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► **To cite this version:**

Metod Saniga, Peter Levay, Petr Pracna, Peter Vrana. The Veldkamp Space of  $GQ(2,4)$ . 2009. hal-00365656v1

**HAL Id: hal-00365656**

**<https://hal.science/hal-00365656v1>**

Preprint submitted on 4 Mar 2009 (v1), last revised 6 Jul 2009 (v2)

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# The Veldkamp Space of $GQ(2,4)$

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(4 March 2009)

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## Abstract

It is shown that the Veldkamp space of the unique generalized quadrangle  $GQ(2,4)$  is isomorphic to  $PG(5,2)$ . Since the  $GQ(2,4)$  features only two kinds of geometric hyperplanes, namely point's perp-sets and  $GQ(2,2)$ s, the 63 points of  $PG(5,2)$  split into two families; 27 being represented by perp-sets and 36 by  $GQ(2,2)$ s. The 651 lines of  $PG(5,2)$  are found to fall into four distinct classes: in particular, 45 of them feature only perp-sets, 216 comprise two perp-sets and one  $GQ(2,2)$ , 270 consist of one perp-set and two  $GQ(2,2)$ s and the remaining 120 ones are composed solely of  $GQ(2,2)$ s, according as the intersection of two distinct hyperplanes determining the (Veldkamp) line is, respectively, a line, an ovoid, a perp-set and a grid (i. e.,  $GQ(2,1)$ ) of a copy of  $GQ(2,2)$ . Surmised relevance of this finding for quantum (information) theory and the so-called black hole analogy is also outlined.

**MSC Codes:** 51Exx, 81R99

**PACS Numbers:** 02.10.Ox, 02.40.Dr, 03.65.Ca

**Keywords:** Generalized Quadrangle of Order (2,4) – Veldkamp Space – 3-Qubits/2-Qutrits  
– 5-D Black Holes

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## 1 Introduction

There are a number of beautiful mathematical concepts, objects and constructions that have so far evaded attention of theoretical physicists, being also of virtually no, or very little, interest to the mathematics community itself. One of such concepts is, in our opinion, that of the *Veldkamp space* of a point-line incidence geometry [1, 2]. Two of us became familiar with this notion some two years ago; it was the Veldkamp space of the smallest thick generalized quadrangle, isomorphic to  $PG(4,2)$ , whose structure and properties were immediately recognized to be of relevance to quantum physics [3]. The sole purpose of this short note is to bring to the limelight a close ally of the former and briefly outline its physical relevance (to be worked out and discussed in detail in a separate paper).

## 2 Generalized Quadrangles, Geometric Hyperplanes and Veldkamp spaces

We will first highlight the basics of the theory of finite generalized quadrangles [4] and then introduce the concept of a geometric hyperplane [5] and that of the Veldkamp space of a point-line incidence geometry [1, 2].

A *finite generalized quadrangle* of order  $(s, t)$ , usually denoted  $\text{GQ}(s, t)$ , is an incidence structure  $S = (P, B, I)$ , where  $P$  and  $B$  are disjoint (non-empty) sets of objects, called respectively points and lines, and where  $I$  is a symmetric point-line incidence relation satisfying the following axioms [4]: (i) each point is incident with  $1 + t$  lines ( $t \geq 1$ ) and two distinct points are incident with at most one line; (ii) each line is incident with  $1 + s$  points ( $s \geq 1$ ) and two distinct lines are incident with at most one point; and (iii) if  $x$  is a point and  $L$  is a line not incident with  $x$ , then there exists a unique pair  $(y, M) \in P \times B$  for which  $xIMyIL$ ; from these axioms it readily follows that  $|P| = (s+1)(st+1)$  and  $|B| = (t+1)(st+1)$ . It is obvious that there exists a point-line duality with respect to which each of the axioms is self-dual. Interchanging points and lines in  $S$  thus yields a generalized quadrangle  $S^D$  of order  $(t, s)$ , called the dual of  $S$ . If  $s = t$ ,  $S$  is said to have order  $s$ . The generalized quadrangle of order  $(s, 1)$  is called a grid and that of order  $(1, t)$  a dual grid. A generalized quadrangle with both  $s > 1$  and  $t > 1$  is called thick.

Given two points  $x$  and  $y$  of  $S$  one writes  $x \sim y$  and says that  $x$  and  $y$  are collinear if there exists a line  $L$  of  $S$  incident with both. For any  $x \in P$  denote  $x^\perp = \{y \in P | y \sim x\}$  and note that  $x \in x^\perp$ ; obviously,  $|x^\perp| = 1 + s + st$ . Given an arbitrary subset  $A$  of  $P$ , the *perp(-set)* of  $A$ ,  $A^\perp$ , is defined as  $A^\perp = \bigcap \{x^\perp | x \in A\}$  and  $A^{\perp\perp} := (A^\perp)^\perp$ . A triple of pairwise non-collinear points of  $S$  is called a *triad*; given any triad  $T$ , a point of  $T^\perp$  is called its center and we say that  $T$  is *acentric*, *centric* or *unicentric* according as  $|T^\perp|$  is, respectively, zero, non-zero or one. An *ovoid* of a generalized quadrangle  $S$  is a set of points of  $S$  such that each line of  $S$  is incident with exactly one point of the set; hence, each ovoid contains  $st + 1$  points.

A *geometric hyperplane*  $H$  of a point-line geometry  $\Gamma(P, B)$  is a proper subset of  $P$  such that each line of  $\Gamma$  meets  $H$  in one or all points [5]. For  $\Gamma = \text{GQ}(s, t)$ , it is well known that  $H$  is one of the following three kinds: (i) the perp-set of a point  $x$ ,  $x^\perp$ ; (ii) a (full) subquadrangle of order  $(s, t')$ ,  $t' < t$ ; and (iii) an ovoid.

Finally, we shall introduce the notion of the *Veldkamp space* of a point-line incidence geometry  $\Gamma(P, B)$ ,  $\mathcal{V}(\Gamma)$  [1].  $\mathcal{V}(\Gamma)$  is the space in which (i) a point is a geometric hyperplane of  $\Gamma$  and (ii) a line is the collection  $H_1H_2$  of all geometric hyperplanes  $H$  of  $\Gamma$  such that  $H_1 \cap H_2 = H_1 \cap H = H_2 \cap H$  or  $H = H_i$  ( $i = 1, 2$ ), where  $H_1$  and  $H_2$  are distinct points of  $\mathcal{V}(\Gamma)$ . In the above definition it is assumed that any three distinct hyperplanes  $H_1, H_2$  and  $H_3$  of  $\Gamma$  satisfy the following two conditions [2]: (i)  $H_1$  is not properly contained in  $H_2$  and (ii)  $H_1 \cap H_2 \subseteq H_3$  implies  $H_1 \cap H_2 = H_1 \cap H_3$ .

## 3 GQ(2,4) and its Veldkamp space

The smallest thick generalized quadrangle is obviously the (unique)  $\text{GQ}(2,2)$ , often dubbed the ‘‘doily.’’ This quadrangle is endowed with 15 points/lines, with each line containing three points and, dually, each point being on three lines; moreover, it is a self-dual object, i. e., isomorphic to its dual. It features all the three kinds of geometric hyperplanes, of the following cardinalities [1]: 15 perp-sets,  $x^\perp$ , seven points each; 10 grids (i. e.  $\text{GQ}(2,1)s$ ), nine points each; and six ovoids, five points each. The quadrangle also exhibits two distinct kinds of triads, viz. unicentric and tricentric. Its Veldkamp space is isomorphic to  $\text{PG}(4,2)$  whose detailed description, together with its important physical applications, can be found in [3].

The next case in the hierarchy is  $\text{GQ}(2,4)$ , the unique generalized quadrangle of order  $(2,4)$ , which possesses 27 points and 45 lines, with lines of size three and five lines through a point. Its full group of automorphisms is of order 51840, being isomorphic to the Weyl group  $W(E_6)$ . Consider a nonsingular *elliptic* quadric,  $\mathcal{Q}^-(5, 2)$ , in  $\text{PG}(5,2)$ ; then the points and

Table 1: The properties of the four different types of the lines of  $\mathcal{V}(\text{GQ}(2,4))$  in terms of the common intersection and the types of geometric hyperplanes featured by a generic line of a given type. The last column gives the total number of lines per the corresponding type.

Type	Intersection	Perps	Doilies	(Ovoids)	Total
I	Line	3	0	(-)	45
II	Ovoid	2	1	(-)	216
III	Perp-set	1	2	(-)	270
IV	Grid	0	3	(-)	120

the lines of such a quadric form a  $\text{GQ}(2,4)$ .  $\text{GQ}(2,4)$  is obviously not a self-dual structure; its dual,  $\text{GQ}(4,2)$ , features 45 points and 27 lines, with lines of size five and three lines through a point. Unlike its dual, which exhibits all the three kinds of geometric hyperplanes,  $\text{GQ}(2,4)$  is endowed only with perps<sup>1</sup> (of cardinality 11 each) and  $\text{GQ}(2,2)$ s, *not* admitting ovoids [4, 6]. This last property, being a particular case of the general theorem stating that a  $\text{GQ}(s, t)$  with  $s > 1$  and  $t > s^2 - s$  has no ovoids [4], substantially facilitates construction of its Veldkamp space,  $\mathcal{V}(\text{GQ}(2,4))$ . Also, every triad in  $\text{GQ}(2,4)$  is tricentric.

Obviously, there are 27 distinct perps in  $\text{GQ}(2,4)$ . Since  $\text{GQ}(2,4)$  is rather small, its diagrams/drawings given in [7] were employed to check by hand that it contains 36 different copies of  $\text{GQ}(2,2)$ . It thus follows that  $\mathcal{V}(\text{GQ}(2,4))$  is endowed with 63 points. As the only projective space having this number of points is the five-dimensional projective space over  $\text{GF}(2)$ ,  $\text{PG}(5,2)$ , one is immediately tempted to the conclusion that  $\mathcal{V}(\text{GQ}(2,4)) \cong \text{PG}(5,2)$ . To demonstrate that this is really the case we only have to show that  $\mathcal{V}(\text{GQ}(2,4))$  features 651 lines as well, each represented by three hyperplanes.

This task was also accomplished by hand. That is, we took the pictures of all the 63 different copies of geometric hyperplanes of  $\text{GQ}(2,4)$  and looked for every possible intersection between pairs of them. We have found that the intersection of two perps is either a line or an ovoid of  $\text{GQ}(2,2)$  according as their centers are collinear or not, whereas that of two  $\text{GQ}(2,2)$ s is a perp-set or a grid — as sketchily illustrated in Figure 1 and Figure 2, respectively.<sup>2</sup> This enabled us to verify that: a) the complement of the symmetric difference of any two geometric hyperplanes is also a geometric hyperplane and, so, the hyperplanes indeed form a  $\text{GF}(2)$ -vector space; b) that the total number of lines is 651; and c) that they split into four qualitatively distinct classes, as summarized in Table 1. The cardinality of type I class is obviously equal to the number of lines of  $\text{GQ}(2,4)$ . The number of Veldkamp lines of type II stems from the fact that  $\text{GQ}(2,4)$  contains (number of  $\text{GQ}(2,2)$ s)  $\times$  (number of ovoids per a  $\text{GQ}(2,2)$ ) =  $36 \times 6 = 216$  ovoids (of  $\text{GQ}(2,2)$ ) and that an ovoid sits in a unique  $\text{GQ}(2,2)$ . Since each copy of  $\text{GQ}(2,2)$  contains 15 perp-sets and any of them is shared by two  $\text{GQ}(2,2)$ s, we have  $36 \times 15/2 = 270$  Veldkamp lines of type III. Finally, with 10 grids per a  $\text{GQ}(2,2)$  and three  $\text{GQ}(2,2)$ s through a grid, we arrive at  $36 \times 10/3 = 120$  lines of type IV. We also note in passing that the fact that three  $\text{GQ}(2,2)$ s share a grid is closely related with the property that there exist triples of pairwise disjoint grids partitioning the point set of  $\text{GQ}(2,4)$ ; the number of such triples is 40 [4].

The above-given chain of arguments can be recast into a more rigorous and compact form as follows. We return to the representation of  $\text{GQ}(2,4)$  as an elliptic quadric  $Q^-(5,2)$  in  $\text{PG}(5,2)$  and let  $H$  be a hyperplane (i. e.,  $\text{PG}(4,2)$ ) of  $\text{PG}(5,2)$ . Then there are two cases:

<sup>1</sup>In what follows, the perp-set of a point of  $\text{GQ}(2,4)$  will simply be referred to as a perp in order to avoid any confusion with the perp-set of a point of  $\text{GQ}(2,2)$ .

<sup>2</sup>In both the figures, each picture depicts all 27 points (circles) but only 19 lines (line segments and arcs of circles) of  $\text{GQ}(2,4)$ , with the two points located in the middle of the doily being regarded as lying one above and the other below the plane the doily is drawn in. 16 out of the missing 26 lines can be obtained in each picture by its successive rotations through 72 degrees around the center of the pentagon. For the illustration of the remaining 10 lines, half of which pass through either of the two points located off the doily's plane, and further details about this pictorial representation of  $\text{GQ}(2,4)$ , see [7].

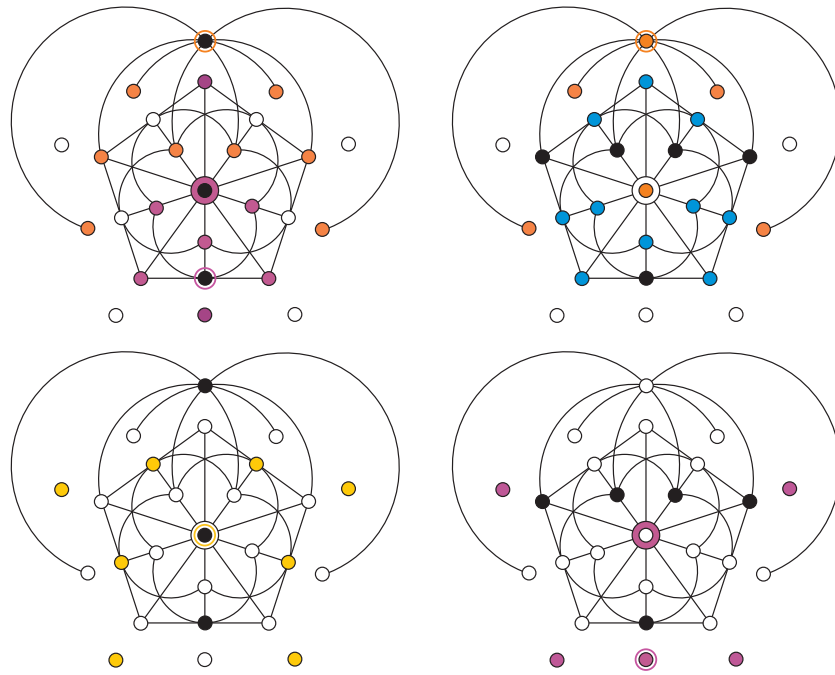


Figure 1: A pictorial illustration of the structure of the Veldkamp lines of  $\mathcal{V}(\text{GQ}(2,4))$ . *Left:* – A line of Type I, comprising three distinct perps (distinguished by three different colours) having collinear centers (encircled). *Right:* – A line of Type II, featuring two perps with non-collinear centers (orange and purple) and a doily (blue). In both the cases the black bullets represent the common elements of the three hyperplanes.

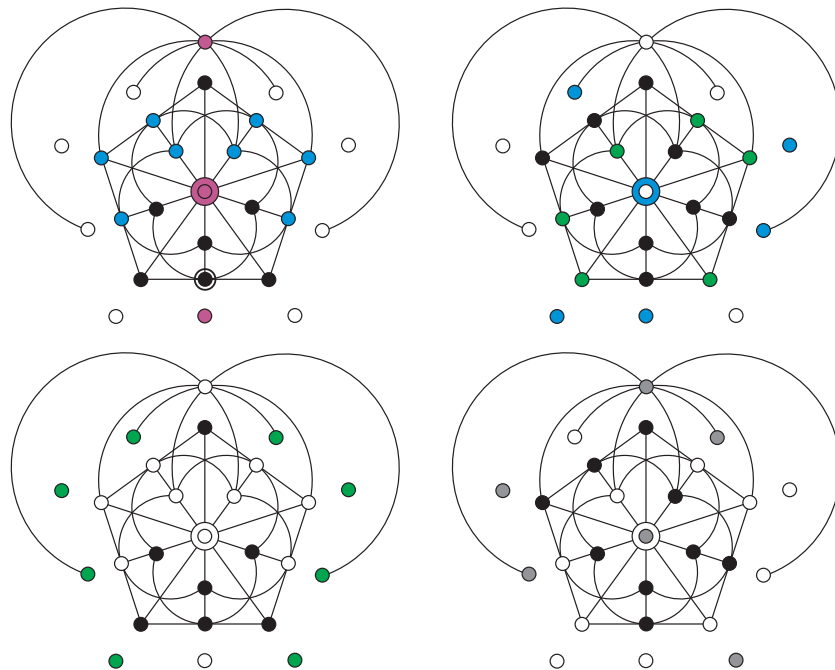


Figure 2: *Left:* – A line of Type III, endowed with two doilies (blue and green) and a perp (purple). *Right:* – A line of Type IV, composed of three doilies (blue, green and gray).

a)  $H$  is not tangent to  $Q^-(5, 2)$ . Then  $H \cap Q^-(5, 2)$  is a (parabolic) quadric of  $H$ . Such a quadric has 15 points and these 15 points generate the geometric hyperplane isomorphic to  $GQ(2, 2)$ . b)  $H$  is tangent to  $Q^-(5, 2)$  at a point  $P$ , say. Then  $H \cap Q^-(5, 2)$  is a quadratic cone with vertex  $P$  whose “base” is an elliptic quadric in a  $PG(3, 2)$  contained in  $H$  and not containing  $P$ . The “base” has 5 points, so that the cone has  $2 \times 5 + 1 = 11$  points. These 11 points generate the hyperplane isomorphic to a perp-set. (The base cannot be a hyperbolic quadric, since on such a quadric there are lines and the join of such a line with  $P$  would be a plane contained in  $Q^-(5, 2)$ , a contradiction.) By the above, two distinct hyperplanes  $H, H'$  of  $PG(5, 2)$  have distinct intersections  $H \cap Q^-(5, 2), H' \cap Q^-(5, 2)$ . These intersections are therefore distinct geometric hyperplanes of the  $GQ(2, 4)$ . There are 63 hyperplanes in  $PG(5, 2)$ , so that we obtain 63 geometric hyperplanes of the  $GQ(2, 4)$  or, in other words, all its geometric hyperplanes. Now we turn to the Veldkamp space. Its points are the hyperplanes of  $PG(5, 2)$ , as we use the one-one correspondence from above for an identification. Given distinct hyperplanes  $H, H'$  we have to ask for all hyperplanes containing  $H \cap H' \cap Q^-(5, 2)$  to get all points of the Veldkamp line joining  $H$  and  $H'$ . Clearly, the third hyperplane  $H''$  through  $H \cap H'$  is of this kind. If  $H \cap H' \cap Q^-(5, 2)$  generates the three-dimensional subspace  $H \cap H'$ , then the Veldkamp line is  $\{H, H', H''\}$ . This is the case whenever  $H \cap H' \cap Q^-(5, 2)$  is an elliptic quadric, a hyperbolic quadric, or a quadratic cone of  $H \cap H'$  (that is, an ovoid, a grid, or a perp-set of  $GQ(2, 4)$ , respectively). In general,  $H \cap H' \cap Q^-(5, 2)$  need not generate  $H \cap H'$  but it still may be a Veldkamp line (obviously of Type I). In this case the argument from above cannot be applied, but one can check by hand that the corresponding Veldkamp line has indeed only three elements. All in all, one finds that the  $\mathcal{V}(GQ(2, 4))$  is just the dual space of  $PG(5, 2)$ .

Let us assume now that our  $PG(5, 2)$  is provided with a non-degenerate *elliptic* quadric  $Q^-(5, 2)$  [8]; then the 27/36 points lying on/off such a quadric correspond to 27 perps/36 doilies of  $GQ(2, 4)$ . If, instead, one assumes  $PG(5, 2)$  to be equipped with a preferred *hyperbolic* quadric,  $Q^+(5, 2)$ , which induces an orthogonal  $O^+(6, 2)$  polarity in it [9], then under this polarity the set of 651 lines decomposes into 315 isotropic and 336 hyperbolic ones. The former are readily found to be made of the Veldkamp lines of Type I and III (odd number of perps – see Table 1), whilst the latter consist of those of Type II and IV (odd number of  $GQ(2, 2)$ s).

To conclude the section, it is worth mentioning that the collinearity, or point graph of  $GQ(2, 4)$ , i. e. the graph whose vertices are the points of  $GQ(2, 4)$  and two vertices are adjacent iff the corresponding points are collinear, is a strongly regular graph with parameters  $v = (s + 1)(st + 1) = 27, k = s(t + 1) = 10, \lambda = s - 1 = 1$  and  $\mu = t + 1 = 5$  [4]. The complement of this graph is the *Schläfli* graph, which is intimately connected with the configuration of 27 lines lying on a non-singular complex cubic surface [10]. Moreover, taking any triple of pairwise disjoint  $GQ(2, 1)$ s and removing their lines from  $GQ(2, 4)$  one gets a  $27_3$  configuration whose point-line incidence graph is the *Gray* graph — the smallest cubic graph which is edge-transitive and regular, but not vertex-transitive [11].

## 4 Conclusion

We have demonstrated that  $\mathcal{V}(GQ(2, 4))$  is isomorphic to  $PG(5, 2)$ , i. e., to the projective space where  $GQ(2, 4)$  itself lives. Although this result might seem to be of rather marginal importance from a mathematical point of view, it has a few remarkable physical implications. The Veldkamp space of the doily,  $\mathcal{V}(GQ(2, 2))$ , has already been recognized to be an important tool in revealing some intricacies of the commutation algebra of the generalized Pauli group of *two*-qubits [3]. We surmise that  $\mathcal{V}(GQ(2, 4))$  will play an analogous role as per the generalized Pauli group of *three*-qubits. This because the geometry behind the latter group is that of the split Cayley hexagon of order two [12] and of the symplectic polar space of rank three and order two [13], *both* of which live in  $PG(5, 2)$ ! There may even be a link to a *two-qutrit* case, for  $GQ(2, 4)$  can be viewed as a derived geometry at a point of the symplectic generalized quadrangle of order three,  $W(3)$  [4], the dual of which was found to mimic the algebraic

combinatorial properties of the maximum sets of mutually commuting operators of the two-qutrit Pauli group [14]. In this connection it is worth noting that the above-mentioned Gray graph is the edge residual of the generalized quadrangle  $W(3)$  [15]. It is, however, the so-called Black Hole Analogy (for a recent review, see [16] and references therein) where  $GQ(2,4)$  (and implicitly also  $\mathcal{V}(GQ(2,4))$ ) has just been found to play by far the most prominent role; as elaborated in detail in a separate paper [17], not only does its point-line incidence geometry fully encode the 5-D extremal black hole entropy formula, but also naturally leads to an intriguing kind of *non-commutative* labelling of the associated operators/observables.

### Acknowledgement

This work was partially supported by the VEGA grant agency projects Nos. 2/0092/09 and 2/7012/27. We thank Hans Havlicek for providing us with more rigorous arguments for  $\mathcal{V}(GQ(2,4)) \cong PG(5,2)$ .

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