

A direct proof of the functional Santaló inequality

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Abstract

We give a simple proof of a functional version of the Blaschke-Santaló inequality due to Artstein, Klartag and Milman. The proof is by induction on the dimension and does not use the Blaschke-Santaló inequality.

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1 Introduction

For $x, y \in \mathbb{R}^n$, we denote their inner product by $\langle x, y \rangle$ and the Euclidean norm of x by $|x|$. If A is a subset of \mathbb{R}^n , we let $A^\circ = \{x \in \mathbb{R}^n \mid \forall y \in A, \langle x, y \rangle \leq 1\}$ be its polar body. The Blaschke-Santaló inequality states that any convex body K in \mathbb{R}^n with center of mass at 0 satisfies

$$\text{vol}_n(K) \text{vol}_n(K^\circ) \leq \text{vol}_n(D) \text{vol}_n(D^\circ) = v_n^2, \quad (1)$$

where vol_n stands for the volume, D for the Euclidean ball and v_n for its volume. Let g be a non-negative Borel function on \mathbb{R}^n satisfying $0 < \int g < \infty$ and $\int |x|g(x) dx < \infty$, then $\text{bar}(g) = (\int g)^{-1} (\int g(x)x dx)$ denotes its center of mass (or barycenter). The center of mass (or centroid) of a measurable subset of \mathbb{R}^n is by definition the barycenter of its indicator function.

Let us state a functional form of (1) due to Artstein, Klartag and Milman [1]. If f is a non-negative Borel function on \mathbb{R}^n , the polar function of f is the log-concave function defined by

$$f^\circ(x) = \inf_{y \in \mathbb{R}^n} (e^{-\langle x, y \rangle} f(y)^{-1})$$

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Theorem 1 (Artstein, Klartag, Milman). *If f is a non-negative integrable function on \mathbb{R}^n such that f° has its barycenter at 0, then*

$$\int_{\mathbb{R}^n} f(x) dx \int_{\mathbb{R}^n} f^\circ(y) dy \leq \left(\int_{\mathbb{R}^n} e^{-\frac{1}{2}|x|^2} dx \right)^2 = (2\pi)^n.$$

In the special case where the function f is even, this result follows from an earlier inequality of Keith Ball [2]; and in [4], Fradelizi and Meyer prove something more general (see also [5]). In the present note we prove the following:

Theorem 2. *Let f and g be non-negative Borel functions on \mathbb{R}^n satisfying the duality relation*

$$\forall x, y \in \mathbb{R}^n, \quad f(x)g(y) \leq e^{-\langle x, y \rangle}. \quad (2)$$

If f (or g) has its barycenter at 0 then

$$\int_{\mathbb{R}^n} f(x) dx \int_{\mathbb{R}^n} g(y) dy \leq (2\pi)^n. \quad (3)$$

This is slightly stronger than Theorem 1 in which the function that has its barycenter at 0 should be log-concave. The point of this note is not really this improvement, but rather to present a simple proof of Theorem 1. Theorem 2 yields an improved Blaschke-Santaló inequality, obtained by Lutwak in [6], with a completely different approach.

Corollary 3. *Let S be a star-shaped (with respect to 0) body in \mathbb{R}^n having its centroid at 0. Then*

$$\text{vol}_n(S) \text{vol}_n(S^\circ) \leq v_n^2. \quad (4)$$

Proof. Let $N_S(x) = \inf\{r > 0 \mid x \in rS\}$ be the gauge of S and $\phi_S = \exp(-\frac{1}{2}N_S^2)$. Integrating ϕ_S and the indicator function of S on level sets of N_S , it is easy to see that $\int_{\mathbb{R}^n} \phi_S = c_n \text{vol}_n(S)$ for some constant c_n depending only on the dimension. Replacing S by the Euclidean ball in this equality yields $c_n = (2\pi)^{n/2} v_n^{-1}$. Therefore it is enough to prove that

$$\int \phi_S \int \phi_{S^\circ} \leq (2\pi)^n. \quad (5)$$

Similarly, it is easy to see that $\text{bar}(\phi_S) = c'_n \text{bar}(S) = 0$. Besides, we have $\langle x, y \rangle \leq N_S(x)N_{S^\circ}(y) \leq \frac{1}{2}N_S(x)^2 + \frac{1}{2}N_{S^\circ}(y)^2$, for all $x, y \in \mathbb{R}^n$. Thus ϕ_S and ϕ_{S° satisfy (2), then by Theorem 2 we get (5). \square

2 Main results

Theorem 4. *Let f be a non-negative Borel function on \mathbb{R}^n having a barycenter. Let H be an affine hyperplane splitting \mathbb{R}^n into two half-spaces H_+ and H_- . Define $\lambda \in [0, 1]$ by $\lambda \int_{\mathbb{R}^n} f = \int_{H_+} f$. Then there exists $z \in \mathbb{R}^n$ such that for every non-negative Borel function g*

$$(\forall x, y \in \mathbb{R}^n, f(z+x)g(y) \leq e^{-\langle x, y \rangle}) \Rightarrow \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq \frac{1}{4\lambda(1-\lambda)} (2\pi)^n. \quad (6)$$

In particular, in every median H ($\lambda = \frac{1}{2}$) there is a point z such that for all g

$$(\forall x, y \in \mathbb{R}^n, f(z+x)g(y) \leq e^{-\langle x, y \rangle}) \Rightarrow \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq (2\pi)^n. \quad (7)$$

A similar result concerning convex bodies (instead of functions) was obtained by Meyer and Pajor in [7].

Let us derive Theorem 2 from the latter. Let f, g satisfy (2). Assume for example that $\text{bar}(g) = 0$, then 0 cannot be separated from the support of g by a hyperplane, so there exists $x_1, \dots, x_{n+1} \in \mathbb{R}^n$ such that 0 belongs to the interior of $\text{conv}\{x_1 \dots x_{n+1}\}$ and $g(x_i) > 0$ for $i = 1 \dots n+1$. Then (2) implies that $f(x) \leq Ce^{-\|x\|}$, for some $C > 0$, where $\|x\| = \max(\langle x, x_i \rangle \mid i \leq n+1)$. Assume also that $\int f > 0$, then f has a barycenter. Apply the “ $\lambda = 1/2$ ” part of Theorem 4 to f . There exists $z \in \mathbb{R}^n$ such that (7) holds. On the other hand, by (2)

$$f(z+x)g(y)e^{\langle y, z \rangle} \leq e^{-\langle z+x, y \rangle} e^{\langle y, z \rangle} = e^{-\langle x, y \rangle}$$

for all $x, y \in \mathbb{R}^n$. Therefore

$$\int_{\mathbb{R}^n} f(x) dx \int_{\mathbb{R}^n} g(y) e^{\langle y, z \rangle} dy \leq (2\pi)^n. \quad (8)$$

Integrating with respect to $g(y)dy$ the inequality $1 \leq e^{\langle y, z \rangle} - \langle y, z \rangle$ we get

$$\int_{\mathbb{R}^n} g(y) dy \leq \int_{\mathbb{R}^n} g(y) e^{\langle y, z \rangle} dy - \int_{\mathbb{R}^n} \langle y, z \rangle g(y) dy.$$

Since $\text{bar}(g) = 0$, the latter integral is 0 and together with (8) we obtain (3). Observe also that this proof shows that Theorem 4 in dimension n implies Theorem 2 in dimension n .

In order to prove Theorem 4, we need the following logarithmic form of the Prékopa-Leindler inequality. For details on Prékopa-Leindler, we refer to [3].

Lemma 5. *Let ϕ_1, ϕ_2 be non-negative Borel functions on \mathbb{R}_+ . If $\phi_1(s)\phi_2(t) \leq e^{-st}$ for every s, t in \mathbb{R}_+ , then*

$$\int_{\mathbb{R}_+} \phi_1(s) ds \int_{\mathbb{R}_+} \phi_2(t) dt \leq \frac{\pi}{2}. \quad (9)$$

Proof. Let $f(s) = \phi_1(e^s)e^s$, $g(t) = \phi_2(e^t)e^t$ and $h(r) = \exp(-e^{2r}/2)e^r$. For all $s, t \in \mathbb{R}$ we have $\sqrt{f(s)g(t)} \leq h(\frac{t+s}{2})$, hence by Prékopa-Leindler $\int_{\mathbb{R}} f \int_{\mathbb{R}} g \leq (\int_{\mathbb{R}} h)^2$. By change of variable, this is the same as $\int_{\mathbb{R}_+} \phi_1 \int_{\mathbb{R}_+} \phi_2 \leq (\int_{\mathbb{R}_+} e^{-u^2/2} du)^2$ which is the result. \square

3 Proof of Theorem 4

Clearly we can assume that $\int f = 1$. Let μ be the measure with density f . In the sequel we let $f_z(x) = f(z+x)$ for all x, z .

We prove the theorem by induction on the dimension. Let f be a non-negative Borel function on the line, let $r \in \mathbb{R}$ and $\lambda = \mu([r, \infty)) \in [0, 1]$. Let g satisfy $f(r+s)g(t) \leq e^{-st}$, for all s, t . Apply Lemma 5 twice: first to $\phi_1(s) = f(r+s)$ and $\phi_2(t) = g(t)$ then to $\phi_1(s) = f(r-s)$ and $\phi_2(t) = g(-t)$. Then

$$\int_{\mathbb{R}_+} f_r \int_{\mathbb{R}_+} g \leq \frac{\pi}{2} \quad \text{and} \quad \int_{\mathbb{R}_-} f_r \int_{\mathbb{R}_-} g \leq \frac{\pi}{2}.$$

Therefore $\int_{\mathbb{R}_+} g \leq \frac{\pi}{2\lambda}$ and $\int_{\mathbb{R}_-} g \leq \frac{\pi}{2(1-\lambda)}$, which yields the result in dimension 1.

Assume the theorem to be true in dimension $n-1$. Let H be an affine hyperplane splitting \mathbb{R}^n into two half-spaces H_+ and H_- and let $\lambda = \mu(H_+)$. Provided that $\lambda \neq 0, 1$ we can define b_+ and b_- to be the barycenters of $\mu|_{H_+}$ and $\mu|_{H_-}$, respectively. Since $\mu(H) = 0$, the point b_+ belongs to the interior of H_+ , and similarly for b_- . Hence the line passing through b_+ and b_- intersects H at one point, which we call z . Let us prove that z satisfies (6), for all g . Clearly, replacing f by f_z and H by $H - z$, we can assume that $z = 0$. Let g satisfy

$$\forall x, y \in \mathbb{R}^n, \quad f(x)g(y) \leq e^{-\langle x, y \rangle}. \quad (10)$$

Let e_1, \dots, e_n be an orthonormal basis of \mathbb{R}^n such that $H = e_n^\perp$ and $\langle b_+, e_n \rangle > 0$. Let $v = b_+ / \langle b_+, e_n \rangle$ and A be the linear operator on \mathbb{R}^n that maps e_n to v and e_i to itself for $i = 1 \dots n-1$ and let $B = (A^{-1})^t$. Define

$$F_+ : y \in H \mapsto \int_{\mathbb{R}_+} f(y+sv) ds \quad \text{and} \quad G_+ : y' \in H \mapsto \int_{\mathbb{R}_+} g(By'+te_n) dt.$$

By Fubini, and since A has determinant 1, $\int_H F_+ = \int_{H_+} f \circ A = \mu(H_+) = \lambda$. Also, letting P be the projection with range H and kernel $\mathbb{R}v$, we have

$$\text{bar}(F_+) = \frac{1}{\lambda} \int_{H_+} P(Ax) f(Ax) dx = \frac{1}{\lambda} P \left(\int_{H_+} x f(x) dx \right) = P(b_+),$$

and this is 0 by definition of P . Since $\langle Ax, Bx' \rangle = \langle x, x' \rangle$ for all $x, x' \in \mathbb{R}^n$, we have $\langle y + sv, By' + te_n \rangle = \langle y, y' \rangle + st$ for all $s, t \in \mathbb{R}$ and $y, y' \in H$. So (10) implies

$$f(y + sv)g(By' + te_n) \leq e^{-st - \langle y, y' \rangle}.$$

Applying Lemma 5 to $\phi_1(s) = f(y + sv)$ and $\phi_2(t) = g(By' + te_n)$ we get $F_+(y)G_+(y') \leq \frac{\pi}{2} e^{-\langle y, y' \rangle}$ for every $y, y' \in H$. Recall that $\text{bar}(F_+) = 0$, then by the induction assumption (which implies Theorem 2 in dimension $n - 1$)

$$\int_H F_+ \int_H G_+ \leq \frac{\pi}{2} (2\pi)^{n-1}. \quad (11)$$

hence $\int_{H_+} g(Bx) dx \leq \frac{1}{4\lambda} (2\pi)^n$. In the same way $\int_{H_-} g(Bx) dx \leq \frac{1}{4(1-\lambda)} (2\pi)^n$, adding these two inequalities, we obtain

$$\int_{\mathbb{R}^n} g(Bx) dx \leq \frac{1}{4\lambda(1-\lambda)} (2\pi)^n$$

which is the result since B has determinant 1.

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