Control of constrained $(max, +)$-linear systems minimizing delays

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Introduction

Linear system theory over dioids [1]

- Discrete Event Dynamic Systems (DEDS) involving synchronization phenomena

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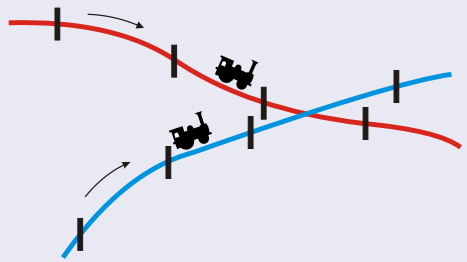

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Dioid theory

Dioid

A dioid $(\mathcal{D}, \oplus, \otimes)$ is a set endowed with two inner operations.

- $\oplus$: commutative, associative, idempotent and admits a neutral element ε
- $\otimes$: associative, distributive with respect to the sum and admits a neutral element e.

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$(\mathcal{D}, \oplus, \otimes)$ is complete if it is closed for infinite sums and if $\mathcal{B} \subseteq \mathcal{D}$

$$c \otimes \left(\bigoplus_{x \in \mathcal{B}} x\right) = \bigoplus_{x \in \mathcal{B}} (c \otimes x) \quad \text{and} \quad \left(\bigoplus_{x \in A} x\right) \otimes c = \bigoplus_{x \in A} (x \otimes c).$$

Dioid theory

Lattice structure

The order relation induces a lattice structure where $a \oplus b$ is the least upper bound of a and b.

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A complete dioid is a complete lattice.
We denote $\top$ the greatest element of the complete dioid.

Example

$\overline{\mathbb{Z}}_{max} = (\mathbb{Z} \cup \{+\infty, -\infty\}, max, +)$

Residuation theory[2]

Residuation

f is residuated $\Leftrightarrow$ the greatest element of set $\{x \in \mathcal{D} | f(x) \preceq b\}$ exists and is denoted $f^{\sharp}(b)$

$f^{\sharp}$ is called *residual* of f.

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Dual residuation

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Example

$T_a : x \mapsto a \oplus x$ is dually residuated. The dual residual is denoted $T_a^{\flat}(b) = b \oslash a$.

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Fixed points of mappings defined over dioids

Notations

for $f : \mathcal{D} \rightarrow \mathcal{C}$

fixed points	$\mathcal{F}_f = \{x \mid f(x) = x\}$
post-fixed points	$\mathcal{P}_f = \{x \mid f(x) \succeq x\}$
pre-fixed points	$\mathcal{Q}_f = \{x \mid f(x) \preceq x\}$

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Definitions

for $f : \mathcal{D} \rightarrow \mathcal{C}$

f *isotone*	$a, b \in \mathcal{D},\ a \preceq b \Rightarrow f(a) \preceq f(b)$
f *antitone*	$a, b \in \mathcal{D},\ a \preceq b \Rightarrow f(a) \succeq f(b)$

Fixed points of mappings defined over dioids

case of f isotone[3]

Theorem

$\mathcal{P}_f$, $\mathcal{F}_f$ and $\mathcal{Q}_f$ are complete lattices.

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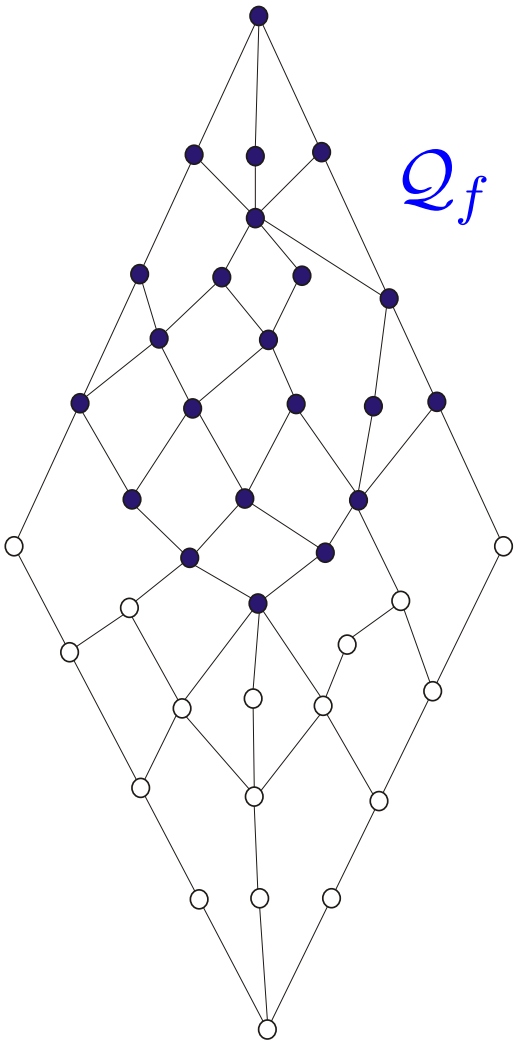
Fixed points of mappings defined over dioids



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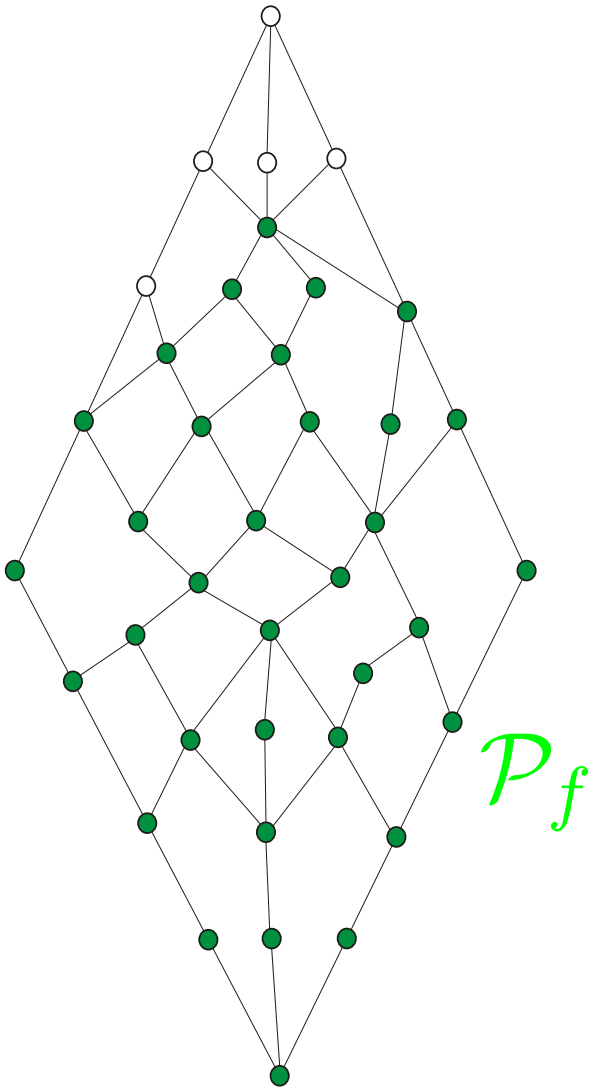

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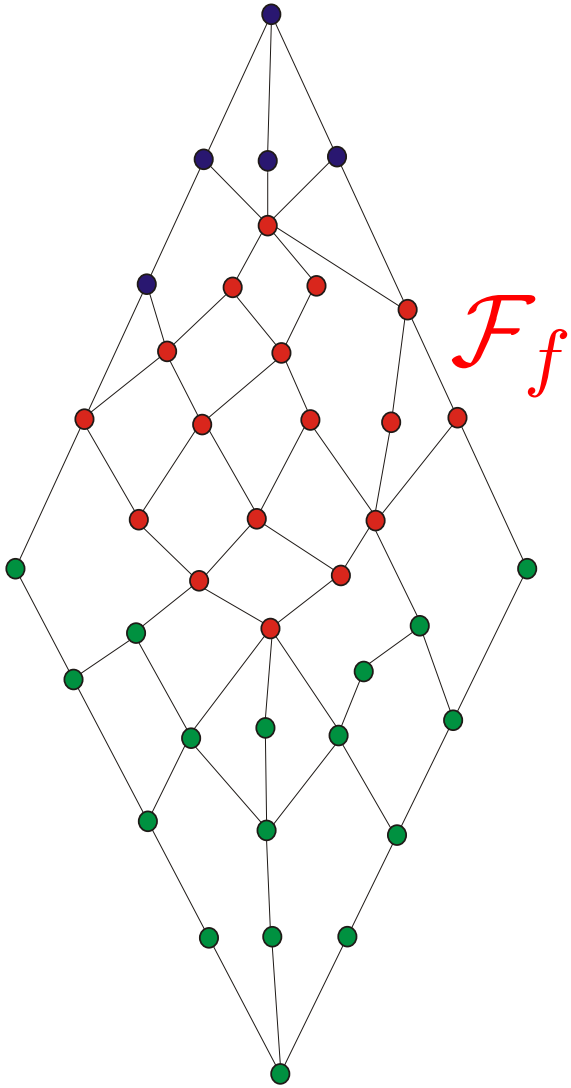

Fixed points of mappings defined over dioids

Proposition : greatest fixed point of an isotone mapping

If the following iterative computation

$$\begin{aligned} y_0 &= \top \\ y_{k+1} &= f(y_k) \end{aligned}$$

converges in a finite number k_e of iterations, then y_{k_e} is the greatest fixed point of f.

Fixed points of mappings defined over dioids

case of f antitone

Remark

f antitone $\Longrightarrow$ $f \circ f = f^2$ isotone.

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$\mathcal{P}_f$ is a complete lower semi-lattice. $\mathcal{Q}_f$ is a complete upper semi-lattice. No indication on the structure of $\mathcal{F}_f$.

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Denoting $u = Inf\ \mathcal{F}_{f^2}$ and $v = Sup\ \mathcal{F}_{f^2}$, we have $u \in \mathcal{P}_f$ and $v \in \mathcal{Q}_f$.

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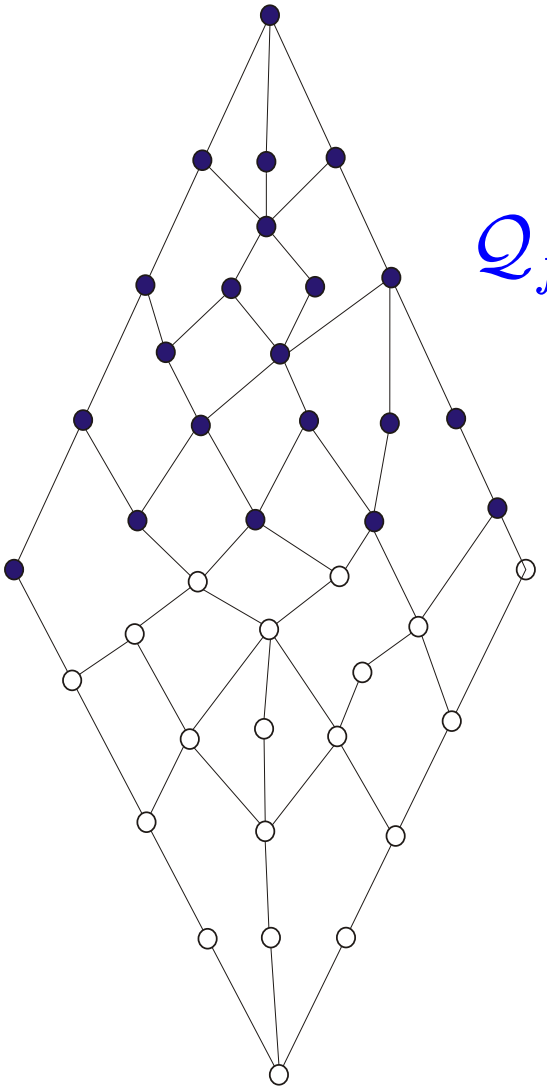
Remark

Since $\forall x \succeq v$, we have $x \in \mathcal{Q}_f$ and any $x' \in \mathcal{F}_f$ is such that $u \preceq x' \preceq v$, we can consider v as an approximation of a minimal element of $\mathcal{Q}_f$.

Fixed points of mappings defined over dioids



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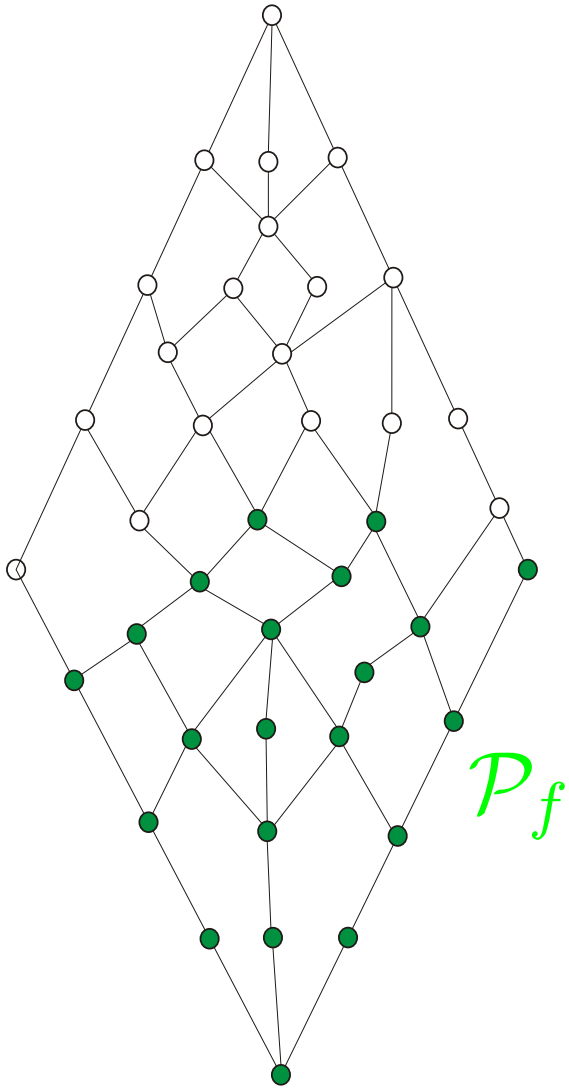

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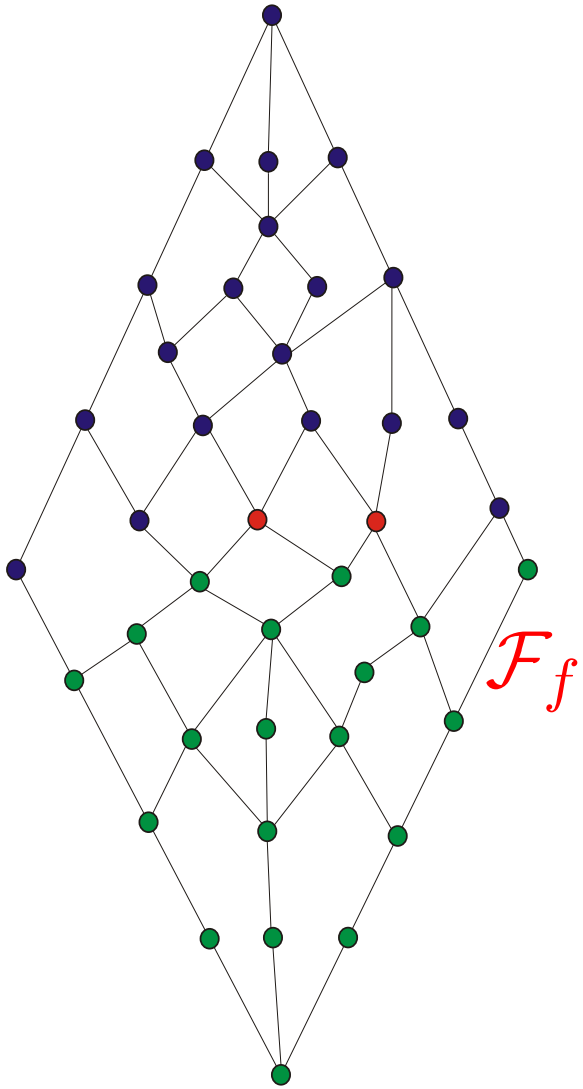

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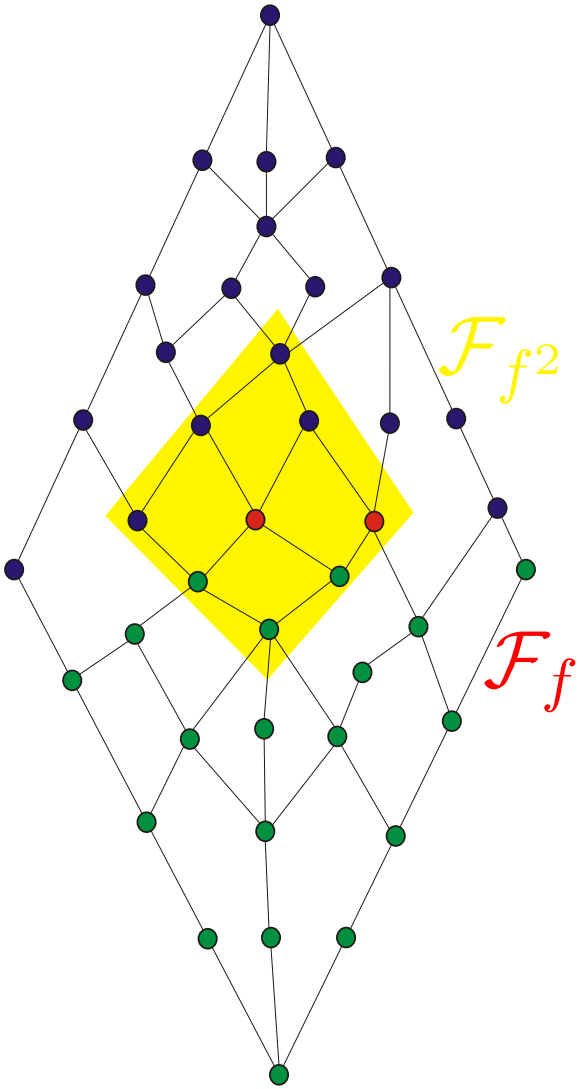

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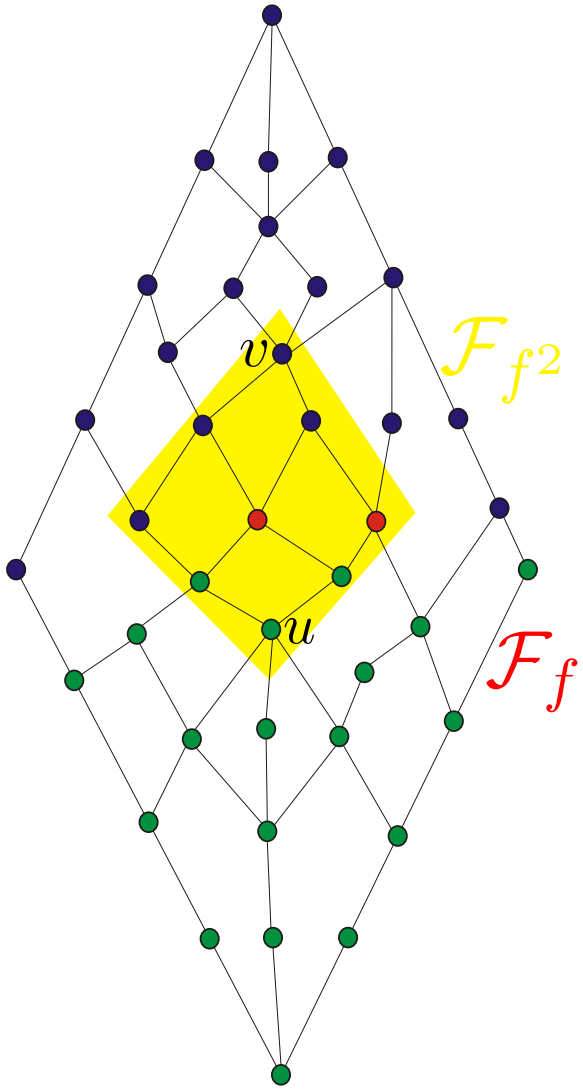

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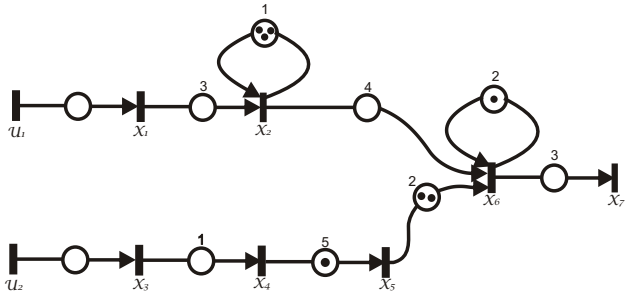
Representation of TEG using dioids



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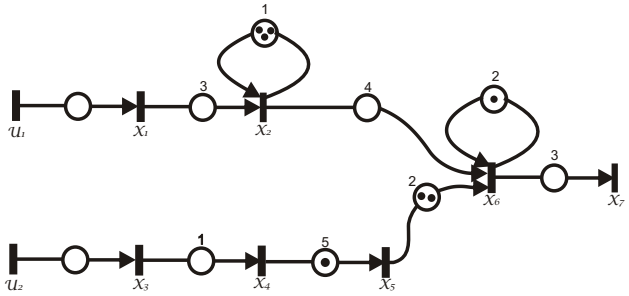

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- *dater* $x(k)$: date of the $k+1$-st occurrence

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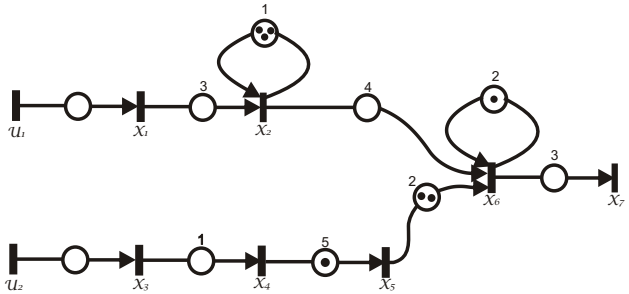

Representation of TEG using dioids

- *dater* $x(k)$: date of the $k+1$-st occurrence
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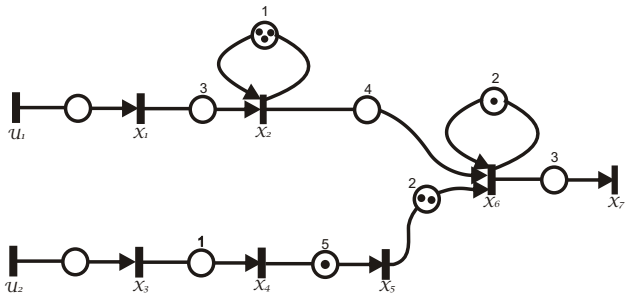

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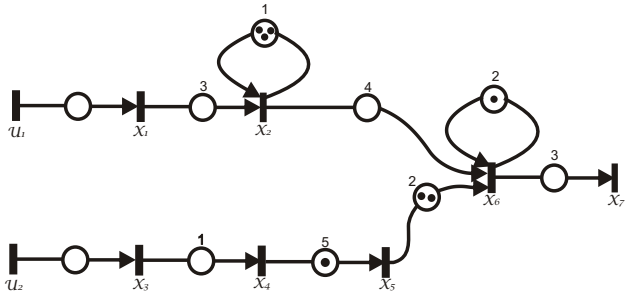
  $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$: dioid of power series in γ, δ

Representation of TEG using dioids



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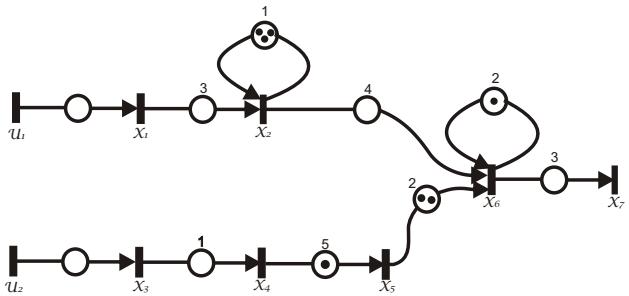

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- state representation in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$: $\begin{cases} x = Ax \oplus Bu \\ y = Cx. \end{cases}$
- input/output representation : $y = CA^*B \otimes u = H \otimes u$

  H : transfer matrix

Causality

Definitions

for $x \in \mathcal{M}_{in}^{max}[\![\gamma, \delta]\!]$

x causal : all exponents of x are in $\mathbb{N}$

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- no negative delay (no anticipation)

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Definitions

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- no negative delay (no anticipation)
- no negative event shift

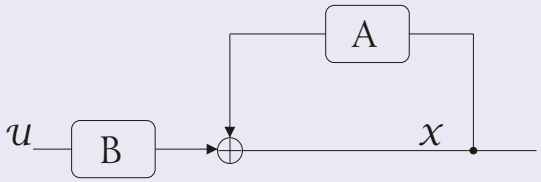
Causality

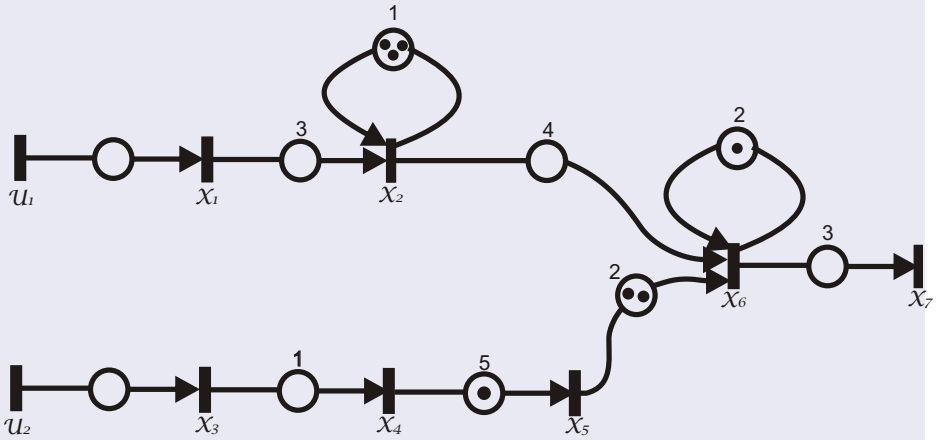
Proposition

x has no negative exponent in γ $\Leftrightarrow$ there exists a least causal approximation of x

Control problem statement

Control structure





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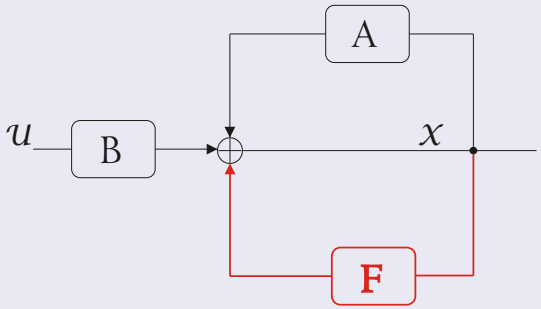

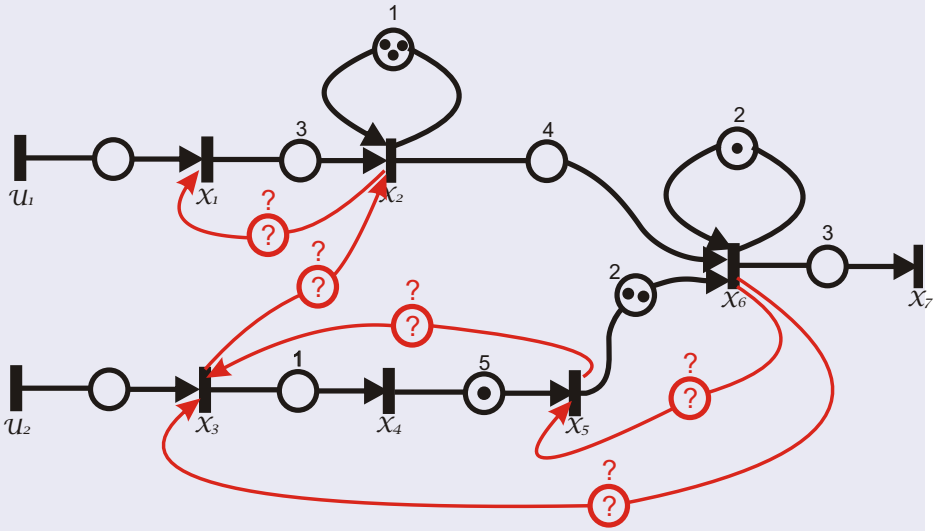

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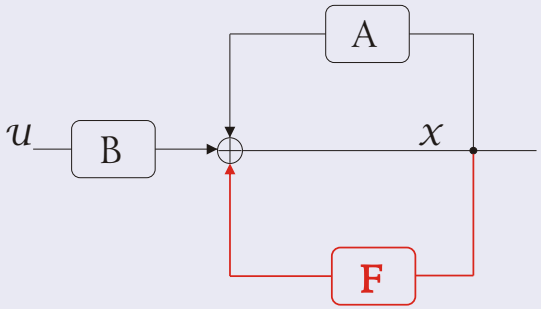

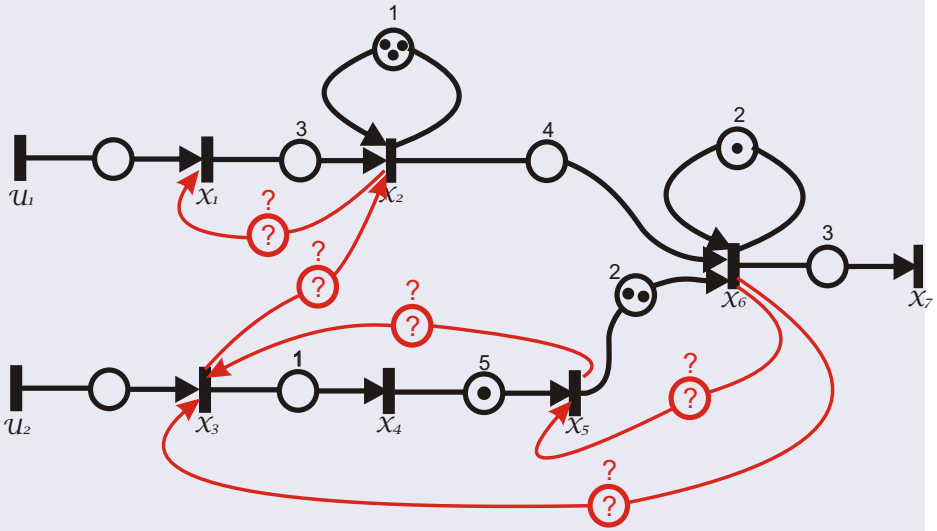

$$x = Ax \oplus Fx \oplus Bu$$

earliest functioning : $x = (A \oplus F)^* Bu$

Criterion and control objective

Control criterion

- delaying as less as possible the system

$$\Longleftrightarrow$$

finding the least feedback F

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Constraints on state of the system :

- minimum time separations : $x_i(k+1) \succeq \Delta_{min}\ x_i(k)$

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- limit number of tokens : $\kappa \succeq x_i(t) - x_j(t) + \alpha$

Formalization

Control objective

$$x \succeq \phi x \qquad \forall u$$

We seek the least controlled state x which satisfies

$$x \succeq \phi x \oplus Ax \oplus Bu$$

Formalization

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We seek the least controlled state x which satisfies

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we seek the least controlled state x, that is the least x greater than the least solution : $x \succeq (A \oplus \phi)^* Bu$

$$\begin{array}{ccc} (A \oplus F)^* Bu & \succeq & (A \oplus \phi)^* Bu \qquad \forall u \\ & \Longleftrightarrow & \\ (A \oplus F)^* B & \succeq & (A \oplus \phi)^* B \end{array}$$

Formalization

Proposition

each entry of $(A \oplus \phi)^* B$ has no negative exponent in γ $\Leftrightarrow$ there exists a causal feedback F

G : causal approximation of $(A \oplus \phi)^* B$.

Formalization

Proposition

each entry of $(A \oplus \phi)^* B$ has no negative exponent in γ $\Leftrightarrow$ there exists a causal feedback F

G : causal approximation of $(A \oplus \phi)^* B$.
We then seek for the least F $s.t.$

$$(A \oplus F)^* B \succeq G \tag{1}$$

Feedback computation

Proposition

Solutions of (1) are elements of $\mathcal{Q}_f$ with $f : F \mapsto G \oslash (A \oplus F)^*$.

Feedback computation

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Solutions of (1) are elements of $\mathcal{Q}_f$ with $f : F \mapsto G \oslash (A \oplus F)^*$.

Sketch of proof

$$\begin{array}{lrcll} & (A \oplus F)^* B & \succeq & G & \\ \Leftrightarrow & G \oslash (A \oplus F)^* & \preceq & F & \text{using dual residuation of } T_{(A \oplus F)^*} \end{array}$$

Feedback computation

Computation

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Computation

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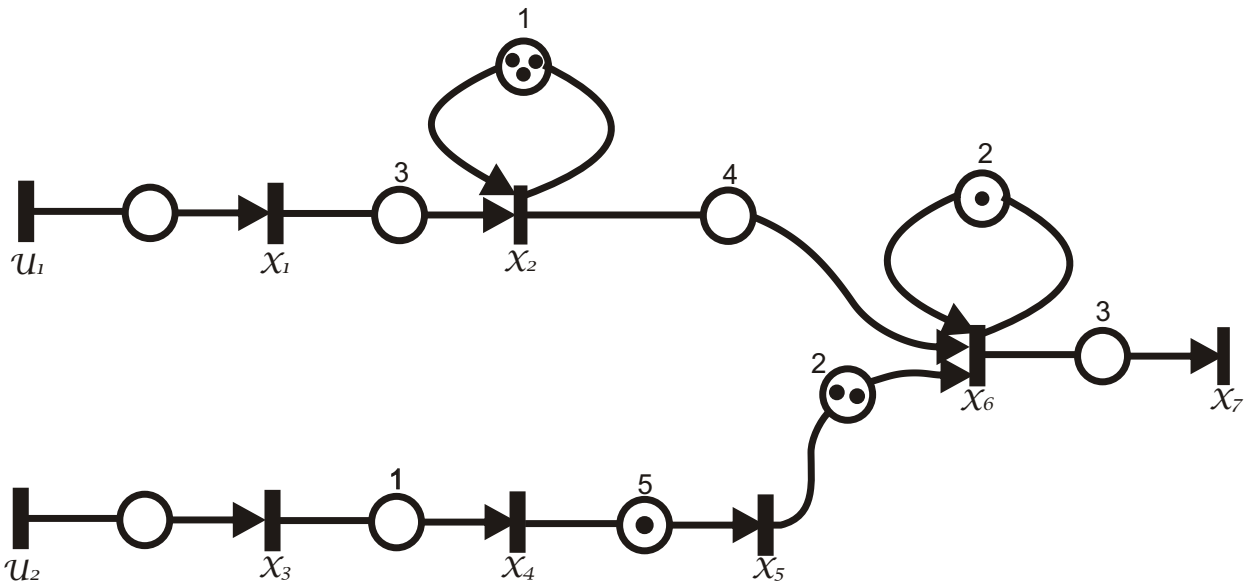
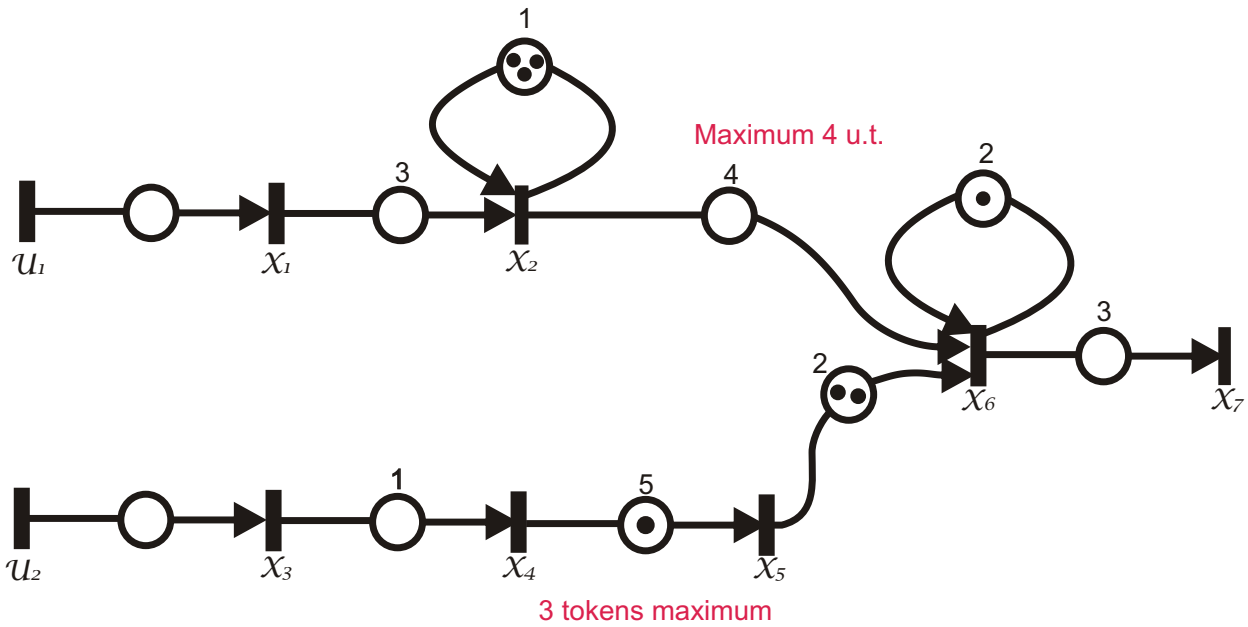
v can be used to approximate a minimal feedback satisfying (1).

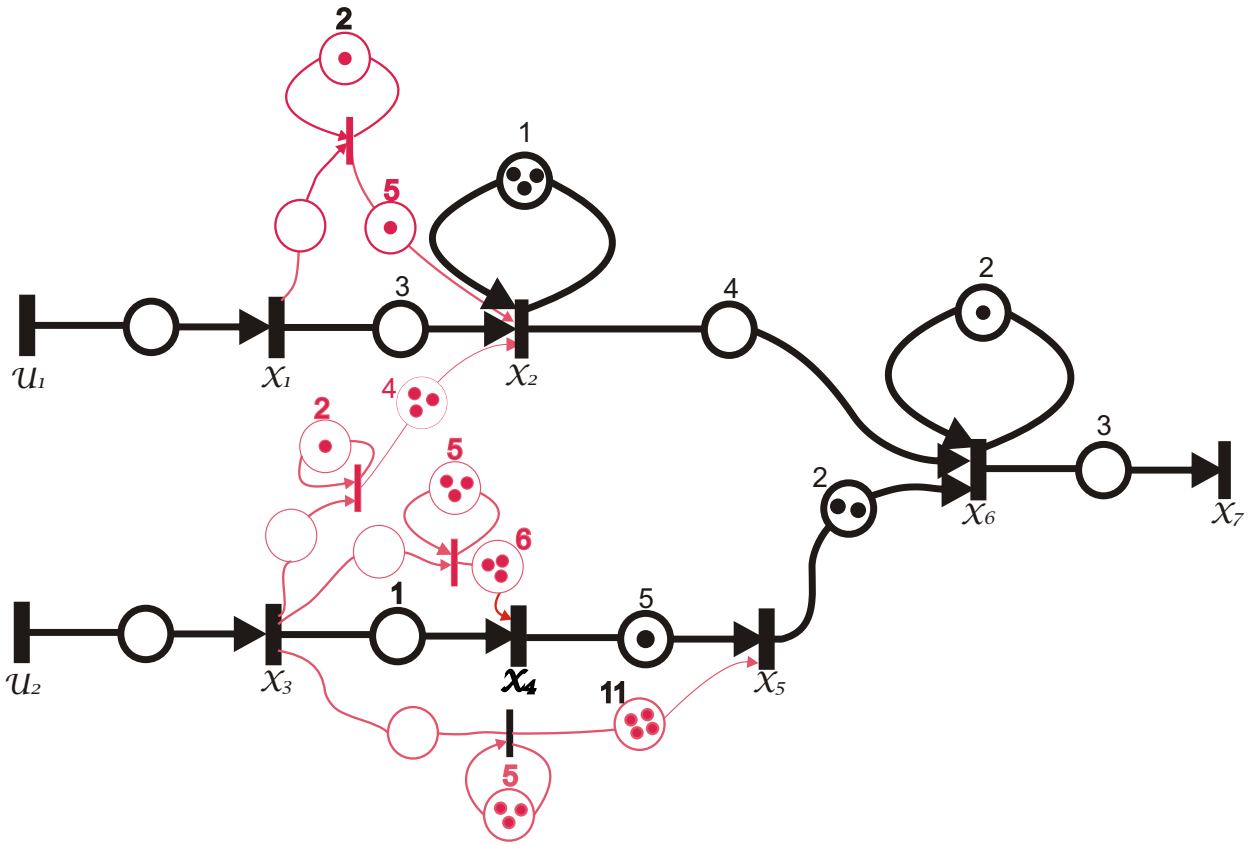
Example



Example



Example



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Original control problem in $(max, +)$-linear system theory :

- new criterion : delaying as less as possible the system
- new control objective : constraints on the state of the system

Use of results concerning isotone and antitone mappings.
Feedback not necessarily minimal.

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Future works

- a minimal solution.
- other feedback structures.