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# TANGENCY AMONG THREE CYLINDERS A HYPERBOLOID AND A TORUS

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A challenging exercise in solid modeling is examined and developed as an opportunity to raise students' consciousness in "geometric thinking" by considering how to extract additional, required design parameters from an initial specification. This involves the formulation and solution of a variety of problems in analytical geometry and inevitable digression into some simple application of polynomial discriminants and the notion that pitfalls as well as advantages accompany symmetric configuration.

**Keywords:** Cylinder, Hyperbola, Torus, Tangency

## 1. SOLID MODELING A TABLE

Look at Fig. 1. The problem at hand is to extract the necessary design information from the following specifications so that a 3D model of this small patio cocktail table may be assembled from five simple parts, very easily constructed individually with "Solid Works" once the design information is available.

- The table top is a 500mm diameter by 25mm thick disc and sits on
- Three legs that are cylinders of 50mm outer diameter and inclined at the angle of the sloping edges of a regular tetrahedron with one base horizontal.
- Legs mutually touch, each pair on a point at half their total sloping leg height, *i.e.*, legs are beveled at an angle so their elliptical ends lie in horizontal planes on the bottom of the table top and the floor.
- Vertical distance between floor and under surface of table top is 500mm.
- A ring in the shape of a torus or "perfect doughnut" holds the legs together and touches them. The smallest circular section of this torus has a diameter of 50mm.

The task will be to map the elliptical footprints of the legs onto the under surface of the table top and to the floor and to find the inner radius of the "doughnut hole".



Figure 1: A Round Table with Three Legs Held by a Ring

### 1.1 Expectation

The student is expected to produce and deliver:-

- A nice CAD drawing of the hyperbola and the minor circular torus section, of radius  $R$ , in tangency as was done in the circle and ellipse in Fig. 5,

- An analysis to find the minor (meridian) circle centre radius  $x_2$  of the torus and the points of tangency of this circular section with the hyperbola on the axial section of the hyperboloid swept by the table legs and
- Of course, the entire five piece solid model.

## 2. NEAT DESCRIPTIVE GEOMETRY

The region where the three legs and torus touch are described constructively in the top and elevation view drawing shown in Fig. 2 while Fig. 3 shows one of the leg axes in end or point view. These three illustrations, including Fig. 1, were taken from an article that appeared recently in **IBDG** [1]. These may help if you can read multiview scaled drawings and have learned a little descriptive geometry.

### 2.1 Touching Legs and Torus

Now the leg axis geometry and that of the hyperbolic external envelope of the legs will be addressed so that the legs and the torus can be constructed. First the mutual tangency of the three legs will be considered using two approaches. In the first, a conic coefficient matrix and its dual, cofactored form will be used. In the second, the relation between tangency and double roots is shown thus providing a simple introduction to discriminant methods. The enveloping hyperboloid will be found by identifying the ruling line, on a leg cylinder, that is furthest from the hyperboloid axis.

### 2.2 Slope of the Legs

To find the slope of the legs, from the specification that their cylindrical axes are parallel to three intersecting edges of a regular tetrahedron and their slope is the angle  $\theta$  between an edge and the plane formed by the remaining two intersecting edges, imagine a unit cube with the three intersecting edges on its face diagonals thus.

$$OA, OB, OC, O(0,0,0)$$

$$A(1,1,0), B(0,1,1), C(1,0,1)$$

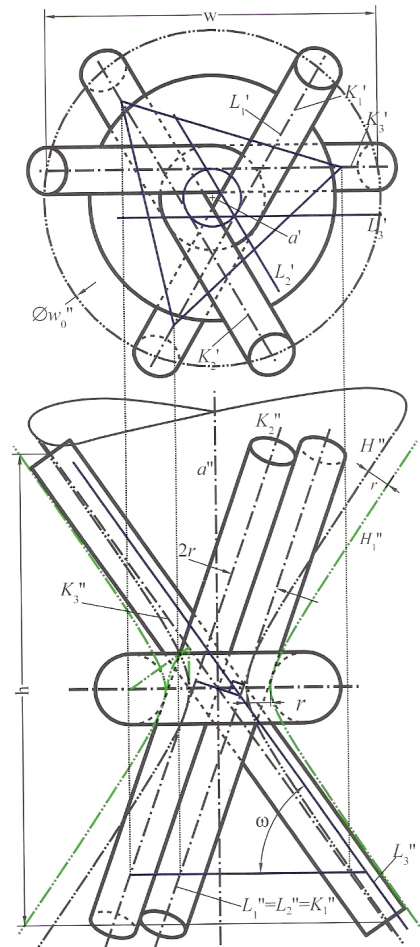


Figure 2: Leg in True View and One-Sheet External Hyperboloid of Revolution

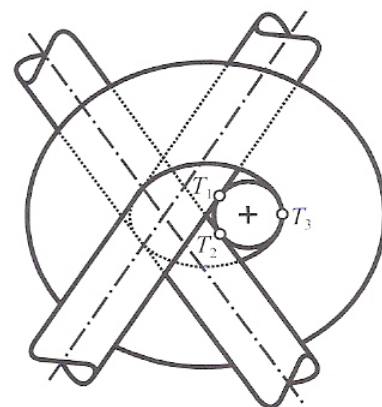


Figure 3: End View of a Leg

$O, A, B, C$  are cube vertices and the unit normal to plane  $ABC$  is  $\mathbf{n}$  while  $\mathbf{u}$  is the unit vector along  $\overline{AO}$ .

$$\mathbf{n} = \frac{\overline{AB} \times \overline{AC}}{|\overline{AB}||\overline{AC}|} = \frac{1}{\sqrt{3}}[-1 \ -1 \ -1]^T$$

$$\mathbf{u} = \frac{1}{\sqrt{2}}[-1 \ -1 \ 0]^T$$

The inner product  $\mathbf{n} \cdot \mathbf{u} = \sin \theta = \sqrt{2/3}$  .

### 2.3 Leg Cotangency Conditions

Imagine that the three legs are already in their proper positions. Examining the layout in a top view projection like the upper illustration in Fig. 2. The equilateral triangle in the centre is reproduced in Fig. 4 and represents the three leg axis. Sectioning the legs with successive horizontal planes proceeding from the floor up one will eventually arrive at a level, exactly mid-way up, where the elliptical sections of the legs are cotangential. Only one such ellipse is shown in Fig. 4 but two other, congruent ones may be visualized with their major axes on the other two sides of the equilateral triangle. It is obvious that one need not consider ellipse tangency since the right bisectors of the triangle will more simply serve the same purpose. Now consider the ellipse in standard form as shown in Fig. 4 in the Cartesian frame on origin  $O$  and axes  $x_1, x_2$ . One knows the leg cylinder radius is  $b = 25\text{mm}$  and

$$\sin \theta = \frac{b}{a}$$

where  $a$  is the ellipse semi-major axis length and  $\theta$  is the slope angle of legs parallel to the concurrent edges of a regular tetrahedron, *i.e.*,  $\sin \theta = \sqrt{2/3}$  and  $a = 25\sqrt{3/2}\text{mm}$ . What is not known is  $w$  the distance between hyperboloid's axis of symmetry and any leg axis.

### 2.4 Coefficient Matrix and Cofactors

We have a situation where the five conditions necessary to define a conic, the ellipse in this case, include four tangent lines,  $p$  on points  $S$

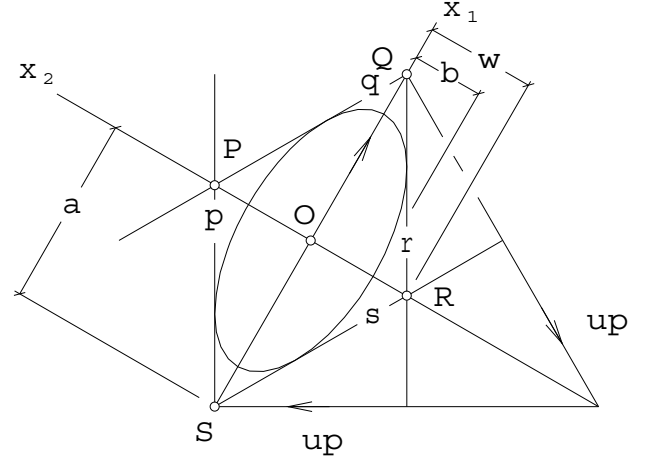


Figure 4: Triangle and Ellipse

and  $P, q$  on points  $P$  and  $Q, r$  on points  $Q$  and  $R$  and  $s$  on points  $R$  and  $S$ . Furthermore one knows that the point (not shown)  $T(0, -b)$  is on the ellipse, too. In order to review principles used below, refer to [2]. Consider the three conic coefficient matrices in the following expression.

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{01} & A_{11} & A_{12} \\ A_{02} & A_{12} & A_{22} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} A_{11}A_{22} - A_{12}^2 & A_{02}A_{12} - A_{01}A_{22} \\ A_{02}A_{12} - A_{01}A_{22} & A_{00}A_{22} - A_{02}^2 \\ A_{01}A_{12} - A_{02}A_{11} & A_{01}A_{02} - A_{00}A_{12} \\ A_{01}A_{12} - A_{02}A_{11} \\ A_{01}A_{02} - A_{00}A_{12} \\ A_{00}A_{11} - A_{01}^2 \end{bmatrix}$$

$$\equiv \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix}$$

The first coefficient matrix is that of a conic tangent line equation, *i.e.*, pre-multiplying it with a row vector of planar line coordinates and post-multiplying by the column vector produces the scalar line equation of the conic. Cofactoring

produces the middle matrix, the coefficient matrix of the corresponding point conic with coefficients represented as  $a_{ij}$ . With the four tangent line corner point homogeneous coordinates from Fig. 4 we get the four tangent line equation coefficients.

$$\begin{aligned} &P\{1 : 0 : w\}, \quad Q\{1 : \sqrt{3}w : 0\} \\ &R\{1 : 0 : -w\}, \quad S\{1 : -\sqrt{3} : 0\} \rightarrow \\ &p\{\sqrt{3}w : 1 : -\sqrt{3}\}, \quad q\{\sqrt{3}w : -1 : -\sqrt{3}\} \\ &r\{\sqrt{3}w : -1 : \sqrt{3}\}, \quad s\{\sqrt{3}w : 1 : \sqrt{3}\} \end{aligned} \quad (1)$$

Now the five constraint equations, Eq. 2, define the ellipse in terms of coefficients  $A_{ij}$ .

$$\begin{aligned} p : & 3w^2A_{00} + 2\sqrt{3}wA_{01} - 6wA_{02} + A_{11} \\ & -2\sqrt{3}A_{12} + 3A_{22} = 0 \\ q : & 3w^2A_{00} - 2\sqrt{3}wA_{01} - 6wA_{02} + A_{11} \\ & +2\sqrt{3}A_{12} + 3A_{22} = 0 \\ r : & 3w^2A_{00} - 2\sqrt{3}wA_{01} + 6wA_{02} + A_{11} \\ & -2\sqrt{3}A_{12} + 3A_{22} = 0 \\ s : & 3w^2A_{00} + 2\sqrt{3}wA_{01} + 6wA_{02} + A_{11} \\ & +2\sqrt{3}A_{12} + 3A_{22} = 0 \\ T : & A_{11}A_{22} - A_{12}^2 + 2bA_{01}A_{12} - 2bA_{02}A_{11} \\ & + b^2A_{00}A_{11} - b^2A_{01}^2 = 0 \end{aligned} \quad (2)$$

Solving for  $A_{ij}$  and cofactoring produces the ellipse point coefficient matrix, in  $a_{ij}$ , on the right in expression 3, not surprisingly in standard form.

$$\begin{aligned} &\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3(w^2 - b^2) & 0 \\ 0 & 0 & b^2 \end{bmatrix} \rightarrow \\ &\begin{bmatrix} -3(w^2 - b^2)b^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & 3(w^2 - b^2) \end{bmatrix} \end{aligned} \quad (3)$$

With row and column vectors of variable homogeneous coordinates of point  $X\{x_0 : x_1 : x_2\}$  and with  $x_0 = 1$  in Euclidean space the ellipse is given by Eq. 4.

$$-1 + \frac{x_1^2}{3(w^2 - b^2)} + \frac{x_2^2}{b^2} = 0 \quad (4)$$

Equating denominators of  $x_1^2$  by definition as  $a^2 = 3(w^2 - b^2)$  gives  $w = \pm\sqrt{(a^2 + 3b^2)}/3$ .

## 2.5 Standard Form and Discriminant

An alternate approach to symmetrically placing the three coplanar, cotangential elliptical sections on equilateral triangle sides, formed by a top view projection of the leg cylinder axes, is now discussed. Since the solution is easily seen to simplify to a standard form ellipse with given minor axis length  $b$  and a tangent line, say,  $QR \rightarrow r$  with given principal axis intercepts  $Q(\sqrt{3}w, 0)$  and  $R(0, -w)$ , the ellipse and line, using the negative reciprocal intercept formulation described in [3], are presented in Eq. 5 in terms of three unknowns,  $x_1, x_2, a$ , where  $a$  is the semi-major axis length.

$$-1 + \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 0, \quad 1 - \frac{x_1}{\sqrt{3}w} + \frac{x_2}{w} = 0 \quad (5)$$

Eliminating  $x_2$  between the ellipse and line expressed in Eq. 5 produces a quadratic in  $x_1$  only.

$$(9b^2 + 3a^2)x_1^2 - 6\sqrt{3}a^2wx_1 - 9a^2(b^2 + w^2) = 0$$

This is differentiated with respect to  $x_1$

$$(a^2 + 3b^2)x_1 - \sqrt{3}a^2w = 0$$

and  $x_1$  is eliminated between the quadratic and its derivative yielding a function  $f(a, b, w) = 0$ .

$$324a^2b^2(a^2 + 3b^2)(a^2 + 3b^2 - 3w^2) = 0$$

The factor containing  $w$  is solved to again arrive at  $w = \pm\sqrt{(a^2 + 3b^2)}/3$  and with the given specifications  $w = a = 25\sqrt{3}/2\text{mm}$ .

## 2.6 From Ellipse to Hyperboloid Axis

Fig. 5 shows the tangent circle of maximum radius, centred on the minor axis of the ellipse at distance  $w$  from its centre. A leg cylinder generator on such a tangent point is also a generator of the one sheet hyperboloid of revolution to which the torus is tangent. In fact, the circle is the throat circle of that hyperboloid.

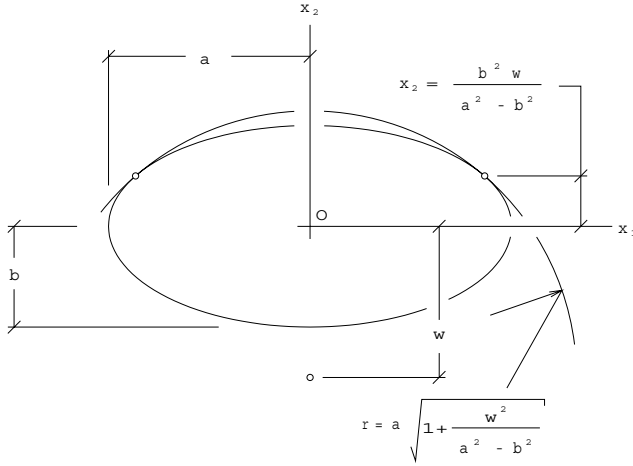


Figure 5: Hyperboloid Throat Circle

The standard form ellipse and tangent circle equations are written as Eq. 6.

$$-1 + \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 0, \quad -r^2 + x_1^2 + (x_2 + w)^2 = 0 \quad (6)$$

Eliminating  $x_2$  produces a quadratic in  $x_1$ . Taking its derivative with respect to  $x_1$  yields a second, linear equation in  $x_1$ . Solving these to eliminate  $x_1$  gives the circle radius  $r$  and then simultaneous solution of Eq. 6 finds the two points that each support a cylinder-and-hyperboloid generator. The radius and tangent points are given by Eq. 7 where results are expressed as dimensionless ratios.

$$\frac{r^2}{a^2} = 1 + \frac{w^2}{a^2 - b^2}, \quad \frac{x_1}{a} = \pm \sqrt{1 - \left( \frac{wb}{a^2 - b^2} \right)^2}$$

$$\frac{x_2}{w} = \frac{b^2}{a^2 - b^2} \quad (7)$$

Alas, although the solution above has been simplified so as to deal only with a quadratic equation, the intersection between two conics admits four solutions. Referring to [4] one sees a comprehensive development of this topic. In the solution Eq. 7 above the quartic

$$Px_1^4 + Qx_1^3 + Rx_1^2 + Sx_1 + T = 0$$

where  $Q = S = T = 0$  has been represented by only the quadratic factor  $Px^2 + R = 0$  that leads to a pair of imaginary solutions. The real, significant, double root  $(x_1 - 0)^2 = 0$  is omitted. In such cases where the circle centre point is on the ellipse minor axis  $x_2 = b$ . A pair of real roots emerges from the quadratic factor if  $\theta$  is sufficiently small as in the arbitrary case illustrated in Fig. 5. The actual situation with the given design parameters that produce  $x_1 = 0$  is summed up in Fig. 6 where  $r = b + w = b + a = 25(1 + \sqrt{3/2})$ .

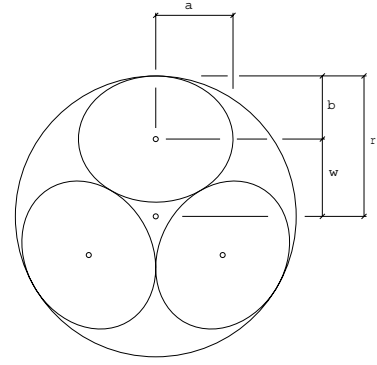


Figure 6: Three Leg Sections on Torus Equator

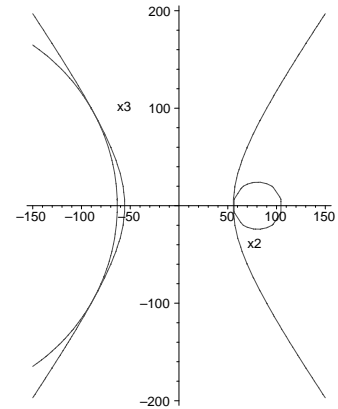


Figure 7: Tori and Hyperboloid Sections

## 2.7 Torus and Hyperboloid Tangency

First the axial hyperbolic section of the standard form one sheet hyperboloid of revolution with throat radius  $r$  given by Eq. 7 is configured as a planar projection on, say,  $x_1 = 0$ . This is done using the point  $X\{x_0 : x_2 : x_3\} = \{1 : r : 0\}$  and an absolute point  $A\{a_0 : a_2 : a_3\} = \{0 : \cos \theta : \sin \theta\}$  on the chosen plane of projection,  $x_1 = 0$ . The chosen hyperbola is written in Eq. 8 along with the coefficients derived with points  $X$  and  $A$ . Notice that  $\alpha$  and  $\beta$  are used to represent the unknown conic coefficients since the more usual  $a$  and  $b$  were already used for the ellipse.

$$\begin{aligned} \beta^2 x_2^2 - \alpha^2 x_3^2 - \alpha^2 \beta^2 x_0^2 &= 0 \rightarrow \\ \frac{x_2^2}{\alpha^2} - \frac{x_3^2}{\beta^2} - 1 &= 0 \\ x_0 &= 1, \quad \alpha = r, \quad \beta = r \tan \theta \end{aligned} \quad (8)$$

As in the case of finding the radius  $r$  of the circle that circumscribes the three mutually tangential elliptical sections of the leg cylinders, as shown in Fig. 6, one must find the location of the circular axis of the torus, a distance  $v$  from the hyperbola axis on  $x_2 = 0$ . The meridial circular section of the torus is of given radius  $R = 25\text{mm}$  and this circle  $(x_2 - v)^2 + x_3^2 - R^2 = 0$  must be tangential to the hyperbola given by Eq. 8. If one chooses to eliminate  $x_3$  between Eqs. 8 and the circle, two ‘‘right’’ answers are obtained, the first of the two pairs  $\pm(\alpha + R)$ ,  $\pm(\alpha - R)$ ,  $\alpha = r$ . The second pair represents tangency of circles inside the hyperbola. If however  $R$  is sufficiently large or  $\theta$  is sufficiently small then the solution produced by eliminating  $x_2$ , that admits two separate points of tangency, applies, *i.e.*,  $v = \sqrt{r^2(1 + \tan^2 \theta) + R^2(1 + \cot^2 \theta)}$ . Sections of an  $R = 25\text{mm}$  and an  $R = 200\text{mm}$  torus are shown on the right and left of the hyperbola pertaining to  $b = 25\text{mm}$  legs at slopes of  $\theta = \sin^{-1} \sqrt{2/3}$  are shown in Fig. 7.

## 2.8 Conic and Circle Tangency

Applying the analysis outlined in [4] to the configurations depicted in Figs. 5,6,7, where the circle is centred at  $P(0, \pm p_2, 0)$  and  $P(\pm p_1, 0, 0)$ ,

respectively, reveals the condition where the two contact points at  $\pm x_1$  and  $\pm x_3$  collapse to  $x_1 = 0$  or  $x_3 = 0$ , respectively. This is expressed in terms of appropriate dimensionless ratios.

$$\left(\frac{x_1}{a}\right)^2 = 1 - \left(\frac{bw}{a^2 - b^2}\right)^2 \quad (9)$$

Eq. 9, essentially a reprise of Eq. 7, is devoid of radius  $r$  or  $R$ . As  $w$  is increased so that

$$\left|\frac{w/a}{(a/b) - (b/a)}\right| > 1$$

then the single point contact applies and the inner radius of the torus is  $r = b + w$  and the circular axis of the torus is at  $x_2 = r + R$  for the given dimensional specification of the table leg configuration, *i.e.*,  $b = R = 25\text{mm}$  and  $w = a = 25b$ .

## 3. GENERAL CONFIGURATION

Consider placing in mutual contact three cylinders of revolution of various given radii and axial directions. There is no loss in generality if one chooses the first cylinder axis to lie on the Cartesian frame axis  $x_1$ . This axis is shown in Fig. 8 as segment  $AB$  of line  $\mathcal{P}$ . The second cylinder axis on segment  $CD$  of line  $\mathcal{Q}$  is separated from  $AB$  by common normal  $AC$  along the Cartesian frame axis  $x_3$ . The third cylinder axis cannot be placed immediately but its direction is given by line segment  $AX$ . When  $AX$  is projected on a plane normal to this direction the two pairs of parallel ruling lines on the first two cylinders are shown to define the limits of these two cylinders. There are four possible locations for the circular section of the third cylinder. These are located by intersecting line pairs that are respectively parallel to a limit ruling of each of the first two cylinders and offset by a distance equal to the radius of the third cylinder. The third cylinder axis chosen is shown as segment  $EF$  of line  $\mathcal{R}$ . Now the other two common normals  $GH$  and  $JK$ , connecting  $EF$  to  $AB$  and to  $CD$ , can be found and points  $P, Q, R$  placed on these and on  $AC$ . These three points are where cylinder pairs make contact.

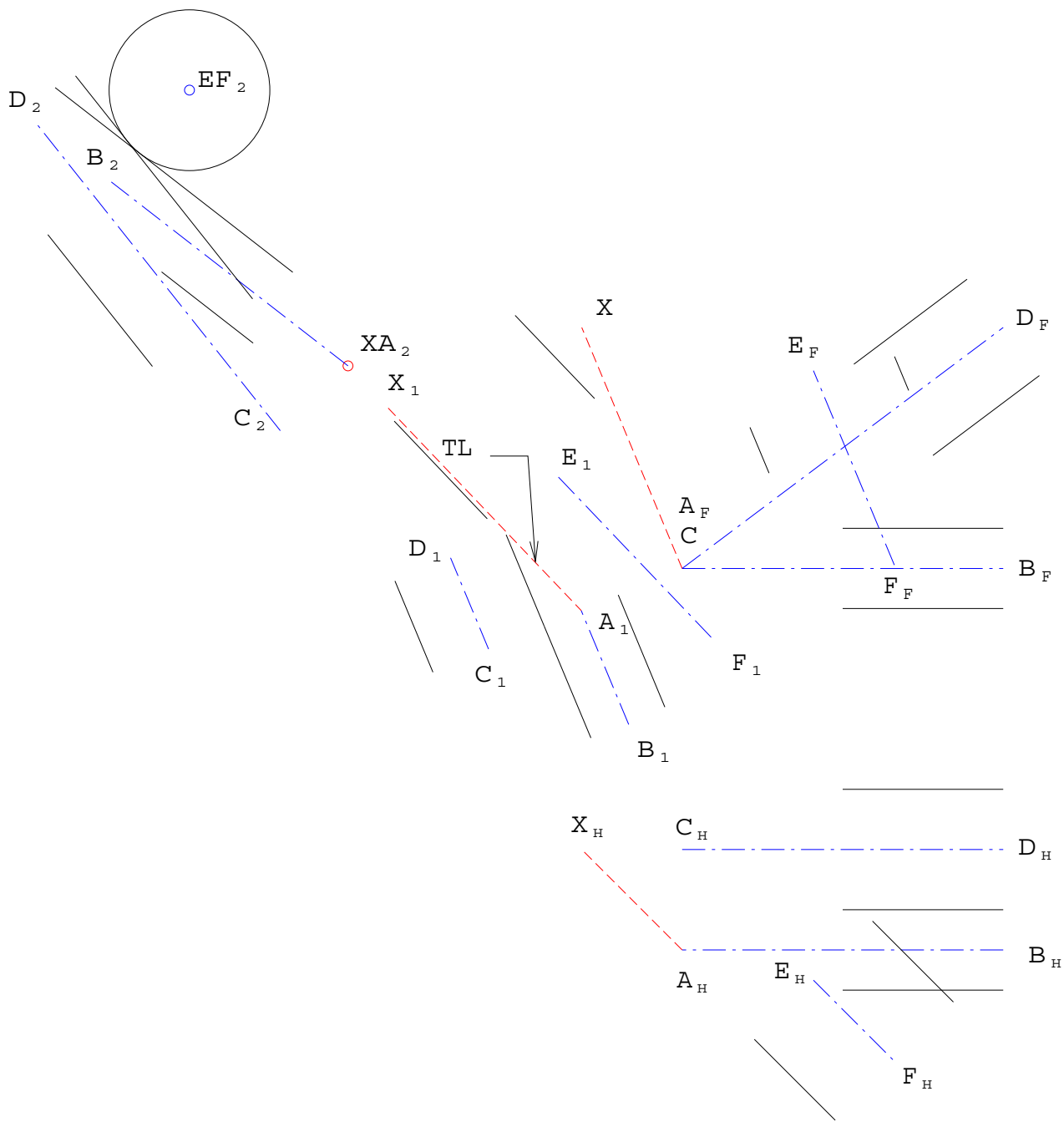


Figure 8: Three Contacting Cylinders of Various Radius and Axial Direction



### 3.1 Analytic Geometry

In what follows the axis of the third cylinder will be defined on the intersection of a plane pair. As in the case of offset lines on the plane of projection, above, there are four such planes that may be paired to locate four possible axes of the third cylinder. Fig. 9 shows three touching cylinders of different radius and, for clarity, with their axes mutually orthogonal. A sphere of radius equal to the sum of the radii of the second and third cylinders is placed anywhere on the axis of the second. Then a plane is shown tangent to the sphere and parallel to the second and third cylinder axes. Again, for clarity, this plane is not on the axis of the third cylinder but it is easy to imagine the two planes which are.

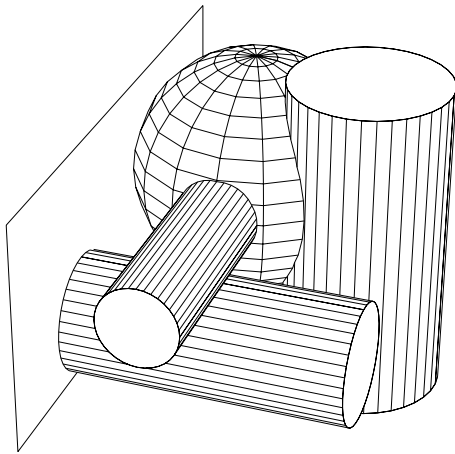


Figure 9: Three Contacting Cylinders, a Sphere and a Plane

### 4. CONCLUSION

Apart from providing an interesting and challenging exercise in solid modeling this problem goes well beyond the issues of contacting cylinders and tori. Specifically, the robust and efficient computation of shortest distance between points in the plane and in space and conics and quadrics, respectively, is an active subject of modern research in image processing for precise camera aided inspection of industrial products, *e.g.*, [5].

### ACKNOWLEDGEMENT

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