# A direct proof of the functional Santalo inequality

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**Abstract**

We give a simple proof of a functional version of the Blaschke-Santaló inequality due to Artstein, Klartag and Milman. The proof is by induction on the dimension and does not use the Blaschke-Santaló inequality.

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## 1 Introduction

For $x, y \in \mathbb{R}^n$, we denote their inner product by $\langle x, y\rangle$ and the Euclidean norm of $x$ by $|x|$. If $A$ is a subset of $\mathbb{R}^n$, we let $A^\circ = \{x \in \mathbb{R}^n \,|\, \forall y \in A,\ \langle x, y\rangle \leq 1\}$ be its polar body. The Blaschke-Santaló inequality states that any convex body $K$ in $\mathbb{R}^n$ with center of mass at 0 satisfies

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^\circ) \leq \mathrm{vol}_n(D)\,\mathrm{vol}_n(D^\circ) = v_n^2, \tag{1}$$

where $\mathrm{vol}_n$ stands for the volume, $D$ for the Euclidean ball and $v_n$ for its volume. Let $g$ be a non-negative Borel function on $\mathbb{R}^n$ satisfying $0 < \int g < \infty$ and $\int |x| g(x)\, dx < \infty$, then $\mathrm{bar}(g) = \left(\int g\right)^{-1}\left(\int g(x) x\, dx\right)$ denotes its center of mass (or barycenter). The center of mass (or centroid) of a measurable subset of $\mathbb{R}^n$ is by definition the barycenter of its indicator function.

Let us state a functional form of (1) due to Artstein, Klartag and Milman [1]. If $f$ is a non-negative Borel function on $\mathbb{R}^n$, the polar function of $f$ is the log-concave function defined by

$$f^\circ(x) = \inf_{y \in \mathbb{R}^n} \left(\mathrm{e}^{-\langle x, y\rangle} f(y)^{-1}\right)$$

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**Theorem 1** (Artstein, Klartag, Milman)**.** *If $f$ is a non-negative integrable function on $\mathbb{R}^n$ such that $f^\circ$ has its barycenter at $0$, then*

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} f^\circ(y)\,dy \le \Big(\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}|x|^2}\,dx\Big)^2 = (2\pi)^n.$$

In the special case where the function $f$ is even, this result follows from an earlier inequality of Keith Ball [2]; and in [4], Fradelizi and Meyer prove something more general (see also [5]). In the present note we prove the following:

**Theorem 2.** *Let $f$ and $g$ be non-negative Borel functions on $\mathbb{R}^n$ satisfying the duality relation*

$$\forall x, y \in \mathbb{R}^n, \qquad f(x)g(y) \le \mathrm{e}^{-\langle x,y\rangle}. \tag{2}$$

*If $f$ (or $g$) has its barycenter at $0$ then*

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(y)\,dy \le (2\pi)^n. \tag{3}$$

This is slightly stronger than Theorem 1 in which the function that has its barycenter at 0 should be log-concave. The point of this note is not really this improvement, but rather to present a simple proof of Theorem 1. Theorem 2 yields an improved Blaschke-Santaló inequality, obtained by Lutwak in [6], with a completely different approach.

**Corollary 3.** *Let $S$ be a star-shaped (with respect to $0$) body in $\mathbb{R}^n$ having its centroid at $0$. Then*

$$\mathrm{vol}_n(S)\,\mathrm{vol}_n(S^\circ) \le v_n^2. \tag{4}$$

*Proof.* Let $N_S(x) = \inf\{r > 0 \,|\, x \in rS\}$ be the gauge of $S$ and $\phi_S = \exp\big(-\frac{1}{2}N_S^2\big)$. Integrating $\phi_S$ and the indicator function of $S$ on level sets of $N_S$, it is easy to see that $\int_{\mathbb{R}^n} \phi_S = c_n \,\mathrm{vol}_n(S)$ for some constant $c_n$ depending only on the dimension. Replacing $S$ by the Euclidean ball in this equality yields $c_n = (2\pi)^{n/2} v_n^{-1}$. Therefore it is enough to prove that

$$\int \phi_S \int \phi_{S^\circ} \le (2\pi)^n. \tag{5}$$

Similarly, it is easy to see that $\mathrm{bar}(\phi_S) = c'_n \,\mathrm{bar}(S) = 0$. Besides, we have $\langle x, y\rangle \le N_S(x)N_{S^\circ}(y) \le \frac{1}{2}N_S(x)^2 + \frac{1}{2}N_{S^\circ}(y)^2$, for all $x, y \in \mathbb{R}^n$. Thus $\phi_S$ and $\phi_{S^\circ}$ satisfy (2), then by Theorem 2 we get (5). $\square$

## 2 Main results

**Theorem 4.** *Let $f$ be a non-negative Borel function on $\mathbb{R}^n$ having a barycenter. Let $H$ be an affine hyperplane splitting $\mathbb{R}^n$ into two half-spaces $H_+$ and $H_-$. Define $\lambda \in [0,1]$ by $\lambda \int_{\mathbb{R}^n} f = \int_{H_+} f$. Then there exists $z \in \mathbb{R}^n$ such that for every non-negative Borel function $g$*

$$\big(\forall x, y \in \mathbb{R}^n,\ f(z+x)g(y) \leq \mathrm{e}^{-\langle x,y\rangle}\big) \quad \Rightarrow \quad \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq \frac{1}{4\lambda(1-\lambda)}(2\pi)^n. \tag{6}$$

*In particular, in every median $H$ ($\lambda = \frac{1}{2}$) there is a point $z$ such that for all $g$*

$$\big(\forall x, y \in \mathbb{R}^n,\ f(z+x)g(y) \leq \mathrm{e}^{-\langle x,y\rangle}\big) \quad \Rightarrow \quad \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq (2\pi)^n. \tag{7}$$

A similar result concerning convex bodies (instead of functions) was obtained by Meyer and Pajor in [7].

Let us derive Theorem 2 from the latter. Let $f, g$ satisfy (2). Assume for example that $\mathrm{bar}(g) = 0$, then 0 cannot be separated from the support of $g$ by a hyperplane, so there exists $x_1, \ldots, x_{n+1} \in \mathbb{R}^n$ such that 0 belongs to the interior of $\mathrm{conv}\{x_1 \ldots x_{n+1}\}$ and $g(x_i) > 0$ for $i = 1 \ldots n+1$. Then (2) implies that $f(x) \leq C\mathrm{e}^{-\|x\|}$, for some $C > 0$, where $\|x\| = \max\big(\langle x, x_i\rangle \,|\, i \leq n+1\big)$. Assume also that $\int f > 0$, then $f$ has a barycenter. Apply the "$\lambda = 1/2$" part of Theorem 4 to $f$. There exists $z \in \mathbb{R}^n$ such that (7) holds. On the other hand, by (2)

$$f(z+x)g(y)\mathrm{e}^{\langle y,z\rangle} \leq \mathrm{e}^{-\langle z+x,y\rangle}\mathrm{e}^{\langle y,z\rangle} = \mathrm{e}^{-\langle x,y\rangle}$$

for all $x, y \in \mathbb{R}^n$. Therefore

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(y)\mathrm{e}^{\langle y,z\rangle}\,dy \leq (2\pi)^n. \tag{8}$$

Integrating with respect to $g(y)dy$ the inequality $1 \leq \mathrm{e}^{\langle y,z\rangle} - \langle y, z\rangle$ we get

$$\int_{\mathbb{R}^n} g(y)\,dy \leq \int_{\mathbb{R}^n} g(y)\mathrm{e}^{\langle y,z\rangle}\,dy - \int_{\mathbb{R}^n} \langle y,z\rangle g(y)\,dy.$$

Since $\mathrm{bar}(g) = 0$, the latter integral is 0 and together with (8) we obtain (3). Observe also that this proof shows that Theorem 4 in dimension $n$ implies Theorem 2 in dimension $n$.

In order to prove Theorem 4, we need the following logarithmic form of the Prékopa-Leindler inequality. For details on Prékopa-Leindler, we refer to [3].

**Lemma 5.** *Let $\phi_1, \phi_2$ be non-negative Borel functions on $\mathbb{R}_+$. If $\phi_1(s)\phi_2(t) \leq \mathrm{e}^{-st}$ for every $s, t$ in $\mathbb{R}_+$, then*

$$\int_{\mathbb{R}_+} \phi_1(s)\, ds \int_{\mathbb{R}_+} \phi_2(t)\, dt \leq \frac{\pi}{2}. \tag{9}$$

*Proof.* Let $f(s) = \phi_1(\mathrm{e}^s)\mathrm{e}^s$, $g(t) = \phi_2(\mathrm{e}^t)\mathrm{e}^t$ and $h(r) = \exp(-\mathrm{e}^{2r}/2)\mathrm{e}^r$. For all $s, t \in \mathbb{R}$ we have $\sqrt{f(s)g(t)} \leq h(\frac{t+s}{2})$, hence by Prékopa-Leindler $\int_{\mathbb{R}} f \int_{\mathbb{R}} g \leq \big(\int_{\mathbb{R}} h\big)^2$. By change of variable, this is the same as $\int_{\mathbb{R}_+} \phi_1 \int_{\mathbb{R}_+} \phi_2 \leq \big(\int_{\mathbb{R}_+} \mathrm{e}^{-u^2/2}\, du\big)^2$ which is the result. □

# 3 Proof of Theorem 4

Clearly we can assume that $\int f = 1$. Let $\mu$ be the measure with density $f$. In the sequel we let $f_z(x) = f(z + x)$ for all $x, z$.

We prove the theorem by induction on the dimension. Let $f$ be a non-negative Borel function on the line, let $r \in \mathbb{R}$ and $\lambda = \mu\big([r, \infty)\big) \in [0, 1]$. Let $g$ satisfy $f(r + s)g(t) \leq \mathrm{e}^{-st}$, for all $s, t$. Apply Lemma 5 twice: first to $\phi_1(s) = f(r+s)$ and $\phi_2(t) = g(t)$ then to $\phi_1(s) = f(r-s)$ and $\phi_2(t) = g(-t)$. Then

$$\int_{\mathbb{R}_+} f_r \int_{\mathbb{R}_+} g \leq \frac{\pi}{2} \qquad \text{and} \qquad \int_{\mathbb{R}_-} f_r \int_{\mathbb{R}_-} g \leq \frac{\pi}{2}.$$

Therefore $\int_{\mathbb{R}_+} g \leq \frac{\pi}{2\lambda}$ and $\int_{\mathbb{R}_-} g \leq \frac{\pi}{2(1-\lambda)}$, which yields the result in dimension 1.

Assume the theorem to be true in dimension $n - 1$. Let $H$ be an affine hyperplane splitting $\mathbb{R}^n$ into two half-spaces $H_+$ and $H_-$ and let $\lambda = \mu(H_+)$. Provided that $\lambda \neq 0, 1$ we can define $b_+$ and $b_-$ to be the barycenters of $\mu_{|H_+}$ and $\mu_{|H_-}$, respectively. Since $\mu(H) = 0$, the point $b_+$ belongs to the interior of $H_+$, and similarly for $b_-$. Hence the line passing through $b_+$ and $b_-$ intersects $H$ at one point, which we call $z$. Let us prove that $z$ satisfies (6), for all $g$. Clearly, replacing $f$ by $f_z$ and $H$ by $H - z$, we can assume that $z = 0$. Let $g$ satisfy

$$\forall x, y \in \mathbb{R}^n, \qquad f(x)g(y) \leq \mathrm{e}^{-\langle x,y\rangle}. \tag{10}$$

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $H = e_n^\perp$ and $\langle b_+, e_n\rangle > 0$. Let $v = b_+/\langle b_+, e_n\rangle$ and $A$ be the linear operator on $\mathbb{R}^n$ that maps $e_n$ to $v$ and $e_i$ to itself for $i = 1 \ldots n - 1$ and let $B = (A^{-1})^t$. Define

$$F_+ : y \in H \mapsto \int_{\mathbb{R}_+} f(y+sv)\, ds \qquad \text{and} \qquad G_+ : y' \in H \mapsto \int_{\mathbb{R}_+} g(By'+te_n)\, dt.$$

By Fubini, and since $A$ has determinant 1, $\int_H F_+ = \int_{H_+} f \circ A = \mu(H_+) = \lambda$. Also, letting $P$ be the projection with range $H$ and kernel $\mathbb{R}v$, we have

$$\mathrm{bar}(F_+) = \frac{1}{\lambda}\int_{H_+} P(Ax)f(Ax)\,dx = \frac{1}{\lambda}P\Big(\int_{H_+} xf(x)\,dx\Big) = P(b_+),$$

and this is 0 by definition of $P$. Since $\langle Ax, Bx'\rangle = \langle x, x'\rangle$ for all $x, x' \in \mathbb{R}^n$, we have $\langle y + sv, By' + te_n\rangle = \langle y, y'\rangle + st$ for all $s, t \in \mathbb{R}$ and $y, y' \in H$. So (10) implies

$$f(y + sv)g(By' + te_n) \leq \mathrm{e}^{-st-\langle y,y'\rangle}.$$

Applying Lemma 5 to $\phi_1(s) = f(y + sv)$ and $\phi_2(t) = g(By' + te_n)$ we get $F_+(y)G_+(y') \leq \frac{\pi}{2}\mathrm{e}^{-\langle y,y'\rangle}$ for every $y, y' \in H$. Recall that $\mathrm{bar}(F_+) = 0$, then by the induction assumption (which implies Theorem 2 in dimension $n - 1$)

$$\int_H F_+ \int_H G_+ \leq \frac{\pi}{2}(2\pi)^{n-1}. \tag{11}$$

hence $\int_{H_+} g(Bx)\,dx \leq \frac{1}{4\lambda}(2\pi)^n$. In the same way $\int_{H_-} g(Bx)\,dx \leq \frac{1}{4(1-\lambda)}(2\pi)^n$, adding these two inequalities, we obtain

$$\int_{\mathbb{R}^n} g(Bx)\,dx \leq \frac{1}{4\lambda(1-\lambda)}(2\pi)^n$$

which is the result since $B$ has determinant 1.

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