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***Abstract*— In this paper, dynamic practical stability properties of infinite-dimensional sampled-data systems are discussed. A family of finite-dimensional discrete-time controllers are first designed to uniformly exponentially stabilize numerical approximate models that are obtained from space and time discretization. Sufficient conditions are provided to ensure that these controllers can be used to drive trajectories of infinite-dimensional sampled-data systems to a neighborhood of the origin by properly tuning the sampling period, space and time discretization parameters and choosing an appropriate filtering process for initial conditions.**

I. INTRODUCTION

Linear distributed parameter systems (LDPS) arise in a range of different processes such as optical telecommunications, fluid flows, thermal processes, biology, chemistry, environmental sciences, mechanical systems, and so on. LDPS are modelled by linear partial differential equations (PDEs) or abstract differential equations in an infinite-dimensional space, as opposed to linear lumped parameter systems (LLPS) that are modelled by linear ordinary differential equations (ODEs) in a finite-dimensional space.

In this paper, we consider systems governed by partial differential equations with appropriate initial and boundary conditions that can be represented by the following abstract differential equation,

$$\dot{x}(t) \quad = \quad Ax(t) + Bu(t), x(0) = x_0 \in X, \forall t \geq 0. \quad (1)$$

Here the state $x(t)$ belongs to a Banach X and the control input $u(t)$ belongs to a subset of Banach space U. The operator A maps from $D(A)$ to X, $D(A)$ is the domain of A, which is a subset in X. The operator B is a control operator (in general, unbounded) on U.

Nowadays most control systems are implemented using digital technology since it is very cheap, fast, relatively easy to operate, flexible and reliable. This motivates the investigation of the so called *sampled-data systems* that consist of a continuous-time plant or process controlled by a discrete-time controller as discussed in [5], [22], [23]. The plant and the controller are interconnected via the analog-to-digital (A-D) and digital-to-analog (D-A) converters. Consequently, the designed controller needs to be time-discretized in order to be implemented using the digital technology. Due to prevalence of the computer controlled systems, it is often assumed that the system (1) is between a sampler (A-D converter) and a zero-order-hold (D-A converter). Let $T > 0$ denote a sampling period. The control signal is assumed to be piecewise constant,

$$u(t) = u(kT) \stackrel{\triangle}{=} u(k), \forall t \in [kT, (k+1)T), k \in \mathbb{N}, \quad (2)$$

where $\mathbb{N}$ is the set of integers. In the sequel, the following "sampled-data system" is obtained.

$$\dot{x}(t) \quad = \quad Ax(t) + Bu(kT), \qquad x(0) = x_0 \in X, \quad (3)$$

for all $t \in [kT, (k+1)T), k \in \mathbb{N}$. The control input $u(kT)$ needs to be designed so that trajectories of the sampled-data system (3) converge to the origin, or a neighborhood of the origin.

Sampled-data control of linear infinite-dimensional system (1) has been discussed in [21], [26], and references cited therein. In these references, *an infinite-dimensional continuous-time* feedback controller was first designed to stabilize the system (1) without consideration of sampling in time, followed by a time-discretization in order to implement digitally. This is so called an "indirect method", which consists in designing a controller on the continuous model, and then in discretizing the closed-loop system.

Given a controlled PDE (1), it is however not always possible to guess an expression of a feedback controller stabilizing the system. In this article, we rather propose a direct approach, which consists in designing such a controller from finite-dimensional approximations that are obtained from space and time discretization. The reason for designing controllers for approximate models of (1) lies in several aspects. First of all, in general, analytical solutions of the infinite-dimensional system (1) or (3) are not possible. In various engineering applications, it is very natural to use numerical solutions. These numerical solutions are generated by numerical approximate models that come from numerical algorithms such as finite difference methods, finite element methods, Galerkin approximations and so on. As these numerical algorithms are available, engineers just need to pick up one that is applicable to the particular application. Furthermore, it can be very efficient for some applications in which very accurate discrete models are available. Secondly, it is appealing for engineers to design controllers for discretization models. Although there is a large number of publications on stabilization of systems like (1), see Russell [33], Lions [19], Komornik [13], Curtain

This research was supported by the Australia Research Council under the Discovery Project DP0664501

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and Zwart [9], Lasiecka and Triggiani [17] and references therein, it may be a very difficult task to find control laws for infinite-dimensional system (1) or (3). At last, the family of finite-dimensional discrete-time controllers are easy to be implemented. Indeed, while infinite-dimensional controllers are theoretically very important and often arise naturally in theory, the controller has to be finite-dimensional to be implemented digitally.

It is by now well known that the scheme "control design/ discretization" is not commutative (see e.g. [38]). Whereas it is quite easy to prove convergence results for an indirect method, with standard assumptions and a standard Lax procedure (see [18]), obtaining a convergence result for a direct approach may be very challenging, due to a possible loss of uniformity. Actually, when implementing a direct approach, the standard assumptions which are usually ensuring the convergence of a given scheme, namely, consistency plus uniform boundedness (or stability), are not enough in general to ensure the convergence of the family of controls designed from the approximate models towards the control of the continuous model. As explained in [38], this phenomenon is due to an interference of high frequencies with the mesh of the discretization; this interference may create spurious high frequency oscillations which, as in a resonance phenomenon, infer the divergence of the direct procedure.

In this paper we assume that there exists a family of finite-dimensional discrete-time approximate models in the vector spaces X_h and U_h after space and time discretization. Here h is a parameter for the space approximation and Δt is a parameter for time-discretization. Both h and Δt are sufficiently small. This family of approximate models, represented in terms of $x_h^a(j\Delta t)$[1], take the following form,

$$x_h^a\left((j+1)\Delta t\right) = A_{h,\Delta t}^a x_h^a\left(j\Delta t\right) + B_{h,\Delta t}^a u_{h,\Delta t}\left(j\Delta t\right), \qquad (4)$$

with the initial condition $x_h^a(0) \in X_h$ and $j \in \mathbb{N}$. Here $x_h^a \in X_h$, $u_h \in U_h$ and $A_{h,\Delta t}^a : X_h \to X_h$ and $B_{h,\Delta t}^a : U_h \to X_h$.

Furthermore, we assume that the family of controllers $u_{h,\Delta t}$ are carefully designed in the sense that they can uniformly exponentially stabilize approximate models (4). More specifically, the solutions of the approximate models (4) satisfy

$$\left\|x_h^a\left((j+1)\Delta t\right)\right\|_{X_h} \leq M_a e^{-\lambda_a j\Delta t}\left\|x_h^a(0)\right\|_{X_h},$$

for some positive constants M_a and λ_a that are independent of the choice h and Δt.

As mentioned in [25], it is not always possible to construct proper control sequence $u_{h,\Delta t}$ to uniformly exponentially stabilize a family of finite-dimensional approximations (4) due to the existence of spurious high frequency modes. Uniform stability properties (controllability and/or observability) of the family of approximation control systems have been investigated in [36], [38], [25], [37], [40], [41] for different discretization processes, on different systems. In this paper, for simplicity of the presentation, we just assume the existence of such "good" controllers. How to design them is outside the scope of this paper. We however refer readers to references in [2], [3], [10], [12], [17] to results concerning the design of control laws having such uniform properties, based on a Riccati procedure.

Once a "good" control sequence $u_{h,\Delta t}$ is available, our aim is to find (sharp) sufficient conditions that can ensure this control sequence can be used to drive trajectories of the infinite-dimensional sampled-data system (3) to the origin (or a small neighborhood of the origin).

It is important to note that the control input applied to the sampled-data system (3) is computed from approximate models (4). In other words, the controller $u(kT)$ in (3) is generated from a family of finite-dimensional discrete-time controllers $u_{h,\Delta t}(j\Delta t)$. Thus $u(kT)$ is not in a typical continuous-time state-feedback form that is obtained from state measurement $x(t)$ of the system (1). As $u_{h,\Delta t}(j\Delta t)$ can be treated as a kind of "memory" variable, we adopt the terminology introduced in [30], [31], "dynamic (practical) stabilization" is used in this paper.

We stress that we do not assume that there exists an infinite-dimensional controller that can practically exponentially stabilize the exact infinite-dimensional continuous-time system (1) and prove that finite-dimensional discrete-time controllers computed from numerical approximations converge uniformly to the desired one as the discretization parameters tend to zero as done in the literature, in particular, in the context of the Riccati theory, see, for example, [3], [2], [10], [20], [12], [17], [15] as reference therein. Our result (Theorem 1) provides sufficient conditions to ensure the "practical stability properties" of a general class of sampled-data LDPS by using a dynamic feedback that can "uniformly exponentially stabilize" numerical approximate models (4). More precisely, we obtain sufficient conditions to ensure that for any given positive pair (Δ, ν), there exist a filtering process depending on (Δ, ν) such that for any filtered initial condition, trajectories of the infinite-dimensional sampled-data system (3) with the control input sequence $u(kT)$ that is generated from $u_{h,\Delta}(j\Delta t)$ will converge to a ν-neighborhood of origin by properly tuning the sampling period T, numerical discretization parameters (h and Δt) and choosing an appropriate filtering process. These sufficient conditions include

1 The trajectories from numerical approximations (4) have to be "good" enough to well approximate the trajectories of the exact system (1).
2 $u_{h,\Delta t}$ has to be uniformly bounded with respect to space and time discretization parameters.
3 The proper filtering process determined by (Δ, ν) is needed to filter out high frequency components of the initial data.
4 The filtering process has to be compatible with the uniform stability properties of the approximate models.

To the best of our knowledge, it is the first time to address the *practical exponential stability properties* of a general class of *sampled-data LDPS* by using a dynamic feedback that is generated from numerical approximate models (4). Our result can provide useful guideline to choose the sampling period, appropriate numerical schemes including discretization parameters as well as the proper filtering process. It is also worthwhile to highlight that though conditions in

[1]In this paper, $x_h^a(\cdot)$ represents trajectories generated from approximate models in the vector space X_h.

the main result are sufficient, they are "sharp". If one of the conditions is not satisfied then the conclusion of the main result may fail (counterexamples can be found in literature).

This paper is organized as follows. Section 2 presents preliminaries and problem formulation. Sufficient conditions and main results of this paper are stated and discussed in Section 3 followed by the conclusions in Section 4.

II. Preliminaries and Problem Formulation

In this paper, X and U are Banach spaces with their norms denoted as $\|\cdot\|_X$ and $\|\cdot\|_U$ respectively. X' is the dual space of X. Let $S(t)$ denote a strongly continuous semigroup (C_0-semigroup) on X, of generator $(A, (D(A))$ in the system (1). Let $\alpha > 0$, $X_{-\alpha}$ denote the completion of X for the norm $\|x\|_{-\alpha} = \|(\beta I_X - A)^{-\alpha} x\|$, where a real number $\beta \in \rho(A)$ is fixed, $\rho(A)$ is the resolvent set of A and I_X is the identity in X. The semigroup $S(t)$ can be extended to a C_0-semigroup on $X_{-\alpha}$, denoted by the same symbol, and the generator of this extended semigroup is an extension of A, still denoted A. With this notation, A is a linear operator from X to $X_{-\alpha}$. Since A generates a C_0-semigroup, there exists a real number $\omega \in \rho(A)$ such that $A - \omega I_X$ is invertible. Denote $\hat{A} = A - \omega I_X$ and the fraction powers of $(-\hat{A})^\alpha$ are well-defined. A^* is the adjoint of A.

We denote $\mathcal{L}(X, Y)$ as the space of all linear bounded operators from X to Y where both X and Y are Banach spaces and $\mathcal{L}(X) \triangleq \mathcal{L}(X, X)$.

The set of integers is denoted as $\mathbb{N}$, the set of real numbers is denoted as $\mathbb{R}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}_\infty$ if $\gamma(0) = 0$ and strictly increasing and $\lim_{s \to \infty} \gamma(s) = \infty$. The set B_Δ is defined as $B_\Delta \triangleq \{x \in X \,|\, \|x\|_X \leq \Delta\}$.

The control operator B in (1) is not necessarily bounded.[2]. However it is assumed that $B \in \mathcal{L}(U, X_{-\alpha})$ and B in (1) is admissible (see Definition 1).

Definition 1: An unbounded linear control operator $B \in \mathcal{L}(U, X_{-\alpha})$ is called admissible for the semigroup $S(t)$ if the weak solution of (1) with $x(0) = x_0 \in X$ belongs to X for every $t \geq 0$, whenever $u \in \mathcal{L}^2([0, \infty], U)$ and (1) holds true in $X_{-\alpha}$. The weak solution can be represented in the following form

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau \in X, \forall t \geq 0. \quad (5)$$

Define $L_t u \triangleq \int_0^t S(t-s)Bu(s)ds$, the admissible controller operator B is equivalent to requiring that $L_t \in \mathcal{L}(\mathcal{L}^2(0, t; U), X)$. The admissible control operator implies that the system (1) is well-posed in the sense that the weak solutions of (1) exist.

With the sampled-data controller defined in (2), the following weak solutions of the system (1) are obtained:

$$x(t) = S(t - kT)x(kT) + \int_{kT}^t S(t-\tau)Bd\tau u(kT), \quad (6)$$

for all $t \in [kT, (k+1)T)$ and $k \in \mathbb{N}$.

[2]The control operator B is called bounded if it maps boundedly into the state space X. Otherwise B is called "unbounded" (with respect to the state space X). Unbounded control operators appear naturally when dealing with boundary or pointwise controls.

A. Numerical approximation

A family of finite-dimensional discrete-time approximations (4) are obtained by discretizing (1) in both time Δt and space h. In this paper, time-discretization is after space discretization and the numerical approximations have to be "good" enough to well-approximate the behavior of the system (1). First we introduce adapted assumptions on space approximations, inspired by [17], [15]. Consider two families $(X_h)_{0<h<h_o}$ and $(U_h)_{0<h<h_o}$ of finite-dimensional spaces.

Assumption 1: [Consistency of the space semi-discretization scheme] For every $h \in (0, h_o)$, there exist mappings $R_h : D\left((-\hat{A}^*)^{-\alpha}\right)' \to X_h$, $P_h : X_h \to D\left((-\hat{A}^*)^{-\alpha}\right)$, $\bar{R}_h : U \to U_h$ and $\bar{P}_h : U_h \to U$ such that the following conditions hold.

1 For every $h \in (0, h_o)$, the following holds

$$R_h P_h = I_{X_h}, \quad \bar{R}_h \bar{P}_h = I_{U_h}, \quad (7)$$

where I_{X_h} and I_{U_h} are identities in X_h and U_h respectively.

2 For any $\phi \in D(A^*)$ or any $\psi \in U$, we have

$$\|\{I_X - P_h R_h\}\phi\|_X \xrightarrow[h \to 0]{} 0, \quad (8)$$

$$\left\|\left\{I_U - \bar{P}_h \bar{R}_h\right\}\psi\right\|_U \xrightarrow[h \to 0]{} 0, \quad (9)$$

where I_U is the identity in U. ∘

Remark 1: Equation (8) (or (9)) means that for each $\phi \in X$ (or $\psi \in U$), as $h \to 0$, $P_h R_h \phi$ (or $\bar{P}_h \bar{R}_h \psi$) approaches ϕ (or ψ). However, the convergence is not uniform, i.e., for different ϕ (or ψ), the convergence speed will be different.

For every $h \in (0, h_o)$, the vector spaces X_h and U_h are endowed with the norm $\|\cdot\|_{X_h}$ and $\|\cdot\|_{U_h}$ defined as follows

$$\|\phi_h\|_{X_h} \triangleq \|P_h \phi_h\|_X, \forall \phi_h \in X_h; \quad (10)$$

$$\|\psi_h\|_{U_h} \triangleq \left\|\bar{P}_h \psi_h\right\|_U, \forall \psi_h \in U_h. \quad (11)$$

Remark 2: With the endowed norms defined in (10) and (11), it is obvious that P_h and $\bar{P}_h$ are both linear bounded operators satisfying

$$\|P_h\|_{\mathcal{L}(X_h, X)} = \left\|\bar{P}_h\right\|_{\mathcal{L}(U_h, U)} = 1. \quad (12)$$

Remark 3: By using the Banach-Steinhaus Theorem, Condition 2 in Assumption 1 implies that both R_h and $\bar{R}_h$ are linear bounded operators, whose bounds are uniform in h. That is, for all $h \in (0, h_o)$, there exists $B_R > 0$, independent of h, such that

$$\|R_h\|_{\mathcal{L}(X, X_h)} \leq B_R, \quad \left\|\bar{R}_h\right\| \leq B_R. \quad (13)$$

For every $h \in (0, h_o)$, we define the approximation operator $A_h^* : X_h \to X_h$ of A^* and $B_h^* : X_h \to U_h$ of B^*, by $A_h^* = R_h A^* P_h$ and $B_h^* = R_h B^* \bar{P}_h$. We set $A_h = (A_h^*)^*$ and $B_h = (B_h^*)^*$ with respect to the pivot space X and U.

Other than good space-discretization, a good time-discretization is also needed.

Assumption 2: [Time approximation] Let A_h and B_h come from Assumption 1. For any $h \in (0, h_o)$, under an appropriate Courant-Friedrichs-Lewy (CFL) condition, there exists $\Delta t_o^*(h) > 0$ and $\rho_h(\cdot) \in \mathcal{K}_\infty$ such that for all

$\Delta t \in (0, \Delta t_o^*(h))$ such that for any $\varphi_h \in X_h$ and $\nu_h \in U_h$, the following conditions hold

$$\left\|\left\{e^{A_h \Delta t} - A_{h,\Delta t}^a\right\}\varphi_h\right\|_{X_h} \leq \Delta t \rho_h(\Delta t)\left\|\varphi_h\right\|_{X_h} \quad (14)$$

$$\left\|\left\{B_h \Delta t - B_{h,\Delta t}^a\right\}\nu_h\right\|_{X_h} \leq \Delta t \rho_h(\Delta t)\left\|\nu_h\right\|_{U_h}. \quad (15)$$

Moreover, for any $t > 0$, there exists $\bar{B}_{A,a} = \bar{B}_{A,a}(t,h) > 0$ such that

$$\left\|(A_{h,\Delta t})^j\right\|_{\mathcal{L}(X_h)} \leq \bar{B}_{A,a}, \quad (16)$$

for all $j\Delta t \in [0,t]$ and $\Delta t \in (0, \Delta t_o^*(h))$. ∘

When A_h is obtained after space semi-discretization, if a finite difference method is used, $A_{h,\Delta t}^a$ and $B_{h,\Delta t}^a$ can be

$$A_{h,\Delta t}^a = \Delta t A_h + I_{X_h}, B_{h,\Delta t} = \Delta t B_h,$$

satisfying Assumption 2 with $\rho_h(\Delta t) = \max\left\{\|A_h\|_{\mathcal{L}(X_h)}^2 \Delta t, \Delta t\right\}$.

Remark 4: Time discretization is done after the space discretization parameter h is fixed. In general, the CFL condition [8], [11] is needed to ensure that numerical approximations after space and time discretization are "uniformly bounded" on compact intervals (or stable). By using the well-known Lax-Richtmyer Equivalence Theorem [18], the CFL condition ensures that solutions of numerical approximations can well approximate solutions of the exact continuous-time infinite-dimensional system. The CFL condition requires that the discretization of time Δt should be sufficiently smaller than the discretization of space h. Therefore, in numerical discretization schemes, h is chosen first and Δt is chosen accordingly.

Remark 5: Assumption 1 is very general. It holds for almost all of the classic numerical space semi-discretization approximation schemes such as finite-difference methods, finite-element methods, Galerkin methods, spectral methods and so on. Assumption 2 is also very general. ∘

B. Controller design

Once a family of finite-dimensional discrete-time numerical approximation systems (4) are obtained, the control input $u_{h,\Delta t}$ is designed to stabilize the approximation system (4). A family of "feedback" controllers are used in this paper. That is $u_{h,\Delta t}(j\Delta t) = K_{h,\Delta t} x_h^a(j\Delta t)$. The closed-loop of the approximation system (4) becomes

$$\begin{aligned} x_h^a((j+1)\Delta t) &= A_{h,\Delta t}^a x_h^a(j\Delta t) + B_{h,\Delta t}^a u_{h,\Delta t}(j\Delta t) \\ &= (A_{h,\Delta t}^a + B_{h,\Delta t}^a K_{h,\Delta t}) x_h^a(j\Delta t), \end{aligned} \quad (17)$$

where $x_h^a(0) \in X_h, \forall j \in \mathbb{N}$.

The feedback gain operator $K_{h,\Delta t}$ maps from X_h to U_h and is parameterized by discretization parameters $(h, \Delta t)$. As discussed in Introduction, it is assumed that we have "good" controllers that can achieve some nice "uniform properties" of approximate models.

Assumption 3: The family of finite-dimensional discrete-time linear approximate models (17) are exponentially stable, uniformly in small h. That is, let M_a and λ_a be positive constants, there exists $h_1^* > 0$ such that for any $h \in (0, h_1^*)$, there exists $\Delta t_1^*(h) > 0$ such that for any $\Delta t \in (0, \Delta t_1^*(h))$, solutions of systems (17), denoted as $x_h^a(j\Delta t) = x_h^a(j\Delta t; x_h^a(0))$, satisfy

$$\left\|x_h^a(j\Delta t; x_h^a(0))\right\|_{X_h} \leq M_a e^{-\lambda_a j \Delta t}\left\|x_h^a(0)\right\|_{X_h}, \quad (18)$$

for all $j \in \mathbb{N}$. ∘

As indicated in Introduction, Assumption 3 is a basic assumption in the proposed controller design method.

For simplicity, $x_h^a(j; x_h(0))$, trajectories of the closed-loop system (17) can be represented as $\tilde{S}_{h,\Delta t}(j\Delta t)x_h(0)$ where $\tilde{S}_{h,\Delta t}$ is a semigroup generated by (17). Using Assumption 3, it can be derived that

$$\left\|\tilde{S}_{h,\Delta t}(j\Delta t)\right\|_{\mathcal{L}(X_h)} \leq M_a e^{-\lambda_a j \Delta t}. \quad (19)$$

Remark 6: Assumption 3 is consistent with numerical discretization (Assumption 1 and Assumption 2). Since in numerical schemes, the choice of Δt_1^* depends on the choice of h, the stability property is only uniform in small h, (see definition of uniform stability in small parameters in [34, Defintion 1]), though the choices of M_a and λ_a depend on neither Δt nor h. ∘

Remark 7: In [25], [38], [40], [41], numerical viscosity was employed in the numerical schemes to ensure uniform stability of a family of finite-dimensional approximations. Adding such a viscosity in numerical schemes can ensure that Assumption 3 holds. ∘

It is worthwhile to highlight that the time-discretization parameter Δt has to be "different" from the sampling period T. The CFL condition requires that the choice of proper Δt depends on the choice of h in order to ensure that trajectories of approximate models can well approximate trajectories of the exact model. Usually, Δt is much smaller than h and can be very small. Since the sampling period cannot be arbitrarily small due to the hardware limitation, the time-discretization parameter and the sampling period have to be different. T is typically much larger than Δt. To simplify the presentation, we assume that the ratio between T and Δt is an integer, i.e. $\frac{T}{\Delta t} = N, N \in \mathbb{N}, N \geq 1$.

After mapping from U_h to U by using the operator $\bar{P}_h$ (see, Assumption 1) and with the consideration of the sampling period T, we have

$$u_T = u_{T,h,\Delta t}(k; x_0) = \bar{P}_h K_{h,\Delta t} x_h^a(jN\Delta t; x_h^a(0)), \quad (20)$$

where $x_h^a(j\Delta t; x_h^a(0)) \in X_h$ is the solutions obtained from (17) and they satisfy Assumption 3. With the pre-designed control sequences from (20), the sampled-data system (3) thus becomes

$$\dot{x}(t) = Ax(t) + B\bar{P}_h K_{h,\Delta t} u_{T,h,\Delta t}(k; x_0). \quad (21)$$

The controller $u_{T,h,\Delta t}(k; x_0)$ in (20) is generated from a family of finite-dimensional discrete-time approximate models, it is easy to be implemented in practice. On the other hand, it is well-known in literature that when wave-like systems are considered, numerical approximations may generate spurious solutions that do not exist in exact model, leading to divergent trajectories as discussed in [38], [40], [41] and reference therein. Therefore, Assumptions 1-3 are not enough to ensure that trajectories of the infinite-dimensional

sampled-data system (21) will converge. Other sufficient conditions are needed.

Control objective. Assume that Assumptions 1-3 hold. This paper aims at providing sufficient conditions to ensure that the trajectories of (21) will converge to a neighborhood of the origin.

Remark 8: The sampled-data system (21) can be re-written as

$$\dot{x}(t) = Ax(t) + \bar{B}\bar{u}_h(kT), \tag{22}$$

with $\bar{B} \triangleq B\bar{P}_h$ and $\bar{u}_h \triangleq K_{h,\Delta t}u_{T,h,\Delta t}(k;x_0)$. Since the controller $\bar{u}_h(kT)$ is in a finite-dimensional space X_h, the control objective becomes to find sufficient conditions to ensure that a finite-dimensional controller can drive an infinite-dimensional sampled-data system (3) to a neighborhood of the origin exponentially. ∘

If there are infinitely many unstable modes, it is not possible to design a finite-dimensional controller that can drive trajectories of general infinite-dimensional systems (22) to the origin (or a neighborhood of the origin) as pointed out in [28]. Therefore, other sufficient conditions are indeed needed.

Our main result provides sufficient conditions to ensure that for any initial condition x_0 that is well-behaved, the control input $u_{T,h,\Delta t}(k;x_0)$ can drive trajectories of the sampled-data system (21) to some neighborhood of the origin. The size of the neighborhood can be chosen arbitrarily small by tuning the parameters $(T,h,\Delta t)$ properly and choosing the appropriate filtering process with respect to the initial condition.

III. Sufficient Conditions and Main Result

This section discusses sufficient conditions that can ensure that the controller (20) can gradually move trajectories of the sampled-data system (21) to some neighborhood of the origin. It is followed by the statement of the main results (Theorem 1 and Corollary 1). Due to space limitation, the proof is omitted. Moreover, the necessity of conditions in main result is also discussed.

A. The existence of ϵ-filtering

Assumption 4: Let $\epsilon > 0$ be arbitrary. There exist a linear continuous operator $\mathcal{F} = \mathcal{F}(\epsilon)$ satisfying $\mathcal{F} : X \to X$ and a subspace $E = E(\epsilon) \subset X$ with $dim(E) < \infty$, such that the following conditions hold

- (a) $X = E \bigoplus F$ for some closed $F = F(\epsilon)$, where $\bigoplus$ denotes the direct sum.
- (b) $\forall x \in X$, $\mathcal{F}x = x^E + x^F$ with $\left\|x^F\right\|_X \leq \epsilon \|x\|_X$, where $x^E \in E$ and $x^F \in F$.
- (c) For any $x \in E$ (or $x \in F$), we have $Ax \in E$ (or $Ax \in F$), that is, A induces operators $A_E : E \to E$ and $A_F : F \to F$. A_E and A_F generate C_0-semigroups $S_E(t)$ and $S_F(t)$ respectively.
- (d) $S_F(t)$ is a uniformly bounded linear operator, i.e., there exists $B_F > 0$ such that $\|S_F(t)\|_{\mathcal{L}(X)} \leq B_F$.

We denote $\left\|x^E\right\|_E \triangleq \left\|P_E x^E\right\|_X$, where P_E is a linear operator mapping from E to X. Another linear operator R_E is a map from X to E. Both P_E and R_E are bounded linear operators satisfying $\|P_E\|_{\mathcal{L}(E,X)} = 1$ and $\|R_E\|_{\mathcal{L}(X,E)} \leq B_E$ and $R_E P_E = I_E$ where I_E is the identity on E. For any $x \in X$, it is said filtered if $\mathcal{F}(\epsilon)x = x$, i.e, $x \in Ker(\mathcal{F}(\epsilon) - I_X)$.

A_E is the restriction of A on E while A_F is the restriction of A on F. Condition (c) in Assumption 4 implies that the infinitesimal generator A can be decomposed as $A = \begin{bmatrix} A_E & 0 \\ 0 & A_F \end{bmatrix}$. In particular, the system (1) can be re-written as follows (see [33, Page 711] for more details)

$$\dot{x}^E = A_E x^E + B_E u(t) \tag{23}$$

$$\dot{x}^F = A_F x^F + B_F u(t), \tag{24}$$

where $B_E = P_E R_E B$ and $B_F = (I_X - P_E R_E)B$. Intuitively, when diagonalizing, if possible, the C_0-semigroup $S(t)$, E contains a finite number of unstable modes and F contains an infinite number of stable (non-positive) modes. Note that, such a decomposition method is widely used in the controller design of PDEs. For example, it was used in a pole shifting process in [33] and in structural assignment for parabolic equations using Dirichlet boundary feedback in [16] and sampled-data control of infinite-dimensional systems in [21]. It was also used in [6] for stabilization of semilinear heat equations and in [7] for stabilization of semilinear wave equations.

As discussed in [33], A_E can be generated by spanning first M unstable eigenvalues of the operator A and A_F can be generated by spanning infinitely many stable eigenvalues of A. If A_E is known exactly, by constructing appropriate feedback control laws to stabilize M unstable modes without moving the others, it is possible to obtain the practical stability properties of the system (21) when the initial condition is filtered. However E may not be known a *priori* in most applications. In most cases, engineering practitioners can "guess" what E should be with some uncertainty measure (noise on estimation) up to certain precision level ϵ as indicated in Condition (b) in Assumption 4.

Assumption 4 also requires the existence of ϵ-filtering for any given ϵ (the quality of the filter). That is, for any $\epsilon > 0$ arbitrary, it is possible to construct a well-designed filter so that Condition (b) holds. As will be shown in Theorem 1, how to choose ϵ for the filter depends on the set in which the initial condition stays and the offset the neighborhood of the origin to which the solutions of the system (6) converge. The smaller the ν (the size of the neighborhood) and the larger the Δ (the set containing the initial condition), the smaller the ϵ is needed, requiring a better filtering process (see Theorem 1). This is a strong filtering requirement.

In practice, it is not always possible to obtain a filtering process that can achieve the given precision requirement ϵ. A weaker version of Assumption 4 can be used when practitioners know roughly on finite-dimensional space E and can guess the size of ϵ with respect to the E. Then a weaker result for the system (21) will be obtained, in which ν, the size of the neighborhood of the origin (or offset) will be determined by ϵ. The better is the filter (the smaller ϵ), the smaller ν will be obtained in practical stabilization (see Corollary 1). Designing a proper filtering process is natural in the control of the PDEs and the performance of PDEs

depends on the choice of filtering processes. Due to space limitation, how to implement this ϵ-filtering is outside of the scope of this paper.

B. High frequency filtering property

If A_E is known, the system (1) can be practically stabilized by a finite-dimensional controller on E. However, A_E is not completely known. The approximation of E is known according to Assumption 4 with the approximation error up to ϵ. Since B is an unbounded control operator, the existence of ϵ can perturb stable modes of the sampled-data system (3) through B and may lead to divergent trajectories. Therefore, the following assumption is also needed.

Assumption 5: "High frequency filtering property" (HFFP) with respect to ϵ-filtering is satisfied for the system (1). That is, let ϵ be from Assumption 4, for any $y(\cdot) \in \mathcal{L}^2([0,t],U)$ satisfying $\|y(t)\|_U \leq e^{-\lambda_y t}, \forall t \geq 0$ for some positive λ_y, the following inequality holds

$$\left\| \int_0^t S_F(t-s)(I_X - P_E R_E) B y(s) ds \right\|_X \leq \epsilon, \qquad (25)$$

for all $t \geq 0$.

"HFFP" assumption is new and it plays a crucial role in our general problem setting. It reflects what has been done by E. Zuazua and his coauthors in their work in which filtering out high frequencies components are needed (see discussion in [38], [39], [40], [41]). It is also consistent with what engineers are always doing in practice, when applying filters to their process to regularize their data.

Remark 9: The HFFP assumption is not restrictive in the control of PDEs. When A is analytic and B is admissible, for example, when we consider parabolic PDEs, 'HFFP" is always satisfied as possible high frequency components are automatically damped out (see more discussion in [41]). When A is not analytic, for example when hyperbolic PDEs are considered, "uniform gap assumption" used in [25] combined with a moment method as in [33], [7] can ensure that "HFFP" condition holds. ◦

C. Uniform boundedness of the feedback control operator $K_{h,\Delta t}$

The following assumption is always needed when the direct method is employed.

Assumption 6: There exists $h_o > 0$ such that for any $h \in (0, h_o)$, there exists $\Delta t_K^*(h) > 0$ such that for all $\Delta t \in (0, \Delta t_K^*(h))$, there exists $B_K > 0$ such that the family of feedback control operators satisfy $\|K_{h,\Delta t}\|_{\mathcal{L}(X_h, U_h)} \leq B_K$.

Remark 10: Assumptions 3 and 6 are satisfied with the Riccati procedure that appears in the LQR optimal control problems (See discussion and results in [3], [2], [10], [12], [17], [15], [25]). ◦

D. Uniform Hurwitzian property

Assumptions 1 and 2 ensure the existence of "good" numerical algorithms. Assumptions 3 and 6 guarantee the existence of good finite-dimensional controllers $u_{h,\Delta t}$. Assumptions 4 and 5 ensure that "good" filtering process is available. However, it is still not enough to ensure that trajectories of the sampled-data system (21) do converge. The following compatibility requirement (or uniform Hurwitzian property) is also needed.

Assumption 7: "Uniform Hurwitzian property" (UHP) is satisfied for system (21). That is, let M_E and λ_E be positive constants with $M_E > 1$. Let the feedback control operators $K_{h,\Delta t}$ be from (17). There exists $T_E^* > 0$ and $h_E^* > 0$ such that for all $T \in (0, T_E^*)$, for any $h \in (0, h_E^*)$, there exists $\Delta t_E^*(h)$ such that for all $\Delta t \in (0, \Delta t_E^*(h))$, for all $z_0 \in E$, solutions of the following system, denoted as $z(t; z_0)$,

$$\dot{z}(t) = R_E P_h A_h R_h P_E z(t) + R_E P_h B_h K_{h,\Delta t} R_h P_E z(kT), \quad (26)$$

with $z(0) = z_0 \in E$ satisfy

$$\begin{aligned} &\|z(t; z_0)\|_E = \|P_E z(t; z_0)\|_X \\ = \;& \left\| P_E S_{T,h}^E(t) z_0 \right\|_E \leq M_E e^{-\lambda_E t} \|z_0\|_E, \end{aligned} \qquad (27)$$

where $S_{T,h}^E$ represents a family of semigroups generated by the system (26).

Remark 11: When there is no sampling, (26) becomes a continuous-time system on a finite-dimensional space E in the following form

$$\dot{z}(t) = R_E P_h \left(A_h + B_h K_{h,\Delta t}\right) R_h P_E z(t). \qquad (28)$$

Under such a situation, Assumption 7 is ensured when $A_h + B_h K_{h,\Delta t}$ is Hurwitz for any selected h and Δt as well as uniform boundedness of $K_{h,\Delta t}$ (see Assumption 6). When one filters with E, the filtered feedback matrix:

$$R_E P_h (A_h + B_h K_{h,\Delta t}) R_h P_E,$$

should be also uniformly Hurwitz. This filtered feedback matrix can be treated as the matrix $A_h + B_h K_{h,\Delta t}$ viewed "through E". ◦

Remark 12: As $R_E P_h (A_h + B_h K_{h,\Delta t}) R_h P_E$ is uniformly Hurwitz in small h, there exists a sufficiently small $T_E^* > 0$ such that the stability properties of (26) can be obtained for all $T \in (0, T_E^*)$. This is a general emulation result on the sampled-data control of finite-dimensional system with a sampling period T. Note that since the stability properties of $R_E P_h (A_h + B_h K_{h,\Delta t}) R_h P_E$ are uniform in h, the choice of T^* is independent of the choice h. ◦

Remark 13: Assumption 7 is satisfied when the discretization scheme is built on E and such numerical schemes are called spectral, or modal, that is, $E \subset Im(R_h)$, for all h small enough as discussions in [6], [7]. ◦

Although the controller (21) is not in a feedback form, Assumption 7 ensures that all unstable modes can be exponentially stabilized by using controllers designed for numerical approximations.

E. Main result

The main result of this paper is stated as follows.

Theorem 1: Let (Δ, ν) be positive constants. Assume Assumptions 1- 7 hold true with ϵ in Assumptions 4-5 determined by Δ, ν. Then there exist $T^* > 0$, $h^* > 0$ such that for all $T \in (0, T^*)$, for any $h \in (0, h^*)$, there exists $\Delta t^*(h) > 0$ and positive constants M and λ, such that for all $\Delta t \in (0, \Delta t^*(h))$, for every $x_0 \in B_\Delta \cap Ker(\mathcal{F}(\epsilon) - I_X)$, the trajectories of (21), denoted

as $x(t) = x(t; x_0, u_{T,h,\Delta t}(k; x_0))$, with the so constructed parameters $(T, h, \Delta t)$ exist and satisfy

$$\|x(t; x_0, u_{T,h,\Delta t}(k; x_0))\|_X \leq Me^{-\lambda t} \|x_0\|_X + \nu, \quad (29)$$

for all $t \in [kT, (k+1)T), k \in \mathbb{N}$. ∘

Normally one cannot get the suitable filtering process to achieve the required precision (ϵ) that is determined by ν and Δ (see Theorem 1). In most of situations, engineers have a filter with known "ϵ", then the size of the neighborhood depends on the quality of the filter. Thus, a weaker result can be obtained in the following corollary.

Corollary 1: Let (Δ, ϵ) be positive constants. Assume Assumptions 1- 7 hold true. Then there exist $T^* > 0$, $h^* > 0$ such that for all $T \in (0, T^*)$, for any $h \in (0, h^*)$, there exists $\Delta t^*(h) > 0$ and positive constants M and λ and $\gamma \in \mathcal{K}_\infty$, such that for all $\Delta t \in (0, \Delta t^*(h))$, for every $x_0 \in B_\Delta \cap Ker(\mathcal{F}(\epsilon) - I_X)$, the trajectories of (21), denoted as $x(t) = x(t; x_0, u_{T,h,\Delta t}(k; x_0))$, with the so constructed parameters $(T, h, \Delta t)$ exist and satisfy

$$\|x(t; x_0, u_{T,h,\Delta t}(k; x_0))\|_X \leq Me^{-\lambda t} \|x_0\|_X + \gamma(\epsilon), \quad (30)$$

for all $t \in [kT, (k+1)T), k \in \mathbb{N}$. ∘

The result is "practical exponential stability" implying that asymptotic approximate controllability when T tends to ∞. In general, obtaining "practical stability" is much easier than "stability" (see discussion in [38]). It is not possible to obtain exponentially stability properties when the direct method is applied to general sampled-data infinite-dimensional systems. Due the existence of the sampling mechanism, numerical approximation errors as well as estimation error ϵ on E, practical stability properties are best result that can be obtained.

The result in Theorem 1 is much more general than results obtained in [21], in which only sampling is considered and controller is designed for continuous time system. Moreover, the system considered in this paper is much more general (A is not analytic and B is unbounded) than that in [21]. With the help of the analysis tools in this paper (ϵ-filtering and "HFFP"), it is possible to extend the result in [21] to a more general setting.

Remark 14: In [15], Labbé and Trélat discussed the necessary and sufficient conditions under which uniform controllability properties of a family space semi-discretized approximations implies the controllability properties of the exact model (1) for parabolic systems. The main result in their paper [15, Theorem 3.1] implies presented result (Theorem 1) in the parabolic case, though the proof of this implication would require several developments. ∘

Remark 15: When the infinitesimal generator A in (1) is analytic, i.e. the system (1) is parabolic, HFFP condition is automatically satisfied. Assumption 3 and Assumption 6 are satisfied with the Riccati procedure (see discussion in[3], [2], [10], [12], [17], [15] and references therein). Under such a situation, the filtering process is not required since high frequency components are naturally damped. Sharp estimates of convergence of $S_h(t)$ to $S(t)$ therefore exist (see[17, Chapter 4]), which can greatly simplify the proof of our main result. ∘

Remark 16: If suitable numerical viscosity terms are added in the numerical schemes in our main result, Assumptions 3 and 6, 7 and 5 hold (see discussion in [25] for more details). Therefore, by choosing appropriate numerical schemes, the assumptions in Theorem 1 (Corollary 1) are not restrictive. ∘

Theorem 1 provides a useful guideline for engineers to choose appropriate numerical schemes (h and Δt), sampling period T and filtering processes. There are four design parameters $(T, h, \Delta t, \epsilon)$ that are determined by the performance requirement (Δ, ν). Therefore, how to design these parameters is of great importance to ensure that the proposed method can work for a very general sampled-data LDPS.

F. Discussion on necessity of assumptions

Although the proposed method (designing controllers for approximate models) is widely used in engineering applications, assumptions in Theorem 1 may be very hard to check in practice. However, our result shed the light on how to properly design sampled-data controllers for infinite-dimensional systems by using numerical approximate models. Furthermore, Assumptions 4–7 are not only sufficient, but also necessary. If one of them fails, counterexamples can be found in literature.

Assumption 4. The initial condition x_0 has to be filtered, i.e., $x_0 \cap Ker(\mathcal{F}(\epsilon) - I_X)$ and be bounded by B_Δ. The role of ϵ-filtering is to filter out the high frequency components in space for the given initial condition x_0 of the exact system (1). It is important to note that the assumption of ϵ-filtering (Assumption 4) and filtering out high frequency components of initial conditions are necessary conditions to ensure that Theorem 1 holds. Counterexamples can be found when wave-like equations are considered. It is well-known that for hyperbolic systems, high frequency components of initial conditions may not be damped and will interfere with the numerical discretization, generating spurious oscillation and leading to divergent trajectories. Such discussion can be found in [38] (space semi-discretization schemes), in [40] (time semi-discretization schemes) and in [41] (full discretization schemes). ∘

Assumption 5. Assumption 5 cannot be removed in general, as A is not restricted to be analytic and B is an unbounded operator. For example, when the hyperbolic PDEs are considered, "resonance" phenomena may occur if "HFFP" condition is not satisfied. Unstable trajectories thus could be obtained (see discussion in [38] and references therein).

Assumption 6. If Assumption 6 is not satisfied, when h and Δt tend to zero, $\|K_{h,\Delta t}\|_{\mathcal{L}(X_h)}$ may tend to infinity. The control input $u_{T,h,\Delta t}(k; x_0)$ become divergent. Therefore uniform boundedness of $K_{h,\Delta t}$ is necessary in main result.

Assumption 7. Assumption 7 is necessary and plays a crucial role to ensure that Theorem 1 holds. Intuitively, if the numerical schemes are not compatible, that is, some unstable modes cannot be detected and stabilized by numerical approximations, it is not possible to obtain convergent trajectories of the sampled-data system (3). It is indicated in [36] that when the discretization scheme is far from spectral (that is, Assumption 7 does not hold), the trajectories of the sampled-data system (3) may diverge.

IV. Conclusions

In this paper, practical exponential stability properties of the sampled-data infinite-dimensional systems using controllers generated from numerical approximations are discussed. The controllers are first designed uniformly exponentially stabilize a family of finite-dimensional discrete-time approximations. Then they are used in sampled-data infinite-dimensional systems. Under some tight sufficient conditions, given any positive pair (Δ, ν), by tuning the parameters $(T, h, \Delta t)$, for any initial condition $x_0 \in B_\Delta$ that is properly filtered, the obtained controllers will gradually move trajectories of the sampled-data system to ν-neighborhood of the origin.

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