Geometric Invariant Theory and Generalized Eigenvalue Problem II

Nicolas Ressayre

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Abstract

Let G be a connected reductive subgroup of a complex connected reductive group $\hat{G}$. Fix maximal tori and Borel subgroups of G and $\hat{G}$. Consider the cone $\mathcal{LR}^\circ(\hat{G}, G)$ generated by the pairs $(\nu, \hat{\nu})$ of strictly dominant characters such that V_ν is a submodule of $V_{\hat{\nu}}$. The main result of this article is a bijective parametrisation of the faces of $\mathcal{LR}^\circ(\hat{G}, G)$. We also explain when such a face is contained in another one.

In way, we obtain results about the faces of the Dolgachev-Hu's G-ample cone. We also apply our results to reprove known results about the moment polytopes.

1 Introduction

Let G be a connected reductive subgroup of a complex connected reductive group $\hat{G}$. Fix maximal tori and Borel subgroups of G and $\hat{G}$. Consider the cone $\mathcal{LR}^\circ(\hat{G}, G)$ generated by the pairs $(\nu, \hat{\nu})$ of strictly dominant characters such that V_ν is a submodule of $V_{\hat{\nu}}$. This work is a continuation of [Res07]. We obtain results about general GIT-cones and apply it to obtain a bijective parametrisation of the faces of the cone $\mathcal{LR}^\circ(\hat{G}, G)$.

Consider a connected reductive group G acting on a projective variety X. To any G-linearized line bundle $\mathcal{L}$ on X we associate the following open subset $X^{\rm ss}(\mathcal{L})$ of X:

$$X^{\rm ss}(\mathcal{L}) = \left\{ x \in X \,:\, \exists n > 0 \text{ and } \sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0 \right\}.$$

The points of $X^{\rm ss}(\mathcal{L})$ are said to be *semistable* for $\mathcal{L}$. Note that if $\mathcal{L}$ is not ample, this notion of semistability is not the standard one. In particular, the quotient $\pi_\mathcal{L} \,:\, X^{\rm ss}(\mathcal{L}) \longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$ is a good quotient, if $\mathcal{L}$ is ample. In this context, we ask for:

What are the $\mathcal{L}$'s with non empty set $X^{\rm ss}(\mathcal{L})$?

Let us fix a freely finitely generated subgroup Λ of the group $\rm Pic^G(X)$ of G-linearized line bundles on X. Let $\Lambda_{\mathbb{Q}}$ denote the $\mathbb{Q}$-vector space containing Λ as a lattice. Consider the convex cones $\mathcal{TC}^G_\Lambda(X)$ (resp. $\mathcal{AC}^G_\Lambda(X)$) generated in $\Lambda_{\mathbb{Q}}$ by the $\mathcal{L}$'s (resp. the ample $\mathcal{L}$'s) in Λ which have non zero G-invariant sections. By [DH98] (see also [Res00]), $\mathcal{AC}^G_\Lambda(X)$ is a closed convex rational polyhedral cone in the dominant cone of $\Lambda_{\mathbb{Q}}$. We are interested in the faces of $\mathcal{AC}^G_\Lambda(X)$ and $\mathcal{TC}^G_\Lambda(X)$.

We need to introduce a definition due to D. Luna. Assume that X is smooth. Let $\mathcal{O}$ be an orbit of G in X. For $x\in\mathcal{O}$, we consider the action of the isotropy G_x on the normal space N_x of $\mathcal{O}$ in X at x. The pair (G_x, N_x) is called the *type* of the orbit $\mathcal{O}$ and is defined up to conjugacy by G. The main part of Theorem 3 is:

Theorem A *We assume that X is smooth. Let $\mathcal{F}$ be a face of $\mathcal{AC}^G_\Lambda(X)$.*

Then, the type of the closed orbit in $\pi_{\mathcal{L}}^{-1}(\xi)$ for $\xi\in X^{\rm ss}(\mathcal{L})/\!/G$ general does not depends on the choice of an ample G-linearized line bundle $\mathcal{L}$ in the relative interior of $\mathcal{F}$.

We will call this type the type of $\mathcal{F}$.

Let $\mathcal{F}$ be a face of $\mathcal{AC}^G(X)$. Let $\mathcal{L}_0$ be any point in the relative interior of $\mathcal{F}$. The local geometry $\mathcal{AC}^G(X)$ around $\mathcal{F}$ is described by the convex cone $\mathcal{C}_{\mathcal{F}}$ generated by the vectors $p-\mathcal{L}_0$ for $p\in\mathcal{AC}^G(X)$.

We now introduce some notation to describe this cone. Consider the quotient $\pi : X^{\rm ss}(\mathcal{L}) \longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$. Let x be any point in $X^{\rm ss}(\mathcal{L})$ with closed orbit in $X^{\rm ss}(\mathcal{L})$, and so, reductive isotropy G_x. Then, the fiber $\pi^{-1}(\pi(x))$ is isomorphic to a fiber product $G\times_{G_x} L$, for an affine G_x-variety L with a fixed point as unique closed orbit. Let $X(G_x)$ denote the group of characters of G_x. Consider the semicone $\mathcal{C}_x$ in $X(G_x)\otimes\mathbb{Q}$ generated by the weights of G_x on the set of regular functions on L. Finally, we consider is the linear map $\mu : \Lambda_{\mathbb{Q}} \longrightarrow X(G_x)\otimes\mathbb{Q}$ obtained by considering the action of G_x on the fibers $\mathcal{L}_x$ in $\mathcal{L}\in\Lambda$ over x.

Theorem B *With above notation, we have:*

$$\mathcal{C}_{\mathcal{F}} = \mu^{-1}(\mathcal{C}_x).$$

In the symplectic setting, S. Sjamaar obtained a description of the local structure of the moment polytope (see [Sja98]) which is closed from Theorem B. In Sjamaar's situation, G_x° is a torus which simplifies a little bit.

Now, assume that the variety X equals $Y \times G/B$, for a G-variety Y. Let $\mathcal{L}$ be an ample G-linearized line bundle on Y. Let Λ be the subgroup of $\mathrm{Pic^G(X)}$ generated by the pullback of $\mathcal{L}$ and the pullbacks of the G-linearized line bundles on G/B. Then, $\mathcal{TC}^G_\Lambda(X)$ is a cone over the moment polytope $P(Y,\mathcal{L})$ defined in [Bri99]; in particular, the faces of $\mathcal{TC}^G_\Lambda(X)$ correspond bijectively to the faces of $P(Y,\mathcal{L})$.

Following [Bri99], we show in Proposition 7 below, that any moment polytope $P(Y,\mathcal{L})$ can be describe in terms of one which intersects the interior of the dominant chamber. We now assume that $P(Y,\mathcal{L})$ intersects the interior of the dominant chamber and that Y is smooth. In Proposition 8 below, we associate to each face of $P(Y,\mathcal{L})$ which intersects the interior of the dominant chamber a well B-covering pair (see Definition 7.2) of Y improving (with stronger assumptions) [Bri99, Theorem 1 and 2].

Now, G is assumed to be embedded in another connected reductive group $\hat{G}$. We fix maximal tori $T \subset \hat{T}$ and Borel subgroups $B \supset T$ and $\hat{B} \supset \hat{T}$ of G and $\hat{G}$. Consider the diagonal action of G on $\hat{G}/\hat{B} \times G/B$ and the associated GIT-cone $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$. Actually, $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$ identifies with the cone generated by pairs $(\nu, \hat{\nu})$ of strictly dominant character of $T \times \hat{T}$ such that the dual of the G-module associated to ν can be G-equivariantly embedded in the $\hat{G}$-module associated to $\hat{\nu}$. The interior of $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$ is non empty if and only if no non trivial connected normal subgroup of G is normal in $\hat{G}$: we assume, from now on that $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$ has non empty interior. Theorem 5 below gives a bijective parametrisation of the faces of $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$. Moreover, we can read very easily the inclusions between faces using this parametrisation. To avoid too many notation, in this introduction, we will only state our results in the case when $\hat{G} = G^2$.

For any standard parabolic subgroup P of G, we consider the cohomology group $\mathrm{H}^*(G/P,\mathbb{Z})$ and its basis consisting in classes of Schubert varieties. We consider on this group the Belkale-Kumar product $\odot_0$ defined in [BK06]. The coefficient-structure of this product in this basis are either 0 or the coefficient-structure of the usual cup product. These coefficients are parametrized by the triple of Schubert classes.

Theorem C *The group G is assumed to be semi-simple. The set of faces of $\mathcal{AC}^G((G/B)^3)$ correspond bijectively to the set of structure coefficient of $(\mathrm{H}^*(G/P,\mathbb{Z}),\odot_0)$ equal to one, for the various standard parabolic subgroups P of G.*

We will now explain how to read the inclusion off this parametrization. Let P and P' be two standard parabolic subgroups. Let Λ_1, Λ_2 and Λ_3

(resp. Λ'_1, Λ'_2 and Λ'_3) three Schubert varieties in G/P (resp. G/P') such the corresponding coefficients structure for $\odot_0$ equal to one. Let $\mathcal{F}$ and $\mathcal{F}'$ denote the corresponding faces of $\mathcal{AC}^G((G/B)^3)$.

Theorem D *Let $\mathcal{F}$ and $\mathcal{F}'$ be two faces of $\mathcal{AC}^G((G/B)^3)$. The following are equivalent:*

(i) $\mathcal{F} \subset \mathcal{F}'$;

(ii) $P \subset P'$ *and* $\pi(\Lambda_i) = \Lambda'_i$ *for* $i = 1, 2$ *and* 3*, where* $\pi : G/P \longrightarrow G/P'$ *is the* G*-equivariant morphism mapping* P/P *on* P'/P'*.*

Convention. The ground field $\mathbb{K}$ is assumed to be algebraically closed of characteristic zero. The notation introduced in the environments "**Notation.**" are fixed for all the sequence of the article.

2 An example of GIT-cone

Let us fix a connected reductive group G acting on an irreducible projective algebraic variety X.

2.1 An Ad Hoc notion of semistability

As in the introduction, for any G-linearized line bundle $\mathcal{L}$ on X, we consider the following set of *semistable points*:

$$X^{\rm ss}(\mathcal{L}) = \left\{ x \in X \,:\, \exists n > 0 \text{ and } \sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0 \right\}.$$

To precise the acting group, we sometimes denote $X^{\rm ss}(\mathcal{L})$ by $X^{\rm ss}(\mathcal{L}, G)$.

The subset $X^{\rm ss}(\mathcal{L})$ is open and stable by G. Note that this definition of $X^{\rm ss}(\mathcal{L})$ is the standard one only when $\mathcal{L}$ is ample. Indeed, one usually imposes that the open subset defined by the non vanishing of σ to be affine.

If $\mathcal{L}$ is ample, there exists a categorical quotient:

$$\pi \,:\, X^{\rm ss}(\mathcal{L}) \longrightarrow X^{\rm ss}(\mathcal{L})/\!/G,$$

such that $X^{\rm ss}(\mathcal{L})/\!/G$ is a projective variety and π is affine. A point $x \in X^{\rm ss}(\mathcal{L})$ is said to be *stable* if G_x is finite and $G.x$ is closed in $X^{\rm ss}(\mathcal{L})$. Then, for all stable point x we have $\pi^{-1}(\pi(x)) = G.x$; and the set $X^{\rm s}(\mathcal{L})$ of stable points is open in X.

2.2 Definitions

Let us recall from the introduction that Λ is a freely finitely generated subgroup of $\mathrm{Pic^G(X)}$ and $\Lambda_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector space containing Λ as a lattice. Since $X^{\mathrm{ss}}(\mathcal{L}) = X^{\mathrm{ss}}(\mathcal{L}^{\otimes n})$, for any G-linearized line bundle and any positive integer n, we can define $X^{\mathrm{ss}}(\mathcal{L})$ for any $\mathcal{L} \in \Lambda_{\mathbb{Q}}$. We consider the following *total G-cone*:

$$\mathcal{TC}^G_\Lambda(X) = \{\mathcal{L} \in \Lambda_{\mathbb{Q}} \ : \ X^{\mathrm{ss}}(\mathcal{L}) \text{ is not empty}\}.$$

Since the tensor product of two non zero G-invariant sections is a non zero G-invariant section, $\mathcal{TC}^G_\Lambda(X)$ is a convex cone.

Consider the convex cones $\Lambda^+_{\mathbb{Q}}$ and $\Lambda^{++}_{\mathbb{Q}}$ generated respectively by the semiample and ample elements of Λ. For all $\mathcal{L} \in \Lambda^+_{\mathbb{Q}}$ (resp. $\Lambda^{++}_{\mathbb{Q}}$), there exists a positive integer n such that $\mathcal{L}^{\otimes n}$ is a semiample (resp. ample) G-linearized line bundle on X in Λ. So, any set of semistable points associated to a point in $\Lambda^+_{\mathbb{Q}}$ (resp. $\Lambda^{++}_{\mathbb{Q}}$) is in fact a set of semistable point associated to a semiample (resp. ample) G-linearized line bundle. We consider the following *semiample and ample G-cones*:

$$\mathcal{SAC}^G_\Lambda(X) = \mathcal{TC}^G_\Lambda(X) \cap \Lambda^+_{\mathbb{Q}} \ \text{ and } \ \mathcal{AC}^G_\Lambda(X) = \mathcal{TC}^G_\Lambda(X) \cap \Lambda^{++}_{\mathbb{Q}}.$$

By [DH98] (see also [Res00]), $\mathcal{AC}^G_\Lambda(X)$ is a closed convex rational polyhedral cone in $\Lambda^{++}_{\mathbb{Q}}$. This cone is the central object of this article.

Two points $\mathcal{L}$ and $\mathcal{L}'$ in $\mathcal{AC}^G_\Lambda(X)$ are said to be *GIT-equivalent* if $X^{\mathrm{ss}}(\mathcal{L}) = X^{\mathrm{ss}}(\mathcal{L}')$. An equivalence class is simply called a *GIT-class*.

For $x \in X$, the *stability set of x* is the set of $\mathcal{L} \in \Lambda^{++}_{\mathbb{Q}}$ such that $X^{\mathrm{ss}}(\mathcal{L})$ contains x; it is denoted by $\Omega_\Lambda(x)$ or $\Omega_\Lambda(G.x)$. In [Res00], we have studied the geometry of the GIT-classes and the stability sets with lightly different assumptions (no Λ for example). However all the results and proofs of [Res00] remain valuable here. In particular, there are only finitely many GIT-classes; and each GIT-class is the relative interior of a closed convex polyhedral cone of $\Lambda^{++}_{\mathbb{Q}}$.

2.3 An example of G-ample cone

Notation. If Γ is an affine algebraic group, $[\Gamma, \Gamma]$ will denote its derived subgroup and $X(\Gamma)$ will denote the character group of Γ.

For later use, we consider here a G-ample cone for the action of G over an affine variety. More precisely, let V be an affine G-variety containing a fix point O as unique closed orbit. The action of G over the fiber gives a morphism $\mu^\bullet(O,G)\ :\ \mathrm{Pic^G(V)} \longrightarrow \mathrm{X(G)}$. By [Res00, Lemma 7], $\mu^\bullet(O,G)$ is an isomorphism. We denote by $V^{\rm ss}(\chi)$ the set of semistable points for the G-linearized line bundle $\mathcal{L}_\chi$ associated to $\chi\in X(G)$; that is, the trivial line bundle on V linearized by χ. As in the projective case, we consider the G-ample cone $\mathcal{AC}^G(V)$ in $X(G)\otimes\mathbb{Q}$.

For any $\chi\in X(G)$, we have:

$$\mathrm{H}^0(V,\mathcal{L}_\chi)^G=\{f\in\mathbb{K}[V]\ :\ \forall x\in V\ \ (g.f)(x)=\chi(g)f(x)\}=\mathbb{K}[V]_\chi.$$

Note that $\mathrm{H}^0(V,\mathcal{L}_\chi)^G$ is contained in $\mathbb{K}[V]^{[G,G]}$. Set

$$S=\{\chi\in X(G)\ :\ \mathrm{H}^0(V,\mathcal{L}_\chi)^G\ \text{ is non trivial}\}.$$

It is the set of weights of $G/[G,G]$ in $\mathbb{K}[V]^{[G,G]}$. We have:

Lemma 1 *We assume that V is irreducible. The set S is a finitely generated semigroup in $X(G)$. Moreover, $\mathcal{AC}^G(V)$ is the convex cone generated by S; it is strictly convex.*

Proof. Since $\mathbb{K}[V]^{[G,G]}$ is a finitely generated algebra, S is a finitely generated semigroup. The fact that $\mathcal{AC}^G(V)$ is generated by S is obvious. Finally, $\mathcal{AC}^G(V)$ is strictly convex since $\mathrm{H}^0(V,\mathcal{L}_0)^G=\mathbb{K}$. □

3 Slice Etale Theorem

In this section, we recall some very useful results of D. Luna. We fix an ample G-linearized line bundle $\mathcal{L}$ on the irreducible projective G-variety X.

3.1 Closed orbits in general position

Notation. If H is a subgroup of G, $N_G(H)$ denotes the normalizer of H in G. Consider the quotient $\pi: X^{\rm ss}(\mathcal{L})\longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$. For all $\xi\in X^{\rm ss}(\mathcal{L})/\!/G$, we denote by $T(\xi)$ the unique closed orbit of G in $\pi^{-1}(\xi)$. We denote by $(X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr}$ the set of ξ such that there exists an open neighborhood Ω of ξ in $X^{\rm ss}(\mathcal{L})/\!/G$ such that the orbits $T(\xi')$ are isomorphic to $T(\xi)$, for all $\xi'\in\Omega$.

Since π is a gluing of affine quotients, some results on the actions of G on affine variety remains true for $X^{\rm ss}(\mathcal{L})$. For example, the following theorem is a result of Luna and Richardson (see [LR79, Section 3] and [Lun75, Corollary 4] or [PV91, Section 7]):

Theorem 1 *With above notation, if X is normal, we have:*

(i) The set $(X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$ is a non empty open subset of $X^{\mathrm{ss}}(\mathcal{L})/\!/G$. Let H be the isotropy of a point in $T(\xi_0)$ with $\xi_0 \in (X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$. The group H has fixed points in $T(\xi)$ for any $\xi \in X^{\mathrm{ss}}(\mathcal{L})/\!/G$.

(ii) Let Y be the closure of $\pi^{-1}\left((X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}\right)^H$ in X. It is the union of some components of X^H. Then, H acts trivially on some positive power $\mathcal{L}_{|Y}^{\otimes n}$ of $\mathcal{L}_{|Y}$. Moreover, the natural map

$$Y^{\mathrm{ss}}(\mathcal{L}_{|Y}^{\otimes n})/\!/(N_G(H)/H) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G$$

is an isomorphism. Moreover, Y contains stable points for the action of $N_G(H)/H$ and the line bundle $\mathcal{L}_{|Y}^{\otimes n}$.

A subgroup H as in Theorem 1 will be called *a generic closed isotropy of* $X^{\mathrm{ss}}(\mathcal{L})$. The conjugacy class of H which is obviously unique is called *the generic closed isotropy of* $X^{\mathrm{ss}}(\mathcal{L})$.

3.2 The principal Luna stratum

When X is smooth, the open subset $(X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$ is called the *principal Lana stratum* and has very useful properties (see [Lun73] or [PV91]):

Theorem 2 (Luna) *We assume that X is smooth. Let H be a generic closed isotropy of $X^{\mathrm{ss}}(\mathcal{L})$.*

Then, there exists a H-module L such that for any $\xi \in (X^{\mathrm{ss}}(\mathcal{L})/\!/G)_{\mathrm{pr}}$, there exist points x in $T(\xi)$ satisfying:

(i) $G_x = H$;

(ii) the H-module $T_xX/T_x(G.x)$ is isomorphic to the sum of L and its fix points, in particular, it is independent of ξ and x;

(iii) for any $v \in L$, 0 belongs to the closure of $H.v$;

(iv) the fiber $\pi^{-1}(\xi)$ is isomorphic to $G \times_H L$.

3.3 The fibers of quotient morphisms

Another useful consequence of Luna's Slice Etale Theorem is (see [Lun73] or [PV91]):

Proposition 1 *Let x be a semistable point for $\mathcal{L}$ whose the orbit is closed in $X^{\rm ss}(\mathcal{L})$. Then, there exists an affine G_x-variety V containing a unique closed orbit which is a fixed point and such that $\pi^{-1}(\pi(x))$ is isomorphic to $G\times_{G_x} V$.*

4 About faces of the G-ample cone

4.1 Isotropy subgroups associated to faces of $\mathcal{AC}^G_\Lambda(X)$

Let φ be a linear form on $\Lambda_\mathbb{Q}$ which is non negative on $\mathcal{AC}^G_\Lambda(X)$. If the set of $\mathcal{L}\in\mathcal{AC}^G_\Lambda(X)$ such that $\varphi(\mathcal{L})=0$ is non empty it will be called *a face of* $\mathcal{AC}^G_\Lambda(X)$. Now, we associate two invariants to a face $\mathcal{F}$ of $\mathcal{AC}^G_\Lambda(X)$.

Theorem 3 *Let $\mathcal{F}$ be a face of $\mathcal{AC}^G_\Lambda(X)$. Let $\mathcal{L}$ be a point in the relative interior of $\mathcal{F}$. Then, we have:*

(i) The generic closed isotropy of $X^{\rm ss}(\mathcal{L})$ does not depend on the point $\mathcal{L}$ in the relative interior of $\mathcal{F}$, but only in $\mathcal{F}$. We call this isotropy the generic closed isotropy of $\mathcal{F}$.

Let us fix a generic closed isotropy H of $\mathcal{F}$.

(ii) For any $\mathcal{M}\in\mathcal{F}$, H fixes points in any closed orbit of G in $X^{\rm ss}(\mathcal{M})$.

(iii) The closure Y of $\left(\pi_\mathcal{L}^{-1}\left((X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr}\right)\right)^H$ in X does not depends on a choice of $\mathcal{L}$. Let $Y_\mathcal{F}$ denote this subvariety of X^H; it is the union of some components of X^H.

(iv) The group H acts trivially on some positive power $\mathcal{L}^{\otimes n}_{|Y_\mathcal{F}}$ of $\mathcal{L}_{|Y_\mathcal{F}}$. Moreover, the natural map

$$Y^{\rm ss}_\mathcal{F}(\mathcal{L}^{\otimes n}_{|Y_\mathcal{F}})/\!/(N_G(H)/H)\longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$$

is an isomorphism. Moreover, $Y_\mathcal{F}$ contains stable points for the action of $N_G(H)/H$ and the line bundle $\mathcal{L}^{\otimes n}_{|Y_\mathcal{F}}$.

(v) Set $Y_\mathcal{F}^+:=\{x\in X\ :\ \overline{H.x}\cap Y_\mathcal{F}\neq\emptyset\}$. Then $G.Y^+_\mathcal{F}$ contains an open subset of X.

Proof. Let $\mathcal{L}_1$, $\mathcal{L}_2\in\mathcal{AC}^G_\Lambda(X)$. By an easy argument of convexity, to prove Assertion (i) it is sufficient to prove that the generic closed isotropy of $X^{\rm ss}(\mathcal{L})$ does not depend on $\mathcal{L}$ in the open segment $]\mathcal{L}_1,\mathcal{L}_2[$. Let us fix $\mathcal{L}$, $\mathcal{L}'\in$

$]\mathcal{L}_1, \mathcal{L}_2[$. Let $x \in X$ which maps in $(X^{\rm ss}(\mathcal{M})/\!/G)_{\rm pr}$, for $\mathcal{M} = \mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}$ and $\mathcal{L}'$ by the quotient maps.

Recall that $\Omega_\Lambda(x)$ is a polyhedral convex cone. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ belong to $\Omega_\Lambda(x)$, $\mathcal{L}$ and $\mathcal{L}'$ belong to the relative interior of the same face of $\Omega_\Lambda(x)$. By [Res00, Proposition 6, Assertion (iii)], there exists $x' \in \overline{G.x}$ such that this face is $\Omega_\Lambda(x')$. But, [Res00, Proposition 6, Assertion (i)] shows that the closed orbits in $X^{\rm ss}(\mathcal{L}) \cap \overline{G.x'}$ and $X^{\rm ss}(\mathcal{L}') \cap \overline{G.x'}$ are equal. Now, our choice of the point x implies that the generic closed isotropies of $X^{\rm ss}(\mathcal{L})$ and $X^{\rm ss}(\mathcal{L}')$ are equal.

Let H be a generic closed isotropy of $X^{\rm ss}(\mathcal{L})$. Let Y be the closure of $X^H \cap \pi_{\mathcal{L}}^{-1}((X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr})$. By Theorem 1, $N_G(H)$ acts transitively on the set of irreducible components of Y. Let Y_1 be such a component of X^H. Again by Theorem 1, $\pi_{\mathcal{L}}(Y_1 \cap X^{\rm ss}(\mathcal{L})) = X^{\rm ss}(\mathcal{L})/\!/G$; that is, any closed G-orbit in $X^{\rm ss}(\mathcal{L})$ intersects Y_1. Finally, Y is the union of irreducible components of X^H which intersect a general closed G-orbit in $X^{\rm ss}(\mathcal{L})$. But, the above proof of Assertion (i) shows that a general closed orbit in $X^{\rm ss}(\mathcal{L})$ is also a closed orbit in $X^{\rm ss}(\mathcal{L}')$. In particular, Y is the closure of $X^H \cap \pi_{\mathcal{L}'}^{-1}((X^{\rm ss}(\mathcal{L}')/\!/G)_{\rm pr})$. Assertion (iii) follows.

Let us now fix a generic closed isotropy H of $\mathcal{F}$. Let $\mathcal{M}_1 \in \mathcal{F}$. By Assertion (i) of Theorem 1, to prove the second assertion, it is sufficient to prove that the generic closed isotropy of $X^{\rm ss}(\mathcal{M}_1)$ contains H. By [Res00, Theorem 4], there exists a point $\mathcal{M}_2$ in the relative interior of $\mathcal{F}$ such that $X^{\rm ss}(\mathcal{M}_1)$ contains $X^{\rm ss}(\mathcal{M}_2)$. The inclusion $X^{\rm ss}(\mathcal{M}_2) \subset X^{\rm ss}(\mathcal{M}_1)$ induces a surjective morphism $\eta : X^{\rm ss}(\mathcal{M}_2)/\!/G \longrightarrow X^{\rm ss}(\mathcal{M}_1)/\!/G$. Let $\xi' \in (X^{\rm ss}(\mathcal{M}_2)/\!/G)_{\rm pr}$ such that $\xi = \eta(\xi') \in (X^{\rm ss}(\mathcal{M}_1)/\!/G)_{\rm pr}$. Let x be a point in the closed G-orbit in $X^{\rm ss}(\mathcal{M}_1)$ over ξ. The fiber in $X^{\rm ss}(\mathcal{M}_1)$ over ξ is fibered over $G.x$; hence, for any y in this fiber, G_y is conjugated to a subgroup of G_x. Since this fiber contains the fiber in $X^{\rm ss}(\mathcal{M}_2)$ over ξ', H is conjugated to a subgroup of G_x. The second assertion is proved.

From now on, $\mathcal{L}$ is a point in the relative interior of $\mathcal{F}$. Let Y be the subvariety of X^H of Assertion (iii). By Theorem 1, Y satisfies Assertion (iv). Moreover, $G.Y^+$ contains $\pi_{\mathcal{L}}^{-1}((X^{\rm ss}(\mathcal{L})/\!/G)_{\rm pr})$; and, Assertion (v) is proved.
□

4.2 Local structure of the G-ample cone around a face

Notation. Let E be a prime Cartier divisor on a variety X endowed with a line bundle $\mathcal{L}$ and σ be a rational section for $\mathcal{L}$. We will denote by $\nu_E(\sigma) \in \mathbb{Z}$ the order of zero of σ along E.

Let $\mathcal{P}$ be a polyhedron in a rational vector space V and $\mathcal{F}$ be a face of $\mathcal{P}$. Let x in the relative interior of $\mathcal{F}$. The cone of V generated by the vectors $y - x$ for $y \in \mathcal{P}$ does not depends on the choice of x in the relative interior of $\mathcal{F}$ and will be called *the cone of* $\mathcal{P}$ *viewed from* $\mathcal{F}$. This cone carries the geometry of $\mathcal{P}$ in a neighborhood of x.

Let $\mathcal{F}$ be a face of $\mathcal{AC}^G_\Lambda(X)$. Let $\mathcal{L}$ in the relative interior of $\mathcal{F}$. Let x be a semistable point for $\mathcal{L}$ whose the orbit is closed in $X^{\rm ss}(\mathcal{L})$. Let V be an affine G_x-variety satisfying Proposition 1. Consider the cone $\mathcal{AC}^{G_x}(V)$ as in Section 2.3. Notice that V is not necessarily irreducible, and so $\mathcal{C}^{G_x}(V)$ is not necessarily convex.

The action of G_x on the fiber over x defines a morphism $\mu^\bullet(x, G_x)$: $\Lambda \longrightarrow X(G_x)$; and so, a linear map from $\Lambda_\mathbb{Q}$ on $X(G_x)_\mathbb{Q}$ also denoted by $\mu^\bullet(x, G_x)$.

Theorem 4 *With above notation, the cone of $\mathcal{AC}^G(X)$ viewed from $\mathcal{F}$ equals $(\mu^\bullet(x, G_x))^{-1}(\mathcal{AC}^{G_x}(V))$. In particular, if $\mu^\bullet(x, G_x)$ is surjective then $\mathcal{AC}^{G_x}(V)$ is convex.*

Proof. Let $\mathcal{L}_0$ and $\mathcal{L}$ be two different ample G-linearized line bundles in Λ. We assume that $\mathcal{L}_0$ is the only point in the segment $[\mathcal{L}; \mathcal{L}_0]$ which belongs to $\mathcal{AC}^G_\Lambda(X)$. For convenience, we set $U = X^{\rm ss}(\mathcal{L}_0)$. By assumption, there is no G-invariant rational section of $\mathcal{L}$ which is regular on X; we claim that there is no such section which is regular on U.

Let us prove the claim. Let us fix a non zero regular G-invariant section σ_0 of $\mathcal{L}_0^{\otimes m}$ for a positive integer m. Let σ be a G-invariant rational section of $\mathcal{L}$ which is regular on U. For any positive integer k, $\sigma \otimes \sigma_0^{\otimes k}$ is a rational G-invariant section of $\mathcal{L} \otimes \mathcal{L}_0^{\otimes mk}$ which is regular on U. Let E be an irreducible component of codimension one of $X - U$. By definition of U, σ_0 is zero along E; and, $\nu_E(\sigma_0) > 0$. Then, $\nu_E(\sigma \otimes \sigma_0^{\otimes k}) = \nu_E(\sigma) + k.\nu_E(\sigma_0)$ is positive for k big enough. We deduce that $\sigma \otimes \sigma_0^{\otimes k}$ is regular on X for k big enough. Since by assumption $\mathcal{L} \otimes \mathcal{L}_0^{\otimes mk}$ does not belong to $\mathcal{AC}^G_\Lambda(X)$, this implies that $\sigma \otimes \sigma_0^{\otimes k}$ and finally σ are zero. The claim is proved.

We now fix a point $\mathcal{L}_0$ in the relative interior of $\mathcal{F}$ as in the statement. By an elementary argument of convexity, there exists an open neighborhood

Ω of $\mathcal{L}_0$ in $\Lambda_{\mathbb{Q}}$ such that

(i) for any $\mathcal{L} \in \Omega$, if $\mathcal{L}$ does not belong to $\mathcal{AC}^G_\Lambda(X)$ then $\mathcal{L}_0$ is the only point in $[\mathcal{L}, \mathcal{L}_0] \cap \mathcal{AC}^G_\Lambda(X)$.

By [Res00, Proposition 2.3], we may also assume that for all $\mathcal{L} \in \Omega$, $X^{\rm ss}(\mathcal{L})$ is contained in $X^{\rm ss}(\mathcal{L}_0)$. It remains to prove that for any $\mathcal{L} \in \Omega$, $\mathcal{L} \in \mathcal{AC}^G_\Lambda(X)$ if and only if $\mu^{\mathcal{L}}(x, G_x) \in \mathcal{AC}^{G_x}(V)$.

Let $\mathcal{L} \in \Omega$ which does not belongs to $\mathcal{AC}^G_\Lambda(X)$. Set $\xi = \pi_{\mathcal{L}_0}(x)$. By the beginning of the proof, for any positive n, $\mathrm{H}^0(U, \mathcal{L}^{\otimes n})^G = \{0\}$. Since $\pi^{-1}_{\mathcal{L}_0}(\xi)$ is closed in U, this implies that $\mathrm{H}^0(\pi^{-1}_{\mathcal{L}_0}(\xi), \mathcal{L}^{\otimes n})^G = \{0\}$ for all positive n. So, for all positive n, $\mathrm{H}^0(V, \mathcal{L}^{\otimes n}_{|V})^{G_x} = \{0\}$; and, so $\mu^{\mathcal{L}}(x, G_x)$ does not belong to $\mathcal{AC}^{G_x}(V)$.

Let now $\mathcal{L} \in \Omega \cap \mathcal{AC}^G_\Lambda(X)$. Since the map $\phi : X^{\rm ss}(\mathcal{L})/\!/G \longrightarrow X^{\rm ss}(\mathcal{L}_0)/\!/G$ induced by the inclusion $X^{\rm ss}(\mathcal{L}) \subset X^{\rm ss}(\mathcal{L}_0)$ is surjective, there exists $y \in X^{\rm ss}(\mathcal{L})$ such that $\phi \circ \pi_{\mathcal{L}}(y) = \xi$. Up to changing y by $g.y$ for some $g \in G$, one may assume that $y \in V$. Let σ be a G-invariant section of $\mathcal{L}$ which is non zero at y. Obviously, the restriction of σ is a G_x-invariant section of $\mathcal{L}_{|V}$ which is non zero. It follows that $\mu^{\mathcal{L}}(x, G_x)$ belongs to $\mathcal{AC}^{G_x}(V)$.

The last assertion follows from an obvious argue of convexity. $\square$

5 Well covering pairs

5.1 The functions $\mu^\bullet(x, \lambda)$

Let $\mathcal{L} \in \mathrm{Pic^G(X)}$. Let x be a point in X and λ be a one parameter subgroup of G. Since X is complete, $\lim_{t\to 0} \lambda(t)x$ exists; let z denote this limit. The image of λ fixes z and so the group $\mathbb{K}^*$ acts via λ on the fiber $\mathcal{L}_z$. This action defines a character of $\mathbb{K}^*$, that is, an element of $\mathbb{Z}$ denoted by $\mu^{\mathcal{L}}(x, \lambda)$.

The numbers $\mu^{\mathcal{L}}(x, \lambda)$ are used in [MFK94] to give a numerical criterion for stability with respect to an ample $G-$linearized line bundle $\mathcal{L}$:

$$x \in X^{\rm ss}(\mathcal{L}) \iff \mu^{\mathcal{L}}(x, \lambda) \leq 0 \text{ for all one parameter subgroup } \lambda,$$
$$x \in X^{\rm s}(\mathcal{L}) \iff \mu^{\mathcal{L}}(x, \lambda) < 0 \text{ for all non trivial } \lambda.$$

5.2 Definition

Notation. The set of fix points of the image of λ will be denoted by X^λ; the centralizer of this image will be denoted by G^λ. We consider the following parabolic subgroup of G:

$$P(\lambda) = \left\{ g \in G \;:\; \lim_{t \to 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

Let C be an irreducible component of X^λ. Since G^λ is connected, C is a G^λ-stable closed subvariety of X. We set:

$$C^+ := \{x \in X \;:\; \lim_{t \to 0} \lambda(t)x \in C\}.$$

Then, C^+ is a locally closed subvariety of X stable by $P(\lambda)$. Moreover, the map $p_\lambda \;:\; C^+ \longrightarrow C$, $x \longmapsto \lim_{t \to 0} \lambda(t)x$ is a morphism satisfying:

$$\forall (l, u) \in G^\lambda \times U(\lambda) \qquad p_\lambda(lu.x) = lp_\lambda(x).$$

Consider over $G \times C^+$ the action of $G \times P(\lambda)$ given by the formula (with obvious notation):

$$(g, p).(g', y) = (gg'p^{-1}, py).$$

Since the quotient map $G \longrightarrow G/P(\lambda)$ is a Zariski-locally trivial principal $P(\lambda)$-bundle; one can easily construct a quotient $G \times_{P(\lambda)} C^+$ of $G \times C^+$ by the action of $\{e\} \times P(\lambda)$. The action of $G \times \{e\}$ induces an action of G on $G \times_{P(\lambda)} C^+$.

Definition. Consider the following G-equivariant map

$$\begin{array}{cccc} \eta \;: & G \times_{P(\lambda)} C^+ & \longrightarrow & X \\ & [g : x] & \longmapsto & g.x. \end{array}$$

The pair (C, λ) is said to be *covering* (resp. *dominant*) if η is birational (resp. dominant). It is said to be *well covering* if η induces an isomorphism from $G \times_{P(\lambda)} \Omega$ onto a $P(\lambda)$-stable open subset of X for an open subset Ω of C^+ intersecting C.

5.3 Face associated to (C, λ)

Let us denote by $\mu^\bullet(C, \lambda)$, the common value of the $\mu^\bullet(x, \lambda)$, for $x \in C^+$.

Lemma 2 *Let (C, λ) be a dominant pair. The set of $\mathcal{L} \in \mathcal{AC}^G_\Lambda(X)$ such that $\mu^{\mathcal{L}}(C, \lambda) = 0$ is either empty or a face $\mathcal{F}$ of $\mathcal{AC}^G_\Lambda(X)$. Moreover, $\mathcal{F}$ is the set of $\mathcal{L} \in \mathcal{AC}^G_\Lambda(X)$ such that $X^{\rm ss}(\mathcal{L})$ intersects C.*

From now on, $\mathcal{F}$ which only depends on C is denoted by $\mathcal{F}(C)$.

Proof. The first assertion is [Res07, Lemma 7]. If $\mathcal{L} \in \mathcal{F}$, then there exists $x \in C^+$ semistable for $\mathcal{L}$. By [Res07, Lemma 4], $\lim_{t\to 0} \lambda(t)x$ is semistable and belongs to C. Conversely, assume that $X^{\rm ss}(\mathcal{L}) \cap C$ contains z. Since z is fixed by λ, $\mu^{\mathcal{L}}(z, -\lambda) = -\mu^{\mathcal{L}}(z, \lambda)$. But z is semistable, so $\mu^{\mathcal{L}}(z, -\lambda) = 0$. □

6 The case $X = Y \times G/B$

In this section, we assume that $X = Y \times G/B$, with a normal projective G-variety Y. Moreover, we assume that Λ is abundant (see [Res07]).

6.1 General closed isotropy and well covering pairs

Let S be a torus contained in G. Let C be an irreducible component of X^S.
Definition. The pair (C, S) is said to be *admissible* if there exists $x \in C$ such that $G_x^\circ = S$. The pair is said to be *well covering* if there exists a one parameter subgroup λ of S, such that C is an irreducible component of X^λ and (C, λ) is well covering.

A rephrasing of [Res07, Corollary 3] is

Proposition 2 *We assume that $X = Y \times G/B$ with a normal projective G-variety Y and Λ is abundant (see [Res07, Section 7.4]). Let $\mathcal{F}$ be a face of $\mathcal{AC}_\Lambda^G(X)$ of codimension r. Then, there exists an admissible well covering pair (C, S) with S of dimension r such that $\mathcal{F} = \mathcal{F}(C)$.*

We are now interested in the generic closed isotropy of faces of $\mathcal{AC}_\Lambda^G(X)$:

Proposition 3 *Let $\mathcal{F}$ be a face of codimension codimension r. Let (C, S) be an admissible well covering pair with a r-dimensional torus S such that $\mathcal{F}(C) = \mathcal{F}$.*

There exists a generic closed isotropy H of $\mathcal{F}$ such that $H^\circ = S$.

Proof. By Lemma 2, $\mathcal{F}$ is an union of GIT-classes. By [Res00], there are only finitely many such classes and they are convex; so, they exists a GIT-class F which spans $\mathcal{F}$. Let $\mathcal{L} \in F$. Let $\xi \in (X^{\rm ss}(\mathcal{L})//G)_{\rm pr}$ and $\mathcal{O}_\xi$ be the corresponding closed G-orbit in $X^{\rm ss}(\mathcal{L})$.

By [Res07, Theorem 5], $\mathcal{O}_\xi$ intersects C. Let us fix $x \in O_\xi \cap C$. Now, consider the morphism $\mu^\bullet(x, G_x) : \Lambda_{\mathbb{Q}} \longrightarrow X(G_x) \otimes \mathbb{Q}$ induced by restriction and the isomorphism $X(G_x) \simeq {\rm Pic}^G(\mathcal{O}_\xi)$. By Theorem 4, ${\rm Ker}\mu^\bullet(x, G_x)$ is contained in Span($\mathcal{F}$). On the other hand, the GIT-class of $\mathcal{L}$ is contained in Ker$\mu^\bullet(x, G_x)$. Finally, Ker$\mu^\bullet(x, G_x)$ = Span($\mathcal{F}$). Since Λ is abundant, this implies that the rank of $X(G_x)$ equals r.

Since $G.x$ is affine, G_x is reductive. Since $X = Y \times G/B$, G_x is contained in a Borel subgroup of G. Finally, G_x is diagonalisable. But the rank of $X(G_x)$ equals r; and, G_x° is a r-dimensional torus.

Since $x \in C$, S is contained in G_x; it follows that $G_x^\circ = S$. □

6.2 Unicity

Notation. Let S be a torus. We will denote $Y(S)$ the group of one parameter subgroups of S. There is a natural perfect paring $Y(S) \times X(S) \longrightarrow \mathbb{Z}$ denoted by $\langle \cdot, \cdot \rangle$.

The following proposition is a first statement of unicity.

Lemma 3 *We assume that Y (and so X) is smooth. Let $\mathcal{F}$ be a face of codimension r. Let (C_1, S_1) and (C_2, S_2) be two well covering pairs with two r-dimensional tori S_1 and S_2 such that $\mathcal{F}(C_1) = \mathcal{F}(C_2) = \mathcal{F}$.*

Then, the there exists $g \in G$ such that $g.C_2 = C_1$ and $g.S_2.g^{-1} = S_1$.

Proof. Starting the proof as Proposition 3, we obtain that $\mathcal{O}_\xi$ intersects C_1 and C_2. Up to conjugacy, we may assume that x belongs to $\mathcal{O}_\xi \cap C_1 \cap C_2$. So, we obtain that $G_x^\circ = S_1 = S_2$. Then, C_1 equals C_2 since they are the irreducible component of $X^{S_1} = X^{S_2}$ containing x. □

Let us fix a face $\mathcal{F}$ of codimension r. The set of linear forms $\varphi \in$ Hom$(\Lambda_\mathbb{Q}, \mathbb{Q})$ such that φ is non negative on $\mathcal{AC}^G_\Lambda(X)$ and zero on $\mathcal{F}$ is denote by $\mathcal{F}^\vee$.

Let (C, S) be an admissible pair where S has dimension r and set $\mathcal{F} = \mathcal{F}(C)$. Let $\mathcal{C}$ denote the set of $\lambda \in Y(S) \otimes \mathbb{Q}$ such that for some positive n, the pair $(C_{n\lambda}, n\lambda)$ is dominant; where, $C_{n\lambda}$ denote the irreducible component of $X^{n\lambda}$ containing C.

Lemma 4 *We assume that Y is smooth.*

Then, $\lambda \in \mathcal{C}$ if and only if $\mu^\bullet(C, \lambda) \in \mathcal{F}^\vee$.

Proof. Let $\lambda \in \mathcal{C}$ and n be a positive integer such that $n\lambda \in Y(S)$. Since $(C_{n\lambda}, n\lambda)$ is dominant, [Res07, Lemma 7] imply that $\mu^\bullet(C, \lambda)$ is non negative on $\mathcal{AC}^G_\Lambda(X)$. Moreover, for any $\mathcal{L} \in \mathcal{F}$, $X^{\rm ss}(\mathcal{L})$ intersects C. This implies that $\mu^{\mathcal{L}}(C, \lambda) = 0$. Finally, $\mu^\bullet(C, \lambda) \in \mathcal{F}^\vee$.

Conversely, let λ be a rational one parameter subgroup and n be a positive integer such that $n\lambda \in Y(S)$ and $\mu^\bullet(C,\lambda) \in \mathcal{F}^\vee$. Set $C^+ = \{x \in X : \lim_{t\to 0} n\lambda(t)x \in C_{n\lambda}\}$ and $\eta : G\times_{P(n\lambda)} C^+ \longrightarrow X$. Let us fix a generic point $x \in C$. Then, G_x is the generic closed isotropy of $\mathcal{F}$, its neutral component is S and the G_x-module $T_xX/T_xG.x$ is the type of $\mathcal{F}$. Theorem 4 implies that $\mu^\bullet(C,\lambda) \in \mathcal{F}^\vee$ if and only if $\langle n\lambda, \cdot\rangle$ is non negative on all weights of S in $T_xX/T_xG.x$. We deduce that $T\eta_x$ is surjective. Since Y is smooth, this implies that η is dominant. □

We can now state our main result of unicity:

Proposition 4 *We assume that Y is smooth. Let $p_{G/B} : X \longrightarrow G/B$ denote the projection. Let us fix a Borel subgroup B of G and a maximal torus T of B.*

Let $\mathcal{F}$ be a face of codimension r. Then there exists a unique well covering pair (C,S) where S is a r-dimensional subtorus of T, $p_{G/B}(C)$ contains B/B and $\mathcal{F}(C) = \mathcal{F}$.

Let λ_1 and λ_2 be two one parameter subgroups of S such that (C,λ_i) is dominant and $\mathcal{F}$ equals the set of $\mathcal{L} \in \mathcal{AC}^G_\Lambda(X)$ such that $\mu^{\mathcal{L}}(C,\lambda_i) = 0$ for $i = 1,2$. Then, $P(\lambda_1) = P(\lambda_2)$, C is an irreducible component of X^{λ_1} and X^{λ_2} and $C^+(\lambda_1) = C^+(\lambda_2)$.

Proof. Let (C_1,S_1) and (C_2,S_2) be two admissible well covering pairs with two r-dimensional tori S_1 and S_2 such that $\mathcal{F}(C_1) = \mathcal{F}(C_2) = \mathcal{F}$. We also assume that $p_{G/B}(C_1)$ and $p_{G/B}(C_2)$ contain B/B. By Lemma 3, there exists $g \in G$ such that $gS_2g^{-1} = S_1$ and $gC_2 = C_1$. Note that $g^{-1}Tg$ and T contain S_2 and are maximal tori of G^{S_2}: there exists $h \in G^{S_2}$ such that $hTh^{-1} = g^{-1}Tg$. Set $\tilde{w} = gh$. One easily checks that $\tilde{w}$ normalize T, $\tilde{w}S_2\tilde{w}^{-1} = S_1$ and $\tilde{w}C_2 = C_1$. Now, $G^{S_1}B/B = p_{G/B}(C_1) = \tilde{w}p_{G/B}(C_2) \ni \tilde{w}B/B$. We deduce that $\tilde{w} \in G^{S_1}$. So, $S_2 = \tilde{w}^{-1}S_1\tilde{w} = S_1$ and $C_2 = \tilde{w}^{-1}C_1 = C_1$. The first assertion is proved.

Let now C, S, λ_1 and λ_2 be as in the statement. Let us first prove that $P(\lambda_1) = P(\lambda_2)$. By Lemma 4, the set of the one parameter subgroups λ of S as in the proposition is convex. So, if it is possible to have $P(\lambda_1) \neq P(\lambda_2)$; it is possible to have $P(\lambda_1) \subset P(\lambda_2)$ and $P(\lambda_1) \neq P(\lambda_2)$. In other words, we may assume that $P(\lambda_1) \subset P(\lambda_2)$. Let C_1 (resp. C_2) denote the irreducible component of X^{λ_1} (resp. X^{λ_2}) containing C. By Lemma 3, we have $C_1 = C_2$. In particular, $G^{\lambda_1}B/B = p_{G/B}(C_1) = p_{G/B}(C_2) = G^{\lambda_2}B/B$;

so, $G^{\lambda_1} = G^{\lambda_2}$. It follows that $P(\lambda_1) = P(\lambda_2)$. Let P denote this parabolic subgroup of G.

Let $x \in C$ be general. Since λ_1 fixes x, it acts on the tangent space $T_x X$. Consider the subspaces $(T_x X)_{>0}$ and $(T_x X)_{<0}$ of the $\xi \in T_x X$ such that $\lim_{t\to 0} \lambda_1(t)\xi = 0$ and $\lim_{t\to 0} \lambda_1(t^{-1})\xi = 0$ respectively. We have: $T_x X = (T_x X)_{<0} \oplus (T_x X)_0 \oplus (T_x X)_{>0}$. The first part identify with $T_e G/P$ and the second one with $T_x C$ as S-modules. But, the third part is the unique S-stable supplementary of the sum of the two first one. In particular, the same construction with λ_2 in place of λ_1 gives the same decomposition $T_x X = (T_x X)_{<0} \oplus (T_x X)_0 \oplus (T_x X)_{>0}$. It follows that $C^+(\lambda_1) = C^+(\lambda_2)$. □

6.3 Inclusion of faces

Proposition 5 *We assume that Y is smooth. Let us fix a maximal torus T of G and a Borel subgroup B containing T. Let (C_1, S_1) and (C_2, S_2) be two admissible well covering pairs with two subtori S_1 and S_2 of T of dimension r_1 and r_2 such that $B/B \in p_{G/B}(C_i)$ for $i = 1, 2$. We assume that $\mathcal{F}(C_1)$ and $\mathcal{F}(C_2)$ have respectively codimension r_1 and r_2.*

Then, the following are equivalent:

(i) $\mathcal{F}(C_1) \subset \mathcal{F}(C_2)$;

(ii) $C_1 \subset C_2$ *and* $S_2 \subset S_1$.

Proof. The second assertion implies the first one by Lemma 2. Conversely, let us assume that $\mathcal{F}(C_1) \subset \mathcal{F}(C_2)$.

By Proposition 3, there exists $\mathcal{L} \in \mathcal{F}(C_1)$ and $x \in C_1$ such that $G_x^\circ = S_1$ and $G.x$ is closed in $X^{\rm ss}(\mathcal{L})$. Since C_2 intersects $G.x$, there exists $g \in G$ such that $g.x \in C_2$. Since S_2 fixes $g.x$, $S_2 \subset gS_1g^{-1}$. In particular, S_2 is contained in T and gTg^{-1}; so, T and gTg^{-1} are maximal tori in G^{S_2}. There exists $l \in G^{S_2}$ such that $lTl^{-1} = gTg^{-1}$. The element $n = l^{-1}g$ normalizes T. Since C_2 is stable by G^{S_2} (which is connected), x belongs to $n^{-1}C_2$. Applying $p_{G/B}$ we obtain that $p_{G/B}(x)$ belongs to $n^{-1}G^{S_2}B/B \cap G^{S_1}B/B$.

Since $n^{-1}S_2n = g^{-1}S_2g \subset S_1$, we have $G^{n^{-1}S_2n} \subset G^{S_1}$. In particular, $n^{-1}G^{S_2}B/B \subset G^{S_1}n^{-1}B/B$. It follows that $G^{S_1}n^{-1}B/B = G^{S_1}B/B$.

Since n normalizes T, this implies that $n \in G^{S_1}$. So, $S_2 \subset nS_1n^{-1} = S_1$.

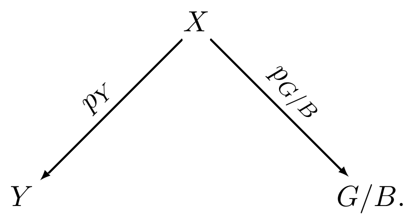
Since $n \in G^{S_1}$, $nx \in C_1$. It follows that C_1 is the irreducible component of X^{S_1} containing nx. On the other hand, C_2 is the irreducible component of X^{S_2} containing nx. It follows that C_1 is contained in C_2. □

7 GIT-cone and moment polytope

Let us now explain the relation mentioned in the introduction between the moment polytopes of Y and some total G-cones of $X = Y \times G/B$.

Let us fix a maximal torus T of G and a Borel subgroup B containing T. Let $\mathcal{L}$ be an ample G-linearized line bundle on Y. We consider the set $P_G(Y, \mathcal{L})$ of the points $p \in X(T)_{\mathbb{Q}}$ such that for some positive integer n, np is a dominant character of T and the dual V_{np}^* of V_{np} is a submodule of $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n})$. In fact, $P_G(Y, \mathcal{L})$ is a polytope, called *moment polytope.* Notice that "the dual" is not usual in the definition; but it will be practical in our context.

Consider the two projections:



In Section 7, Λ will always denote the subgroup of $\mathrm{Pic^G(X)}$ generated by $p_{G/B}^*(\mathrm{Pic^G(G/B)})$ and $p_Y^*(\mathcal{L})$. Note that $\mathrm{Pic^G(G/B)}$ identifies with $X(T)$; we will denote $\mathcal{L}_\nu$ the element of $\mathrm{Pic^G(G/B)}$ associated to $\nu \in X(T)$

Proposition 6 *With above notation, we have:*

(i) $\mathcal{TC}_\Lambda^G(X) = \mathcal{SAC}_\Lambda^G(X)$*;*

(ii) $\mathcal{SAC}_\Lambda^G(X)$ *is a cone over* $P_G(Y, \mathcal{L})$*; more precisely, for all positive rational number* m *and* $\nu \in X(T)$*, we have:*

$$mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu) \in \mathcal{SAC}_\Lambda^G(X) \iff \frac{\nu}{m} \in P_G(Y, \mathcal{L}).$$

Proof. Let n be a non negative integer and ν be a character of T. As a G-module, $\mathrm{H}^0(X, p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu))$ is isomorphic to $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n}) \otimes \mathrm{H}^0(G/B, \mathcal{L}_\nu)$. In particular, if $p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$ has non zero global sections then $n \geq 0$ and ν is dominant; in this case, it is semiample. The first assertion follows.

Assume now that ν is dominant. Then, $\mathrm{H}^0(G/B, \mathcal{L}_\nu) = V_\nu$. Hence, $p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$ has non zero G-invariant sections if and only if $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n}) \otimes$

V_ν contains non zero G-invariant vectors; that is, if and only if V_ν^* is a submodule of $\mathrm{H}^0(Y, \mathcal{L}^{\otimes n})$. The second assertion follows. $\square$

Remark. To each face $\mathcal{F}$ of $\mathcal{AC}_\Lambda^G(X)$, one can associate a face of $P_G(Y, \mathcal{L})$ (by intersecting and taking a closure) which intersects the interior of the dominant chamber. By this way, we obtain a bijection between the set of faces of $\mathcal{AC}_\Lambda^G(X)$ and the faces of $P_G(Y, \mathcal{L})$ which intersects the interior of the dominant chamber.

7.1 A reduction

It is possible that $\mathcal{AC}_\Lambda^G(X)$ is empty. In this case, our results cannot be applied directly.

Let $mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$ in the relative interior of $\mathcal{SAC}^G(X)$ such that $\mathrm{H}^0(X, mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu))^G$ is non zero, that is such that V_ν^* can be equivariantly embedded in $\mathrm{H}^0(Y, \mathcal{L}^{\otimes m})$. Let P be the standard parabolic subgroup of G associated to the face of the dominant chamber containing ν. Let L denote the Levi subgroup of P containing T and D denote its derived subgroup.

The next proposition shows that $\mathcal{SAC}^G(X)$ is always equal to such a cone satisfying $\mathcal{AC}^G(X)^n eq\emptyset$. The proof which is essentially extracted from [Bri99, Section 5] is included for completeness.

Proposition 7 *With above notation, there exists an irreducible component C_Y of Y^D such that a point $\mathcal{L} \in \Lambda_\mathbb{Q}$ belongs to $\mathcal{SAC}^G(X)$ if and only if $\mathcal{L}_{|C}$ belongs to $\mathcal{SAC}^{G^D}(C_Y \times G^D.B/B)$.*

Moreover, $G^D.B/B$ is isomorphic to the variety of complete flags of G^D and $\mathcal{AC}^{G^D}(C_Y \times G^D.B/B)$ is non empty.

Proof. The inclusion $V_\nu^* \subset \mathrm{H}^0(Y, \mathcal{L}^{\otimes m})$ gives a G-equivariant rational map $\phi : Y \longrightarrow \mathbb{P}(V_\nu)$. Let v_ν be a vector of highest weight in V_ν; P is the stabilizer in G of $[v_\nu] \in \mathbb{P}(V_\nu)$. Let $\sigma \in V_\nu^*$ be an eigenvector of the Borel subgroup B^- opposite to B and containing T. Let Q be the stabilizer in G of $[\sigma] \in \mathbb{P}(V_\nu^*)$ and Q^u be its unipotent radical.

Let Y_σ denote the set of $y \in Y$ such that $\sigma(y) \neq 0$. Let W be a L-stable supplementary subspace of $\mathbb{K}.v_\nu$ in V_ν. By $w \longmapsto [v_\nu + w]$, we identify W with an open subspace of $\mathbb{P}(V_\nu)$. Then ϕ induces by restriction $\widetilde{\phi} : Y_\sigma \longrightarrow W$.

Let S be a L-stable supplementary to $T_{[v_\nu]}G.[v_\nu]$ in W (actually, W canonically identify with $T_{[v_\nu]}\mathbb{P}(V_\nu)$). Set $Z = \widetilde{\phi}^{-1}(S)$. By [Bri99, Remark in Section 5], Z is point wise fixed by D and the action of Q^u induces an isomorphism $Q^u \times Z \simeq Y_\sigma$.

Consider $X' = Y \times G/P$. Let Λ' be the subgroup of $\mathrm{Pic^G(X')}$ generated by $p_Y^*(\mathcal{L})$ and $p_{G/P}^*(\mathrm{Pic^G(G/P)})$ (with obvious notation). It is clear that $\mathcal{TC}_\Lambda^G(X)$ identifies with $\mathcal{TC}_{\Lambda'}^G(X')$. Moreover, $\mathcal{AC}_{\Lambda'}^G(X')$ is not empty. Consider a generic closed isotropy H of $\mathcal{TC}_{\Lambda'}^G(X')$ viewed as a face $\mathcal{F}$ of itself. Since $Q^u \times Z \simeq Y_\sigma \subset X^{\rm ss}(mp_Y^*(\mathcal{L}) \otimes p_{G/P}^*(\mathcal{L}_\nu))$, up to conjugacy, one may assume that $D \subset H \subset L$. Since $Y_\sigma \times \{P/P\} \subset X^{\rm ss}(mp_Y^*(\mathcal{L}) \otimes p_{G/P}^*(\mathcal{L}_\nu))$, $X'_\mathcal{F}$ intersects $Z \times \{P/P\}$.

Consider the irreducible component C_Y of Y^D which contains Z. By Theorem 3, for any ample $\mathcal{L} \in \Lambda'$, $X'^{\rm ss}(\mathcal{L})$ intersects $C_Y \times \{P/P\}$ if it is non empty. By continuity, this is also true if $\mathcal{L}$ is only semiample. The proposition follows easily. □

7.2 Faces of $\mathcal{SAC}^G(X)$ if $\mathcal{AC}^G(Y)$ is non empty

From now on, we assume that Y is smooth. We will first adapt the notion of covering and well covering pairs for the situation.

Recall that $T \subset B$ are fixed. Let λ be a one parameter subgroup of T. Set $B(\lambda) = B \cap P(\lambda)$. Let C be an irreducible component of Y^λ and C^+ the associated Bialinicki-Birula cell.

Definition. The pair (C, λ) is said to be B-*covering* if the natural map $\eta\ :\ B \times_{B(\lambda)} C^+ \longrightarrow Y$ is birational. It is said to be *well B-covering* is η induces an isomorphism over an open subset of Y intersecting C.

The proof of the following lemma is obvious.

Lemma 5 *With above notation, the pair (C, λ) is B-covering (resp. well B-covering) if and only if $(C \times G^\lambda B/B)$ is covering (resp. well covering).*

Let us recall that the subtori of T correspond bijectively to the linear subspaces of $X(T)_\mathbb{Q}$. If V is a linear subspace of $X(T)_\mathbb{Q}$, the associated torus is the neutral component of the intersection of kernels of elements in $X(T) \cap V$. If F is a convex part of $X(T)_\mathbb{Q}$, the direction $\mathrm{dir}(F)$ of F is the linear subspace spanned by the differences of two elements of F.

We will denote by $\mathcal{C}^+$ the convex cone in $X(T)_\mathbb{Q}$ generated by the dominant weights. The next proposition is an improvement of [Bri99, Theorem 1]:

Proposition 8 *We keep the above notation and assume that Y is smooth and $P_G(Y,\mathcal{L})$ intersects the interior of the dominant chamber. Let $\mathcal{F}$ be a face of codimension d of $P_G(Y,\mathcal{L})$ which intersect the interior of the dominant chamber. Let S the subtorus of T associated to* $\mathrm{dir}(\mathcal{F})$.

There exists a unique irreducible component C of Y^S and a one parameter subgroup λ of S such that $G^\lambda = G^S$ and (C,λ) is a well B-covering pair such that $\mathcal{F} = P_{G^S}(C,\mathcal{L}_{|C}) \cap \mathcal{C}^+$.

Proof. Let $\widetilde{\mathcal{F}}$ be the face of $\mathcal{AC}^G_\Lambda(X)$ corresponding to $\mathcal{F}$ and r denote its codimension. By Proposition 2, there exists an admissible well covering pair (C_X, S') such that $\widetilde{\mathcal{F}} = \mathcal{F}(C_X)$ and S' is a r-dimensional torus. Up to conjugacy, we may assume that C_X intersects $Y \times B/B$, and S' is contained in T. Let λ be a one parameter subgroup of S' such that (C_X,λ) is well covering. Then, $C_X = C \times G^\lambda B/B$ for some irreducible component C of $Y^{S'}$.

The fact $\widetilde{\mathcal{F}} = \mathcal{F}(C_X)$ readily means that $\mathcal{F} = P_{G^{S'}}(C,\mathcal{L}_{|C}) \cap \mathcal{C}^+$. Since the direction of $P_{G^{S'}}(C,\mathcal{L}_{|C})$ is contained in $X(T)^{S'}$, this implies that $X(T)^S$ is contained in $X(T)^{S'}$. But, S and S' have the same rank, it follows that $S = S'$.

The unicity part is a direct consequence of Proposition 4. $\square$

8 The case $X = \hat{G}/\hat{B} \times G/B$

8.1 Interpretations of the G-cones

From now on, we assume that G is a connected reductive subgroup of a connected reductive group $\hat{G}$. Let us fix maximal tori T (resp. $\hat{T}$) and Borel subgroups B (resp. $\hat{B}$) of G (resp. $\hat{G}$) such that $T \subset B \subset \hat{B} \supset \hat{T} \supset T$.

Let $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ denote the Lie algebras of G and $\hat{G}$ respectively.

We denote by $\mathcal{LR}(\hat{G},G)$ (resp. $\mathcal{LR}^\circ(\hat{G},G)$) the cone of the pairs $(\hat{\nu},\nu) \in X(\hat{T})_\mathbb{Q} \times X(T)_\mathbb{Q}$ such that for a positive integer n, $n\hat{\nu}$ and $n\nu$ are dominant (resp. strictly dominant) weights such that $V_{n\hat{\nu}} \otimes V_{n\nu}$ contains non zero G-invariant vectors.

In this section, X denote the variety $\hat{G}/\hat{B} \times G/B$ endowed with the diagonal action of G. We will apply the results of Section 4 to X with $\Lambda = \mathrm{Pic}^G(X)$. The cones $\mathcal{TC}^G(X)$, $\mathcal{SAC}^G(X)$ and $\mathcal{AC}^G(X)$ will be denoted without the Λ in subscribe. By [Res07, Proposition 9], $\mathcal{LR}^\circ(\hat{G},G) = \mathcal{AC}^G(X) \subset \mathcal{SAC}^G(X) = \mathcal{TC}^G(X) = \mathcal{LR}(\hat{G},G)$. Moreover, if no ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$, by [Res07, Assertion (i) of Theorem 9] $\mathcal{LR}^\circ(\hat{G},G)$ has non empty interior.

8.2 The case $X = \hat{G}/\hat{B} \times G/B$

8.2.1 — Consider the G-module $\hat{\mathfrak{g}}/\mathfrak{g}$. Let $\chi_1, \cdots, \chi_n$ be the set of the non trivial weights of T on $\hat{\mathfrak{g}}/\mathfrak{g}$. For $I \subset \{1, \cdots, n\}$, we will denote by T_I the neutral component of the intersection of the kernels of the χ_i's with $i \in I$. A subtorus of the form T_I is said to be *admissible.*

Let λ be a one parameter subgroup of T. Consider the parabolic subgroups P and $\hat{P}$ of G and $\hat{G}$ associated to λ. Let W_P be the Weyl group of P. The cohomology group $\mathrm{H}^*(G/P, \mathbb{Z})$ is freely generated by the Schubert classes $[\overline{BwP/P}]$ parametrized by the elements $w \in W/W_P$. Since $\hat{P} \cap G = P$, we have a canonical G-equivariant immersion $\iota : G/P(\lambda) \longrightarrow \hat{G}/\hat{P}(\lambda)$; and the corresponding morphism in cohomology ι^*.

Let ρ (resp. ρ^λ) denote the half sum of the positive roots of G (resp. G^λ). Let Φ^+ and $\Phi(P^u)$ denote the set of roots of the groups B and P^u for the torus T. In the same way, we define $\hat{\Phi}^+$ and $\Phi(\hat{P}^u)$. For $\hat{w} \in \hat{W}$, we set:

$$\theta^P := \sum_{\alpha \in \Phi^+ \cap \Phi(P^u)} \alpha \in X(T) \quad \text{and} \quad \theta^{\hat{P}}_{\hat{w}} := \sum_{\alpha \in \hat{w}\hat{\Phi}^+ \cap \Phi(\hat{P}^u)} \alpha \in X(\hat{T}).$$

8.2.2 — Let S be an admissible subtorus of T. All irreducible component C of X^S such that $p_{G/B}(C)$ contains B/B equals $C(\hat{w}) := (\hat{G}^S.\hat{w}^{-1}\hat{B}/\hat{B} \times G^S B/B)$ for a unique element $\hat{w} \in \hat{W}/\hat{W}_{\hat{G}^S}$. Let us fix $\hat{w} \in \hat{W}/\hat{W}_{\hat{G}^S}$. The pair $(S, \hat{w})$ is said to be *admissible* if there exists a parabolic subgroup $\hat{P}$ of $\hat{G}$ such that

(i) there exists $\lambda \in Y(S)$ such that $\hat{P} = \hat{P}(\lambda)$;

(ii) $\hat{G}^S$ is a Levi subgroup of $\hat{P}$;

(iii) G^S is a Levi subgroup of $\hat{P} \cap G =: P$;

(iv) $\iota^*([\overline{\hat{B}\hat{w}\hat{P}/\hat{P}}]).[\overline{BP/P}] = [\mathrm{pt}] \in \mathrm{H}^*(G/P, \mathbb{Z})$;

(v) $(\theta^{\hat{P}}_{\hat{w}})_{|S} = (\theta^P - 2(\rho - \rho^\lambda))_{|S}$.

Lemma 6 *Let S be an admissible subtorus of T and $\hat{w} \in \hat{W}/\hat{W}_{\hat{G}^S}$. The pair $(S, \hat{w})$ is admissible if and only if there exists a one parameter subgroup λ of S such that $C(\hat{w})$ is an irreducible component of X^λ and $(C(\hat{w}), \lambda)$ is a well covering pair.*

Proof. The proof is very analogous to [Res07, Proposition 10]: we leave details to the reader. We prove (using mainly Kleiman's Theorem) that

$\iota^*([\overline{\hat{B}\hat{w}\hat{P}/\hat{P}}]).[\overline{BP/P}] = [\mathrm{pt}] \in \mathrm{H}^*(G/P,\mathbb{Z})$ if and only if η is birational. Now, the condition $(\theta^{\hat{P}}_{\hat{w}})_{|S} = (\theta^P - 2(\rho-\rho^\lambda))_{|S}$ means that S acts trivially on the restriction over C of the determinant bundle of η. □

8.2.3 — To simplify, in the following statement we assume that $\mathcal{AC}^G(X)$ has a non empty interior in $\mathrm{Pic^G(X)}_{\mathbb{Q}}$. In fact, this assumption is equivalent to say that no ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$.

Theorem 5 *We assume that no ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$.*

The map which associates to a pair $(S,\hat{w})$ the set $\mathcal{F}(S,\hat{w}) = \{(\hat{\nu},\nu)\in C^G(X) : \hat{w}\hat{\nu}_{|S} = -\nu_{|S}\}$ is a bijection from the set of admissible pairs onto the set of faces of $\mathcal{AC}^G_\Lambda(X)$. Moreover, the codimension of $\mathcal{F}(S,\hat{w})$ equals the dimension of S.

The following are equivalent:

(i) $\mathcal{F}(S,\hat{w})\subset\mathcal{F}(S',\hat{w}')$;

(ii) $S'\subset S$ *and* $\hat{w}W_{G^{S'}} = \hat{w}'W_{G^{S'}}$.

Proof. Let $(S,\hat{w})$ be an admissible pair. Set $\overline{\mathcal{F}}(S,\hat{w}) = \{(\hat{\nu},\nu)\in\mathcal{LR}(G,\hat{G}) : \hat{w}\hat{\nu}_{|S} = -\nu_{|S}\}$. By [Res07, Theorem 9], $\overline{\mathcal{F}}(S,\hat{w})$ is a face of $\mathcal{LR}(G,\hat{G})$ of codimension $\dim(S)$. In particular, $\widetilde{\mathcal{F}}(S,\hat{w})$ spans the vector subspace of the $(\hat{\nu},\nu)\in X(\hat{T})\times X(T)$ such that $\hat{w}\hat{\nu}_{|S} = -\nu_{|S}\}$. Since $\mathcal{AC}^G_\Lambda(X)$ is the interior of $\mathcal{LR}(G,\hat{G})$, to prove that the map in the theorem is well defined, it is enough to prove that $\overline{\mathcal{F}}(S,\hat{w})$ intersects $\mathcal{AC}^G_\Lambda(X)$. If not, $\overline{\mathcal{F}}(S,\hat{w})$ would be contained in the boundary of the dominant chamber. Its projection on $X(\hat{T})_{\mathbb{Q}}$ or $X(T)_{\mathbb{Q}}$ would be contained in an hyperplane; which is a contradiction.

The surjectivity is a rephrasing of [Res07, Theorem 9 Assertion *(ii)*]. The injectivity is a direct application of Proposition 4.

The last assertion follows from Proposition 5. □

9 Application to the tensor product cone

9.1 — In this section, G is assumed to be semisimple. As above, $T\subset B$ are fixed maximal torus and Borel subgroup of G. We also fix an integer $s\geq 2$ and set $\hat{G} = G^s$, $\hat{T} = T^s$ and $\hat{B} = B^s$. We embed G diagonally in $\hat{G}$. Now, $X = \hat{G}/_hB\times G/B = (G/B)^{s+1}$. Then $\mathcal{AC}^G(X)\cap X(T)^{s+1}$ identifies with the $(s+1)$-uple $(\nu_1,\cdots,\nu_{s+1})\in X(T)^{s+1}$ such that the for n big enough

$n\nu_i$'s are strictly dominant weights and $V_{n\nu_1} \otimes \cdots \otimes V_{n\nu_{s+1}}$ contains a non zero G-invariant vector.

A parabolic subgroup P of G is said to be *standard* if it contains B. We will denote by $Z(P)$ the neutral component of the center of the Levi subgroup of P containing T.

9.2 — In [BK06], Belkale and Kumar defined a new product denoted $\odot_0$ on the cohomology groups $H^*(G/P, \mathbb{Z})$ for any parabolic subgroup P of G. We consider the set Θ of the $(P, \Lambda_{w_0}, \cdots, \Lambda_{w_s})$ where P is a standard parabolic subgroup of G and the Λ_{w_i}'s are $s+1$ Schubert varieties of G/P such that

$$[\Lambda_{w_0}] \odot_0 \cdots \odot_0 [\Lambda_{w_s}] = [\mathrm{pt}].$$

9.3 — In [Res07], Theorem 9 applied to $\hat{G} = G^s$ gives Corollary 5. The same translation of Theorem 5 to this case gives the following:

Theorem 6 *The map which associates to a $(P, \Lambda_{w_0}, \cdots, \Lambda_{w_s}) \in \Theta$ the set $\mathcal{F}(P, \Lambda_{w_0}, \cdots, \Lambda_{w_s})$ of the $(\nu_0, \cdots, \nu_s) \in \mathcal{AC}^G(X)$ such that the restriction of $\sum_i w_i^{-1}\nu_i$ to $Z(P)$ is trivial is a bijection from Θ onto the set of faces of $\mathcal{AC}^G(X)$. Moreover, the codimension of $\mathcal{F}(P, \Lambda_{w_0}, \cdots, \Lambda_{w_s})$ equals the dimension of $Z(P)$.*

The following are equivalent:

(i) $\mathcal{F}(P, \Lambda_{w_0}, \cdots, \Lambda_{w_s}) \subset \mathcal{F}(P', \Lambda_{w'_0}, \cdots, \Lambda_{w'_s}))$;

(ii) $P \subset P'$ and $\pi(\Lambda_{w_i}) = \Lambda_{w'_i}$ for all $i = 0, \cdots, s$ (here, $\pi : G/P \longrightarrow G/P'$ is the natural G-equivariant map).

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N. R.
Université Montpellier II
Département de Mathématiques
Case courrier 051-Place Eugène Bataillon
34095 Montpellier Cedex 5
France
e-mail: `ressayre@math.univ-montp2.fr`