expansion of

$$\frac{1}{\sqrt{1-x}} = \sum_{k=0}^{+\infty} b_k x^k.$$

One can easily see that all the b_k are positive. Moreover

$$0 \leq f, g, h \in L_2^1 \implies P(f, g, h) \geq 0.$$

Thus, the solution f of (21) is nonnegative. Moreover, owing to the definition of $P(\cdot, \cdot, \cdot)$, one can verify that this solution preserves the mass. From that we deduce that f is solution of (18) on $(0, T)$. But it preserves the mass, and it relies in L_2^1 , so that it also preserves the energy. Since the time T depends only on $\|f_0\|_{L^1}, \|f_0\|_{L_2^1}, \|\psi\|_{L^\infty}, \|\beta\|_{L^1(-\frac{\pi}{2}, \frac{\pi}{2})}$, we can use the same arguments on the time intervals $(T, 2T), (2T, 3T)$, etc, to get a solution on $(0, +\infty)$. Finally, in the case $K \neq 1$, it is enough to rescale the time to get the right formulae, and to obtain the same conclusions. \square

The following theorem claims the existence of a solution of (18) in the non-cutoff case in some weak sense.

Theorem 2. Let $f_0 \in L^1_2(\mathbb{R})$ be a nonnegative function. Let β satisfy the assumptions (H1) and (H2). Then, there exists a function $g \in L^\infty(\mathbb{R}_+; C^0 \cap L^\infty(\mathbb{R}))$ which is a solution of (18) in the weak sense defined by the forthcoming equation (22). Moreover, g preserves the mass.

Proof. We introduce, for $n \in \mathbb{N}_*$, the cross-section

$$\beta_n(\theta) = \beta(\theta) \wedge n = \min(\beta(\theta), n),$$

and we denote by f^n the solution of the problem (18) corresponding to the cross-section β_n . This solution exists thanks to Theorem 1. For all n and all $t > 0$, $f_n(t)$ relies in L^1 , so that we can define its Fourier transform. Moreover, $\tilde{Q}_{QBE}(f^n)$ also relies in L^1 . Hence, we can write the following equation:

$$\frac{\partial \hat{f}^n(t, \xi)}{\partial t} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^2} \beta_n(\theta) f^n f_*^n (1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v'_*)) (e^{-iv'\xi} - e^{-iv\xi}) dv dv_* d\theta.$$

We can split the integral on the right into three parts. The first part gives

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^2} \beta_n(\theta) f^n f_*^n (e^{-iv'\xi} - e^{-iv\xi}) dv dv_* d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \left(\hat{f}^n(\xi \cos \theta) \hat{f}^n(\xi \sin \theta) - \hat{f}^n(\xi) \hat{f}^n(0) \right) d\theta. \end{aligned}$$

The second part can be evaluated using the inverse Fourier transform of the function \tilde{f}^n which relies in L^2

$$\begin{aligned} & \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^2} \beta_n(\theta) f^n f_*^n \tilde{f}^n(v') (e^{-iv'\xi} - e^{-iv\xi}) dv dv_* d\theta \\ &= \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \int_{\mathbb{R}^3} f^n f_*^n \mathcal{F}(\tilde{f}^n)(\eta) e^{i\eta(v \cos \theta - v_* \sin \theta)} \\ & \cdot \left(e^{-i\xi(v \cos \theta - v_* \sin \theta)} - e^{-iv\xi} \right) d\eta dv dv_* d\theta \\ &= \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \int_{\mathbb{R}} \hat{f}^n((\xi - \eta) \cos \theta) \hat{f}^n((\xi - \eta) \sin \theta) \mathcal{F}(\tilde{f}^n)(\eta) d\eta d\theta \\ & \quad - \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \int_{\mathbb{R}} \hat{f}^n(\xi - \eta \cos \theta) \hat{f}^n(\eta \sin \theta) \mathcal{F}(\tilde{f}^n)(\eta) d\eta d\theta. \end{aligned}$$

The third term can be computed in the same way. At the end we get that the Fourier transform of f^n satisfies

$$\begin{aligned}
\frac{\partial \hat{f}^n(t, \xi)}{\partial t} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \left(\hat{f}^n(\xi \cos \theta) \hat{f}^n(\xi \sin \theta) - \hat{f}^n(\xi) \hat{f}^n(0) \right) d\theta \\
&+ \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \int_{\mathbb{R}} \left[\hat{f}^n((\xi - \eta) \cos \theta) \hat{f}^n((\xi - \eta) \sin \theta) \right. \\
&\quad \left. - \hat{f}^n(\xi - \eta \cos \theta) \hat{f}^n(\eta \sin \theta) \right] \mathcal{F}(\tilde{f}^n)(\eta) d\eta d\theta \\
&+ \frac{\delta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \int_{\mathbb{R}} \left[\hat{f}^n(\xi \cos \theta - \eta \sin \theta) \hat{f}^n(-\xi \sin \theta - \eta \cos \theta) \right. \\
&\quad \left. - \hat{f}^n(\xi - \eta \sin \theta) \hat{f}^n(-\eta \cos \theta) \right] \mathcal{F}(\tilde{f}^n)(\eta) d\eta d\theta.
\end{aligned} \tag{22}$$

Note that here the mass is the quantity $\hat{f}(0)$. In addition

$$\mathcal{F}(\tilde{f}^n)(\eta) = \hat{f}^n(\eta) \hat{\psi}(\eta).$$

Since the second moment of $f^n(v, t)$ is finite and conserved in time, its Fourier transform is two times differentiable, and satisfies

$$\sup_{t>0} \|\partial_{\xi\xi}^2 \hat{f}^n(t, \xi)\|_{L^\infty} \leq \int_{\mathbb{R}} v^2 f_0(v) dv.$$

Hence, we can use the Taylor formula at the order 2

$$h(\theta) = h(0) + \theta h'(0) + \theta^2 \int_0^1 (1-s) h''(s\theta) ds$$

on the functions $\theta \mapsto \hat{f}^n(\xi \cos \theta) \hat{f}^n(\xi \sin \theta)$, $\theta \mapsto \hat{f}^n((\xi - \eta) \cos \theta) \hat{f}^n((\xi - \eta) \sin \theta)$, $\theta \mapsto \hat{f}^n(\xi - \eta \cos \theta) \hat{f}^n(\eta \sin \theta)$, $\theta \mapsto \hat{f}^n(\xi \cos \theta - \eta \sin \theta) \hat{f}^n(-\xi \sin \theta - \eta \cos \theta)$ and $\theta \mapsto \hat{f}^n(\xi - \eta \sin \theta) \hat{f}^n(-\eta \cos \theta)$.

Using the notations $m = \int_{\mathbb{R}} f_0(v) dv$ and $e = \int_{\mathbb{R}} v^2 f_0(v) dv$, we get the following estimates

$$\begin{aligned}
&\left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \left(\hat{f}^n(\xi \cos \theta) \hat{f}^n(\xi \sin \theta) - \hat{f}^n(\xi) \hat{f}^n(0) \right) d\theta \right| \\
&\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^2 \beta(\theta) d\theta \left(4|\xi|^2 m e + 2|\xi| m^{3/2} e^{1/2} \right),
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \left(\hat{f}^n((\xi - \eta) \cos \theta) \hat{f}^n((\xi - \eta) \sin \theta) \right. \right. \\
&\quad \left. \left. - \hat{f}^n(\xi - \eta \cos \theta) \hat{f}^n(\eta \sin \theta) \right) \mathcal{F}(\tilde{f}^n)(\eta) d\theta \right| \\
&\leq |\hat{\psi}(\eta)| m \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^2 \beta(\theta) d\theta \left(4(|\xi - \eta|^2 + |\eta|^2) m e + 2(|\xi - \eta| + |\eta|) m^{3/2} e^{1/2} \right)
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
& \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \left(\hat{f}^n(\xi \cos \theta - \eta \sin \theta) \hat{f}^n(-\xi \sin \theta - \eta \cos \theta) \right. \right. \\
& \quad \left. \left. - \hat{f}^n(\xi - \eta \sin \theta) \hat{f}^n(-\eta \cos \theta) \right) \mathcal{F}(\tilde{f}^n)(\eta) d\theta \right| \\
& \leq |\hat{\psi}(\eta)| m \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^2 \beta(\theta) d\theta \left(4(|\xi| + |\eta|)^2 + |\eta|^2 \right) m e + 2(|\xi| + 2|\eta|) m^{3/2} e^{1/2}
\end{aligned} \tag{25}$$

The right member of (23) is integrable in time on any interval $[t_1, t_2] \subset \mathbb{R}_+$, and the right-members of (24) and (25) are integrable in (t, η) on any $[t_1, t_2] \times \mathbb{R}$ with $0 < t_1 < t_2$, since $\psi \in C_c^\infty(\mathbb{R})$. Therefore, to pass to the limit in (22), it is enough for \hat{f}^n to converge pointwise on $\mathbb{R}_+ \times \mathbb{R}$. But inequalities (23), (24) and (25) ensure that for all compact set $\mathcal{K} \subset \mathbb{R}$, there exists a constant C depending only on \mathcal{K} , m , e , ψ , β , such that

$$|\hat{f}^n(t_1, \xi) - \hat{f}^n(t_2, \xi)| \leq C |t_1 - t_2| \quad \forall 0 < t_1 < t_2, \quad \forall \xi \in \mathcal{K}.$$

Then, thanks to Ascoli's theorem, there exists a function $g \in L^\infty([t_1, t_2] \times \mathcal{K})$ such that, up to the extraction of a subsequence,

$$\|\hat{f}^n - g\|_{L^\infty([t_1, t_2] \times \mathcal{K})} \xrightarrow{n \rightarrow +\infty} 0.$$

All this being true for every t_1, t_2, \mathcal{K} , we deduce that g is well defined on $\mathbb{R}_+ \times \mathbb{R}$ and that $g \in L^\infty(\mathbb{R}_+; L^\infty \cap C^0(\mathbb{R}))$. We can therefore pass to the limit in (22). Finally, we obtained the existence of a function $g(\xi, t)$ which satisfies (22) with the original cross-section β , and such that

$$g(t, 0) = m \quad \forall t > 0.$$

□

4 Moments of the cutoff solutions, regularity of the non cutoff solution

The second step of our analysis is the study of the regularity of the Fourier transform of the solution obtained in the previous Section. This can be done by investigating the moments of the solution to the cut-off equation. Since it is enough for our needs, we will limit ourselves to the fourth moment. Our result is the following

Theorem 3. *Let $\beta(\theta)$ satisfy the assumptions (H1) and (H2), and let $f^n(t, v)$ be the solution of the problem (18) with cross-section β_n (as defined in the proof of Theorem 2) and with nonnegative initial datum $f_0 \in L^1_4(\mathbb{R})$. Then there exists a positive constant λ such that we have*

$$\sup_{t \in \mathbb{R}_+} \|f^n(t)\|_{L^1_4} \leq \max\{\lambda, \|f_0\|_{L^1_4}\}. \tag{26}$$

Moreover, λ depends only on $\|f_0\|_{L^1}$, $\|f_0\|_{L^2}$, δ , $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^{1+\frac{\nu}{2}} \beta(\theta) d\theta$ and ψ .

Proof. Let us take $\phi(v) = v^4$ as test function. Thanks to the symmetries of the kernel we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} v^4 f^n(v, t) dv \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \int_{\mathbb{R}^2} (v'^4 - v^4 + v_*'^4 - v_*^4) f^n f_*^n (1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v_*')) dv dv_* d\theta. \end{aligned}$$

From the collision rule (11) it follows

$$\begin{aligned} v'^4 - v^4 &= v^4(\cos^4 \theta - 1) - 4v^3 v_* \cos^3 \theta \sin \theta + 2v^2 v_*^2 \sin^2 \theta \cos^2 \theta \\ &\quad - 4v v_*^3 \cos \theta \sin^3 \theta + v_*^4 \sin^4 \theta \end{aligned}$$

and

$$\begin{aligned} v_*'^4 - v_*^4 &= v_*^4(\cos^4 \theta - 1) + 4v^3 v_* \cos \theta \sin^3 \theta + 2v^2 v_*^2 \sin^2 \theta \cos^2 \theta \\ &\quad + 4v v_*^3 \cos^3 \theta \sin \theta + v_*^4 \sin^4 \theta. \end{aligned}$$

Consequently

$$\frac{d}{dt} \int_{\mathbb{R}} v^4 f^n(v, t) dv \leq -c_1 \int_{\mathbb{R}} v^4 f^n(t, v) dv + I_2 + I_3$$

where

$$c_1 = \|f\|_{L^1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) (1 - \cos^4 \theta - \sin^4 \theta) d\theta > 0,$$

$$\begin{aligned} I_2 &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \cos^2 \theta \sin^2 \theta \int_{\mathbb{R}^2} v^2 v_*^2 f^n f_*^n (1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v_*')) dv dv_* d\theta \\ &\quad + 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \cos \theta \sin^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f^n f_*^n (1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v_*')) dv dv_* d\theta \\ &= I_2^1 + I_2^2 \end{aligned}$$

and

$$I_3 = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v_*^2 - v^2) f^n f_*^n (1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v_*')) dv dv_* d\theta. \quad (27)$$

The last term is more difficult to handle, because the presence of $\sin \theta$ at power one is not enough to guarantee that the integral in $d\theta$ is finite.

Let us first take care of I_2 . We have the bound

$$I_2^1 \leq c_2,$$

where, if $m = \|f_0\|_{L^1}$ and $e = \int_{\mathbb{R}} v^2 f_0(v) dv$ as before,

$$c_2 = 2(1 + 2\delta \|\psi\|_{L^\infty} m) e^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \sin^2 \theta \cos^2 \theta d\theta.$$

This bound follows from the conservation of the energy, and from the smoothing of f , which implies $\|\tilde{f}(t)\|_{L^\infty} \leq \|\psi\|_{L^\infty} m$. Moreover, the change of variable $(v, v_*, \theta) \mapsto (v_*, v, -\theta)$ leads us to

$$I_2^2 = 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \cos \theta \sin^3 \theta \int_{\mathbb{R}^2} v^3 v_* f^n f_*^n (1 + \delta \tilde{f}^n(v') + \delta \tilde{f}^n(v_*')) dv dv_* d\theta,$$

and then to

$$|I_2^2| \leq 4(1 + 2\delta\|\psi\|_{L^\infty}m) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \cos \theta |\sin^3 \theta| d\theta \cdot \left(\int_{\mathbb{R}} |v| f^n(t, v) dv \right) \left(\int_{\mathbb{R}} |v|^3 f^n(t, v) dv \right).$$

Then, we apply Hölder's inequality on $|v|$ and $|v|^3$ with the measure $d\mu = f^n(t, \cdot)dv$, to get

$$\int_{\mathbb{R}} |v| f^n(v) dv \leq m^{1/2} e^{1/2}$$

and

$$\int_{\mathbb{R}} |v|^3 f^n(v) dv \leq m^{1/4} \left(\int_{\mathbb{R}} v^4 f(t, v) dv \right)^{3/4}.$$

Finally we can write

$$|I_2^2| \leq c_3 \left(\int_{\mathbb{R}} v^4 f^n(t, v) dv \right)^{3/4}$$

with

$$c_3 = 4(1 + 2\delta\|\psi\|_{L^\infty}m)m^{3/4}e^{1/2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \cos \theta |\sin^3 \theta| d\theta.$$

Using the previous bounds, we obtained that the fourth moment satisfies the differential inequality

$$\frac{d}{dt} \int_{\mathbb{R}} v^4 f^n(t, v) dv \leq -c_1 \int_{\mathbb{R}} v^4 f^n(t, v) dv + c_2 + c_3 \left(\int_{\mathbb{R}} v^4 f^n(t, v) dv \right)^{3/4} + I_3.$$

The treatment of I_3 requires more attention. We can split (27) into three pieces. The first one reads

$$I_3^1 = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v_*^2 - v^2) f^n f_*^n d v d v_* d \theta.$$

Since $\beta_n(\theta)$ is an even function,

$$I_3^1 = 0.$$

Let us now evaluate I_3^2 . Then, the same computations can be used for I_3^3 . Let us set $\tilde{v}' = v - v_* \sin \theta$. Then

$$\begin{aligned} I_3^2 &= 2\delta \int \beta_n(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f^n f_*^n (\tilde{f}^n(v') - \tilde{f}^n(\tilde{v}')) d v d v_* d \theta \\ &\quad + 2\delta \int \beta_n(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f^n f_*^n (\tilde{f}^n(\tilde{v}') - \tilde{f}^n(v)) d v d v_* d \theta \\ &= 2\delta(\tilde{I}_1 + \tilde{I}_2) \end{aligned}$$

To bound from above these two integrals, we use the regularity of \tilde{f} . Consider that

$$\begin{aligned} |\tilde{f}^n(v') - \tilde{f}^n(\tilde{v}')| &\leq |\tilde{f}^n(v') - \tilde{f}^n(\tilde{v}')|^{1/2} |\tilde{f}^n(v') - \tilde{f}^n(\tilde{v}')|^{1/2} \\ &\leq \left(2\|\tilde{f}^n(t)\|_{L^\infty}\right)^{1/2} \|\partial_v \tilde{f}^n(t, \cdot)\|_{L^\infty}^{1/2} |v' - \tilde{v}'|^{1/2} \\ &\leq \sqrt{2}\|\psi\|_{L^\infty}^{1/2} \|\psi'\|_{L^\infty}^{1/2} m |v(\cos\theta - 1)|^{1/2}. \end{aligned}$$

Consequently

$$\begin{aligned} \tilde{I}_1 &\leq \tilde{c}_1 \left(\int_{\mathbb{R}} |v|^{3+\frac{1}{2}} f^n(t, v) dv \int_{\mathbb{R}} |v_*| f^n(t, v_*) dv_* + \right. \\ &\quad \left. \int_{\mathbb{R}} |v|^{\frac{3}{2}} f^n(t, v) dv \int_{\mathbb{R}} |v_*|^3 f^n(t, v_*) dv_* \right) \\ &\leq \tilde{c}_1 \left(\left(\int_{\mathbb{R}} v^4 f^n(t, v) dv \right)^{\frac{7}{8}} m^{\frac{5}{8}} e^{\frac{1}{2}} + \left(\int_{\mathbb{R}} v^4 f^n(t, v) dv \right)^{\frac{3}{4}} m^{\frac{1}{2}} e^{\frac{3}{4}} \right), \end{aligned}$$

with

$$\tilde{c}_1 = \sqrt{2}\|\psi\|_{L^\infty}^{1/2} \|\psi'\|_{L^\infty}^{1/2} m \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin\theta| (1 - \cos\theta)^{1/2} \beta(\theta) d\theta.$$

Let $\nu > 0$ be such that

$$\frac{1}{2} < \frac{\nu}{2} < 1.$$

Then we have

$$\begin{aligned} |\tilde{f}^n(\tilde{v}') - \tilde{f}^n(v)| &\leq |\tilde{f}^n(\tilde{v}') - \tilde{f}^n(v)|^{1-\frac{\nu}{2}} |\tilde{f}^n(\tilde{v}') - \tilde{f}^n(v)|^{\frac{\nu}{2}} \\ &\leq (2\|\psi\|_{L^\infty} m)^{1-\frac{\nu}{2}} (\|\psi'\|_{L^\infty} m)^{\frac{\nu}{2}} |\tilde{v}' - v|^{\frac{\nu}{2}} \\ &\leq m (2\|\psi\|_{L^\infty})^{1-\frac{\nu}{2}} (\|\psi'\|_{L^\infty})^{\frac{\nu}{2}} |v_* \sin\theta|^{\frac{\nu}{2}}. \end{aligned}$$

Thus,

$$\begin{aligned} \tilde{I}_2 &\leq \tilde{c}_2 \left(\int_{\mathbb{R}} |v|^3 f^n(t, v) dv \int_{\mathbb{R}} |v_*|^{1+\frac{\nu}{2}} f^n(t, v_*) dv_* + \right. \\ &\quad \left. \int_{\mathbb{R}} |v| f^n(t, v) dv \int_{\mathbb{R}} |v_*|^{3+\frac{\nu}{2}} f^n(t, v_*) dv_* \right) \\ &\leq \tilde{c}_2 \left(\left(\int_{\mathbb{R}} v^4 f^n(t, v) dv \right)^{\frac{3}{4}} m^{\frac{3-\nu}{4}} e^{\frac{1}{2}+\frac{\nu}{4}} + m^{\frac{3}{4}-\frac{\nu}{8}} e^{\frac{1}{2}} \left(\int_{\mathbb{R}} v^4 f^n(t, v) dv \right)^{\frac{3}{4}+\frac{\nu}{8}} \right) \end{aligned}$$

with

$$\tilde{c}_2 = m (2\|\psi\|_{L^\infty})^{1-\frac{\nu}{2}} (\|\psi'\|_{L^\infty})^{\frac{\nu}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) |\sin\theta|^{1+\frac{\nu}{2}} d\theta.$$

Let us set $y_n(t) = \int_{\mathbb{R}} v^4 f^n(t, v) dv$. Then $y_n(t)$ satisfies

$$\frac{dy_n}{dt} \leq -c_1 y_n + K_1 y_n^{\frac{3}{4}} + K_2 y_n^{\frac{7}{8}} + K_3 y_n^{\frac{3}{4}+\frac{\nu}{8}} + c_2$$

with

$$K_1 = c_3 + 4\delta\tilde{c}_1 m^{\frac{1}{2}} e^{\frac{3}{4}} + 4\delta\tilde{c}_2 m^{\frac{3-\nu}{4}} e^{\frac{2+\nu}{4}},$$

$$K_2 = 4\delta\tilde{c}_1 m^{\frac{5}{8}} e^{\frac{1}{2}},$$

and

$$K_3 = 4\delta\tilde{c}_2 m^{\frac{6-\nu}{8}} e^{\frac{1}{2}}.$$

Let λ be the unique positive solution of the equation

$$-c_1\lambda + K_1\lambda^{\frac{3}{4}} + K_2\lambda^{\frac{7}{8}} + K_3\lambda^{\frac{3}{4}+\frac{\nu}{8}} + c_2 = 0. \quad (28)$$

The result follows. \square

Passing to the limit $n \rightarrow +\infty$ in inequality (26) we obtain

Theorem 4. *Let β satisfy (H1) and (H2), and let g be the weak solution of (18) defined in Theorem 2, with nonnegative initial data $f_0 \in L^1_+$. Then, $g(t)$ is C^3 for all t and*

$$\sup_{t>0} \{ \|g(t)\|_{L^\infty} + \|\partial_\xi g(t)\|_{L^\infty} + \|\partial_{\xi^2}^2 g(t)\|_{L^\infty} + \|\partial_{\xi^3}^3 g(t)\|_{L^\infty} + \|\partial_{\xi^4}^4 g(t)\|_{L^\infty} \} < +\infty.$$

Moreover, g conserves the energy, in the sense that $\partial_{\xi\xi}^2 g(t, 0) = -e$ for all $t > 0$.

Proof. The conservation of the mass and inequality (26) imply that there exists a constant $C > 0$ which do not depend on n such that

$$\begin{cases} \|\partial_\xi \hat{f}^n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq C \\ \|\partial_{\xi^2}^2 \hat{f}^n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq C \\ \|\partial_{\xi^3}^3 \hat{f}^n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq C \\ \|\partial_{\xi^4}^4 \hat{f}^n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq C. \end{cases}$$

Since $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is the dual space of the Banach space $L^1(\mathbb{R}_+ \times \mathbb{R})$, the four sequences converge (up to the extraction of a subsequence) in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ weak-*; the limits can only be respectively $\partial_\xi g$, $\partial_{\xi^2}^2 g$, $\partial_{\xi^3}^3 g$ and $\partial_{\xi^4}^4 g$ (since the convergence in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ weak-* implies the convergence in the distributional sense). Moreover, we have the inequalities, for $1 \leq i \leq 4$,

$$\|\partial_{\xi^i}^i g\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq \liminf_{n \rightarrow +\infty} \|\partial_{\xi^i}^i \hat{f}^n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})}.$$

Finally, we have the embedding

$$W^{4,\infty}(\mathbb{R}) \hookrightarrow C^3(\mathbb{R}),$$

so that $g \in L^\infty(\mathbb{R}_+; C^3(\mathbb{R}))$.

It remains to prove that the energy is conserved. Let us fix some time $t_0 > 0$. It is clear that for all integer n , we have

$$\|\partial_{\xi^2}^2 \hat{f}^n(t_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq e.$$

Therefore, up to the extraction of a subsequence, there exists a function $h \in L^\infty(\mathbb{R})$ such that

$$\partial_{\xi^2}^2 \hat{f}^n(t_0, \cdot) \rightharpoonup h \quad \text{weak} - * \quad L^\infty(\mathbb{R}).$$

But it is clear that $h = \partial_{\xi^2}^2 g(t_0, \cdot)$, since for all function $\phi \in C^\infty(\mathbb{R})$ with compact support,

$$\int_{\mathbb{R}} \hat{f}^n(t_0, \xi) \phi''(\xi) d\xi \rightarrow \int_{\mathbb{R}} g(t_0, \xi) \phi''(\xi) d\xi,$$

or, after two integrations per part on both sides,

$$\int_{\mathbb{R}} \partial_{\xi^2}^2 \hat{f}^n(t_0, \xi) \phi(\xi) d\xi \rightarrow \int_{\mathbb{R}} \partial_{\xi^2}^2 g(t_0, \xi) \phi(\xi) d\xi.$$

Let us define an approximation of the Dirac measure

$$\Phi_p(\xi) = \begin{cases} p & \text{if } -\frac{1}{2p} < \xi < \frac{1}{2p} \\ 0 & \text{otherwise} \end{cases}$$

We have

$$\begin{aligned} \left| \partial_{\xi^2}^2 \hat{f}^n(t_0, 0) - \partial_{\xi^2}^2 g(t_0, 0) \right| &\leq \left| \partial_{\xi^2}^2 \hat{f}^n(t_0, 0) - \int_{\mathbb{R}} \Phi_p(\xi) \partial_{\xi^2}^2 \hat{f}^n(t_0, \xi) d\xi \right| \\ &\quad + \left| \int_{\mathbb{R}} \Phi_p(\xi) \left(\partial_{\xi^2}^2 \hat{f}^n(t_0, \xi) - \partial_{\xi^2}^2 g(t_0, \xi) \right) d\xi \right| \\ &\quad + \left| \int_{\mathbb{R}} \Phi_p(\xi) \partial_{\xi^2}^2 g(t_0, \xi) d\xi - \partial_{\xi^2}^2 g(t_0, 0) \right|. \end{aligned}$$

The first and the third terms converge toward 0 when p converges to infinity, independently of n since

$$\|\partial_{\xi^3}^3 \hat{f}^n(t_0, \cdot)\|_{L^\infty} \leq C$$

with $C > 0$ independent of n . As for the second term, once p has been fixed, it converges to 0 since $\Phi_p \in L^1(\mathbb{R})$, and the result follows. \square

5 The grazing collision limit

We are now in a position to perform the grazing collision limit in equation (18). We can now make precise assumptions on the asymptotics of the grazing collisions, namely in letting the kernel β concentrate on the singularity $\theta = 0$. We will introduce a family of kernels $\{\beta_\epsilon(|\theta|)\}_{\epsilon > 0}$ satisfying hypotheses (H1) and (H2), with

$$\lim_{\epsilon \rightarrow 0^+} \int_0^\pi \beta_\epsilon(|\theta|) \theta^2 d\theta = 1 \quad (29)$$

This can be obtained in several ways, for example taking, for $0 < \mu < 1$

$$\beta_\epsilon(|\theta|) = \frac{2(1-\mu)}{\epsilon|\theta|^{2+\mu}} \quad 0 \leq |\theta| \leq \epsilon^{1/(1-\mu)},$$

$$\beta_\epsilon(|\theta|) = \frac{(1-\mu)\epsilon}{|\theta|^{2+\mu}} \quad \text{elsewhere.}$$

Let g_ϵ be the weak solution of the problem (18) in the sense that it satisfies equation (22), where $\beta(\theta)$ has been replaced by $\beta_\epsilon(\theta)$.

Theorem 5. Let $\beta(\theta)$ satisfy assumptions (H1), (H2), and let $\beta_\varepsilon(\theta)$ satisfy (29). Let g_ε be the weak solution of the problem (18) where $\beta(\theta)$ has been replaced by $\beta_\varepsilon(\theta)$, with the nonnegative initial data f_0 satisfying $f_0 \in L^1_+(\mathbb{R})$.

Then, for all $T > 0$, there exists a function $g \in L^\infty(0, T; W^{4, \infty}(\mathbb{R}))$ such that, up to the extraction of a subsequence,

$$\|g_\varepsilon - g\|_{L^\infty([t_1, t_2] \times \mathcal{K})} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \forall 0 < t_1 < t_2 < T \text{ and } \mathcal{K} \subset \mathbb{R} \text{ compact,}$$

$$\partial_{\xi^i} g_\varepsilon \rightharpoonup \partial_{\xi^i} g \text{ in } L^\infty((0, T) \times \mathbb{R}) \text{ weak-}^*, \quad 1 \leq i \leq 4,$$

and such that g is a solution of the Fourier form of the equation

$$\frac{\partial h}{\partial t} = \left(\int_{\mathbb{R}} h(v) dv \right) \frac{\partial}{\partial v} (vh(1 + \delta \tilde{h})) + \left(\int_{\mathbb{R}} v^2 h(v) (1 + \delta \tilde{h}(v)) dv \right) \frac{\partial^2 h}{\partial v^2}, \quad (30)$$

Remark 6. If $\delta = 0$, equation (30) reduces to the classical linear Fokker-Planck equation.

Proof. To pass to the limit, we need some regularity on the solution g_ε . However, Theorem 4 is not sufficient, since the bound on the fourth derivative of g_ε is given by the constant λ of Theorem 3, which depends on the quantity

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^{1+\frac{\nu}{2}} \beta(\theta) d\theta.$$

Indeed, when replacing β by β_ε , we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^{1+\frac{\nu}{2}} \beta_\varepsilon(\theta) d\theta \cong \frac{1}{\varepsilon^{1-\nu/2}}.$$

and equation (28) shows us that

$$\lambda_\varepsilon \sim \frac{c}{\varepsilon^4} \quad \text{when } \varepsilon \rightarrow 0.$$

Therefore, we need to modify Theorem 3 to make it robust when changing β into β_ε .

Lemma 7. Assume all the hypotheses of Theorem 3. Then, for all $T > 0$, there exists a constant λ_T such that

$$\sup_{t \in (0, T)} \|f^n(t)\|_{L^1_+} \leq \lambda_T. \quad (31)$$

Moreover, λ_T depends only on $\|f_0\|_{L^1}$, $\|f_0\|_{L^{\frac{1}{2}}}$, δ , $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^2 \beta_\varepsilon(\theta) d\theta$, ψ and T .

This time, the bound is only on $(0, T)$, but it is enough for our needs, and the bound will not depend on ε when replacing β by β_ε .

Proof of lemma 7. The proof is the analogue to that of Theorem 3. We just change the bound on \tilde{I}_2 :

$$\begin{aligned}
|\tilde{I}_2| &= \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f^n(v) f^n(v_*) (\tilde{f}^n(\tilde{v}') - \tilde{f}^n(v)) dv dv_* d\theta \right| \\
&\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) |\sin \theta| \cos^3 \theta \int_{\mathbb{R}^2} v v_* |v^2 - v_*^2| f^n(v) f^n(v_*) |\tilde{f}^n(\tilde{v}') - \tilde{f}^n(v)| dv dv_* d\theta \\
&\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) |\sin \theta| \cos^3 \theta \int_{\mathbb{R}^2} v v_* |v^2 - v_*^2| f^n(v) f^n(v_*) \|\partial_v \tilde{f}^n\|_{L^\infty} |\tilde{v}' - v| dv dv_* d\theta \\
&\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \sin^2 \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_*^2 |v^2 - v_*^2| f^n(v) f^n(v_*) \|\psi'\|_{L^\infty} m dv dv_* d\theta \\
&\leq \|\psi'\|_{L^\infty} m \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \sin^2 \theta \cos^3 \theta d\theta \right) \\
&\quad \cdot \left(m^{\frac{1}{4}} e \left(\int_{\mathbb{R}} v^4 f^n(t, v) dv \right)^{\frac{3}{4}} + e^{\frac{1}{2}} m^{\frac{1}{2}} \left(\int_{\mathbb{R}} v^4 f^n(t, v) dv \right) \right).
\end{aligned}$$

Consequently, at the end, the inequality satisfied by $y^n(t) = \int_{\mathbb{R}} v^4 f^n(t, v) dv$ takes the form

$$\frac{dy_n}{dt} \leq (-c_1 + \tilde{c}_1) y_n + \tilde{K}_1 y_n^{\frac{3}{4}} + K_2 y_n^{\frac{7}{8}} + c_2$$

with

$$\tilde{c}_1 = \|\psi'\|_{L^\infty} m^{\frac{3}{2}} e^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \sin^2 \theta d\theta$$

and

$$\tilde{K}_1 = K_1 + \|\psi'\|_{L^\infty} m^{\frac{5}{4}} e \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_n(\theta) \sin^2 \theta d\theta.$$

Then, if $\tilde{c}_1 - c_1 < 0$, we still have the conclusions of Theorem 3.

Otherwise, call $\gamma \in C^1(0, +\infty)$ the unique solution of the problem

$$\begin{cases} \frac{d\gamma}{dt} = (-c_1 + \tilde{c}_1)\gamma + \tilde{K}_1 \gamma^{\frac{3}{4}} + K_2 \gamma^{\frac{7}{8}} + c_2 \\ \gamma(0) = \int_{\mathbb{R}} v^4 f_0(v) dv. \end{cases}$$

Then, we have:

$$y^n(t) \leq \gamma(t) \quad \forall t \geq 0,$$

and consequently, for all $T > 0$,

$$\sup_{0 \leq t \leq T} y^n(t) \leq \lambda_T$$

with

$$\lambda_T = \sup_{0 \leq t \leq T} \gamma(t).$$

□

Thanks to this result, the conclusions of Theorem 4 are still valid on every time interval $(0, T)$; that is, \hat{g}_ε is four times differentiable for almost any time, and these derivatives are bounded uniformly in time and independently of ε , provided the initial datum f_0 relies in $L^1_4(\mathbb{R})$. We now act as if $\beta(\theta)$ was integrable, and then we will obtain the result by an approximation argument. Let us fix some $T > 0$. A Taylor expansion in ε under the integral sign gives

$$\begin{aligned}
& \frac{\partial g_\varepsilon(t, \xi)}{\partial t} \\
&= \frac{1}{\varepsilon^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left[\varepsilon \theta \xi \left(g_\varepsilon(\xi) g'_\varepsilon(0) \right. \right. \\
&\quad \left. \left. - \frac{\delta}{2\pi} \int_{\mathbb{R}} \left(g_\varepsilon(\xi - \eta) g_\varepsilon(\eta) g'_\varepsilon(0) \hat{\psi}(\eta) - g'_\varepsilon(-\eta) g_\varepsilon(\eta) g_\varepsilon(\xi) \hat{\psi}(\eta) \right) d\eta \right) \right. \\
&\quad \left. + \varepsilon^2 \frac{\theta^2}{2} \left(\xi^2 g''_\varepsilon(0) g_\varepsilon(\xi) - \xi g'_\varepsilon(\xi) g_\varepsilon(0) - \frac{\delta}{2\pi} \int_{\mathbb{R}} \hat{\psi}(\eta) \left(g'_\varepsilon(\xi - \eta) g_\varepsilon(\eta) g_\varepsilon(0) \right. \right. \right. \\
&\quad \left. \left. + ((\eta - \xi)^2 - \eta^2) g_\varepsilon(\xi - \eta) g_\varepsilon(\eta) g''_\varepsilon(0) + \xi g'_\varepsilon(\xi) g_\varepsilon(\eta) g_\varepsilon(-\eta) \right. \right. \\
&\quad \left. \left. + \xi^2 g''_\varepsilon(-\eta) g_\varepsilon(\eta) g_\varepsilon(\xi) + 2\eta \xi g'_\varepsilon(\xi) g_\varepsilon(\eta) g'_\varepsilon(-\eta) \right) d\eta \right] + \theta^2 O(\varepsilon^3) \Big] d\theta.
\end{aligned} \tag{32}$$

Since $\beta(\theta)$ is an even function, the first-order terms vanish.

By Lemma 7 there exists a constant $\lambda_T > 0$ which do not depend on ε such that

$$\sup_{0 < t < T} \left\{ \sum_{i=0}^4 \|\partial_{\xi^i} g_\varepsilon(t, \cdot)\|_{L^\infty} \right\} \leq \lambda_T.$$

Using equation (32), we see that the family $(g_\varepsilon)_\varepsilon$ is equicontinuous, so that we can use Ascoli's theorem, which says that there exists a function $g \in L^\infty((0, T) \times \mathbb{R})$ such that, up to the extraction of a subsequence,

$$\|g_\varepsilon - g\|_{L^\infty([t_1, t_2] \times \mathcal{K})} \xrightarrow{\varepsilon \rightarrow 0} 0$$

for all $0 < t_1 < t_2 < T$ and all compact set $\mathcal{K} \subset \mathbb{R}$. In addition, all the results of Theorem 4 are still valid for g on $(0, T)$. Therefore, thanks to both the uniform convergence for g and the convergence in $L^\infty((0, T) \times \mathbb{R})$ weak-* for its derivatives, we can pass to the limit in equation (32), and we get, using classical formulae on the Fourier transform and the conservation of mass and energy, that g satisfies the equation which is the Fourier transform of equation (30). \square

6 Other results

To give a sense to the collision kernel, we have been forced to mollify a part of our original equation

$$\begin{cases} \frac{\partial f}{\partial t} = Q_{QBE}(f)(t, v), & t \in \mathbb{R}_+, \quad v \in \mathbb{R}, \\ f(0, v) = f_0(v), \end{cases} \tag{33}$$

where

$$Q_{QBE}(f)(t, v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}} (f' f'_*(1 + \delta f)(1 + \delta f_*) - f f_*(1 + \delta f')(1 + \delta f'_*)) dv_* d\theta. \quad (34)$$

However, assuming that a solution to this equation exists and is regular enough, we are able to obtain some interesting consequences.

6.1 H -theorem

The first one is the H -theorem. Under the right assumptions on the solution f of equation (33), we can define

$$H(f) = \int_{\mathbb{R}} \left(\frac{1}{\delta} (1 + \delta f) \log(1 + \delta f) - f \log f \right) dv$$

and

$$D(f) = \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} \Gamma \left(f f'_*(1 + \delta f)(1 + \delta f_*), f f_*(1 + \delta f')(1 + \delta f'_*) \right) dv dv_* d\theta$$

with

$$\Gamma(a, b) = \begin{cases} (a - b) \log(a/b), & a > 0, \quad b > 0; \\ +\infty, & a > 0, \quad b = 0 \quad \text{or} \quad a = 0, \quad b > 0; \\ 0, & a = b = 0. \end{cases} \quad (35)$$

Then, f satisfies the so-called H -theorem:

Theorem 8. *Let f be a solution of the problem (33), with $f_0 \in L \log L(\mathbb{R})$, and assume that $H(f)$ and $D(f)$ are well defined. Then*

$$H(f(t, \cdot)) = H(f_0) + \int_0^t D(f(s, \cdot)) ds \quad \forall t > 0.$$

Consequently, the entropy H is increasing along the solution.

6.2 Regularity of the solution

Using the H -theorem, we can give an *a priori* estimate on the solution f of (33).

Theorem 9. *Let β satisfy the properties (H1), (H2) and (H3), and let $f(t, v)$ be a solution of the problem (33) with the initial datum $f_0 \in L^1_2 \cap L \log L$. Assume that f satisfies the H -theorem. Then we have*

$$\log(1 + \delta f) \in L^2_{loc}(\mathbb{R}_+; H^{\nu/2}(\mathbb{R})).$$

If in addition $f \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$, then we have

$$f \in L^2_{loc}(\mathbb{R}_+; H^{\nu/2}(\mathbb{R})).$$

Proof. We will do the computations as if the cross-section and the function f were smooth. Using the classical changes of variable $(v, v_*, \theta) \mapsto (v', v'_*, -\theta)$ and $(v, v_*) \mapsto (v_*, v)$ which have unit jacobian, we have:

$$\begin{aligned}
D(f) &= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} \left(f' f'_*(1 + \delta f)(1 + \delta f_*) - f f_*(1 + \delta f')(1 + \delta f'_*) \right) \\
&\quad \times \log \frac{f' f'_*(1 + \delta f)(1 + \delta f_*)}{f f_*(1 + \delta f')(1 + \delta f'_*)} dv dv_* d\theta \\
&= -\frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} \left(f' f'_*(1 + \delta f)(1 + \delta f_*) - f f_*(1 + \delta f')(1 + \delta f'_*) \right) \\
&\quad \times \log \left(f f_*(1 + \delta f')(1 + \delta f'_*) \right) dv dv_* d\theta \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} \left(f' f'_*(1 + \delta f)(1 + \delta f_*) - f f_*(1 + \delta f')(1 + \delta f'_*) \right) \\
&\quad \times \log \left(f(1 + \delta f') \right) dv dv_* d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f f_*(1 + \delta f')(1 + \delta f'_*) \log \frac{f(1 + \delta f')}{f'(1 + \delta f)} dv dv_* d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) \left(f(1 + \delta f') \log \frac{f(1 + \delta f')}{f'(1 + \delta f)} \right. \\
&\quad \left. - f(1 + \delta f') + f'(1 + \delta f) \right) dv dv_* d\theta \\
&\quad + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) \left(f(1 + \delta f') - f'(1 + \delta f) \right) dv dv_* d\theta \\
&= I_1 + I_2.
\end{aligned}$$

The term I_2 can be treated easily, because it can be written as

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) (f - f') dv dv_* d\theta,$$

and the presence of $f - f'$ involves strong cancellations. In fact, the term I_2 verifies

$$I_2 = \|f\|_{L^1}^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left(1 - \frac{1}{\cos \theta} \right) d\theta. \quad (36)$$

To prove (36), consider that

$$\begin{aligned}
I_2 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(f - f') dv dv_* d\theta + \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_* f'_*(f - f') dv dv_* d\theta \\
&= I_2^1 + I_2^2
\end{aligned}$$

Thanks to the change of variable $(v, v_*, \theta) \mapsto (v', v'_*, -\theta)$, we see that $I_2^2 = 0$. On the second part of I_2^1 , we use the change of variable $v \mapsto v'$ with v_* and θ fixed, which jacobian is

$$\frac{dv}{dv'} = \frac{1}{\cos \theta}.$$

Therefore

$$I_2^1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left(1 - \frac{1}{\cos \theta}\right) \int_{\mathbb{R}^2} f_* f \, dv dv_* \, d\theta$$

and from this (36) follows.

Using now the inequality

$$x \log \frac{x}{y} - x + y \geq (\sqrt{x} - \sqrt{y})^2, \quad \forall x, y > 0,$$

we obtain

$$I_1 \geq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) \left(\sqrt{f(1 + \delta f')} - \sqrt{f'(1 + \delta f)} \right)^2 \, dv dv_* \, d\theta.$$

Therefore

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) \left(\sqrt{f(1 + \delta f')} - \sqrt{f'(1 + \delta f)} \right)^2 \, dv dv_* \, d\theta \leq D(f) + c_1 \|f\|_{L^1}^2 \quad (37)$$

where

$$c_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \left(\frac{1}{\cos \theta} - 1 \right) \, d\theta > 0.$$

Now, we write

$$\sqrt{f(1 + \delta f')} - \sqrt{f'(1 + \delta f)} = (\log(1 + \delta f') - \log(1 + \delta f)) \frac{\sqrt{f(1 + \delta f')} - \sqrt{f'(1 + \delta f)}}{\log(1 + \delta f') - \log(1 + \delta f)}.$$

For $0 < a < x$, let

$$\phi(x) = \frac{\sqrt{x(1 + \delta a)} - \sqrt{a(1 + \delta x)}}{\log(1 + \delta a) - \log(1 + \delta x)}.$$

Then, there exists some constant $c_2 > 0$ that does not depend on x or a such that

$$|\phi(x)| > c_2 \quad \forall x > a.$$

From this inequality we deduce that

$$c_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_*(1 + \delta f'_*) \left(\log(1 + \delta f') - \log(1 + \delta f) \right)^2 \leq D(f) + c_1 \|f\|_{L^1}^2$$

It has been shown in [1] that if F is a real function such that $F(f) \in L^2(\mathbb{R})$ satisfying

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \int_{\mathbb{R}^2} f_* \left(F(f') - F(f) \right)^2 \leq D(f) + c_1 \|f\|_{L^1}^2,$$

then the following inequality holds:

$$\|F(f)\|_{H^{\nu/2}}^2 \leq D(f) + c_1 \|f\|_{L^1}^2.$$

Taking $F(f) = c_2 \log(1 + \delta f)$ the result follows \square

6.3 Moments of the solution

To study the grazing collision limit, we needed some regularity on the Fourier transform of the solution. This is equivalent to have a uniform bound on some higher moment of the solution. In proving Theorem 3, we used the regularity of the mollified part, more precisely the fact that this part was in C^1 . In fact it could be enough to use the $H^{\frac{\nu}{2}}$ regularity which follows from the H -theorem. Indeed, the terms that raise problems in the proof of Theorem 3 are \tilde{I}_1 and \tilde{I}_2 . Let us see how to treat the first one. We have

$$\begin{aligned} |\tilde{I}_1| &= \left| \int \beta(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f f_* (f(v') - f(\tilde{v}')) dv dv_* d\theta \right| \\ &= \left| \int \beta(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f(v) f(v_*) \left[v(1 - \cos \theta) \right]^\alpha \right. \\ &\quad \left. \frac{f(v') - f(\tilde{v}')}{[v(1 - \cos \theta)]^\alpha} dv dv_* d\theta \right| \\ &\leq \int \beta(\theta) |\sin \theta| |1 - \cos \theta|^\alpha |\cos^3 \theta| \left(\int_{\mathbb{R}^2} (v^{1+\alpha} v_* (v^2 - v_*^2))^p f(v)^p f(v_*)^p dv dv_* \right)^{1/p} \\ &\quad \cdot \left(\int_{\mathbb{R}^2} \frac{|f(v') - f(\tilde{v}')|^q}{|v(1 - \cos \theta)|^{\alpha q}} dv dv_* \right)^{1/q} d\theta. \end{aligned}$$

In order to deal with moments not exceeding the fourth one, we use Hölder's inequality, with p such that $(3 + \alpha)p = 4$. Consequently $q = \frac{4}{1-\alpha}$. Moreover, in order to recognize the semi-norm of f in the Sobolev space $H^{\nu/2}$ in the last term of the product, we need to set

$$\alpha q = 1 + \nu,$$

and thus

$$\alpha = \frac{1 + \nu}{5 + \nu}$$

(note that $0 < \alpha < 1$). With these constants, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^2} \frac{|f(v') - f(\tilde{v}')|^q}{|v(1 - \cos \theta)|^{\alpha q}} dv dv_* \right)^{1/q} &= \frac{1}{|1 - \cos \theta|^{1/q}} \left(\int_{\mathbb{R}^2} \frac{|f(v') - f(\tilde{v}')|^q}{|v(1 - \cos \theta)|^{\alpha q}} dv d(1 - \cos \theta) v_* \right)^{1/q} \\ &\leq \frac{2^{1-\frac{2}{q}}}{|1 - \cos \theta|^{\frac{1}{q}} |\sin \theta|^{\frac{1}{q}}} \|f(t)\|_{L^\infty}^{1-\frac{2}{q}} |f(t)|_{H^{\nu/2}}^{\frac{2}{q}}. \end{aligned}$$

Finally

$$|\tilde{I}_1| \leq C \int \beta(\theta) |\cos^3 \theta| |\sin \theta|^{1-\frac{1}{q}} |1 - \cos \theta|^{\alpha-\frac{1}{q}} d\theta \left(\int_{\mathbb{R}} v^4 f(t, v) dv \right)^{\frac{2}{p}} \|f(t)\|_{L^\infty} |f(t)|_{H^{\nu/2}}^{\frac{2}{q}}$$

The integral in θ is finite if and only if

$$2 \left(\alpha - \frac{1}{q} \right) - \nu - \frac{1}{q} > -1.$$

Recalling that

$$q = \frac{4}{1 - \alpha}$$

and that

$$\alpha = \frac{1 + \nu}{5 + \nu},$$

this request is equivalent to

$$\nu^2 + 2\nu - 4 < 0,$$

which is verified for $-1 - \sqrt{5} < \nu < \sqrt{5} - 1$. Combining this with the previous constraints on the parameter ν , we obtain that our bound of the fourth moment works for all ν verifying

$$1 < \nu < \sqrt{5} - 1.$$

The treatment of \tilde{I}_2 is quite more simple, since it requires only some changes of variable. Indeed, using the changes of variable $v \mapsto v - v_* \sin \theta$ and $\theta \mapsto -\theta$

$$\begin{aligned} \tilde{I}_2 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f(v) f(v_*) (f(\tilde{v}') - f(v)) dv dv_* d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta \cos^3 \theta \int_{\mathbb{R}^2} v v_* (v^2 - v_*^2) f(v) f(v_*) f(\tilde{v}') dv dv_* d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta \cos^3 \theta \cdot \\ &\quad \cdot \int_{\mathbb{R}^2} (v + v_* \sin \theta) v_* ((v + v_* \sin \theta)^2 - v_*^2) f(v + v_* \sin \theta) f(v_*) f(v) dv dv_* d\theta \\ &= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta \cos^3 \theta \cdot \\ &\quad \cdot \int_{\mathbb{R}^2} (v - v_* \sin \theta) v_* ((v - v_* \sin \theta)^2 - v_*^2) f(\tilde{v}') f(v_*) f(v) dv dv_* d\theta \end{aligned}$$

This implies

$$\tilde{I}_2 \leq C(m, e) \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \theta d\theta \right) \|f(t)\|_{L^\infty} \left(e^2 + m^{\frac{1}{4}} \left(\int_{\mathbb{R}} v^4 f(t, v) dv \right)^{\frac{3}{4}} \right).$$

We proved

Theorem 10. *Assume that the cross-section β satisfies (H1), (H2), with $1 < \nu < \sqrt{5} - 1$. Assume that there exists a solution f to the problem (33). Assume that f is regular, in the sense that $\|f(t)\|_{L^\infty(\mathbb{R})}$ and $\|f(t)\|_{H^{\frac{p}{2}}}$ are in $L^p(\mathbb{R}_+)$ for some p big enough; then there exists some constant $\lambda > 0$ such that f satisfies*

$$\|f(t)\|_{L^4_4} \leq \max \left\{ \lambda, \|f_0\|_{L^4_4} \right\}.$$

7 Conclusions

In this paper we investigated the asymptotic equivalence between the (mollified) Kac caricature of a Bose-Einstein gas and a nonlinear Fokker-Planck type equation in the so-called grazing collision limit. The limit equation differs from the analogous one present in the literature [12], since in our case the linear diffusion

has a diffusivity which depends on the solution itself, in order to guarantee the conservation of energy. Our analysis refers to a mollified version of the equation, due to the difficulties of handle the third order nonlinearity present in the Bose-Einstein correction. A further inside on the true model, done in the last part of the paper, shows that a proof of the boundedness of the solution would be sufficient to avoid the presence of the mollifier.

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