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# Partitions and functional Santaló inequalities

Joseph Lehec\*

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## Abstract

We give a direct proof of a functional Santaló inequality due to Fradelizi and Meyer. This provides a new proof of the Blaschke-Santaló inequality. The argument combines a logarithmic form of the Prékopa-Leindler inequality and a partition theorem of Yao and Yao.

## Introduction

If  $A$  is a subset of  $\mathbb{R}^n$  we let  $A^\circ$  be the polar of  $A$ :

$$A^\circ = \{x \in \mathbb{R}^n \mid \forall y \in A, x \cdot y \leq 1\},$$

where  $x \cdot y$  denotes the scalar product of  $x$  and  $y$ . We denote the Euclidean norm of  $x$  by  $|x| = \sqrt{x \cdot x}$ . Let  $K$  be a subset of  $\mathbb{R}^n$  with finite measure. The Blaschke-Santaló inequality states that there exists a point  $z$  in  $\mathbb{R}^n$  such that

$$\text{vol}_n(K) \text{vol}_n(K - z)^\circ \leq \text{vol}_n(B_2^n) \text{vol}_n(B_2^n)^\circ = v_n^2, \quad (1)$$

where  $\text{vol}_n$  stands for the Lebesgue measure on  $\mathbb{R}^n$ ,  $B_2^n$  for the Euclidean ball and  $v_n$  for its volume. It was first proved by Blaschke in dimension 2 and 3 and Santaló [7] extended the result to any dimension. We say that an element  $z$  of  $\mathbb{R}^n$  satisfying (1) is a *Santaló point* for  $K$ .

Throughout the paper a *weight* is a measurable function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for any  $n$ , the function  $x \in \mathbb{R}^n \mapsto \rho(|x|)$  is integrable.

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**Definition 1.** Let  $f$  be a non-negative integrable function on  $\mathbb{R}^n$ , and  $\rho$  be a weight. We say that  $c \in \mathbb{R}^n$  is a Santaló point for  $f$  with respect to  $\rho$  if the following holds: for all non-negative Borel function  $g$  on  $\mathbb{R}^n$ , if

$$\forall x, y \in \mathbb{R}^n, \quad x \cdot y \geq 0 \Rightarrow f(c+x)g(y) \leq \rho(\sqrt{x \cdot y})^2, \quad (2)$$

then

$$\int_{\mathbb{R}^n} f(x) dx \int_{\mathbb{R}^n} g(y) dy \leq \left( \int_{\mathbb{R}^n} \rho(|x|) dx \right)^2. \quad (3)$$

Heuristically, the choice of the weight  $\rho$  gives a notion of duality (or polarity) for non-negative functions. Our purpose is give a new proof of the following theorem, due to Fradelizi and Meyer [4].

**Theorem 2.** *Let  $f$  be non-negative and integrable. There exists  $c \in \mathbb{R}^n$  such that  $c$  is a Santaló point for  $f$  with respect to any  $\rho$ . Moreover, if  $f$  is even then 0 is a Santaló point for  $f$  with respect to any weight.*

The even case goes back to Keith Ball in [2], this was the first example of a functional version of (1). Later on, Artstein, Klartag and Milman [1] proved that any integrable  $f$  admits a Santaló point with respect to the weight  $t \mapsto e^{-t^2/2}$ . Moreover in this case the barycenter of  $f$  suits (see [5]). Unfortunately this is not true in general; indeed, taking

$$\begin{aligned} f &= \mathbf{1}_{(-2,0)} + 4\mathbf{1}_{(0,1)} \\ g &= \mathbf{1}_{(-0.5,0]} + \frac{1}{4}\mathbf{1}_{(0,1)} \\ \rho &= \mathbf{1}_{[0,1]}, \end{aligned}$$

it is easy to check that  $f$  has its barycenter at 0, and that  $f(s)g(t) \leq \rho(\sqrt{st})^2$  as soon as  $st \geq 0$ . However

$$\int_{\mathbb{R}} f(s) ds \int_{\mathbb{R}} g(t) dt = \frac{9}{2} > 4 = \left( \int_{\mathbb{R}} \rho(|r|) dr \right)^2.$$

To prove the existence of a Santaló point, the authors of [4] use a fixed point theorem and the usual Santaló inequality (for convex bodies). Our proof is direct, in the sense that it does not use the Blaschke Santaló inequality; it is based on a special form of the Prékopa-Leindler inequality and on a partition theorem due to Yao and Yao [8].

Lastly, the Blaschke-Santaló inequality follows very easily from Theorem 2: we let the reader check that if  $c$  is a Santaló point for  $\mathbf{1}_K$  with respect to the weight  $\mathbf{1}_{[0,1]}$  then  $c$  is a Santaló point for  $K$ .

# 1 Yao-Yao partitions

In the sequel we consider real affine spaces of finite dimension. If  $E$  is such a space we denote by  $\vec{E}$  the associated vector space. We say that  $\mathcal{P}$  is a partition of  $E$  if  $\cup \mathcal{P} = E$  and if the interiors of two distinct elements of  $\mathcal{P}$  do not intersect. For instance, with this definition, the set  $\{(-\infty, a], [a, +\infty)\}$  is a partition of  $\mathbb{R}$ . We define by induction on the dimension a class of partitions of an  $n$ -dimensional affine space.

**Definition 3.** If  $E = \{c\}$  is an affine space of dimension 0, the only possible partition  $\mathcal{P} = \{c\}$  is a Yao-Yao partition of  $E$ , and its center is defined to be  $c$ .

Let  $E$  be an affine space of dimension  $n \geq 1$ . A set  $\mathcal{P}$  is said to be a Yao-Yao partition of  $E$  if there exists an affine hyperplane  $F$  of  $E$ , a vector  $v \in \vec{E} \setminus \vec{F}$  and two Yao-Yao partitions  $\mathcal{P}_+$  and  $\mathcal{P}_-$  of  $F$  having the same center  $c$  such that

$$\mathcal{P} = \{A + \mathbb{R}_-v \mid A \in \mathcal{P}_-\} \cup \{A + \mathbb{R}_+v \mid A \in \mathcal{P}_+\},$$

The center of  $\mathcal{P}$  is then  $x$ .

If  $A$  is a subset of  $\vec{E}$  we denote by  $\text{pos}(A)$  the positive hull of  $A$ , that is to say the smallest convex cone containing  $A$ .

A Yao-Yao partition  $\mathcal{P}$  of an  $n$ -dimensional space  $E$  has  $2^n$  elements and for each  $A$  in  $\mathcal{P}$  there exists a basis  $v_1, \dots, v_n$  of  $\vec{E}$  such that

$$A = c + \text{pos}(v_1, \dots, v_n), \tag{4}$$

where  $c$  is the center of  $\mathcal{P}$ . Indeed, assume that  $\mathcal{P}$  is defined by  $F, v, \mathcal{P}_+$  and  $\mathcal{P}_-$  (see Definition 3). Let  $A \in \mathcal{P}_+$  and assume inductively that there is a basis  $v_1, \dots, v_{n-1}$  of  $\vec{F}$  such that  $A = c + \text{pos}(v_1, \dots, v_{n-1})$ . Then  $A + \mathbb{R}_+v = c + \text{pos}(v, v_1, \dots, v_{n-1})$ .

A fundamental property of this class of partitions is the following

**Proposition 4.** *Let  $\mathcal{P}$  be a Yao-Yao partition of  $E$  and  $c$  its center. Let  $\ell$  be an affine form on  $E$  such that  $\ell(c) = 0$ . Then there exists  $A \in \mathcal{P}$  such that  $\ell(x) \geq 0$  for all  $x \in A$ . Moreover there is at most one element  $A$  of  $\mathcal{P}$  such that  $\ell(x) > 0$  for all  $x \in A \setminus \{c\}$ .*

*Proof.* By induction on the dimension  $n$  of  $E$ . When  $n = 0$  it is obvious, we assume that  $n \geq 1$  and that the result holds for all affine spaces of dimension

$n - 1$ . Let  $\ell$  be an affine form on  $E$  such that  $\ell(c) = 0$ . We introduce  $F, v, \mathcal{P}_+, \mathcal{P}_-$  given by Definition 3. By the induction assumption, there exists  $A_+ \in \mathcal{P}_+$  and  $A_- \in \mathcal{P}_-$  such that

$$\forall y \in A_+ \cup A_- \quad \ell(y) \geq 0.$$

If  $\ell(c + v) \geq 0$  then  $\ell(x + tv) \geq 0$  for all  $x \in A_+$  and  $t \in \mathbb{R}_+$ , thus  $\ell(x) \geq 0$  for all  $x \in A_+ + \mathbb{R}_+v$ . If on the contrary  $\ell(c + v) \leq 0$  then  $\ell(x) \geq 0$  for all  $x \in A_- + \mathbb{R}_-v$ , which proves the first part of the proposition. The proof of the ‘moreover’ part is similar.  $\square$

The latter proposition yields the following corollary, which deals with dual cones: if  $C$  is cone of  $\mathbb{R}^n$  the dual cone of  $C$  is by definition

$$C^* = \{y \in \mathbb{R}^n \mid \forall x \in C, x \cdot y \geq 0\}.$$

**Corollary 5.** *Let  $\mathcal{P}$  be a Yao-Yao partition of  $\mathbb{R}^n$  centered at 0. Then*

$$\mathcal{P}^* := \{A^* \mid A \in \mathcal{P}\}$$

*is also a partition of  $\mathbb{R}^n$ .*

Actually the dual partition is also a Yao-Yao partition centered at 0 but we will not use this fact.

*Proof.* Let  $x \in \mathbb{R}^n$  and  $\ell : y \in \mathbb{R}^n \mapsto x \cdot y$ . By the previous proposition there exists  $A \in \mathcal{P}$  such that  $\ell(y) \geq 0$  for all  $y \in A$ . Then  $x \in A^*$ . Thus  $\cup \mathcal{P}^* = \mathbb{R}^n$ . Moreover if  $x$  belongs to the interior of  $A^*$ , then for all  $y \in A \setminus \{0\}$  we have  $\ell(y) > 0$ . Again by the proposition above there is at most one such  $A$ . Thus the interiors of two distinct elements of  $\mathcal{P}^*$  do not intersect.  $\square$

We now let  $\mathcal{M}(E)$  be the set of Borel measure  $\mu$  on  $E$  which are finite and which satisfy  $\mu(F) = 0$  for any affine hyperplane  $F$ .

**Definition 6.** Let  $\mu \in \mathcal{M}(E)$ , a Yao-Yao equipartition  $\mathcal{P}$  for  $\mu$  is a Yao-Yao partition of  $E$  satisfying

$$\forall A \in \mathcal{P}, \quad \mu(A) = 2^{-n} \mu(E). \tag{5}$$

We say that  $c \in E$  is a Yao-Yao center of  $\mu$  if  $c$  is the center of a Yao-Yao equipartition for  $\mu$ .

Here is the main result concerning those partitions.

**Theorem 7.** *Let  $\mu \in \mathcal{M}(\mathbb{R}^n)$ , there exists a Yao-Yao equipartition for  $\mu$ . Moreover, if  $\mu$  is even then 0 is a Yao-Yao center for  $\mu$ .*

It is due to Yao and Yao [8]. They have some extra hypothesis on the measure and their paper is very sketchy, so we refer to [6] for a proof of this very statement.

## 2 Proof of the Fradelizi-Meyer inequality

In this section, all integrals are taken with respect to the Lebesgue measure. Let us recall the Prékopa-Leindler inequality, which is a functional form of the famous Brunn-Minkowski inequality, see for instance [3] for a proof and selected applications. If  $\varphi_1, \varphi_2, \varphi_3$  are non-negative and integrable functions on  $\mathbb{R}^n$  satisfying  $\varphi_1(x)^\lambda \varphi_2(y)^{1-\lambda} \leq \varphi_3(\lambda x + (1-\lambda)y)$  for all  $x, y$  in  $\mathbb{R}^n$  and for some fixed  $\lambda \in (0, 1)$ , then

$$\left( \int_{\mathbb{R}^n} \varphi_1 \right)^\lambda \left( \int_{\mathbb{R}^n} \varphi_2 \right)^{1-\lambda} \leq \int_{\mathbb{R}^n} \varphi_3.$$

The following lemma is a useful (see [4, 2]) logarithmic version of Prékopa-Leindler. We recall the proof for completeness.

**Lemma 8.** *Let  $f_1, f_2, f_3$  be non-negative Borel functions on  $\mathbb{R}_+^n$  satisfying*

$$f_1(x)f_2(y) \leq \left( f(\sqrt{x_1y_1}, \dots, \sqrt{x_ny_n}) \right)^2$$

for all  $x, y$  in  $\mathbb{R}_+^n$ . Then

$$\int_{\mathbb{R}_+^n} f_1 \int_{\mathbb{R}_+^n} f_2 \leq \left( \int_{\mathbb{R}_+^n} f_3 \right)^2. \quad (6)$$

*Proof.* For  $i = 1, 2, 3$  we let

$$g_i(x) = f_i(e^{x_1}, \dots, e^{x_n})e^{x_1 + \dots + x_n}.$$

Then by change of variable we have

$$\int_{\mathbb{R}^n} g_i = \int_{\mathbb{R}_+^n} f_i.$$

On the other hand the hypothesis on  $f_1, f_2, f_3$  yields

$$g_1(x)g_2(y) \leq g_3\left(\frac{x+y}{2}\right),$$

for all  $x, y$  in  $\mathbb{R}^n$ . Then by Prékopa-Leindler

$$\int_{\mathbb{R}^n} g_1 \int_{\mathbb{R}^n} g_2 \leq \left( \int_{\mathbb{R}^n} g_3 \right)^2. \quad \square$$

**Theorem 9.** *Let  $f$  be a non-negative Borel integrable function on  $\mathbb{R}^n$ , and let  $c$  be a Yao-Yao center for the measure with density  $f$ . Then  $c$  is a Santaló point for  $f$  with respect to any weight.*

Combining this result with Theorem 7 we obtain a complete proof of the Fradelizi-Meyer inequality.

*Proof.* It is enough to prove that if 0 is a Yao-Yao center for  $f$  then 0 is a Santaló point. Indeed, if  $c$  is a center for  $f$  then 0 is a center for

$$f_c : x \mapsto f(c + x).$$

And if 0 is a Santaló point for  $f_c$  then  $c$  is a Santaló point for  $f$ .

Let  $\mathcal{P}$  be a Yao-Yao equipartition for  $f$  with center 0. Let  $g$  and  $\rho$  be such that (2) holds (with  $c = 0$ ). Let  $A \in \mathcal{P}$ , by (4), there exists an operator  $T$  on  $\mathbb{R}^n$  with determinant 1 such that  $A = T(\mathbb{R}_+^n)$ . Let  $S = (T^{-1})^*$ , then  $S(\mathbb{R}_+^n) = A^*$ . Let  $f_1 = f \circ T$ ,  $f_2 = g \circ S$  and  $f_3(x) = \rho(|x|)$ . Since for all  $x, y$  we have  $T(x) \cdot S(y) = x \cdot y$ , we get from (2)

$$f_1(x)f_2(y) \leq \rho(\sqrt{x \cdot y})^2 = f_3(\sqrt{x_1y_1}, \dots, \sqrt{x_ny_n})^2,$$

for all  $x, y$  in  $\mathbb{R}_+^n$ . Applying the previous lemma we get (6). By change of variable it yields

$$\int_A f \int_{A^*} g \leq \left( \int_{\mathbb{R}_+^n} \rho(|x|) dx \right)^2.$$

Therefore

$$\sum_{A \in \mathcal{P}} \int_A f \int_{A^*} g \leq 2^n \left( \int_{\mathbb{R}_+^n} \rho(|x|) dx \right)^2. \quad (7)$$

Since  $\mathcal{P}$  is an equipartition for  $f$  we have for all  $A \in \mathcal{P}$

$$\int_A f = 2^{-n} \int_{\mathbb{R}^n} f.$$

By Corollary 5, the set  $\{A^*, A \in \mathcal{P}\}$  is a partition of  $\mathbb{R}^n$ , thus

$$\sum_{A \in \mathcal{P}} \int_{A^*} g = \int_{\mathbb{R}^n} g.$$

Inequality (7) becomes

$$\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq 4^n \left( \int_{\mathbb{R}_+^n} \rho(|x|) dx \right)^2,$$

and of course the latter is equal to  $\left( \int_{\mathbb{R}^n} \rho(|x|) dx \right)^2$ . □

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