

Timetables synthesis using $(max, +)$ algebra


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18 may 2006

Linear system theory over dioids ¹

- Discrete Event Dynamic Systems (DEDS) involving synchronization phenomena

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 - urban traffic

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 - urban bus networks

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Dioid

A dioid $(\mathcal{D}, \oplus, \otimes)$ is a set endowed with two inner operations.

- \oplus : commutative, associative, idempotent and admits a neutral element ε
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Order relation

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$(\mathcal{D}, \oplus, \otimes)$ is complete if it is closed for infinite sums and if $\mathcal{B} \subseteq \mathcal{D}$

$$c \otimes \left(\bigoplus_{x \in \mathcal{B}} x \right) = \bigoplus_{x \in \mathcal{B}} (c \otimes x) \quad \text{and} \quad \left(\bigoplus_{x \in \mathcal{A}} x \right) \otimes c = \bigoplus_{x \in \mathcal{A}} (x \otimes c).$$

Lattice structure

The order relation induces a **lattice structure** where $a \oplus b$ is the least upper bound of a and b .

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Example

$$\overline{\mathbb{Z}}_{max} = (\mathbb{Z} \cup \{+\infty, -\infty\}, max, +)$$

Residuation

f is residuated \Leftrightarrow the greatest element of set $\{x \in \mathcal{D} \mid f(x) \preceq b\}$ exists and is denoted $f^\#(b)$

$f^\#$ is called *residual* of f .

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Example

$L_a : x \mapsto a \otimes x$ is residuated,
greatest solution of $L_a(x) \preceq b$ is $L_a^\sharp(b) = \frac{b}{a}$.

▶ JIT control

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Greatest fixed point of isotone mappings defined over dioids ³⁴

$$\mathcal{F}_f = \{x \mid f(x) = x\} \quad \mathcal{P}_f = \{x \mid f(x) \succeq x\}$$

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$$\mathcal{F}_f = \{x | f(x) = x\} \quad \mathcal{P}_f = \{x | f(x) \succeq x\}$$

Theorem : Sets \mathcal{P}_f and \mathcal{F}_f are **complete lattices**.

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Theorem :

$$\text{Sup } \mathcal{P}_f = \text{Sup } \mathcal{F}_f$$

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Proposition : greatest fixed point of an isotone mapping

If the following iterative computation

$$\begin{aligned}y_0 &= \top \\ y_{k+1} &= f(y_k)\end{aligned}$$

converges in a finite number k_e of iterations, then y_{k_e} is the greatest fixed point of f .

◀ Back

Timetables synthesis⁵⁶

The behavior of such systems is controlled by a timetable.

Timetables synthesis is a part of an optimization which consists in :

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- model for urban bus networks in $(max, +)$ algebra

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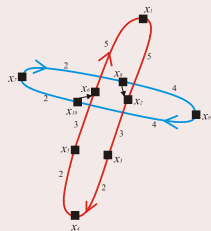
- model for urban bus networks in $(max, +)$ algebra
- timetables synthesis problem
 - ▶ formulated as a constraint problem on this problem
 - ▶ solved using previous proposition on fixed points computation

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Modeling of a bus network

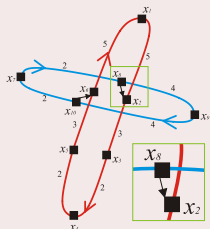
travelling times and connections



$x_i(k)$: departure time of the $k+1$ -st bus at stop i .

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$$x(k) \geq Ax(k-1)$$

A_{ij} : travelling time from stop j to i
connection : $x_2(k) = 5 \otimes x_1(k) \oplus x_8(k)$

timetables

$$x(k) \geq Bu(k)$$

$B_{ij} = e$ if buses respect timetable at stop i , $B_{ij} = \varepsilon$ otherwise

state representation

$$x(k) = Ax(k - 1) \oplus Bu(k)$$

Initial and final conditions

For each x_i

- Initial conditions : k_{d_i} (index of the first bus for which timetable is generated at stop x_i)

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- Initial conditions : k_{d_i} (index of the first bus for which timetable is generated at stop x_i)
- Final conditions : k_{f_i} (index of the last bus for which timetable is generated at stop x_i)

Constraints

- ▷ Expiries at strategic stops :
the k -th departure at stop x_i must occur before a date t .

$$C \otimes x(k) \preceq z(k) \quad \text{for } k_{d_i} \leq k \leq k_{f_i},$$

Formulation of the timetables synthesis problem

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the k -th departure at stop x_i must occur before a date t .

$$C \otimes x(k) \preceq z(k) \quad \text{for } k_{d_i} \leq k \leq k_{f_i},$$

- ▷ Maximum headways :
maximum time separation between two buses departures at a stop x_i .

$$\begin{cases} x_i(k) \preceq \Delta_i^{\max} \otimes x_i(k-1) & \text{for } k_{d_i} < k \leq k_{f_i}, \\ x_i(k) = x_i(k_{d_i}) & \text{for } k = k_{d_i}, \end{cases}$$

$$\iff x_i(k) \preceq \Delta_i^{\max} \otimes x_i(k-1) \oplus x_i(k_{d_i}) \quad \text{for } k_{d_i} \leq k \leq k_{f_i},$$

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$$x_i(k) \succeq \Delta_i^{\min} \otimes x_i(k-1) \quad \text{for } k_{d_i} \leq k \leq k_{f_i}.$$

- ▷ Peaks of charge

$$\exists k \in [k_{d_i}, k_{f_i}] \text{ such that } x_i(k) \succeq t_j \text{ and } x_i(k+m) \prec t_j + r,$$

where m is the expected number of departures at stop i during interval $[t_j, t_j + r]$ in order to absorb the peak.

Constraints

- ▶ Maximum waiting times :
limit waiting times between x_j and x_i to d_{ji}^{max} .

$$\begin{cases} x_j(k) \preceq \phi_{ji}^{max} \otimes x_i(k-1) & \text{for } k_{d_j} < k \leq k_{f_j}, \\ x_j(k) = x_j(k_{d_i}) & \text{for } k = k_{d_j}, \end{cases}$$

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$$\text{where } \phi_{ji}^{max} = A_{ji} \otimes d_{ji}^{max}.$$

▶ extension

We can express these constraints in $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$, we obtain

$$u \preceq f(u)$$

in which u denotes the timetable and f is an isotone mapping formulating all the considered constraints.

Solving the timetables synthesis problem

We can express these constraints in $\mathcal{M}_{in}^{ax}[[\gamma, \delta]]$, we obtain

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in which u denotes the timetable and f is an isotone mapping formulating all the considered constraints.

The **iterative computation** presented previously converges in a finite number k_e of iterations and u_{k_e} is the optimal timetables.

- timetables set per period

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- Both company-oriented and passenger-oriented quality criteria :
 - increase passengers confort
 - limit costs for the company

We have introduced a new method to compute timetables for bus networks. Originality is the possibility to take into account constraints which can be formulated as an implicit inequality.

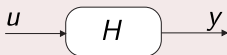
Conclusion

We have introduced a new method to compute timetables for bus networks. Originality is the possibility to take into account constraints which can be formulated as an implicit inequality.

Future works will refine the control method and the model in order to take into account several connections modes between lines.

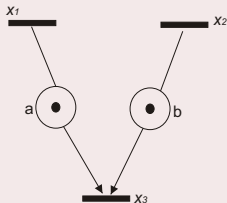
JIT control

For a given target $z(k)$, we seek the greatest u such that $y = Hu \preceq z$.
→ residuation provides the greatest solution $u_{opt} = H \setminus z$.



◀ Back

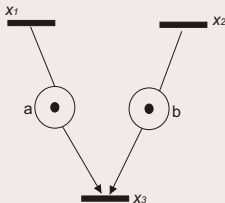
Temps d'attente maximum



Dans le cas de plusieurs bus en correspondance :

$$x_3(k) \preceq ax_1(k-1) \wedge bx_2(k-1)$$

Temps d'attente maximum



Dans le cas de plusieurs bus en correspondance :

$$x_3(k) \preceq ax_1(k-1) \wedge bx_2(k-1)$$

$$\Leftrightarrow x_3(k) \preceq \Phi \backslash x,$$

$$\text{avec } \Phi = \begin{pmatrix} -a \\ -b \\ \varepsilon \end{pmatrix}.$$