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# Geometric Invariant Theory and Generalized Eigenvalue Problem II

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## Abstract

Let  $G$  be a connected reductive subgroup of a complex connected reductive group  $\hat{G}$ . Fix maximal tori and Borel subgroups of  $G$  and  $\hat{G}$ . Consider the cone  $\mathcal{LR}^\circ(\hat{G}, G)$  generated by the pairs  $(\nu, \hat{\nu})$  of strictly dominant characters such that  $V_\nu$  is a submodule of  $V_{\hat{\nu}}$ . The main result of this article is a bijective parametrisation of the faces of  $\mathcal{LR}^\circ(\hat{G}, G)$ . We also explain when such a face is contained in another one.

In way, we obtain results about the faces of the Dolgachev-Hu's  $G$ -ample cone. We also apply our results to reprove known results about the moment polytopes.

## 1 Introduction

Let  $G$  be a connected reductive subgroup of a complex connected reductive group  $\hat{G}$ . Fix maximal tori and Borel subgroups of  $G$  and  $\hat{G}$ . Consider the cone  $\mathcal{LR}^\circ(\hat{G}, G)$  generated by the pairs  $(\nu, \hat{\nu})$  of strictly dominant characters such that  $V_\nu$  is a submodule of  $V_{\hat{\nu}}$ . This work is a continuation of [Res07]. We obtain results about general GIT-cones and apply it to obtain a bijective parametrisation of the faces of the cone  $\mathcal{LR}^\circ(\hat{G}, G)$ .

Consider a connected reductive group  $G$  acting on a projective variety  $X$ . To any  $G$ -linearized line bundle  $\mathcal{L}$  on  $X$  we associate the following open subset  $X^{\text{ss}}(\mathcal{L})$  of  $X$ :

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \sigma \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\}.$$

The points of  $X^{\text{ss}}(\mathcal{L})$  are said to be *semistable* for  $\mathcal{L}$ . Note that if  $\mathcal{L}$  is not ample, this notion of semistability is not the standard one. In particular, the quotient  $\pi_{\mathcal{L}} : X^{\text{ss}}(\mathcal{L}) \longrightarrow X^{\text{ss}}(\mathcal{L})//G$  is a good quotient, if  $\mathcal{L}$  is ample. In this context, we ask for:

What are the  $\mathcal{L}$ 's with non empty set  $X^{\text{ss}}(\mathcal{L})$  ?

Let us fix a freely finitely generated subgroup  $\Lambda$  of the group  $\text{Pic}^G(X)$  of  $G$ -linearized line bundles on  $X$ . Let  $\Lambda_{\mathbb{Q}}$  denote the  $\mathbb{Q}$ -vector space containing  $\Lambda$  as a lattice. Consider the convex cones  $\mathcal{TC}_{\Lambda}^G(X)$  (resp.  $\mathcal{AC}_{\Lambda}^G(X)$ ) generated in  $\Lambda_{\mathbb{Q}}$  by the  $\mathcal{L}$ 's (resp. the ample  $\mathcal{L}$ 's) in  $\Lambda$  which have non zero  $G$ -invariant sections. By [DH98] (see also [Res00]),  $\mathcal{AC}_{\Lambda}^G(X)$  is a closed convex rational polyhedral cone in the dominant cone of  $\Lambda_{\mathbb{Q}}$ . We are interested in the faces of  $\mathcal{AC}_{\Lambda}^G(X)$  and  $\mathcal{TC}_{\Lambda}^G(X)$ .

We need to introduce a definition due to D. Luna. Assume that  $X$  is smooth. Let  $\mathcal{O}$  be an orbit of  $G$  in  $X$ . For  $x \in \mathcal{O}$ , we consider the action of the isotropy  $G_x$  on the normal space  $N_x$  of  $\mathcal{O}$  in  $X$  at  $x$ . The pair  $(G_x, N_x)$  is called the *type* of the orbit  $\mathcal{O}$  and is defined up to conjugacy by  $G$ . The main part of Theorem 3 is:

**Theorem A** *We assume that  $X$  is smooth. Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_{\Lambda}^G(X)$ .*

*Then, the type of the closed orbit in  $\pi_{\mathcal{L}}^{-1}(\xi)$  for  $\xi \in X^{\text{ss}}(\mathcal{L})//G$  general does not depends on the choice of an ample  $G$ -linearized line bundle  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ .*

*We will call this type the type of  $\mathcal{F}$ .*

Let  $\mathcal{F}$  be a face of  $\mathcal{AC}^G(X)$ . Let  $\mathcal{L}_0$  be any point in the relative interior of  $\mathcal{F}$ . The local geometry  $\mathcal{AC}^G(X)$  around  $\mathcal{F}$  is described by the convex cone  $\mathcal{C}_{\mathcal{F}}$  generated by the vectors  $p - \mathcal{L}_0$  for  $p \in \mathcal{AC}^G(X)$ .

We now introduce some notation to describe this cone. Consider the quotient  $\pi : X^{\text{ss}}(\mathcal{L}) \longrightarrow X^{\text{ss}}(\mathcal{L})//G$ . Let  $x$  be any point in  $X^{\text{ss}}(\mathcal{L})$  with closed orbit in  $X^{\text{ss}}(\mathcal{L})$ , and so, reductive isotropy  $G_x$ . Then, the fiber  $\pi^{-1}(\pi(x))$  is isomorphic to a fiber product  $G \times_{G_x} L$ , for an affine  $G_x$ -variety  $L$  with a fixed point as unique closed orbit. Let  $X(G_x)$  denote the group of characters of  $G_x$ . Consider the semicone  $\mathcal{C}_x$  in  $X(G_x) \otimes \mathbb{Q}$  generated by the weights of  $G_x$  on the set of regular functions on  $L$ . Finally, we consider is the linear map  $\mu : \Lambda_{\mathbb{Q}} \longrightarrow X(G_x) \otimes \mathbb{Q}$  obtained by considering the action of  $G_x$  on the fibers  $\mathcal{L}_x$  in  $\mathcal{L} \in \Lambda$  over  $x$ .

**Theorem B** *With above notation, we have:*

$$\mathcal{C}_{\mathcal{F}} = \mu^{-1}(\mathcal{C}_x).$$

In the symplectic setting, S. Sjamaar obtained a description of the local structure of the moment polytope (see [Sja98]) which is closed from Theorem B. In Sjamaar's situation,  $G_x^{\circ}$  is a torus which simplifies a little bit.

Now, assume that the variety  $X$  equals  $Y \times G/B$ , for a  $G$ -variety  $Y$ . Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $Y$ . Let  $\Lambda$  be the subgroup of  $\text{Pic}^G(X)$  generated by the pullback of  $\mathcal{L}$  and the pullbacks of the  $G$ -linearized line bundles on  $G/B$ . Then,  $\mathcal{TC}_\Lambda^G(X)$  is a cone over the moment polytope  $P(Y, \mathcal{L})$  defined in [Bri99]; in particular, the faces of  $\mathcal{TC}_\Lambda^G(X)$  correspond bijectively to the faces of  $P(Y, \mathcal{L})$ .

Following [Bri99], we show in Proposition 7 below, that any moment polytope  $P(Y, \mathcal{L})$  can be describe in terms of one which intersects the interior of the dominant chamber. We now assume that  $P(Y, \mathcal{L})$  intersects the interior of the dominant chamber and that  $Y$  is smooth. In Proposition 8 below, we associate to each face of  $P(Y, \mathcal{L})$  which intersects the interior of the dominant chamber a well  $B$ -covering pair (see Definition 7.2) of  $Y$  improving (with stronger assumptions) [Bri99, Theorem 1 and 2].

Now,  $G$  is assumed to be embedded in another connected reductive group  $\hat{G}$ . We fix maximal tori  $T \subset \hat{T}$  and Borel subgroups  $B \supset T$  and  $\hat{B} \supset \hat{T}$  of  $G$  and  $\hat{G}$ . Consider the diagonal action of  $G$  on  $\hat{G}/\hat{B} \times G/B$  and the associated GIT-cone  $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$ . Actually,  $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$  identifies with the cone generated by pairs  $(\nu, \hat{\nu})$  of strictly dominant character of  $T \times \hat{T}$  such that the dual of the  $G$ -module associated to  $\nu$  can be  $G$ -equivariantly embedded in the  $\hat{G}$ -module associated to  $\hat{\nu}$ . The interior of  $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$  is non empty if and only if no non trivial connected normal subgroup of  $G$  is normal in  $\hat{G}$ : we assume, from now on that  $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$  has non empty interior. Theorem 5 below gives a bijective parametrisation of the faces of  $\mathcal{AC}^G(\hat{G}/\hat{B} \times G/B)$ . Moreover, we can read very easily the inclusions between faces using this parametrisation. To avoid too many notation, in this introduction, we will only state our results in the case when  $\hat{G} = G^2$ .

For any standard parabolic subgroup  $P$  of  $G$ , we consider the cohomology group  $H^*(G/P, \mathbb{Z})$  and its basis consisting in classes of Schubert varieties. We consider on this group the Belkale-Kumar product  $\odot_0$  defined in [BK06]. The coefficient-structure of this product in this basis are either 0 or the coefficient-structure of the usual cup product. These coefficients are parametrized by the triple of Schubert classes.

**Theorem C** *The group  $G$  is assumed to be semi-simple. The set of faces of  $\mathcal{AC}^G((G/B)^3)$  correspond bijectively to the set of structure coefficient of  $(H^*(G/P, \mathbb{Z}), \odot_0)$  equal to one, for the various standard parabolic subgroups  $P$  of  $G$ .*

We will now explain how to read the inclusion off this parametrization. Let  $P$  and  $P'$  be two standard parabolic subgroups. Let  $\Lambda_1, \Lambda_2$  and  $\Lambda_3$

(resp.  $\Lambda'_1, \Lambda'_2$  and  $\Lambda'_3$ ) three Schubert varieties in  $G/P$  (resp.  $G/P'$ ) such the corresponding coefficients structure for  $\odot_0$  equal to one. Let  $\mathcal{F}$  and  $\mathcal{F}'$  denote the corresponding faces of  $\mathcal{AC}^G((G/B)^3)$ .

**Theorem D** *Let  $\mathcal{F}$  and  $\mathcal{F}'$  be two faces of  $\mathcal{AC}^G((G/B)^3)$ . The following are equivalent:*

- (i)  $\mathcal{F} \subset \mathcal{F}'$ ;
- (ii)  $P \subset P'$  and  $\pi(\Lambda_i) = \Lambda'_i$  for  $i = 1, 2$  and  $3$ , where  $\pi : G/P \longrightarrow G/P'$  is the  $G$ -equivariant morphism mapping  $P/P$  on  $P'/P'$ .

**Convention.** The ground field  $\mathbb{K}$  is assumed to be algebraically closed of characteristic zero. The notation introduced in the environments “**Notation.**” are fixed for all the sequence of the article.

## 2 An example of GIT-cone

Let us fix a connected reductive group  $G$  acting on an irreducible projective algebraic variety  $X$ .

### 2.1 An Ad Hoc notion of semistability

As in the introduction, for any  $G$ -linearized line bundle  $\mathcal{L}$  on  $X$ , we consider the following set of *semistable points*:

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \sigma \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\}.$$

To precise the acting group, we sometimes denote  $X^{\text{ss}}(\mathcal{L})$  by  $X^{\text{ss}}(\mathcal{L}, G)$ .

The subset  $X^{\text{ss}}(\mathcal{L})$  is open and stable by  $G$ . Note that this definition of  $X^{\text{ss}}(\mathcal{L})$  is the standard one only when  $\mathcal{L}$  is ample. Indeed, one usually imposes that the open subset defined by the non vanishing of  $\sigma$  to be affine.

If  $\mathcal{L}$  is ample, there exists a categorical quotient:

$$\pi : X^{\text{ss}}(\mathcal{L}) \longrightarrow X^{\text{ss}}(\mathcal{L})//G,$$

such that  $X^{\text{ss}}(\mathcal{L})//G$  is a projective variety and  $\pi$  is affine. A point  $x \in X^{\text{ss}}(\mathcal{L})$  is said to be *stable* if  $G_x$  is finite and  $G.x$  is closed in  $X^{\text{ss}}(\mathcal{L})$ . Then, for all stable point  $x$  we have  $\pi^{-1}(\pi(x)) = G.x$ ; and the set  $X^s(\mathcal{L})$  of stable points is open in  $X$ .

## 2.2 Definitions

Let us recall from the introduction that  $\Lambda$  is a freely finitely generated subgroup of  $\text{Pic}^G(X)$  and  $\Lambda_{\mathbb{Q}}$  is the  $\mathbb{Q}$ -vector space containing  $\Lambda$  as a lattice. Since  $X^{\text{ss}}(\mathcal{L}) = X^{\text{ss}}(\mathcal{L}^{\otimes n})$ , for any  $G$ -linearized line bundle and any positive integer  $n$ , we can define  $X^{\text{ss}}(\mathcal{L})$  for any  $\mathcal{L} \in \Lambda_{\mathbb{Q}}$ . We consider the following *total  $G$ -cone*:

$$\mathcal{TC}_{\Lambda}^G(X) = \{\mathcal{L} \in \Lambda_{\mathbb{Q}} : X^{\text{ss}}(\mathcal{L}) \text{ is not empty}\}.$$

Since the tensor product of two non zero  $G$ -invariant sections is a non zero  $G$ -invariant section,  $\mathcal{TC}_{\Lambda}^G(X)$  is a convex cone.

Consider the convex cones  $\Lambda_{\mathbb{Q}}^+$  and  $\Lambda_{\mathbb{Q}}^{++}$  generated respectively by the semiample and ample elements of  $\Lambda$ . For all  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^+$  (resp.  $\Lambda_{\mathbb{Q}}^{++}$ ), there exists a positive integer  $n$  such that  $\mathcal{L}^{\otimes n}$  is a semiample (resp. ample)  $G$ -linearized line bundle on  $X$  in  $\Lambda$ . So, any set of semistable points associated to a point in  $\Lambda_{\mathbb{Q}}^+$  (resp.  $\Lambda_{\mathbb{Q}}^{++}$ ) is in fact a set of semistable point associated to a semiample (resp. ample)  $G$ -linearized line bundle. We consider the following *semiample and ample  $G$ -cones*:

$$\mathcal{SAC}_{\Lambda}^G(X) = \mathcal{TC}_{\Lambda}^G(X) \cap \Lambda_{\mathbb{Q}}^+ \quad \text{and} \quad \mathcal{AC}_{\Lambda}^G(X) = \mathcal{TC}_{\Lambda}^G(X) \cap \Lambda_{\mathbb{Q}}^{++}.$$

By [DH98] (see also [Res00]),  $\mathcal{AC}_{\Lambda}^G(X)$  is a closed convex rational polyhedral cone in  $\Lambda_{\mathbb{Q}}^{++}$ . This cone is the central object of this article.

Two points  $\mathcal{L}$  and  $\mathcal{L}'$  in  $\mathcal{AC}_{\Lambda}^G(X)$  are said to be *GIT-equivalent* if  $X^{\text{ss}}(\mathcal{L}) = X^{\text{ss}}(\mathcal{L}')$ . An equivalence class is simply called a *GIT-class*.

For  $x \in X$ , the *stability set of  $x$*  is the set of  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^{++}$  such that  $X^{\text{ss}}(\mathcal{L})$  contains  $x$ ; it is denoted by  $\Omega_{\Lambda}(x)$  or  $\Omega_{\Lambda}(G.x)$ . In [Res00], we have studied the geometry of the GIT-classes and the stability sets with lightly different assumptions (no  $\Lambda$  for example). However all the results and proofs of [Res00] remain valuable here. In particular, there are only finitely many GIT-classes; and each GIT-class is the relative interior of a closed convex polyhedral cone of  $\Lambda_{\mathbb{Q}}^{++}$ .

## 2.3 An example of $G$ -ample cone

**Notation.** If  $\Gamma$  is an affine algebraic group,  $[\Gamma, \Gamma]$  will denote its derived subgroup and  $X(\Gamma)$  will denote the character group of  $\Gamma$ .

For later use, we consider here a  $G$ -ample cone for the action of  $G$  over an affine variety. More precisely, let  $V$  be an affine  $G$ -variety containing a

fix point  $O$  as unique closed orbit. The action of  $G$  over the fiber gives a morphism  $\mu^\bullet(O, G) : \text{Pic}^G(V) \longrightarrow X(G)$ . By [Res00, Lemma 7],  $\mu^\bullet(O, G)$  is an isomorphism. We denote by  $V^{\text{ss}}(\chi)$  the set of semistable points for the  $G$ -linearized line bundle  $\mathcal{L}_\chi$  associated to  $\chi \in X(G)$ ; that is, the trivial line bundle on  $V$  linearized by  $\chi$ . As in the projective case, we consider the  $G$ -ample cone  $\mathcal{AC}^G(V)$  in  $X(G) \otimes \mathbb{Q}$ .

For any  $\chi \in X(G)$ , we have:

$$H^0(V, \mathcal{L}_\chi)^G = \{f \in \mathbb{K}[V] : \forall x \in V \quad (g.f)(x) = \chi(g)f(x)\} = \mathbb{K}[V]_\chi.$$

Note that  $H^0(V, \mathcal{L}_\chi)^G$  is contained in  $\mathbb{K}[V]^{[G, G]}$ . Set

$$S = \{\chi \in X(G) : H^0(V, \mathcal{L}_\chi)^G \text{ is non trivial}\}.$$

It is the set of weights of  $G/[G, G]$  in  $\mathbb{K}[V]^{[G, G]}$ . We have:

**Lemma 1** *We assume that  $V$  is irreducible. The set  $S$  is a finitely generated semigroup in  $X(G)$ . Moreover,  $\mathcal{AC}^G(V)$  is the convex cone generated by  $S$ ; it is strictly convex.*

**Proof.** Since  $\mathbb{K}[V]^{[G, G]}$  is a finitely generated algebra,  $S$  is a finitely generated semigroup. The fact that  $\mathcal{AC}^G(V)$  is generated by  $S$  is obvious. Finally,  $\mathcal{AC}^G(V)$  is strictly convex since  $H^0(V, \mathcal{L}_0)^G = \mathbb{K}$ .  $\square$

### 3 Slice Etale Theorem

In this section, we recall some very useful results of D. Luna. We fix an ample  $G$ -linearized line bundle  $\mathcal{L}$  on the irreducible projective  $G$ -variety  $X$ .

#### 3.1 Closed orbits in general position

**Notation.** If  $H$  is a subgroup of  $G$ ,  $N_G(H)$  denotes the normalizer of  $H$  in  $G$ . Consider the quotient  $\pi : X^{\text{ss}}(\mathcal{L}) \longrightarrow X^{\text{ss}}(\mathcal{L})//G$ . For all  $\xi \in X^{\text{ss}}(\mathcal{L})//G$ , we denote by  $T(\xi)$  the unique closed orbit of  $G$  in  $\pi^{-1}(\xi)$ . We denote by  $(X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$  the set of  $\xi$  such that there exists an open neighborhood  $\Omega$  of  $\xi$  in  $X^{\text{ss}}(\mathcal{L})//G$  such that the orbits  $T(\xi')$  are isomorphic to  $T(\xi)$ , for all  $\xi' \in \Omega$ .

Since  $\pi$  is a gluing of affine quotients, some results on the actions of  $G$  on affine variety remains true for  $X^{\text{ss}}(\mathcal{L})$ . For example, the following theorem is a result of Luna and Richardson (see [LR79, Section 3] and [Lun75, Corollary 4] or [PV91, Section 7]):

**Theorem 1** *With above notation, if  $X$  is normal, we have:*

- (i) *The set  $(X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$  is a non empty open subset of  $X^{\text{ss}}(\mathcal{L})//G$ . Let  $H$  be the isotropy of a point in  $T(\xi_0)$  with  $\xi_0 \in (X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$ . The group  $H$  has fixed points in  $T(\xi)$  for any  $\xi \in X^{\text{ss}}(\mathcal{L})//G$ .*
- (ii) *Let  $Y$  be the closure of  $\pi^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})^H$  in  $X$ . It is the union of some components of  $X^H$ . Then,  $H$  acts trivially on some positive power  $\mathcal{L}_{|Y}^{\otimes n}$  of  $\mathcal{L}_{|Y}$ . Moreover, the natural map*

$$Y^{\text{ss}}(\mathcal{L}_{|Y}^{\otimes n})/(N_G(H)/H) \longrightarrow X^{\text{ss}}(\mathcal{L})//G$$

*is an isomorphism. Moreover,  $Y$  contains stable points for the action of  $N_G(H)/H$  and the line bundle  $\mathcal{L}_{|Y}^{\otimes n}$ .*

A subgroup  $H$  as in Theorem 1 will be called a *generic closed isotropy* of  $X^{\text{ss}}(\mathcal{L})$ . The conjugacy class of  $H$  which is obviously unique is called the *generic closed isotropy* of  $X^{\text{ss}}(\mathcal{L})$ .

### 3.2 The principal Luna stratum

When  $X$  is smooth, the open subset  $(X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$  is called the *principal Luna stratum* and has very useful properties (see [Lun73] or [PV91]):

**Theorem 2 (Luna)** *We assume that  $X$  is smooth. Let  $H$  be a generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$ .*

*Then, there exists a  $H$ -module  $L$  such that for any  $\xi \in (X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$ , there exist points  $x$  in  $T(\xi)$  satisfying:*

- (i)  $G_x = H$ ;
- (ii) *the  $H$ -module  $T_x X/T_x(G.x)$  is isomorphic to the sum of  $L$  and its fix points, in particular, it is independent of  $\xi$  and  $x$ ;*
- (iii) *for any  $v \in L$ ,  $0$  belongs to the closure of  $H.v$ ;*
- (iv) *the fiber  $\pi^{-1}(\xi)$  is isomorphic to  $G \times_H L$ .*

### 3.3 The fibers of quotient morphisms

Another useful consequence of Luna's Slice Etale Theorem is (see [Lun73] or [PV91]):



**Proposition 1** *Let  $x$  be a semistable point for  $\mathcal{L}$  whose orbit is closed in  $X^{\text{ss}}(\mathcal{L})$ . Then, there exists an affine  $G_x$ -variety  $V$  containing a unique closed orbit which is a fixed point and such that  $\pi^{-1}(\pi(x))$  is isomorphic to  $G \times_{G_x} V$ .*

## 4 About faces of the $G$ -ample cone

### 4.1 Isotropy subgroups associated to faces of $\mathcal{AC}_\Lambda^G(X)$

Let  $\varphi$  be a linear form on  $\Lambda_{\mathbb{Q}}$  which is non negative on  $\mathcal{AC}_\Lambda^G(X)$ . If the set of  $\mathcal{L} \in \mathcal{AC}_\Lambda^G(X)$  such that  $\varphi(\mathcal{L}) = 0$  is non empty it will be called a *face* of  $\mathcal{AC}_\Lambda^G(X)$ . Now, we associate two invariants to a face  $\mathcal{F}$  of  $\mathcal{AC}_\Lambda^G(X)$ .

**Theorem 3** *Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_\Lambda^G(X)$ . Let  $\mathcal{L}$  be a point in the relative interior of  $\mathcal{F}$ . Then, we have:*

- (i) *The generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$  does not depend on the point  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ , but only in  $\mathcal{F}$ . We call this isotropy the generic closed isotropy of  $\mathcal{F}$ .*

*Let us fix a generic closed isotropy  $H$  of  $\mathcal{F}$ .*

- (ii) *For any  $\mathcal{M} \in \mathcal{F}$ ,  $H$  fixes points in any closed orbit of  $G$  in  $X^{\text{ss}}(\mathcal{M})$ .*

- (iii) *The closure  $Y$  of  $\left( \pi_{\mathcal{L}}^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}) \right)^H$  in  $X$  does not depends on a choice of  $\mathcal{L}$ . Let  $Y_{\mathcal{F}}$  denote this subvariety of  $X^H$ ; it is the union of some components of  $X^H$ .*

- (iv) *The group  $H$  acts trivially on some positive power  $\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n}$  of  $\mathcal{L}_{|Y_{\mathcal{F}}}$ . Moreover, the natural map*

$$Y_{\mathcal{F}}^{\text{ss}}(\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n}) // (N_G(H)/H) \longrightarrow X^{\text{ss}}(\mathcal{L}) // G$$

*is an isomorphism. Moreover,  $Y_{\mathcal{F}}$  contains stable points for the action of  $N_G(H)/H$  and the line bundle  $\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n}$ .*

- (v) *Set  $Y_{\mathcal{F}}^+ := \{x \in X : \overline{H.x} \cap Y_{\mathcal{F}} \neq \emptyset\}$ . Then  $G.Y_{\mathcal{F}}^+$  contains an open subset of  $X$ .*

**Proof.** Let  $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{AC}_\Lambda^G(X)$ . By an easy argument of convexity, to prove Assertion (i) it is sufficient to prove that the generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$  does not depend on  $\mathcal{L}$  in the open segment  $] \mathcal{L}_1, \mathcal{L}_2[$ . Let us fix  $\mathcal{L}, \mathcal{L}' \in$

$]\mathcal{L}_1, \mathcal{L}_2[$ . Let  $x \in X$  which maps in  $(X^{\text{ss}}(\mathcal{M})//G)_{\text{pr}}$ , for  $\mathcal{M} = \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}$  and  $\mathcal{L}'$  by the quotient maps.

Recall that  $\Omega_\Lambda(x)$  is a polyhedral convex cone. Since  $\mathcal{L}_1$  and  $\mathcal{L}_2$  belong to  $\Omega_\Lambda(x)$ ,  $\mathcal{L}$  and  $\mathcal{L}'$  belong to the relative interior of the same face of  $\Omega_\Lambda(x)$ . By [Res00, Proposition 6, Assertion (iii)], there exists  $x' \in \overline{G.x}$  such that this face is  $\Omega_\Lambda(x')$ . But, [Res00, Proposition 6, Assertion (i)] shows that the closed orbits in  $X^{\text{ss}}(\mathcal{L}) \cap \overline{G.x'}$  and  $X^{\text{ss}}(\mathcal{L}') \cap \overline{G.x'}$  are equal. Now, our choice of the point  $x$  implies that the generic closed isotropies of  $X^{\text{ss}}(\mathcal{L})$  and  $X^{\text{ss}}(\mathcal{L}')$  are equal.

Let  $H$  be a generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$ . Let  $Y$  be the closure of  $X^H \cap \pi_{\mathcal{L}}^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})$ . By Theorem 1,  $N_G(H)$  acts transitively on the set of irreducible components of  $Y$ . Let  $Y_1$  be such a component of  $X^H$ . Again by Theorem 1,  $\pi_{\mathcal{L}}(Y_1 \cap X^{\text{ss}}(\mathcal{L})) = X^{\text{ss}}(\mathcal{L})//G$ ; that is, any closed  $G$ -orbit in  $X^{\text{ss}}(\mathcal{L})$  intersects  $Y_1$ . Finally,  $Y$  is the union of irreducible components of  $X^H$  which intersect a general closed  $G$ -orbit in  $X^{\text{ss}}(\mathcal{L})$ . But, the above proof of Assertion (i) shows that a general closed orbit in  $X^{\text{ss}}(\mathcal{L})$  is also a closed orbit in  $X^{\text{ss}}(\mathcal{L}')$ . In particular,  $Y$  is the closure of  $X^H \cap \pi_{\mathcal{L}'}^{-1}((X^{\text{ss}}(\mathcal{L}')//G)_{\text{pr}})$ . Assertion (iii) follows.

Let us now fix a generic closed isotropy  $H$  of  $\mathcal{F}$ . Let  $\mathcal{M}_1 \in \mathcal{F}$ . By Assertion (i) of Theorem 1, to prove the second assertion, it is sufficient to prove that the generic closed isotropy of  $X^{\text{ss}}(\mathcal{M}_1)$  contains  $H$ . By [Res00, Theorem 4], there exists a point  $\mathcal{M}_2$  in the relative interior of  $\mathcal{F}$  such that  $X^{\text{ss}}(\mathcal{M}_1)$  contains  $X^{\text{ss}}(\mathcal{M}_2)$ . The inclusion  $X^{\text{ss}}(\mathcal{M}_2) \subset X^{\text{ss}}(\mathcal{M}_1)$  induces a surjective morphism  $\eta : X^{\text{ss}}(\mathcal{M}_2)//G \rightarrow X^{\text{ss}}(\mathcal{M}_1)//G$ . Let  $\xi' \in (X^{\text{ss}}(\mathcal{M}_2)//G)_{\text{pr}}$  such that  $\xi = \eta(\xi') \in (X^{\text{ss}}(\mathcal{M}_1)//G)_{\text{pr}}$ . Let  $x$  be a point in the closed  $G$ -orbit in  $X^{\text{ss}}(\mathcal{M}_1)$  over  $\xi$ . The fiber in  $X^{\text{ss}}(\mathcal{M}_1)$  over  $\xi$  is fibered over  $G.x$ ; hence, for any  $y$  in this fiber,  $G_y$  is conjugated to a subgroup of  $G_x$ . Since this fiber contains the fiber in  $X^{\text{ss}}(\mathcal{M}_2)$  over  $\xi'$ ,  $H$  is conjugated to a subgroup of  $G_x$ . The second assertion is proved.

From now on,  $\mathcal{L}$  is a point in the relative interior of  $\mathcal{F}$ . Let  $Y$  be the subvariety of  $X^H$  of Assertion (iii). By Theorem 1,  $Y$  satisfies Assertion (iv). Moreover,  $G.Y^+$  contains  $\pi_{\mathcal{L}}^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})$ ; and, Assertion (v) is proved.  $\square$

## 4.2 Local structure of the $G$ -ample cone around a face

**Notation.** Let  $E$  be a prime Cartier divisor on a variety  $X$  endowed with a line bundle  $\mathcal{L}$  and  $\sigma$  be a rational section for  $\mathcal{L}$ . We will denote by  $\nu_E(\sigma) \in \mathbb{Z}$  the order of zero of  $\sigma$  along  $E$ .

Let  $\mathcal{P}$  be a polyhedron in a rational vector space  $V$  and  $\mathcal{F}$  be a face of  $\mathcal{P}$ . Let  $x$  in the relative interior of  $\mathcal{F}$ . The cone of  $V$  generated by the vectors  $y - x$  for  $y \in \mathcal{P}$  does not depend on the choice of  $x$  in the relative interior of  $\mathcal{F}$  and will be called *the cone of  $\mathcal{P}$  viewed from  $\mathcal{F}$* . This cone carries the geometry of  $\mathcal{P}$  in a neighborhood of  $x$ .

Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_\Lambda^G(X)$ . Let  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ . Let  $x$  be a semistable point for  $\mathcal{L}$  whose orbit is closed in  $X^{\text{ss}}(\mathcal{L})$ . Let  $V$  be an affine  $G_x$ -variety satisfying Proposition 1. Consider the cone  $\mathcal{AC}^{G_x}(V)$  as in Section 2.3. Notice that  $V$  is not necessarily irreducible, and so  $\mathcal{C}^{G_x}(V)$  is not necessarily convex.

The action of  $G_x$  on the fiber over  $x$  defines a morphism  $\mu^\bullet(x, G_x) : \Lambda \rightarrow X(G_x)$ ; and so, a linear map from  $\Lambda_\mathbb{Q}$  on  $X(G_x)_\mathbb{Q}$  also denoted by  $\mu^\bullet(x, G_x)$ .

**Theorem 4** *With above notation, the cone of  $\mathcal{AC}^G(X)$  viewed from  $\mathcal{F}$  equals  $(\mu^\bullet(x, G_x))^{-1}(\mathcal{AC}^{G_x}(V))$ . In particular, if  $\mu^\bullet(x, G_x)$  is surjective then  $\mathcal{AC}^{G_x}(V)$  is convex.*

**Proof.** Let  $\mathcal{L}_0$  and  $\mathcal{L}$  be two different ample  $G$ -linearized line bundles in  $\Lambda$ . We assume that  $\mathcal{L}_0$  is the only point in the segment  $[\mathcal{L}; \mathcal{L}_0]$  which belongs to  $\mathcal{AC}_\Lambda^G(X)$ . For convenience, we set  $U = X^{\text{ss}}(\mathcal{L}_0)$ . By assumption, there is no  $G$ -invariant rational section of  $\mathcal{L}$  which is regular on  $X$ ; we claim that there is no such section which is regular on  $U$ .

Let us prove the claim. Let us fix a non zero regular  $G$ -invariant section  $\sigma_0$  of  $\mathcal{L}_0^{\otimes m}$  for a positive integer  $m$ . Let  $\sigma$  be a  $G$ -invariant rational section of  $\mathcal{L}$  which is regular on  $U$ . For any positive integer  $k$ ,  $\sigma \otimes \sigma_0^{\otimes k}$  is a rational  $G$ -invariant section of  $\mathcal{L} \otimes \mathcal{L}_0^{\otimes mk}$  which is regular on  $U$ . Let  $E$  be an irreducible component of codimension one of  $X - U$ . By definition of  $U$ ,  $\sigma_0$  is zero along  $E$ ; and,  $\nu_E(\sigma_0) > 0$ . Then,  $\nu_E(\sigma \otimes \sigma_0^{\otimes k}) = \nu_E(\sigma) + k \cdot \nu_E(\sigma_0)$  is positive for  $k$  big enough. We deduce that  $\sigma \otimes \sigma_0^{\otimes k}$  is regular on  $X$  for  $k$  big enough. Since by assumption  $\mathcal{L} \otimes \mathcal{L}_0^{\otimes mk}$  does not belong to  $\mathcal{AC}_\Lambda^G(X)$ , this implies that  $\sigma \otimes \sigma_0^{\otimes k}$  and finally  $\sigma$  are zero. The claim is proved.

We now fix a point  $\mathcal{L}_0$  in the relative interior of  $\mathcal{F}$  as in the statement. By an elementary argument of convexity, there exists an open neighborhood

$\Omega$  of  $\mathcal{L}_0$  in  $\Lambda_{\mathbb{Q}}$  such that

- (i) for any  $\mathcal{L} \in \Omega$ , if  $\mathcal{L}$  does not belong to  $\mathcal{AC}_{\Lambda}^G(X)$  then  $\mathcal{L}_0$  is the only point in  $[\mathcal{L}, \mathcal{L}_0] \cap \mathcal{AC}_{\Lambda}^G(X)$ .

By [Res00, Proposition 2.3], we may also assume that for all  $\mathcal{L} \in \Omega$ ,  $X^{\text{ss}}(\mathcal{L})$  is contained in  $X^{\text{ss}}(\mathcal{L}_0)$ . It remains to prove that for any  $\mathcal{L} \in \Omega$ ,  $\mathcal{L} \in \mathcal{AC}_{\Lambda}^G(X)$  if and only if  $\mu^{\mathcal{L}}(x, G_x) \in \mathcal{AC}^{G_x}(V)$ .

Let  $\mathcal{L} \in \Omega$  which does not belong to  $\mathcal{AC}_{\Lambda}^G(X)$ . Set  $\xi = \pi_{\mathcal{L}_0}(x)$ . By the beginning of the proof, for any positive  $n$ ,  $H^0(U, \mathcal{L}^{\otimes n})^G = \{0\}$ . Since  $\pi_{\mathcal{L}_0}^{-1}(\xi)$  is closed in  $U$ , this implies that  $H^0(\pi_{\mathcal{L}_0}^{-1}(\xi), \mathcal{L}^{\otimes n})^G = \{0\}$  for all positive  $n$ . So, for all positive  $n$ ,  $H^0(V, \mathcal{L}_{|V}^{\otimes n})^{G_x} = \{0\}$ ; and, so  $\mu^{\mathcal{L}}(x, G_x)$  does not belong to  $\mathcal{AC}^{G_x}(V)$ .

Let now  $\mathcal{L} \in \Omega \cap \mathcal{AC}_{\Lambda}^G(X)$ . Since the map  $\phi : X^{\text{ss}}(\mathcal{L})//G \longrightarrow X^{\text{ss}}(\mathcal{L}_0)//G$  induced by the inclusion  $X^{\text{ss}}(\mathcal{L}) \subset X^{\text{ss}}(\mathcal{L}_0)$  is surjective, there exists  $y \in X^{\text{ss}}(\mathcal{L})$  such that  $\phi \circ \pi_{\mathcal{L}}(y) = \xi$ . Up to changing  $y$  by  $g.y$  for some  $g \in G$ , one may assume that  $y \in V$ . Let  $\sigma$  be a  $G$ -invariant section of  $\mathcal{L}$  which is non zero at  $y$ . Obviously, the restriction of  $\sigma$  is a  $G_x$ -invariant section of  $\mathcal{L}_{|V}$  which is non zero. It follows that  $\mu^{\mathcal{L}}(x, G_x)$  belongs to  $\mathcal{AC}^{G_x}(V)$ .

The last assertion follows from an obvious argue of convexity.  $\square$

## 5 Well covering pairs

### 5.1 The functions $\mu^{\bullet}(x, \lambda)$

Let  $\mathcal{L} \in \text{Pic}^G(X)$ . Let  $x$  be a point in  $X$  and  $\lambda$  be a one parameter subgroup of  $G$ . Since  $X$  is complete,  $\lim_{t \rightarrow 0} \lambda(t)x$  exists; let  $z$  denote this limit. The image of  $\lambda$  fixes  $z$  and so the group  $\mathbb{K}^*$  acts via  $\lambda$  on the fiber  $\mathcal{L}_z$ . This action defines a character of  $\mathbb{K}^*$ , that is, an element of  $\mathbb{Z}$  denoted by  $\mu^{\mathcal{L}}(x, \lambda)$ .

The numbers  $\mu^{\mathcal{L}}(x, \lambda)$  are used in [MFK94] to give a numerical criterion for stability with respect to an ample  $G$ -linearized line bundle  $\mathcal{L}$ :

$$\begin{aligned} x \in X^{\text{ss}}(\mathcal{L}) &\iff \mu^{\mathcal{L}}(x, \lambda) \leq 0 \text{ for all one parameter subgroup } \lambda, \\ x \in X^s(\mathcal{L}) &\iff \mu^{\mathcal{L}}(x, \lambda) < 0 \text{ for all non trivial } \lambda. \end{aligned}$$

### 5.2 Definition

**Notation.** The set of fix points of the image of  $\lambda$  will be denoted by  $X^{\lambda}$ ; the centralizer of this image will be denoted by  $G^{\lambda}$ . We consider the following

parabolic subgroup of  $G$ :

$$P(\lambda) = \left\{ g \in G : \lim_{t \rightarrow 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

Let  $C$  be an irreducible component of  $X^\lambda$ . Since  $G^\lambda$  is connected,  $C$  is a  $G^\lambda$ -stable closed subvariety of  $X$ . We set:

$$C^+ := \{x \in X : \lim_{t \rightarrow 0} \lambda(t)x \in C\}.$$

Then,  $C^+$  is a locally closed subvariety of  $X$  stable by  $P(\lambda)$ . Moreover, the map  $p_\lambda : C^+ \rightarrow C$ ,  $x \mapsto \lim_{t \rightarrow 0} \lambda(t)x$  is a morphism satisfying:

$$\forall (l, u) \in G^\lambda \times U(\lambda) \quad p_\lambda(lu.x) = lp_\lambda(x).$$

Consider over  $G \times C^+$  the action of  $G \times P(\lambda)$  given by the formula (with obvious notation):

$$(g, p).(g', y) = (gg'p^{-1}, py).$$

Since the quotient map  $G \rightarrow G/P(\lambda)$  is a Zariski-locally trivial principal  $P(\lambda)$ -bundle; one can easily construct a quotient  $G \times_{P(\lambda)} C^+$  of  $G \times C^+$  by the action of  $\{e\} \times P(\lambda)$ . The action of  $G \times \{e\}$  induces an action of  $G$  on  $G \times_{P(\lambda)} C^+$ .

**Definition.** Consider the following  $G$ -equivariant map

$$\begin{aligned} \eta : G \times_{P(\lambda)} C^+ &\longrightarrow X \\ [g : x] &\longmapsto g.x. \end{aligned}$$

The pair  $(C, \lambda)$  is said to be *covering* (resp. *dominant*) if  $\eta$  is birational (resp. dominant). It is said to be *well covering* if  $\eta$  induces an isomorphism from  $G \times_{P(\lambda)} \Omega$  onto a  $P(\lambda)$ -stable open subset of  $X$  for an open subset  $\Omega$  of  $C^+$  intersecting  $C$ .

### 5.3 Face associated to $(C, \lambda)$

Let us denote by  $\mu^\bullet(C, \lambda)$ , the common value of the  $\mu^\bullet(x, \lambda)$ , for  $x \in C^+$ .

**Lemma 2** *Let  $(C, \lambda)$  be a dominant pair. The set of  $\mathcal{L} \in \mathcal{AC}_\Lambda^G(X)$  such that  $\mu^\mathcal{L}(C, \lambda) = 0$  is either empty or a face  $\mathcal{F}$  of  $\mathcal{AC}_\Lambda^G(X)$ . Moreover,  $\mathcal{F}$  is the set of  $\mathcal{L} \in \mathcal{AC}_\Lambda^G(X)$  such that  $X^{\text{ss}}(\mathcal{L})$  intersects  $C$ .*

*From now on,  $\mathcal{F}$  which only depends on  $C$  is denoted by  $\mathcal{F}(C)$ .*

**Proof.** The first assertion is [Res07, Lemma 7]. If  $\mathcal{L} \in \mathcal{F}$ , then there exists  $x \in C^+$  semistable for  $\mathcal{L}$ . By [Res07, Lemma 4],  $\lim_{t \rightarrow 0} \lambda(t)x$  is semistable and belongs to  $C$ . Conversely, assume that  $X^{\text{ss}}(\mathcal{L}) \cap C$  contains  $z$ . Since  $z$  is fixed by  $\lambda$ ,  $\mu^{\mathcal{L}}(z, -\lambda) = -\mu^{\mathcal{L}}(z, \lambda)$ . But  $z$  is semistable, so  $\mu^{\mathcal{L}}(z, -\lambda) = 0$ .  $\square$

## 6 The case $X = Y \times G/B$

In this section, we assume that  $X = Y \times G/B$ , with a normal projective  $G$ -variety  $Y$ . Moreover, we assume that  $\Lambda$  is abundant (see [Res07]).

### 6.1 General closed isotropy and well covering pairs

Let  $S$  be a torus contained in  $G$ . Let  $C$  be an irreducible component of  $X^S$ .

**Definition.** The pair  $(C, S)$  is said to be *admissible* if there exists  $x \in C$  such that  $G_x^\circ = S$ . The pair is said to be *well covering* if there exists a one parameter subgroup  $\lambda$  of  $S$ , such that  $C$  is an irreducible component of  $X^\lambda$  and  $(C, \lambda)$  is well covering.

A rephrasing of [Res07, Corollary 3] is

**Proposition 2** *We assume that  $X = Y \times G/B$  with a normal projective  $G$ -variety  $Y$  and  $\Lambda$  is abundant (see [Res07, Section 7.4]). Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_\Lambda^G(X)$  of codimension  $r$ . Then, there exists an admissible well covering pair  $(C, S)$  with  $S$  of dimension  $r$  such that  $\mathcal{F} = \mathcal{F}(C)$ .*

We are now interested in the generic closed isotropy of faces of  $\mathcal{AC}_\Lambda^G(X)$ :

**Proposition 3** *Let  $\mathcal{F}$  be a face of codimension  $r$ . Let  $(C, S)$  be an admissible well covering pair with a  $r$ -dimensional torus  $S$  such that  $\mathcal{F}(C) = \mathcal{F}$ .*

*There exists a generic closed isotropy  $H$  of  $\mathcal{F}$  such that  $H^\circ = S$ .*

**Proof.** By Lemma 2,  $\mathcal{F}$  is an union of GIT-classes. By [Res00], there are only finitely many such classes and they are convex; so, there exists a GIT-class  $F$  which spans  $\mathcal{F}$ . Let  $\mathcal{L} \in F$ . Let  $\xi \in (X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$  and  $\mathcal{O}_\xi$  be the corresponding closed  $G$ -orbit in  $X^{\text{ss}}(\mathcal{L})$ .

By [Res07, Theorem 5],  $\mathcal{O}_\xi$  intersects  $C$ . Let us fix  $x \in \mathcal{O}_\xi \cap C$ . Now, consider the morphism  $\mu^\bullet(x, G_x) : \Lambda_\mathbb{Q} \longrightarrow X(G_x) \otimes \mathbb{Q}$  induced by restriction and the isomorphism  $X(G_x) \simeq \text{Pic}^G(\mathcal{O}_\xi)$ . By Theorem 4,  $\text{Ker} \mu^\bullet(x, G_x)$  is

contained in  $\text{Span}(\mathcal{F})$ . On the other hand, the GIT-class of  $\mathcal{L}$  is contained in  $\text{Ker}\mu^\bullet(x, G_x)$ . Finally,  $\text{Ker}\mu^\bullet(x, G_x) = \text{Span}(\mathcal{F})$ . Since  $\Lambda$  is abundant, this implies that the rank of  $X(G_x)$  equals  $r$ .

Since  $G.x$  is affine,  $G_x$  is reductive. Since  $X = Y \times G/B$ ,  $G_x$  is contained in a Borel subgroup of  $G$ . Finally,  $G_x$  is diagonalisable. But the rank of  $X(G_x)$  equals  $r$ ; and,  $G_x^\circ$  is a  $r$ -dimensional torus.

Since  $x \in C$ ,  $S$  is contained in  $G_x$ ; it follows that  $G_x^\circ = S$ .  $\square$

## 6.2 Unicity

**Notation.** Let  $S$  be a torus. We will denote  $Y(S)$  the group of one parameter subgroups of  $S$ . There is a natural perfect paring  $Y(S) \times X(S) \longrightarrow \mathbb{Z}$  denoted by  $\langle \cdot, \cdot \rangle$ .

The following proposition is a first statement of unicity.

**Lemma 3** *We assume that  $Y$  (and so  $X$ ) is smooth. Let  $\mathcal{F}$  be a face of codimension  $r$ . Let  $(C_1, S_1)$  and  $(C_2, S_2)$  be two well covering pairs with two  $r$ -dimensional tori  $S_1$  and  $S_2$  such that  $\mathcal{F}(C_1) = \mathcal{F}(C_2) = \mathcal{F}$ .*

*Then, there exists  $g \in G$  such that  $g.C_2 = C_1$  and  $g.S_2.g^{-1} = S_1$ .*

**Proof.** Starting the proof as Proposition 3, we obtain that  $\mathcal{O}_\xi$  intersects  $C_1$  and  $C_2$ . Up to conjugacy, we may assume that  $x$  belongs to  $\mathcal{O}_\xi \cap C_1 \cap C_2$ . So, we obtain that  $G_x^\circ = S_1 = S_2$ . Then,  $C_1$  equals  $C_2$  since they are the irreducible component of  $X^{S_1} = X^{S_2}$  containing  $x$ .  $\square$

Let us fix a face  $\mathcal{F}$  of codimension  $r$ . The set of linear forms  $\varphi \in \text{Hom}(\Lambda_\mathbb{Q}, \mathbb{Q})$  such that  $\varphi$  is non negative on  $\mathcal{AC}_\Lambda^G(X)$  and zero on  $\mathcal{F}$  is denoted by  $\mathcal{F}^\vee$ .

Let  $(C, S)$  be an admissible pair where  $S$  has dimension  $r$  and set  $\mathcal{F} = \mathcal{F}(C)$ . Let  $\mathcal{C}$  denote the set of  $\lambda \in Y(S) \otimes \mathbb{Q}$  such that for some positive  $n$ , the pair  $(C_{n\lambda}, n\lambda)$  is dominant; where,  $C_{n\lambda}$  denote the irreducible component of  $X^{n\lambda}$  containing  $C$ .

**Lemma 4** *We assume that  $Y$  is smooth.*

*Then,  $\lambda \in \mathcal{C}$  if and only if  $\mu^\bullet(C, \lambda) \in \mathcal{F}^\vee$ .*

**Proof.** Let  $\lambda \in \mathcal{C}$  and  $n$  be a positive integer such that  $n\lambda \in Y(S)$ . Since  $(C_{n\lambda}, n\lambda)$  is dominant, [Res07, Lemma 7] imply that  $\mu^\bullet(C, \lambda)$  is non negative on  $\mathcal{AC}_\Lambda^G(X)$ . Moreover, for any  $\mathcal{L} \in \mathcal{F}$ ,  $X^{\text{ss}}(\mathcal{L})$  intersects  $C$ . This implies that  $\mu^\mathcal{L}(C, \lambda) = 0$ . Finally,  $\mu^\bullet(C, \lambda) \in \mathcal{F}^\vee$ .

Conversely, let  $\lambda$  be a rational one parameter subgroup and  $n$  be a positive integer such that  $n\lambda \in Y(S)$  and  $\mu^\bullet(C, \lambda) \in \mathcal{F}^\vee$ . Set  $C^+ = \{x \in X : \lim_{t \rightarrow 0} n\lambda(t)x \in C_{n\lambda}\}$  and  $\eta : G \times_{P(n\lambda)} C^+ \longrightarrow X$ . Let us fix a generic point  $x \in C$ . Then,  $G_x$  is the generic closed isotropy of  $\mathcal{F}$ , its neutral component is  $S$  and the  $G_x$ -module  $T_x X / T_x G.x$  is the type of  $\mathcal{F}$ . Theorem 4 implies that  $\mu^\bullet(C, \lambda) \in \mathcal{F}^\vee$  if and only if  $\langle n\lambda, \cdot \rangle$  is non negative on all weights of  $S$  in  $T_x X / T_x G.x$ . We deduce that  $T\eta_x$  is surjective. Since  $Y$  is smooth, this implies that  $\eta$  is dominant.  $\square$

We can now state our main result of unicity:

**Proposition 4** *We assume that  $Y$  is smooth. Let  $p_{G/B} : X \longrightarrow G/B$  denote the projection. Let us fix a Borel subgroup  $B$  of  $G$  and a maximal torus  $T$  of  $B$ .*

*Let  $\mathcal{F}$  be a face of codimension  $r$ . Then there exists a unique well covering pair  $(C, S)$  where  $S$  is a  $r$ -dimensional subtorus of  $T$ ,  $p_{G/B}(C)$  contains  $B/B$  and  $\mathcal{F}(C) = \mathcal{F}$ .*

*Let  $\lambda_1$  and  $\lambda_2$  be two one parameter subgroups of  $S$  such that  $(C, \lambda_i)$  is dominant and  $\mathcal{F}$  equals the set of  $\mathcal{L} \in \mathcal{AC}_\lambda^G(X)$  such that  $\mu^\mathcal{L}(C, \lambda_i) = 0$  for  $i = 1, 2$ . Then,  $P(\lambda_1) = P(\lambda_2)$ ,  $C$  is an irreducible component of  $X^{\lambda_1}$  and  $X^{\lambda_2}$  and  $C^+(\lambda_1) = C^+(\lambda_2)$ .*

**Proof.** Let  $(C_1, S_1)$  and  $(C_2, S_2)$  be two admissible well covering pairs with two  $r$ -dimensional tori  $S_1$  and  $S_2$  such that  $\mathcal{F}(C_1) = \mathcal{F}(C_2) = \mathcal{F}$ . We also assume that  $p_{G/B}(C_1)$  and  $p_{G/B}(C_2)$  contain  $B/B$ . By Lemma 3, there exists  $g \in G$  such that  $gS_2g^{-1} = S_1$  and  $gC_2 = C_1$ . Note that  $g^{-1}Tg$  and  $T$  contain  $S_2$  and are maximal tori of  $G^{S_2}$ : there exists  $h \in G^{S_2}$  such that  $hTh^{-1} = g^{-1}Tg$ . Set  $\tilde{w} = gh$ . One easily checks that  $\tilde{w}$  normalize  $T$ ,  $\tilde{w}S_2\tilde{w}^{-1} = S_1$  and  $\tilde{w}C_2 = C_1$ . Now,  $G^{S_1}B/B = p_{G/B}(C_1) = \tilde{w}p_{G/B}(C_2) \ni \tilde{w}B/B$ . We deduce that  $\tilde{w} \in G^{S_1}$ . So,  $S_2 = \tilde{w}^{-1}S_1\tilde{w} = S_1$  and  $C_2 = \tilde{w}^{-1}C_1 = C_1$ . The first assertion is proved.

Let now  $C, S, \lambda_1$  and  $\lambda_2$  be as in the statement. Let us first prove that  $P(\lambda_1) = P(\lambda_2)$ . By Lemma 4, the set of the one parameter subgroups  $\lambda$  of  $S$  as in the proposition is convex. So, if it is possible to have  $P(\lambda_1) \neq P(\lambda_2)$ ; it is possible to have  $P(\lambda_1) \subset P(\lambda_2)$  and  $P(\lambda_1) \neq P(\lambda_2)$ . In other words, we may assume that  $P(\lambda_1) \subset P(\lambda_2)$ . Let  $C_1$  (resp.  $C_2$ ) denote the irreducible component of  $X^{\lambda_1}$  (resp.  $X^{\lambda_2}$ ) containing  $C$ . By Lemma 3, we have  $C_1 = C_2$ . In particular,  $G^{\lambda_1}B/B = p_{G/B}(C_1) = p_{G/B}(C_2) = G^{\lambda_2}B/B$ ;



so,  $G^{\lambda_1} = G^{\lambda_2}$ . It follows that  $P(\lambda_1) = P(\lambda_2)$ . Let  $P$  denote this parabolic subgroup of  $G$ .

Let  $x \in C$  be general. Since  $\lambda_1$  fixes  $x$ , it acts on the tangent space  $T_x X$ . Consider the subspaces  $(T_x X)_{>0}$  and  $(T_x X)_{<0}$  of the  $\xi \in T_x X$  such that  $\lim_{t \rightarrow 0} \lambda_1(t)\xi = 0$  and  $\lim_{t \rightarrow 0} \lambda_1(t^{-1})\xi = 0$  respectively. We have:  $T_x X = (T_x X)_{<0} \oplus (T_x X)_0 \oplus (T_x X)_{>0}$ . The first part identify with  $T_e G/P$  and the second one with  $T_x C$  as  $S$ -modules. But, the third part is the unique  $S$ -stable supplementary of the sum of the two first one. In particular, the same construction with  $\lambda_2$  in place of  $\lambda_1$  gives the same decomposition  $T_x X = (T_x X)_{<0} \oplus (T_x X)_0 \oplus (T_x X)_{>0}$ . It follows that  $C^+(\lambda_1) = C^+(\lambda_2)$ .  $\square$

### 6.3 Inclusion of faces

**Proposition 5** *We assume that  $Y$  is smooth. Let us fix a maximal torus  $T$  of  $G$  and a Borel subgroup  $B$  containing  $T$ . Let  $(C_1, S_1)$  and  $(C_2, S_2)$  be two admissible well covering pairs with two subtori  $S_1$  and  $S_2$  of  $T$  of dimension  $r_1$  and  $r_2$  such that  $B/B \in p_{G/B}(C_i)$  for  $i = 1, 2$ . We assume that  $\mathcal{F}(C_1)$  and  $\mathcal{F}(C_2)$  have respectively codimension  $r_1$  and  $r_2$ .*

*Then, the following are equivalent:*

- (i)  $\mathcal{F}(C_1) \subset \mathcal{F}(C_2)$ ;
- (ii)  $C_1 \subset C_2$  and  $S_2 \subset S_1$ .

**Proof.** The second assertion implies the first one by Lemma 2. Conversely, let us assume that  $\mathcal{F}(C_1) \subset \mathcal{F}(C_2)$ .

By Proposition 3, there exists  $\mathcal{L} \in \mathcal{F}(C_1)$  and  $x \in C_1$  such that  $G_x^\circ = S_1$  and  $G.x$  is closed in  $X^{\text{ss}}(\mathcal{L})$ . Since  $C_2$  intersects  $G.x$ , there exists  $g \in G$  such that  $g.x \in C_2$ . Since  $S_2$  fixes  $g.x$ ,  $S_2 \subset gS_1g^{-1}$ . In particular,  $S_2$  is contained in  $T$  and  $gTg^{-1}$ ; so,  $T$  and  $gTg^{-1}$  are maximal tori in  $G^{S_2}$ . There exists  $l \in G^{S_2}$  such that  $lTl^{-1} = gTg^{-1}$ . The element  $n = l^{-1}g$  normalizes  $T$ . Since  $C_2$  is stable by  $G^{S_2}$  (which is connected),  $x$  belongs to  $n^{-1}C_2$ . Applying  $p_{G/B}$  we obtain that  $p_{G/B}(x)$  belongs to  $n^{-1}G^{S_2}B/B \cap G^{S_1}B/B$ .

Since  $n^{-1}S_2n = g^{-1}S_2g \subset S_1$ , we have  $G^{n^{-1}S_2n} \subset G^{S_1}$ . In particular,  $n^{-1}G^{S_2}B/B \subset G^{S_1}n^{-1}B/B$ . It follows that  $G^{S_1}n^{-1}B/B = G^{S_1}B/B$ .

Since  $n$  normalizes  $T$ , this implies that  $n \in G^{S_1}$ . So,  $S_2 \subset nS_1n^{-1} = S_1$ .

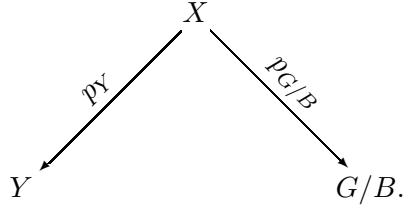
Since  $n \in G^{S_1}$ ,  $nx \in C_1$ . It follows that  $C_1$  is the irreducible component of  $X^{S_1}$  containing  $nx$ . On the other hand,  $C_2$  is the irreducible component of  $X^{S_2}$  containing  $nx$ . It follows that  $C_1$  is contained in  $C_2$ .  $\square$

## 7 GIT-cone and moment polytope

Let us now explain the relation mentioned in the introduction between the moment polytopes of  $Y$  and some total  $G$ -cones of  $X = Y \times G/B$ .

Let us fix a maximal torus  $T$  of  $G$  and a Borel subgroup  $B$  containing  $T$ . Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $Y$ . We consider the set  $P_G(Y, \mathcal{L})$  of the points  $p \in X(T)_{\mathbb{Q}}$  such that for some positive integer  $n$ ,  $np$  is a dominant character of  $T$  and the dual  $V_{np}^*$  of  $V_{np}$  is a submodule of  $H^0(Y, \mathcal{L}^{\otimes n})$ . In fact,  $P_G(Y, \mathcal{L})$  is a polytope, called *moment polytope*. Notice that “the dual” is not usual in the definition; but it will be practical in our context.

Consider the two projections:



In Section 7,  $\Lambda$  will always denote the subgroup of  $\text{Pic}^G(X)$  generated by  $p_{G/B}^*(\text{Pic}^G(G/B))$  and  $p_Y^*(\mathcal{L})$ . Note that  $\text{Pic}^G(G/B)$  identifies with  $X(T)$ ; we will denote  $\mathcal{L}_\nu$  the element of  $\text{Pic}^G(G/B)$  associated to  $\nu \in X(T)$ .

**Proposition 6** *With above notation, we have:*

- (i)  $\mathcal{TC}_\Lambda^G(X) = \mathcal{SAC}_\Lambda^G(X)$ ;
- (ii)  $\mathcal{SAC}_\Lambda^G(X)$  is a cone over  $P_G(Y, \mathcal{L})$ ; more precisely, for all positive rational number  $m$  and  $\nu \in X(T)$ , we have:

$$mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu) \in \mathcal{SAC}_\Lambda^G(X) \iff \frac{\nu}{m} \in P_G(Y, \mathcal{L}).$$

**Proof.** Let  $n$  be a non negative integer and  $\nu$  be a character of  $T$ . As a  $G$ -module,  $H^0(X, p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu))$  is isomorphic to  $H^0(Y, \mathcal{L}^{\otimes n}) \otimes H^0(G/B, \mathcal{L}_\nu)$ . In particular, if  $p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$  has non zero global sections then  $n \geq 0$  and  $\nu$  is dominant; in this case, it is semiample. The first assertion follows.

Assume now that  $\nu$  is dominant. Then,  $H^0(G/B, \mathcal{L}_\nu) = V_\nu$ . Hence,  $p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$  has non zero  $G$ -invariant sections if and only if  $H^0(Y, \mathcal{L}^{\otimes n}) \otimes$

$V_\nu$  contains non zero  $G$ -invariant vectors; that is, if and only if  $V_\nu^*$  is a submodule of  $H^0(Y, \mathcal{L}^{\otimes n})$ . The second assertion follows.  $\square$

**Remark.** To each face  $\mathcal{F}$  of  $\mathcal{AC}_\Lambda^G(X)$ , one can associate a face of  $P_G(Y, \mathcal{L})$  (by intersecting and taking a closure) which intersects the interior of the dominant chamber. By this way, we obtain a bijection between the set of faces of  $\mathcal{AC}_\Lambda^G(X)$  and the faces of  $P_G(Y, \mathcal{L})$  which intersects the interior of the dominant chamber.

### 7.1 A reduction

It is possible that  $\mathcal{AC}_\Lambda^G(X)$  is empty. In this case, our results cannot be applied directly.

Let  $mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$  in the relative interior of  $\mathcal{SAC}^G(X)$  such that  $H^0(X, mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu))^G$  is non zero, that is such that  $V_\nu^*$  can be equivariantly embedded in  $H^0(Y, \mathcal{L}^{\otimes m})$ . Let  $P$  be the standard parabolic subgroup of  $G$  associated to the face of the dominant chamber containing  $\nu$ . Let  $L$  denote the Levi subgroup of  $P$  containing  $T$  and  $D$  denote its derived subgroup.

The next proposition shows that  $\mathcal{SAC}^G(X)$  is always equal to such a cone satisfying  $\mathcal{AC}^G(X) \neq \emptyset$ . The proof which is essentially extracted from [Bri99, Section 5] is included for completeness.

**Proposition 7** *With above notation, there exists an irreducible component  $C_Y$  of  $Y^D$  such that a point  $\mathcal{L} \in \Lambda_\mathbb{Q}$  belongs to  $\mathcal{SAC}^G(X)$  if and only if  $\mathcal{L}|_C$  belongs to  $\mathcal{SAC}^{G^D}(C_Y \times G^D.B/B)$ .*

*Moreover,  $G^D.B/B$  is isomorphic to the variety of complete flags of  $G^D$  and  $\mathcal{AC}^{G^D}(C_Y \times G^D.B/B)$  is non empty.*

**Proof.** The inclusion  $V_\nu^* \subset H^0(Y, \mathcal{L}^{\otimes m})$  gives a  $G$ -equivariant rational map  $\phi : Y \rightarrow \mathbb{P}(V_\nu)$ . Let  $v_\nu$  be a vector of highest weight in  $V_\nu$ ;  $P$  is the stabilizer in  $G$  of  $[v_\nu] \in \mathbb{P}(V_\nu)$ . Let  $\sigma \in V_\nu^*$  be an eigenvector of the Borel subgroup  $B^-$  opposite to  $B$  and containing  $T$ . Let  $Q$  be the stabilizer in  $G$  of  $[\sigma] \in \mathbb{P}(V_\nu^*)$  and  $Q^u$  be its unipotent radical.

Let  $Y_\sigma$  denote the set of  $y \in Y$  such that  $\sigma(y) \neq 0$ . Let  $W$  be a  $L$ -stable supplementary subspace of  $\mathbb{K}.v_\nu$  in  $V_\nu$ . By  $w \mapsto [v_\nu + w]$ , we identify  $W$  with an open subspace of  $\mathbb{P}(V_\nu)$ . Then  $\phi$  induces by restriction  $\tilde{\phi} : Y_\sigma \rightarrow W$ .

Let  $S$  be a  $L$ -stable supplementary to  $T_{[v_\nu]}G.[v_\nu]$  in  $W$  (actually,  $W$  canonically identify with  $T_{[v_\nu]}\mathbb{P}(V_\nu)$ ). Set  $Z = \tilde{\phi}^{-1}(S)$ . By [Bri99, Remark in Section 5],  $Z$  is point wise fixed by  $D$  and the action of  $Q^u$  induces an

isomorphism  $Q^u \times Z \simeq Y_\sigma$ .

Consider  $X' = Y \times G/P$ . Let  $\Lambda'$  be the subgroup of  $\text{Pic}^G(X')$  generated by  $p_Y^*(\mathcal{L})$  and  $p_{G/P}^*(\text{Pic}^G(G/P))$  (with obvious notation). It is clear that  $\mathcal{TC}_\Lambda^G(X)$  identifies with  $\mathcal{TC}_{\Lambda'}^G(X')$ . Moreover,  $\mathcal{AC}_{\Lambda'}^G(X')$  is not empty. Consider a generic closed isotropy  $H$  of  $\mathcal{TC}_{\Lambda'}^G(X')$  viewed as a face  $\mathcal{F}$  of itself. Since  $Q^u \times Z \simeq Y_\sigma \subset X^{\text{ss}}(mp_Y^*(\mathcal{L}) \otimes p_{G/P}^*(\mathcal{L}_\nu))$ , up to conjugacy, one may assume that  $D \subset H \subset L$ . Since  $Y_\sigma \times \{P/P\} \subset X^{\text{ss}}(mp_Y^*(\mathcal{L}) \otimes p_{G/P}^*(\mathcal{L}_\nu))$ ,  $X'_\mathcal{F}$  intersects  $Z \times \{P/P\}$ .

Consider the irreducible component  $C_Y$  of  $Y^D$  which contains  $Z$ . By Theorem 3, for any ample  $\mathcal{L} \in \Lambda'$ ,  $X'^{\text{ss}}(\mathcal{L})$  intersects  $C_Y \times \{P/P\}$  if it is non empty. By continuity, this is also true if  $\mathcal{L}$  is only semiample. The proposition follows easily.  $\square$

## 7.2 Faces of $\mathcal{SAC}^G(X)$ if $\mathcal{AC}^G(Y)$ is non empty

From now on, we assume that  $Y$  is smooth. We will first adapt the notion of covering and well covering pairs for the situation.

Recall that  $T \subset B$  are fixed. Let  $\lambda$  be a one parameter subgroup of  $T$ . Set  $B(\lambda) = B \cap P(\lambda)$ . Let  $C$  be an irreducible component of  $Y^\lambda$  and  $C^+$  the associated Bialinicki-Birula cell.

**Definition.** The pair  $(C, \lambda)$  is said to be *B-covering* if the natural map  $\eta : B \times_{B(\lambda)} C^+ \longrightarrow Y$  is birational. It is said to be *well B-covering* if  $\eta$  induces an isomorphism over an open subset of  $Y$  intersecting  $C$ .

The proof of the following lemma is obvious.

**Lemma 5** *With above notation, the pair  $(C, \lambda)$  is B-covering (resp. well B-covering) if and only if  $(C \times G^\lambda B/B)$  is covering (resp. well covering).*

Let us recall that the subtori of  $T$  correspond bijectively to the linear subspaces of  $X(T)_\mathbb{Q}$ . If  $V$  is a linear subspace of  $X(T)_\mathbb{Q}$ , the associated torus is the neutral component of the intersection of kernels of elements in  $X(T) \cap V$ . If  $F$  is a convex part of  $X(T)_\mathbb{Q}$ , the direction  $\text{dir}(F)$  of  $F$  is the linear subspace spanned by the differences of two elements of  $F$ .

We will denote by  $\mathcal{C}^+$  the convex cone in  $X(T)_\mathbb{Q}$  generated by the dominant weights. The next proposition is an improvement of [Bri99, Theorem 1]:

**Proposition 8** *We keep the above notation and assume that  $Y$  is smooth and  $P_G(Y, \mathcal{L})$  intersects the interior of the dominant chamber. Let  $\mathcal{F}$  be a face of codimension  $d$  of  $P_G(Y, \mathcal{L})$  which intersect the interior of the dominant chamber. Let  $S$  the subtorus of  $T$  associated to  $\text{dir}(\mathcal{F})$ .*

*There exists a unique irreducible component  $C$  of  $Y^S$  and a one parameter subgroup  $\lambda$  of  $S$  such that  $G^\lambda = G^S$  and  $(C, \lambda)$  is a well  $B$ -covering pair such that  $\mathcal{F} = P_{G^S}(C, \mathcal{L}|_C) \cap \mathcal{C}^+$ .*

**Proof.** Let  $\tilde{\mathcal{F}}$  be the face of  $\mathcal{AC}_\Lambda^G(X)$  corresponding to  $\mathcal{F}$  and  $r$  denote its codimension. By Proposition 2, there exists an admissible well covering pair  $(C_X, S')$  such that  $\tilde{\mathcal{F}} = \mathcal{F}(C_X)$  and  $S'$  is a  $r$ -dimensional torus. Up to conjugacy, we may assume that  $C_X$  intersects  $Y \times B/B$ , and  $S'$  is contained in  $T$ . Let  $\lambda$  be a one parameter subgroup of  $S'$  such that  $(C_X, \lambda)$  is well covering. Then,  $C_X = C \times G^\lambda B/B$  for some irreducible component  $C$  of  $Y^{S'}$ .

The fact  $\tilde{\mathcal{F}} = \mathcal{F}(C_X)$  readily means that  $\mathcal{F} = P_{G^{S'}}(C, \mathcal{L}|_C) \cap \mathcal{C}^+$ . Since the direction of  $P_{G^{S'}}(C, \mathcal{L}|_C)$  is contained in  $X(T)^{S'}$ , this implies that  $X(T)^S$  is contained in  $X(T)^{S'}$ . But,  $S$  and  $S'$  have the same rank, it follows that  $S = S'$ .

The unicity part is a direct consequence of Proposition 4.  $\square$

## 8 The case $X = \hat{G}/\hat{B} \times G/B$

### 8.1 Interpretations of the $G$ -cones

From now on, we assume that  $G$  is a connected reductive subgroup of a connected reductive group  $\hat{G}$ . Let us fix maximal tori  $T$  (resp.  $\hat{T}$ ) and Borel subgroups  $B$  (resp.  $\hat{B}$ ) of  $G$  (resp.  $\hat{G}$ ) such that  $T \subset B \subset \hat{B} \supset \hat{T} \supset T$ .

Let  $\mathfrak{g}$  and  $\hat{\mathfrak{g}}$  denote the Lie algebras of  $G$  and  $\hat{G}$  respectively.

We denote by  $\mathcal{LR}(\hat{G}, G)$  (resp.  $\mathcal{LR}^\circ(\hat{G}, G)$ ) the cone of the pairs  $(\hat{\nu}, \nu) \in X(\hat{T})_{\mathbb{Q}} \times X(T)_{\mathbb{Q}}$  such that for a positive integer  $n$ ,  $n\hat{\nu}$  and  $n\nu$  are dominant (resp. strictly dominant) weights such that  $V_{n\hat{\nu}} \otimes V_{n\nu}$  contains non zero  $G$ -invariant vectors.

In this section,  $X$  denote the variety  $\hat{G}/\hat{B} \times G/B$  endowed with the diagonal action of  $G$ . We will apply the results of Section 4 to  $X$  with  $\Lambda = \text{Pic}^G(X)$ . The cones  $\mathcal{TC}^G(X)$ ,  $\mathcal{SAC}^G(X)$  and  $\mathcal{AC}^G(X)$  will be denoted without the  $\Lambda$  in subscribe. By [Res07, Proposition 9],  $\mathcal{LR}^\circ(\hat{G}, G) = \mathcal{AC}^G(X) \subset \mathcal{SAC}^G(X) = \mathcal{TC}^G(X) = \mathcal{LR}(\hat{G}, G)$ . Moreover, if no ideal of  $\mathfrak{g}$  is an ideal of  $\hat{\mathfrak{g}}$ , by [Res07, Assertion (i) of Theorem 9]  $\mathcal{LR}^\circ(\hat{G}, G)$  has non empty interior.

## 8.2 The case $X = \hat{G}/\hat{B} \times G/B$

**8.2.1** — Consider the  $G$ -module  $\hat{\mathfrak{g}}/\mathfrak{g}$ . Let  $\chi_1, \dots, \chi_n$  be the set of the non trivial weights of  $T$  on  $\hat{\mathfrak{g}}/\mathfrak{g}$ . For  $I \subset \{1, \dots, n\}$ , we will denote by  $T_I$  the neutral component of the intersection of the kernels of the  $\chi_i$ 's with  $i \in I$ . A subtorus of the form  $T_I$  is said to be *admissible*.

Let  $\lambda$  be a one parameter subgroup of  $T$ . Consider the parabolic subgroups  $P$  and  $\hat{P}$  of  $G$  and  $\hat{G}$  associated to  $\lambda$ . Let  $W_P$  be the Weyl group of  $P$ . The cohomology group  $H^*(G/P, \mathbb{Z})$  is freely generated by the Schubert classes  $[\overline{BwP/P}]$  parametrized by the elements  $w \in W/W_P$ . Since  $\hat{P} \cap G = P$ , we have a canonical  $G$ -equivariant immersion  $\iota : G/P(\lambda) \longrightarrow \hat{G}/\hat{P}(\lambda)$ ; and the corresponding morphism in cohomology  $\iota^*$ .

Let  $\rho$  (resp.  $\rho^\lambda$ ) denote the half sum of the positive roots of  $G$  (resp.  $G^\lambda$ ). Let  $\Phi^+$  and  $\Phi(P^u)$  denote the set of roots of the groups  $B$  and  $P^u$  for the torus  $T$ . In the same way, we define  $\hat{\Phi}^+$  and  $\Phi(\hat{P}^u)$ . For  $\hat{w} \in \hat{W}$ , we set:

$$\theta^P := \sum_{\alpha \in \Phi^+ \cap \Phi(P^u)} \alpha \in X(T) \quad \text{and} \quad \theta_{\hat{w}}^{\hat{P}} := \sum_{\alpha \in \hat{w}\hat{\Phi}^+ \cap \Phi(\hat{P}^u)} \alpha \in X(\hat{T}).$$

**8.2.2** — Let  $S$  be an admissible subtorus of  $T$ . All irreducible component  $C$  of  $X^S$  such that  $p_{G/B}(C)$  contains  $B/B$  equals  $C(\hat{w}) := (\hat{G}^S \cdot \hat{w}^{-1} \hat{B} / \hat{B} \times G^S B / B)$  for a unique element  $\hat{w} \in \hat{W} / \hat{W}_{\hat{G}^S}$ . Let us fix  $\hat{w} \in \hat{W} / \hat{W}_{\hat{G}^S}$ . The pair  $(S, \hat{w})$  is said to be *admissible* if there exists a parabolic subgroup  $\hat{P}$  of  $\hat{G}$  such that

- (i) there exists  $\lambda \in Y(S)$  such that  $\hat{P} = \hat{P}(\lambda)$ ;
- (ii)  $\hat{G}^S$  is a Levi subgroup of  $\hat{P}$ ;
- (iii)  $G^S$  is a Levi subgroup of  $\hat{P} \cap G =: P$ ;
- (iv)  $\iota^*([\overline{\hat{B}\hat{w}\hat{P}/\hat{P}}]) \cdot [\overline{BP/P}] = [\text{pt}] \in H^*(G/P, \mathbb{Z})$ ;
- (v)  $(\theta_{\hat{w}}^{\hat{P}})|_S = (\theta^P - 2(\rho - \rho^\lambda))|_S$ .

**Lemma 6** *Let  $S$  be an admissible subtorus of  $T$  and  $\hat{w} \in \hat{W} / \hat{W}_{\hat{G}^S}$ . The pair  $(S, \hat{w})$  is admissible if and only if there exists a one parameter subgroup  $\lambda$  of  $S$  such that  $C(\hat{w})$  is an irreducible component of  $X^\lambda$  and  $(C(\hat{w}), \lambda)$  is a well covering pair.*

**Proof.** The proof is very analogous to [Res07, Proposition 10]: we leave details to the reader. We prove (using mainly Kleiman's Theorem) that

$\iota^*([\hat{B}\hat{w}\hat{P}/\hat{P}]) \cdot [\overline{BP/P}] = [\text{pt}] \in H^*(G/P, \mathbb{Z})$  if and only if  $\eta$  is birational. Now, the condition  $(\theta_{\hat{w}}^{\hat{P}})|_S = (\theta^P - 2(\rho - \rho^\lambda))|_S$  means that  $S$  acts trivially on the restriction over  $C$  of the determinant bundle of  $\eta$ .  $\square$

**8.2.3** — To simplify, in the following statement we assume that  $\mathcal{AC}^G(X)$  has a non empty interior in  $\text{Pic}^G(X)_{\mathbb{Q}}$ . In fact, this assumption is equivalent to say that no ideal of  $\mathfrak{g}$  is an ideal of  $\hat{\mathfrak{g}}$ .

**Theorem 5** *We assume that no ideal of  $\mathfrak{g}$  is an ideal of  $\hat{\mathfrak{g}}$ .*

*The map which associates to a pair  $(S, \hat{w})$  the set  $\mathcal{F}(S, \hat{w}) = \{(\hat{\nu}, \nu) \in C^G(X) : \hat{w}\hat{\nu}|_S = -\nu|_S\}$  is a bijection from the set of admissible pairs onto the set of faces of  $\mathcal{AC}_{\Lambda}^G(X)$ . Moreover, the codimension of  $\mathcal{F}(S, \hat{w})$  equals the dimension of  $S$ .*

*The following are equivalent:*

- (i)  $\mathcal{F}(S, \hat{w}) \subset \mathcal{F}(S', \hat{w}')$ ;
- (ii)  $S' \subset S$  and  $\hat{w}W_{GS'} = \hat{w}'W_{GS'}$ .

**Proof.** Let  $(S, \hat{w})$  be an admissible pair. Set  $\overline{\mathcal{F}}(S, \hat{w}) = \{(\hat{\nu}, \nu) \in \mathcal{LR}(G, \hat{G}) : \hat{w}\hat{\nu}|_S = -\nu|_S\}$ . By [Res07, Theorem 9],  $\overline{\mathcal{F}}(S, \hat{w})$  is a face of  $\mathcal{LR}(G, \hat{G})$  of codimension  $\dim(S)$ . In particular,  $\overline{\mathcal{F}}(S, \hat{w})$  spans the vector subspace of the  $(\hat{\nu}, \nu) \in X(\hat{T}) \times X(T)$  such that  $\hat{w}\hat{\nu}|_S = -\nu|_S\}$ . Since  $\mathcal{AC}_{\Lambda}^G(X)$  is the interior of  $\mathcal{LR}(G, \hat{G})$ , to prove that the map in the theorem is well defined, it is enough to prove that  $\overline{\mathcal{F}}(S, \hat{w})$  intersects  $\mathcal{AC}_{\Lambda}^G(X)$ . If not,  $\overline{\mathcal{F}}(S, \hat{w})$  would be contained in the boundary of the dominant chamber. Its projection on  $X(\hat{T})_{\mathbb{Q}}$  or  $X(T)_{\mathbb{Q}}$  would be contained in an hyperplane; which is a contradiction.

The surjectivity is a rephrasing of [Res07, Theorem 9 Assertion (ii)]. The injectivity is a direct application of Proposition 4.

The last assertion follows from Proposition 5.  $\square$

## 9 Application to the tensor product cone

**9.1** — In this section,  $G$  is assumed to be semisimple. As above,  $T \subset B$  are fixed maximal torus and Borel subgroup of  $G$ . We also fix an integer  $s \geq 2$  and set  $\hat{G} = G^s$ ,  $\hat{T} = T^s$  and  $\hat{B} = B^s$ . We embed  $G$  diagonally in  $\hat{G}$ . Now,  $X = \hat{G}/_h B \times G/B = (G/B)^{s+1}$ . Then  $\mathcal{AC}^G(X) \cap X(T)^{s+1}$  identifies with the  $(s+1)$ -uple  $(\nu_1, \dots, \nu_{s+1}) \in X(T)^{s+1}$  such that the for  $n$  big enough

$n\nu_i$ 's are strictly dominant weights and  $V_{n\nu_1} \otimes \cdots \otimes V_{n\nu_{s+1}}$  contains a non zero  $G$ -invariant vector.

A parabolic subgroup  $P$  of  $G$  is said to be *standard* if it contains  $B$ . We will denote by  $Z(P)$  the neutral component of the center of the Levi subgroup of  $P$  containing  $T$ .

**9.2** — In [BK06], Belkale and Kumar defined a new product denoted  $\odot_0$  on the cohomology groups  $H^*(G/P, \mathbb{Z})$  for any parabolic subgroup  $P$  of  $G$ . We consider the set  $\Theta$  of the  $(P, \Lambda_{w_0}, \dots, \Lambda_{w_s})$  where  $P$  is a standard parabolic subgroup of  $G$  and the  $\Lambda_{w_i}$ 's are  $s+1$  Schubert varieties of  $G/P$  such that

$$[\Lambda_{w_0}] \odot_0 \cdots \odot_0 [\Lambda_{w_s}] = [\text{pt}].$$

**9.3** — In [Res07], Theorem 9 applied to  $\hat{G} = G^s$  gives Corollary 5. The same translation of Theorem 5 to this case gives the following:

**Theorem 6** *The map which associates to a  $(P, \Lambda_{w_0}, \dots, \Lambda_{w_s}) \in \Theta$  the set  $\mathcal{F}(P, \Lambda_{w_0}, \dots, \Lambda_{w_s})$  of the  $(\nu_0, \dots, \nu_s) \in \mathcal{AC}^G(X)$  such that the restriction of  $\sum_i w_i^{-1} \nu_i$  to  $Z(P)$  is trivial is a bijection from  $\Theta$  onto the set of faces of  $\mathcal{AC}^G(X)$ . Moreover, the codimension of  $\mathcal{F}(P, \Lambda_{w_0}, \dots, \Lambda_{w_s})$  equals the dimension of  $Z(P)$ .*

*The following are equivalent:*

- (i)  $\mathcal{F}(P, \Lambda_{w_0}, \dots, \Lambda_{w_s}) \subset \mathcal{F}(P', \Lambda_{w'_0}, \dots, \Lambda_{w'_s})$ ;
- (ii)  $P \subset P'$  and  $\pi(\Lambda_{w_i}) = \Lambda_{w'_i}$  for all  $i = 0, \dots, s$  (here,  $\pi : G/P \rightarrow G/P'$  is the natural  $G$ -equivariant map).

## References

- [BK06] Prakash Belkale and Shrawan Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. **166** (2006), no. 1, 185–228.
- [Bri99] Michel Brion, *On the general faces of the moment polytope*, Internat. Math. Res. Notices (1999), no. 4, 185–201.
- [DH98] Igor V. Dolgachev and Yi Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 5–56, With an appendix by Nicolas Ressayre.
- [LR79] D. Luna and R. W. Richardson, *A generalization of the Chevalley restriction theorem*, Duke Math. J. **46** (1979), no. 3, 487–496.



- [Lun73] Domingo Luna, *Slices étales*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33.
- [Lun75] D. Luna, *Adhérences d'orbite et invariants*, Invent. Math. **29** (1975), no. 3, 231–238.
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3d ed., Springer Verlag, New York, 1994.
- [PV91] V. L. Popov and È. B. Vinberg, *Algebraic Geometry IV*, Encyclopedia of Mathematical Sciences, vol. 55, ch. Invariant Theory, pp. 123–284, Springer-Verlag, 1991.
- [Res00] N. Ressayre, *The GIT-equivalence for  $G$ -line bundles*, Geom. Dedicata **81** (2000), no. 1-3, 295–324.
- [Res07] Nicolas Ressayre, *Geometric invariant theory and generalized eigenvalue problem*, Preprint (2007), no. arXiv:0704.2127, 1–45.
- [Sja98] Reyner Sjamaar, *Convexity properties of the moment mapping re-examined*, Adv. Math. **138** (1998), no. 1, 46–91.

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