

Modelling and control of urban bus networks in dioids algebra

Workshop on Discrete Event Systems 2004

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Linear system theory over dioids

- Discrete Event Dynamic Systems (DEDS) involving synchronization
- developed by analogy with conventional theory¹

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¹ F. Baccelli, G. Cohen, G. J. Olsder and J. P. Quadrat, *Synchronization and Linearity*, 1992,

Wiley.

² S. Lahaye, L. Hardouin, J.-L. Boimond, *Models Combination in (Max,+) Algebra for the Implementation of a Simulation and Analysis Software*, 2003

³ J.Y. Le Boudec and P. Thiran, *Network Calculus*, 2001, Springer Verlag.

⁴ H. Braker, *Algorithms and applications in timed discrete event systems*, PhD thesis, 1993.

⁵ P. Lolito, E. Mancinelli and J.P. Quadrat, *A minplus derivation of the fundamental car-traffic law*, 2001.

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Linear system theory over dioids

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Applications

- 1 manufacturing systems²

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Applications

- 1 manufacturing systems²
- 2 communication networks³
- 3 transportation systems:
 - railway networks⁴
 - urban traffic⁵

¹ F. Baccelli, G. Cohen, G. J. Olsder and J. P. Quadrat, *Synchronization and Linearity*, 1992,

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Dioid

A dioid $(\mathcal{D}, \oplus, \otimes)$ is a set with two inner operations.

- \oplus : commutative, associative, idempotent and admits a neutral element ε
- \otimes : associative, distributive with respect to the sum and admits a neutral element e .

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Order relation

The equivalence: $a \succeq b \Leftrightarrow a = a \oplus b$ defines a partial order relation.

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Order relation

The equivalence: $a \succeq b \Leftrightarrow a = a \oplus b$ defines a partial order relation.

Example

$$\overline{\mathbb{Z}}_{max} = (\mathbb{Z} \cup \{+\infty, -\infty\}, max, +)$$

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Residuation theory⁶

f residuated \Leftrightarrow inequation $f(x) \preceq b$ admits
a greatest solution $f^\#(b)$

$f^\#$ is called *residual* of f .

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⁶ T. S. Blyth and M. F. Janowitz, *Residuation theory*, 1972, Pergamon press

⁷ F. Baccelli, G. Cohen, G. J. Olsder and J. P. Quadrat, *Synchronization and Linearity*, §4.4.2,

Residuation theory⁶

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$f^\#$ is called *residual* of f .

Example

$L_a : x \mapsto a \otimes x$ is residuated,
greatest solution of $L_a(x) \preceq b$ is $L_a^\#(b) = \frac{b}{a}$.

▶ Application to JIT control

⁶ T. S. Blyth and M. F. Janowitz, *Residuation theory*, 1972, Pergamon press

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▶ Application to JIT control

Composed mapping residuation⁷

Let $f : \mathcal{D} \rightarrow \mathcal{C}$ and $g : \mathcal{C} \rightarrow \mathcal{B}$. If f and g are residuated
then $g \circ f$ is residuated and $(g \circ f)^\# = f^\# \circ g^\#$.

⁶ T. S. Blyth and M. F. Janowitz, *Residuation theory*, 1972, Pergamon press

⁷ F. Baccelli, G. Cohen, G. J. Olsder and J. P. Quadrat, *Synchronization and Linearity*, §4.4.2,

Mapping restriction

Let $f : \mathcal{D} \rightarrow \mathcal{C}$ a mapping and $\mathcal{A} \subseteq \mathcal{D}$. We denote $f|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{C}$ the mapping defined by $f|_{\mathcal{A}} = f \circ Id_{\mathcal{A}}$ where $Id_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{D}$ is the canonical injection.

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f} & \mathcal{C} \\ Id_{\mathcal{A}} \uparrow & \nearrow & \\ \mathcal{A} & \xrightarrow{f|_{\mathcal{A}}} & \end{array}$$

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Constrained residuation⁸

Such problems consist in finding the greatest solution of $f|_{\mathcal{A}}(x) \preceq b$, with f residuated.

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⁸G. Cohen, *Residuation and applications*, 1998.

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Constrained residuation⁸

Such problems consist in finding the greatest solution of $f|_{\mathcal{A}}(x) \preceq b$, with f residuated.

Proposition

If $Id_{\mathcal{A}}$ is residuated, the constrained residuation problem admits an optimal solution which is

$$f|_{\mathcal{A}}^{\#}(b) = (f \circ Id_{\mathcal{A}})^{\#}(b) = Id_{\mathcal{A}}^{\#} \circ f^{\#}(b).$$

⁸ G. Cohen, *Residuation and applications*, 1998.

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Fixed-point equations⁹

$$x = Ax \oplus b$$

$$x = \underbrace{x}_A \wedge b$$

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⁹ F. Baccelli, G. Cohen, G. J. Olsder and J. P. Quadrat, *Synchronization and Linearity*, §4.5, 1992, Wiley.

Fixed-point equations⁹

$$x = Ax \oplus b$$

$x = A^*b$ least solution of equation $x = Ax \oplus b$,

with $A^* = \bigoplus_{i=0}^{+\infty} A^i$.

$$x = \underbrace{x}_A \wedge b$$

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$$x = \underbrace{x}_A \wedge b$$

$x = \underbrace{b}_{A^*}$ greatest solution of equation $x = \underbrace{x}_A \wedge b$.

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Representation of DEDS using dioids

- ▶ *dater* $x(k)$: date of the $k + 1$ -th occurrence

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▶ *dater* $x(k)$: date of the $k + 1$ -th occurrence

▶ state representation:
$$\begin{cases} x(k) & = & Ax(k-1) \oplus Bu(k) \\ y(k) & = & Cx(k) \end{cases}$$

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Representation of DEDS using dioids

- ▶ *dater* $x(k)$: date of the $k + 1$ -th occurrence
 - ▶ state representation:
$$\begin{cases} x(k) &= Ax(k-1) \oplus Bu(k) \\ y(k) &= Cx(k) \end{cases}$$
 - ▶ γ -transform x of a dater $x(k)$:
$$x = \bigoplus_{k \in \mathbb{Z}} x(k) \gamma^k$$
- $\overline{\mathbb{Z}}_{\max}[[\gamma]]$: dioid of power series in γ

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$\overline{\mathbb{Z}}_{max}[\gamma]$: dioid of power series in γ

▶ state representation in $\overline{\mathbb{Z}}_{max}[\gamma]$:

$$\begin{cases} x &= \gamma Ax \oplus Bu \\ y &= Cx. \end{cases}$$

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- $\overline{\mathbb{Z}}_{max}[\gamma]$: dioid of power series in γ
- ▶ state representation in $\overline{\mathbb{Z}}_{max}[\gamma]$:
$$\begin{cases} x &= \gamma Ax \oplus Bu \\ y &= Cx. \end{cases}$$
- ▶ input/output representation:
$$y = C(\gamma A)^* B \otimes u = H \otimes u$$

 H : transfer matrix

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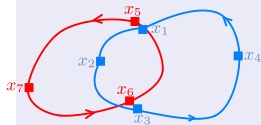
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Modelling of a network: a first model



$x_i(k)$: departure time of the $k+1$ -th bus at stop i .

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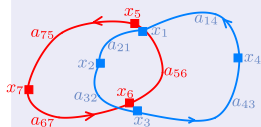
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Modelling of a network: a first model



$$A = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & a_{14} & \varepsilon & \varepsilon & \varepsilon \\ a_{21} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & a_{32} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & a_{43} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & a_{56} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & a_{67} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & a_{75} & \varepsilon & \varepsilon \end{pmatrix}$$

$x_i(k)$: departure time of the $k+1$ -th bus at stop i .

$$x(k) \geq Ax(k-1)$$

A_{ij} : travelling time from stop j to stop i

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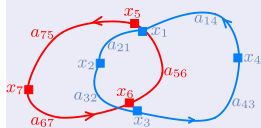
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$x_i(k)$: departure time of the $k+1$ -th bus at stop i .

$$x(k) \geq Ax(k-1)$$

A_{ij} : travelling time from stop j to stop i
 \rightarrow timetable

$$x(k) \geq Bu(k)$$

$B_{ii} = e$ if buses respect timetable at stop i , $B_{ij} = \varepsilon$ otherwise

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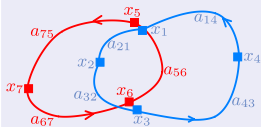
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Functioning of urban bus networks¹⁰

Two stages:

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¹⁰ S. Hayat and S. Maouche, *Régulation du trafic des autobus : amélioration de la qualité des correspondances*, 1997

Functioning of urban bus networks¹⁰

Two stages:

1. Operating schedule

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Functioning of urban bus networks¹⁰

Two stages:

1. Operating schedule

- 1 definition of the bus routes and the bus stops

¹⁰ S. Hayat and S. Maouche, *Régulation du trafic des autobus : amélioration de la qualité des correspondances*, 1997

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Functioning of urban bus networks¹⁰

Two stages:

1. Operating schedule

- 1 definition of the bus routes and the bus stops
- 2 choice of a level-of-service (resources distribution, minimum and maximum headways)

¹⁰ S. Hayat and S. Maouche, *Régulation du trafic des autobus : amélioration de la qualité des correspondances*, 1997

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¹⁰ S. Hayat and S. Maouche, *Régulation du trafic des autobus : amélioration de la qualité des correspondances*, 1997

Functioning of urban bus networks¹⁰

Two stages:

1. Operating schedule

- 1 definition of the bus routes and the bus stops
- 2 choice of a level-of-service (resources distribution, minimum and maximum headways)
- 3 synthesis of timetables

2. Regulation

adjustements of the operating schedule parameters in reaction to the current functioning conditions.

¹⁰ S. Hayat and S. Maouche, *Régulation du trafic des autobus : amélioration de la qualité des correspondances*, 1997

Problem description¹¹

Find timetables which

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¹¹ A. Ceder, B. Golany and O. Tal, *Creating bus timetables with maximal synchronization*, 2000

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Problem description¹¹

Find timetables which

- ensure a level of service at particular stops of the network: $y(k) = C'x(k) \preceq z(k)$

¹¹ A. Ceder, B. Golany and O. Tal, *Creating bus timetables with maximal synchronization*, 2000

Problem description¹¹

Find timetables which

- ensure a level of service at particular stops of the network: $y(k) = C'x(k) \preceq z(k)$
- satisfy maximum and minimum headways:

$$u(k+1) \succeq \Delta^{\min} u(k)$$

$$u(k+1) \preceq \Delta^{\max} u(k)$$

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- minimize waiting times :
we must consider a new model:

$$x(k) = A'x(k-1) \oplus B'u(k)$$

where $A'_{ij} = A_{ij}$ if $A_{ij} \neq \varepsilon$ and
 $A'_{ij} = w_{ij}$ if i in connection with j .

¹¹ A. Ceder, B. Golany and O. Tal, *Creating bus timetables with maximal synchronization*, 2000

Problem formalization

In $\overline{\mathbb{Z}}_{max}[\gamma]$, find the greatest u such that

$$\begin{cases} L_H(u) \preceq z \\ u \in \mathcal{A} \text{ (admissible domain)} \end{cases} \quad (1)$$

where \mathcal{A} is a subset of $\overline{\mathbb{Z}}_{max}[\gamma]$ composed of series satisfying minimum and maximum headways and $H = C'(\gamma A')^* B'$.

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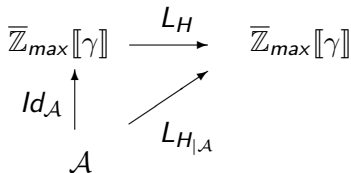
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Problem formalization

In $\overline{\mathbb{Z}}_{max}[\gamma]$, find the greatest u such that

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The constrained problem (1) is equivalent to solve:

$$L_{H|_{\mathcal{A}}}(u) = L_H \circ Id_{\mathcal{A}}(u) \preceq z$$

Characterization of \mathcal{A}

\mathcal{A} is a subset of $\overline{\mathbb{Z}}_{max}[\gamma]$ where

$$u \in \mathcal{A} \Leftrightarrow \begin{cases} \Delta^{min} \gamma u \preceq u \\ (\Delta^{max})^{-1} \gamma^{-1} u \preceq u, \end{cases}$$

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 \end{aligned}$$

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$$\mathcal{A} = \left\{ x \in \overline{\mathbb{Z}}_{max}[\gamma] \mid x = \frac{x}{p^*} \right\}$$

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Residuability of $Id_{\mathcal{A}}$

By quotienting $\overline{\mathbb{Z}}_{max}[\gamma]$ by the following equivalence relation

$$\{x \mathcal{R} y\} \iff \left\{ \frac{x}{p^*} = \frac{y}{p^*} \right\}$$

we can deduce that $\frac{y}{p^*} \in \mathcal{A}$ is the greatest element of \mathcal{A} which is less than or equal to y .

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\Leftrightarrow the inequation $Id_{\mathcal{A}}(x) \preceq x'$ (with $x' \in \overline{\mathbb{Z}}_{max}[\gamma]$) admits a greatest solution which is $\frac{x'}{p^*}$.

► Proposition

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$\Leftrightarrow Id_{\mathcal{A}}$ is residuated and

$$Id_{\mathcal{A}}^{\#}(x') = \bigoplus_{\{x \in \mathcal{A} | x \preceq x'\}} x = \frac{x'}{p^*}$$

► Proposition

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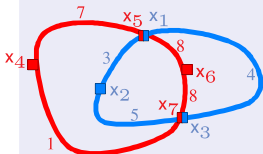
Conclusion

Since $Id_{\mathbb{A}}$ is residuated, $L_{H|_{\mathbb{A}}}$ is residuated too and

$$L_{H|_{\mathbb{A}}}^{\#}(z) = Id_{\mathbb{A}}^{\#} \circ L_H^{\#}(z) = \frac{\langle z \rangle}{p^*}$$

is the optimal solution to the constrained problem.

Example



$$A' = \begin{pmatrix} \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 \\ e & \varepsilon & \varepsilon & 7 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 8 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 8 & \varepsilon \end{pmatrix}$$

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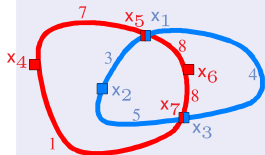
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$$C' = (\varepsilon \quad \varepsilon \quad \varepsilon \quad \varepsilon \quad \varepsilon \quad \varepsilon \quad e) \text{ and } B' = Id$$

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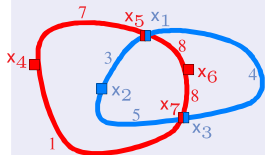
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target for x_7 :

k	1	2	3	4	5
$z(k)$	10	20	24	28	40

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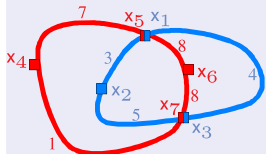
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headway min $\Delta^{min} = 7$

headway max $\Delta^{max} = 10$

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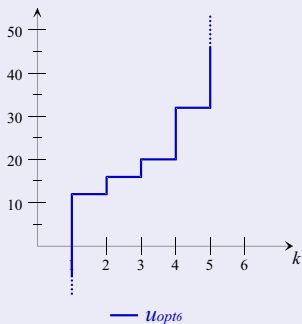
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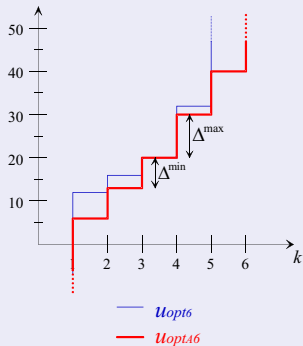
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k	1	2	3	4	5
$y(k)$	10	20	24	28	40
$u_{opt_6}(k)$	12	16	20	32	T

$$u_{opt} = \frac{z}{H}$$





k	1	2	3	4	5
$y(k)$	10	20	24	28	40
$u_{opt_6}(k)$	12	16	20	32	⊔

$$u_{opt} = \frac{z}{H}$$

k	1	2	3	4	5
$y(k)$	7	14	21	28	38
$u_{opt_{A6}}(k)$	6	13	20	30	40

$$u_{opt_A} = \frac{z}{p^*H}$$

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- dioids algebra seems to be appropriate for the study of public transportation networks :

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 - we have proposed a solution for the problem of timetables generation
 - stability of such networks has been studied

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Conclusion

- dioids algebra seems to be appropriate for the study of public transportation networks :
 - we have proposed a solution for the problem of timetables generation
 - stability of such networks has been studied
- extension:
we try currently to control such systems considering more general constraints.

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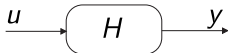
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Just In Time control

For a given target z , find the greatest u such as
 $y = Hu \preceq z$.

→ residuation of L_H leads to $u_{opt} = \underset{H}{\preceq} z$.



$$\left\{ \begin{array}{l} \Delta^{\min} u(k-1) \\ (\Delta^{\max})^{-1} u(k+1) \end{array} \right\} \preceq u(k),$$

$$\begin{cases} \Delta^{\min} u(k-1) \preceq u(k) \\ (\Delta^{\max})^{-1} u(k+1) \preceq u(k), \end{cases}$$

γ -transforms

$$\begin{cases} \Delta^{\min} \gamma u \preceq u \\ (\Delta^{\max})^{-1} \gamma^{-1} u \preceq u, \end{cases}$$

and $(\Delta^{\min} \gamma \oplus (\Delta^{\max})^{-1} \gamma^{-1}) u \preceq u$.

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Since product is residuated, we equivalently have

$$u = \frac{u}{\Delta^{\min} \gamma \oplus (\Delta^{\max})^{-1} \gamma^{-1}} \wedge u.$$

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greatest solution:

$$u = \frac{u}{(\gamma \Delta^{\min} \oplus \gamma^{-1} (\Delta^{\max})^{-1})^*}.$$

proposition

- 1 Each equivalence class of $\overline{\mathbb{Z}}_{max}[\gamma]/\mathcal{R}$ contains one and only one element belonging to $\Pi(\overline{\mathbb{Z}}_{max}[\gamma])$ and this element is explicitly given by $\frac{x}{p^*}$ for any x in the class.
- 2 Element $\frac{x}{p^*}$ is the least element in $[x]_{\mathcal{R}}$, and it is the greatest element of $\Pi(\overline{\mathbb{Z}}_{max}[\gamma])$ which is less than x .