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APPROXIMATE TRANSITIVITY PROPERTY AND LEBESGUE SPECTRUM

E. H. EL ABDALAOUI AND M. LEMAŃCZYK

ABSTRACT. Exploiting a spectral criterion for a system not to be AT we give some new examples of zero entropy systems without the AT property. Our examples include those with finite spectral multiplicity – in particular we show that the system arising from the Rudin-Shapiro substitution is not AT. We also show that some nil-rotations on a quotient of the Heisenberg group as well as some (generalized) Gaussian systems are not AT. All known examples of non AT-automorphisms contain a Lebesgue component in the spectrum.

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1. INTRODUCTION

In this article we deal with the *approximate transitivity* property (AT property for short) in ergodic theory. This property has been introduced by A. Connes and G.J. Woods in [4] in connection with some classification problems of factors of type III_0 in the theory of von Neumann algebras. We recall now the definition and some basic facts.

Let G be an Abelian countable (discrete) group acting as measure preserving transformations: $g \mapsto T_g$, on a standard probability Borel space (X, \mathcal{B}, μ) . This action is called AT (or AT(1)) if for an arbitrary family of nonnegative functions $f_1, \dots, f_l \in L^1_+(X, \mathcal{B}, \mu)$, $l \geq 2$, and any $\varepsilon > 0$, there exist a positive integer s , $g_1, \dots, g_s \in G$, $\lambda_{j,k} \geq 0$, $j = 1, \dots, l$, $k = 1, \dots, s$ and $f \in L^1_+(X, \mathcal{B}, \mu)$ such that

$$(1) \quad \|f_j - \sum_{k=1}^s \lambda_{j,k} f \circ T_{g_k}\|_1 < \varepsilon, \quad 1 \leq j \leq l.$$

In fact, in (1) we can restrict ourselves to take only $l = 2$; indeed, given $\varepsilon > 0$ we apply (1) to f_1, f_2 and obtain $f = f_{12}$, then we apply again (1) to f_{12} and f_3 for some ε' sufficiently small and obtain f_{123} and we conclude after $l - 1$ steps.

Only few general facts about AT-systems in ergodic theory are known. The AT property forces the system to be ergodic and to have zero entropy [7], [4], [8]. Moreover funny rank 1 systems enjoy the AT property [4], [34],[35]. Clearly, the class of AT-systems is closed under taking factors and inverse limits (and roots for \mathbb{Z} -actions).

The action of G on (X, \mathcal{B}, μ) induces a (continuous) unitary representation, called *Koopman representation*, of G in the space $L^2(X, \mathcal{B}, \mu)$ given by $U_{T_g} f = f \circ T_g$, $f \in L^2(X, \mathcal{B}, \mu)$ and $g \in G$. Recall that such a representation is said to have *simple spectrum* if $L^2_0(X, \mathcal{B}, \mu) = G(f)$ where $G(f)$ stands for the *cyclic space* generated by f , i.e. $G(f) = \overline{\text{span}}\{f \circ T_g : g \in G\}$. In view of the definition of the AT property it is natural to ask whether it already implies simplicity of the spectrum – this question appeared (or is treated implicitly) in several papers, see David [7], Hawkins [16], Hawkins-Robinson [17], Golodets [15] and Dooley-Quas [8]. This is still an open problem also for $G = \mathbb{Z}$. A stronger conjecture due to Dooley and Quas [8] is that AT-systems are exactly funny rank-1 systems (this latter class is known to be a subclass of simple spectrum actions). This conjecture is based on the fact that a criterion for a system not be AT given in [8] which we repeat in Section 2 is also sufficient for a system not to be of funny rank-1.

As in the definition of the AT property we deal with L^1 -functions, one can also ask about simplicity of the spectrum in $L^1(X, \mathcal{B}, \mu)$ for the induced action of G on L^1 , that is we ask whether there exists a function $f \in L^1(X, \mathcal{B}, \mu)$ for which the linear span of the functions $f \circ T_g$, $g \in G$, is dense in L^1 . The conjecture of Thouvenot from the 1980th states that each ergodic automorphism has a simple L^1 -spectrum (see also related works on L^p -multiplicities by Iwanik [19, 20] and Iwanik-Sam de Lazaro [21]). Thouvenot himself observed that for $G = \mathbb{Z}$ AT-automorphisms have simple L^1 -spectrum; indeed, all we need to show is that given $g, h \in L^1(X, \mathcal{B}, \mu)$ and $\varepsilon > 0$ we can find $f \in L^1(X, \mathcal{B}, \mu)$ such that

$$d(g, f) < \varepsilon \text{ and } d(h, \mathbb{Z}(f)) < \varepsilon$$

because then the open set $\{f \in L^1(X, \mathcal{B}, \mu) : d(h, \mathbb{Z}(f)) < \varepsilon\}$ is dense, and we can use a Baire type argument. Now by using the AT property, we can easily

arrive at a situation that $d(g, P(U_T)(f')) < \varepsilon'$ and $d(h, Q(U_T)(f')) < \varepsilon'$ where P, Q are trigonometric polynomials. By replacing (in the space $A(\mathbb{T})$) P by another trigonometric polynomial we can assume that P has no zeros on the circle and we simply put $f = P(U_T)f'$ noticing that the cyclic space generated by f is the same as the one generated by f' ; indeed $1/P(z)$ also belongs to the space $A(\mathbb{T})$.

In fact, it was unknown until very recently that a system with zero entropy without AT property could exist ($G = \mathbb{Z}$). In [8] two examples of zero entropy non AT-systems are exhibited. For both of them the associated Koopman operator has a Lebesgue component in the spectrum, moreover the component has infinite multiplicity. In connection with that two natural questions arises. Can we find a non AT ergodic automorphism whose Koopman operator has a finite multiplicity? Does the AT property imply singularity of the spectrum? We will give the positive answer to the first question; we have been unable to answer the second one – recall that even in the class of rank-1 transformations it is unknown whether the spectrum has to be singular.

In the present paper we will prove a criterion (see Proposition 3.4 below) for a system not to be AT which is an elaborated version of an argument implicitly contained in [8]. Our criterion looks spectral and should work in case of an automorphism with a “good” Lebesgue component in the spectrum. However we require for some function of type $\chi_{P_0} - \chi_{P_1}$ ((P_0, P_1) is a partition of X) to have the spectral measure absolutely continuous with a good control of its density which makes the use of the criterion “in practice” a delicate task. We will go through many known constructions of zero entropy dynamical systems having a Lebesgue component in the spectrum and show that they or systems “close” to them are not AT. It should be mentioned that for all known non AT systems the absence of approximate transitivity turned out to be a consequence of the criterion, including the non AT property of positive entropy automorphisms (see Corollary 4.6 below).

We show that the automorphism given by the Rudin-Shapiro substitution is not AT but it has a finite multiplicity (more precisely, the associated Koopman operator has the maximal spectral multiplicity equal to 2 [27], [30], [32]). Recently Giordano and Handelmann [13] introduced and studied the following notion of AT(n).

Definition 1.1. Let (X, \mathcal{B}, μ, T) be a dynamical system and n be a positive integer. The transformation T is AT(n) if for any $\varepsilon > 0$, for any set of $n+1$ nonnegative functions $\{f_i\}_{i=1}^{n+1} \subset L_+^1(X, \mu)$, there exist n nonnegative functions $g_m, m = 1, \dots, n$, a positive integer N , positive coefficients $(\alpha_{i,j}^{(m)})_{i=1, \dots, n+1; j=1, \dots, n}^{m=1, \dots, n}$ and a finite sequence of integers $\{t_j^{(m)}\}_{j=1, \dots, N}^{m=1, \dots, n}$ such that

$$\|f_i - \sum_{m=1}^n \sum_{j=1}^N \alpha_{i,j}^{(m)} g_m \circ T^{t_j^{(m)}}\|_1 < \varepsilon$$

for all $i \in \{1, \dots, n+1\}$.

Clearly, each AT(n) system enjoys AT($n+1$) property. It is easy to see that the transformation with rank n (see e.g. [9] for the relevant definitions and properties) is AT(n) and it is well known that the system determined by a substitution on k symbols has the rank at most k . In the case of Rudin-Shapiro substitution the corresponding automorphism has rank 4 (see [31]). It follows that the automorphism given by the Rudin-Shapiro substitution is a natural example of a system which is AT(4) but not AT(1), see [13] for other examples of that type.

Moreover we show that each ergodic system has an ergodic distal (see [12]) extension which is not AT. Our method also shows that Helson and Parry's "random" construction from [18] of a 2-point extension of an arbitrary (aperiodic) system with a Lebesgue component is "almost" non-AT; actually (on a set of positive measure of parameters) its 4th power is not AT. We also show that some nil-translations on a quotient of the Heisenberg group are not AT. Furthermore we show that the non AT property for affine transformations of the torus enjoy some stability property. Finally we will deal with the non AT property in the class of Gaussian systems. We will give examples of zero entropy (mixing) Gaussian systems T such that $T \times T \times T \times T$ is not AT. Extensions of Gaussian systems via cocycles are treated in the last section.

2. A NECESSARY COMBINATORIAL CONDITION TO HAVE AT PROPERTY

Let G be a countable Abelian (discrete) group acting measurably on a standard probability Borel space (X, \mathcal{B}, μ) by measure-preserving transformations $g \mapsto T_g$, $g \in G$. Let $\mathcal{P} = \{P_0, P_1\}$ be a (measurable) partition of X . Then each point $x \in X$ has its \mathcal{P} -name $\pi(x) = (x_g)_{g \in G} \in \{0, 1\}^G$, where

$$x_g = \begin{cases} 0 & \text{if } T_g(x) \in P_0; \\ 1 & \text{if not.} \end{cases}$$

Let Λ be a finite set in G . We define a *funny word on the alphabet* $\{0, 1\}$ *based on* Λ as a finite sequence $(W_g)_{g \in \Lambda}$, $W_g \in \{0, 1\}$. For any two funny words W, W' based on the same set Λ their *Hamming distance* is given by

$$\bar{d}_\Lambda(W, W') = \frac{1}{|\Lambda|} \text{card} \{i \in \Lambda : W_i \neq W'_i\}.$$

The proposition below gives a necessary condition for an action to be AT. The proof of it follows word by word the proof given by Dooley and Quas [8] in the case of \mathbb{Z} -actions.

Proposition 2.1. ([8]) Let (X, \mathcal{B}, μ, G) be an AT dynamical system. Then for any $\varepsilon > 0$, there exist a finite set $\Lambda \subset G$ of arbitrary large cardinality and a funny word W based on Λ such that

$$|\Lambda| \mu \{x \in X : \bar{d}(\pi(x)|_\Lambda, W) < \varepsilon\} > 1 - \varepsilon.$$

3. CRITERION TO BE NON-AT

In this section we assume that $G = \mathbb{Z}$ and we put $T_1 = T$. We will give a certain criterion for a system not to be AT. It is a spectral extension of a criterion implicitly stated in [8].

3.1. Strongly BH probability measures on the circle. Denote by ε_0 the unique zero in $(0, 0.2)$ of the polynomial $P(t) = 2(1-t)(1-2t)^2 - 1 - t$. Let μ be a probability measure on the circle group. Motivated by Theorem 3.1 below, we call μ a *strongly Blum-Hanson measure* (SBH measure) if the following holds

$$\limsup_{k \rightarrow +\infty} \sup \left\{ \left\| \frac{1}{\sqrt{k}} \sum_{j=1}^k (-1)^{\eta_j} z^{n_j} \right\|_{L^2(\mu)}^2 : n_1 < \dots < n_k, \eta_j \in \{0, 1\}, 1 \leq j \leq k \right\} \leq 1 + \varepsilon_0.$$

Clearly Lebesgue measure is an SBH measure. However more generally each absolutely continuous measure with sufficiently “flat” density g , i.e. the density satisfying $\sup_{z \in \mathbb{T}} g(z) < 1 + \varepsilon_0$ is also an SBH measure.

Recall that a measure μ on the circle is called a *Rajchman measure* if $\lim_{n \rightarrow \infty} \widehat{\mu}(n) = 0$. Using some ideas of Lyons [29] we will prove that each SBH measure is a Rajchman measure. It is well-known that Rajchman measure can be singular (see also Section 4.5 of the paper), however we have been unable to decide whether there exist singular SBH measures. We state also in the following the Blum-Hanson’s theorem [2] in ergodic theory as a characterization of Rajchman measures.

Theorem 3.1. μ is Rajchman measure if and only if for any infinite increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of integers, we have

$$(2) \quad \left\| \frac{1}{k} \sum_{j=1}^k z^{n_j} \right\|_{L^2(\mu)}^2 \xrightarrow[k \rightarrow \infty]{} 0$$

Using some ideas of Lyons [29] we will now show that SBH measures belong to the class of measures that annihilate all so called W^* -sets (this class is known to be a proper subclass of Rajchman measures, [29]). We need to recall some basic definitions. A sequence $\{t_n\} \subset \mathbb{T}$ is said to be *uniformly distributed* if for all arcs $I \subset \mathbb{T}$

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^N \chi_I(n_k x) = |I|.$$

Weyl’s criterion (e.g. [26], pp. 1-3,7-8)) states that a sequence $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{T}$ is uniformly distributed if and only if for every non-zero integer m ,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^N e(mt_k) = 0,$$

where we use notation $e(x)$ for $e^{2i\pi x}$.

Now, following Kahane and Lyons we define the W^* -sets. A Borel set $B \subset \mathbb{T}$ is called a *W^* -set* [23] (or a non-normal set) if there exists an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ such that for every $x \in B$, $\{n_k x\}_{k=1}^{\infty}$ is not uniformly distributed. The maximal W^* -set corresponding to $\{n_k\}_{k \in \mathbb{N}}$ is the set $W^*(\{n_k\}) := \{x \in \mathbb{T} : \{n_k x\} \text{ is not uniformly distributed}\}$ (it is Borel).

By Weyl’s Criterion, in order to show that some probability measure μ vanishes on all W^* -sets, we need to show that for each $m \neq 0$ and each increasing sequence $\{n_k\}$

$$(3) \quad \frac{1}{K} \sum_{k=1}^K e(mn_k t) = 0 \text{ for } \mu\text{-a.e. } t \in \mathbb{T}.$$

Actually, since $\{n_k\}$ is arbitrary, it is enough to establish (3) for $m = 1$. We need the following form of the strong law of large numbers for weakly correlated bounded random variables due to Lyons [29] and for the convenience of the reader we include the proof.

Lemma 3.2. Let $\{X_n\}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$. Suppose that $|X_n| \leq 1$ a.s. and there exists a positive constant $c \geq 1$

such that for any $\{n_k\} \uparrow +\infty$ and N large enough we have

$$(4) \quad \left\| \frac{1}{K} \sum_{k=1}^K X_{n_k} \right\|^2 < \frac{c}{K}.$$

Then the strong law of large numbers holds:

$$\lim_{N \rightarrow +\infty} \frac{1}{K} \sum_{k=1}^K X_{n_k} = 0 \quad \text{a. s.}$$

Proof. Assume that $n_k \uparrow +\infty$. It follows from (4) that

$$\int \sum_{K \geq 1} \left| \frac{1}{K^2} \sum_{k=1}^{K^2} X_{n_k} \right|^2 d\mu < c \sum_{K \geq 1} \frac{1}{K^2} = \frac{c\pi^2}{6}.$$

Therefore \mathbb{P} -a.s.

$$\sum_{K \geq 1} \left| \frac{1}{K^2} \sum_{k=1}^{K^2} X_{n_k} \right|^2 < \infty$$

and hence (\mathbb{P} -a.s.)

$$\lim_{K \rightarrow +\infty} \left| \frac{1}{K^2} \sum_{k=1}^{K^2} X_{n_k} \right| = 0.$$

Now if $m^2 \leq K < (m+1)^2$, then

$$\left| \frac{1}{K} \sum_{k=m^2+1}^K X_{n_k} \right| \leq \frac{1}{K}(K - m^2) \leq \frac{1}{K}(2m+1) \leq \frac{1}{m^2}(2m+1) \rightarrow 0$$

when $m \rightarrow \infty$. But

$$\left| \frac{1}{K^2} \sum_{k=1}^K X_{n_k} \right| \leq \left| \frac{1}{m^2} \sum_{k=1}^{m^2} X_{n_k} \right| + \left| \frac{1}{K} \sum_{k=m^2+1}^K X_{n_k} \right|$$

and hence

$$\lim_{K \rightarrow +\infty} \frac{1}{K} \sum_{k=1}^K X_{n_k} = 0 \quad \text{a.s.}$$

which completes the proof. \square

Proposition 3.3. If μ is an SBH measure then $\mu(E) = 0$ for each W^* -set $E \subset \mathbb{T}$.

Proof. Let μ be an SBH measure. Then, for any increasing sequence $\{n_k\}$ we have

$$\left\| \frac{1}{\sqrt{k}} \sum_{j=1}^k z^{n_j} \right\|_{L^2(\mu)}^2 < 1 + \varepsilon_0$$

for k large enough. It follows from Lemma 3.2 that the strong law of large numbers holds for the sequence $\{X_{n_k}\}$ with $X_{n_k} = e(n_k \cdot)$, $k \geq 1$, and the result follows. \square

We now pass to our criterion for a system not to be an AT-system (we refer the reader to [5] to basic facts about spectral theory of dynamical systems).

Proposition 3.4. Assume that a dynamical system (X, \mathcal{B}, μ, T) is ergodic and that there exists a (measurable) partition $\mathcal{P} = \{P_0, P_1\}$ with the following properties:

- i) There exists S in the centralizer $C(T)$ of T such that $SP_0 = P_1$; in particular $\mu(P_0) = \mu(P_1) = \frac{1}{2}$.
- ii) The spectral measure $\sigma_{\chi_{P_0} - \chi_{P_1}}$ of $\chi_{P_0} - \chi_{P_1}$ is an SBH measure.

Then the system is not AT.

Proof. Let us take W a funny word based on a subset $\Lambda : n_1 < n_2 < \dots < n_k$. For $x \in X$ put

$$\Theta^W(x) = \frac{1}{k} \sum_{j=1}^k A_j^W(x),$$

where A_j^W is defined as

$$A_j^W(x) = \begin{cases} 1 & \text{if } W_{n_j} = x_{n_j} \\ -1 & \text{if not} \end{cases}$$

(recall that $(x_n) = \pi(x)$ is the \mathcal{P} -name of x). Then the distribution Θ_* of Θ is symmetric. Indeed, we have $\pi(x) = -\pi(Sx)$ and therefore

$$\Theta(\pi(x)) = -\Theta(\pi(Sx))$$

and since S is measure-preserving the symmetry of Θ_* follows.

Notice that

$$(5) \quad A_j^W(x) = (-1)^{W_{n_j}} (\chi_{P_0} - \chi_{P_1})(T^{n_j}x)$$

and that

$$(6) \quad \Theta^W(x) = 1 - 2\bar{d}_\Lambda(W, \pi(x)|_\Lambda).$$

In view of (6), the symmetry of Θ_* and the Tchebychev inequality we obtain that

$$(7) \quad \begin{aligned} \mu\{x \in X : \bar{d}_\Lambda(W, \pi(x)|_\Lambda) < \varepsilon\} &= \mu\{x \in X : \Theta^W(x) > 1 - 2\varepsilon\} \\ &= \frac{1}{2} \mu\{x \in X : |\Theta^W(x)| > 1 - 2\varepsilon\} \\ &\leq \frac{1}{2(1 - 2\varepsilon)^2} \|\Theta^W\|_2^2. \end{aligned}$$

But, in view of (5) and the Spectral Theorem

$$\begin{aligned} \|\Theta^W\|_2^2 &= \int_X \left| \frac{1}{k} \sum_{j=1}^k A_j^W \right|^2 d\mu \\ &= \frac{1}{k^2} \sum_{i,j=1}^k \int_X (-1)^{W_{n_i}} (\chi_{P_0} - \chi_{P_1})(T^{n_i}x) \cdot (-1)^{W_{n_j}} (\chi_{P_0} - \chi_{P_1})(T^{n_j}x) d\mu(x) \\ &= \frac{1}{k^2} \sum_{i,j=1}^k (-1)^{W_{n_i} + W_{n_j}} \hat{\sigma}_{\chi_{P_0} - \chi_{P_1}}(n_i - n_j) = \frac{1}{k} \int \left| \frac{1}{\sqrt{k}} \sum_{i=1}^k (-1)^{W_{n_i}} z^{n_i} \right|^2 d\sigma_{\chi_{P_0} - \chi_{P_1}}(z). \end{aligned}$$

It follows that for k large enough we have

$$(8) \quad \|\Theta\|_2^2 < \frac{1}{k}(1 + \varepsilon_0).$$

Combining (7) and (8) we obtain that

$$k \mu\{x \in X : \bar{d}_\Lambda(W, \pi(x)|_\Lambda) < \varepsilon\} \leq \frac{1 + \varepsilon}{2(1 - 2\varepsilon)^2}$$

and since W was arbitrary this contradicts Proposition 2.1. \square

Corollary 3.5. Under the assumptions of Proposition 3.4 assume that the Fourier transform of $\sigma_{\chi_{P_0} - \chi_{P_1}}$ is in l^1 . Then there exists $m_0 \geq 1$ such that T^m is not AT for all $m \geq m_0$.

Proof. In this case $\sigma = \sigma_{\chi_{P_0} - \chi_{P_1}}$ is absolutely continuous with the (continuous) density d given by $d(z) = \sum_{n=-\infty}^{\infty} \hat{\sigma}(n)z^n$ and by the same token if instead of T we consider T^m the spectral measure σ_m of $\chi_{P_0} - \chi_{P_1}$ is also absolutely continuous with the density d_m given by

$$d_m(z) = \sum_{n=-\infty}^{\infty} \hat{\sigma}_m(n)z^n = \sum_{n=-\infty}^{\infty} \hat{\sigma}(mn)z^n,$$

so for m large enough σ_m will be an SBH measure. \square

Remark 3.6. Notice that the L^2 -norm of $\chi_{P_0} - \chi_{P_1}$ is 1. Moreover the spectral measure of $\chi_{P_0} - \chi_{P_1}$ is Lebesgue if and only if the sequence of partitions $\{T^n \mathcal{P}\}$ is pairwise independent. Indeed, for $n \geq 1$, $\hat{\sigma}_{\chi_{P_0} - \chi_{P_1}}(n) = 0$ implies that

$$\mu(T^{-n}P_0 \cap P_0) + \mu(T^{-n}P_1 \cap P_1) = \mu(T^{-n}P_0 \cap P_1) + \mu(T^{-n}P_1 \cap P_0).$$

Thus

$$\mu(T^{-n}P_0 \cap P_0) + \mu(T^{-n}P_1 \cap P_1) = \frac{1}{2}$$

and since $\mu(T^{-n}P_0 \cap P_1) + \mu(T^{-n}P_0 \cap P_0) = \frac{1}{2}$, we have

$$\mu(T^{-n}P_1 \cap P_1) = \mu(T^{-n}P_0 \cap P_1) = \frac{1}{4}$$

and therefore we obtain pairwise independence (see [6], [10]).

However, in general it is unclear that even for transformations with Lebesgue spectrum or the more with a Lebesgue component in the spectrum) we can always find a partition \mathcal{P} satisfying the assumptions of Proposition 3.4 where in addition the spectral measure of $\sigma_{\chi_{P_0} - \chi_{P_1}}$ is exactly Lebesgue. A certain flexibility of Proposition 3.4 consists in the fact that for some natural partitions we need only to show that the corresponding spectral measure is absolutely continuous with the density sufficiently flat.

In the rest of the paper we will show how this can be applied in practice.

Remark 3.7. We can also define the notion of SBH measure for groups G more general than \mathbb{Z} . Based on Proposition 2.1 we can then prove a relevant version of Proposition 3.4 as a criterion for a G -system to be non AT. It would be interesting to know which results of Section 4 have their natural generalizations.

4. APPLICATIONS

Except for the Gaussian case considered in Section 4.5 all examples below of non AT-automorphisms will be given as group extensions. Recall briefly some basic facts. Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be an ergodic automorphism. Let G be a compact metric Abelian group with Haar measure m . Denote by \widehat{G} the character group of G . By a *cocycle* we mean a measurable function $\varphi : X \rightarrow G$; in fact such a φ generates a cocycle $\varphi(n, \cdot) = \varphi^{(n)}(\cdot)$ by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \dots + \varphi(T^{n-1}x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -(\varphi(T^n x) + \dots + \varphi(T^{-1}x)) & \text{if } n < 0. \end{cases}$$

Then we define a $\mu \otimes m$ -preserving automorphism

$$T_\varphi : X \times G \rightarrow X \times G, \quad T_\varphi(x, g) = (Tx, \varphi(x) + g), \quad x \in X, g \in G$$

called a G -extension of T . Notice that $T_\varphi^n(x, g) = (T^n x, \varphi^{(n)}(x) + g)$. The space $L^2(X \times G, \mu \otimes m)$ can be decomposed as

$$(9) \quad L^2(X \times G, \mu \otimes m) = \bigoplus_{\chi \in \widehat{G}} L_\chi,$$

where each subspace $L_\chi = \{f \otimes \chi : f \in L^2(X, \mu)\}$ is U_{T_φ} -invariant, and the restriction of U_{T_φ} to L_χ is unitarily equivalent to $V_{\varphi, T, \chi} : L^2(X, \mu) \rightarrow L^2(X, \mu)$ defined by $V_{\varphi, T, \chi}(f)(x) = \chi(\varphi(x))f(Tx)$, $x \in X$. It follows that to describe spectral properties of T_φ it is sufficient to study spectral properties of $V_{\varphi, T, \chi}$, $\chi \in \widehat{G}$.

4.1. Two point extensions and the AT property, Rudin-Shapiro substitution. Assume that $G = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. In this case we have a natural partition \mathcal{P} given by $P_0 = X \times \{0\}$, $P_1 = X \times \{1\}$. The assumption (i) of Proposition 3.4 is satisfied as $SP_0 = P_1$ where $S(x, i) = (x, i + 1)$ is in the centralizer of T_φ . Notice that the decomposition (9) of $L^2(X \times \{0, 1\})$ is of the form $L_0 \oplus L_1$ where $L_0 = \{f \in L^2(X \times \{0, 1\}) : f \circ S = f\}$ and $L_1 = \{f \in L^2(X \times \{0, 1\}) : f \circ S = -f\}$. Thus $U_{T_\varphi}|_{L_1}$ is spectrally isomorphic to the operator V defined by

$$(V(f))(x) = (-1)^{\varphi(x)} f(Tx).$$

The function $\chi_{P_0} - \chi_{P_1}$ belongs to L_1 and under the above isomorphism it corresponds to the constant function $1 \in L^2(X, \mathcal{B}, \mu)$. It follows that

$$\begin{aligned} \widehat{\sigma}_{\chi_{P_0} - \chi_{P_1}}(n) &= \langle V^n 1, 1 \rangle \\ &= \mu \left(\left\{ x \in X : \sum_{j=0}^{n-1} \varphi(T^j x) = 0 \right\} \right) - \mu \left(\left\{ x \in X : \sum_{j=0}^{n-1} \varphi(T^j x) = 1 \right\} \right). \end{aligned}$$

In case when T is the dyadic odometer such extensions were intensively studied in the 1980th. In particular, Mathew and Nadkarni [30] gave constructions of φ such that

$$(10) \quad \langle V^n 1, 1 \rangle = 0 \quad \text{for all } n \neq 0,$$

that is, the corresponding spectral measure is equal to Lebesgue measure (in fact the Lebesgue component has multiplicity 2). Other examples with Lebesgue component with arbitrary even multiplicity are given by so called Toeplitz extensions in [27] – each time (10) holds. In particular it is shown in [27] that the system given by

Rudin-Shapiro substitution (see [32] where also it is shown that it has a Lebesgue component of multiplicity 2) is a particular member of Mathew-Nadkarni's family.

Corollary 4.1. Mathew-Nadkarni's maps as well as all examples from [27] having even Lebesgue multiplicity are not AT-systems. In particular the automorphism given by the Rudin-Shapiro substitution is not AT.

Remark 4.2. One may also use Ageev's construction [1] to produce a continuum of weakly mixing automorphisms with spectral multiplicity equal to 2 and without AT property.

We now recall Helson and Parry's construction from [18] of "random" 2-point extensions. Given an aperiodic automorphism T of a standard probability Borel space they give a random construction of cocycles $\varphi_\omega : X \rightarrow \{0, 1\}$ (the parameter ω runs over a probability space (Ω, P)) such that for a.e. ω , T_{φ_ω} has absolutely continuous spectrum and on a set of positive P -measure T_{φ_ω} has Lebesgue spectrum. In particular, they prove that

$$(11) \quad \int_{\Omega} \left| \int_X e^{\pi i \varphi_\omega^{(n)}(x)} d\mu(x) \right|^2 dP(\omega) < \gamma_n \text{ for all } n \geq 4,$$

where $\{\gamma_n\}$ is an arbitrary set of positive numbers. Since, by (11), on a set $\Omega_n \subset \Omega$ of measure at least $1 - \frac{1}{2^{n+1}}$ we have

$$\left| \int_X e^{\pi i \varphi_\omega^{(n)}(x)} d\mu(x) \right|^2 < 2^{n+1} \gamma_n, \quad n \geq 4,$$

by selecting $\{\gamma_n\}$ as small as we need we can obtain that the Fourier transform $\{\hat{\sigma}_{\chi_{P_0} - \chi_{P_1}}(n)\}$ is absolutely summable with $\sum_{|n| \geq 4} |\hat{\sigma}_{\chi_{P_0} - \chi_{P_1}}(n)|$ as small as we need on the set $\cap_{n \geq 4} \Omega_n$ of positive measure of parameters. By Corollary 3.5 (or rather its proof) we obtain the following.

Corollary 4.3. For each ergodic (aperiodic) automorphism T there exists an ergodic 2-point extension T_φ such that T_φ^4 is not AT.

Therefore we get a partial answer to the question from [8] (see also next section).

4.2. Distal (ergodic) extension without AT property. The aim of this section is to prove the following.

Proposition 4.4. For each ergodic transformation T acting on a standard Borel probability space (X, \mathcal{B}, μ) there exists a 2-step group extension \overline{T} which is ergodic and such that \overline{T} is not AT.

Proof. We use the idea of affine extension of Glasner [14] combined with Proposition 3.4. Let $\varphi : X \rightarrow \mathbb{T}$, $\mathbb{T} = [0, 1)$, be a cocycle such that T_φ is ergodic and the groups of eigenvalues for T and T_φ are the same; in particular if T is weakly mixing, so is T_φ . Let $\psi : X \times \mathbb{T} \rightarrow \mathbb{T}$ be the cocycle defined by $(x, y) \mapsto y$, and finally put $\overline{T} = (T_\varphi)_\psi$. It is easy to check (by considering the relevant functional equations and applying Fourier series arguments) that \overline{T} is ergodic and it does not change the group of eigenvalues of T .

Consider $P_0 = X \times \mathbb{T} \times [0, \frac{1}{2})$. Then $P_1 \stackrel{\text{def}}{=} P_0^c = R_{\frac{1}{2}} P_0$, where $R_{\frac{1}{2}}(x, y, z) = (x, y, z + \frac{1}{2})$ and $R_{\frac{1}{2}} \in C(\overline{T})$. We have to compute the spectral measure of

$f(x, y, z) = f(z) = \chi_{P_0}(x, y, z) - \chi_{P_1}(x, y, z) = e^{i\pi\chi_A(z)}$, where $A = [0, \frac{1}{2}]$. We have

$$\begin{aligned} \langle U_{\mathbb{T}}^n f, f \rangle &= \int_{X \times \mathbb{T} \times \mathbb{T}} f(T_{\varphi}^{(n)}(x, y), \psi^{(n)}(x, y) + z) \overline{f(x, y, z)} d\mu(x) dy dz \\ &= \int e^{i\pi\chi_A(\psi(x, y) + \psi(Tx, \varphi(x) + y) + \dots + \psi(T^{n-1}x, \varphi^{(n-1)}(x) + y) + z)} e^{-i\pi\chi_A(z)} d\mu(x) dy dz \\ &= \int_{X \times \mathbb{T}} e^{-i\pi\chi_A(z)} \left(\int_{\mathbb{T}} e^{i\pi\chi_A(y + \varphi(x) + y + \dots + \varphi^{(n-1)}(x) + y + z)} dy \right) d\mu(x) dz \\ &= \int_{X \times \mathbb{T}} e^{-i\pi\chi_A(z)} \left(\int_{\mathbb{T}} e^{i\pi\chi_A(ny)} dy \right) d\mu(x) dz. \end{aligned}$$

We will now show that

$$\int_{\mathbb{T}} e^{i\pi\chi_A(ny)} dy = 0 \quad \text{for } n \neq 0.$$

Indeed

$$\int_{\mathbb{T}} e^{i\pi\chi_A(ny)} dy = \sum_{j=0}^{2n-1} \int_{I_j} e^{i\pi\chi_A(ny)} dy \quad \text{where } I_j = \left[\frac{j}{2n}, \frac{j+1}{2n} \right),$$

and we consider the map $y \xrightarrow{\tau_n} ny$ and the images of I_j under this map. We have $\tau_n(I_0) = A, \tau_n(I_1) = A^c, \tau_n(I_2) = A, \dots$. Hence

$$\int_{I_j} e^{i\pi\chi_A(ny)} dy = (-1)^{j+1}.$$

Since $j \in \{0, \dots, 2n-1\}$, it follows that

$$\int_{\mathbb{T}} e^{i\pi\chi_A(ny)} dy = 0.$$

We deduce that the spectral measure of $\chi_{P_0} - \chi_{P_1}$ is exactly Lebesgue measure. \square

Remark 4.5. Instead of $\psi(x, y) = y$ we can take $\psi(x, y) = my, m \neq 0$. In the concluding argument we divide $[0, 1)$ into intervals of length $\frac{1}{2|m|n}$.

Corollary 4.6. ([4],[7]). If the dynamical system (X, \mathcal{B}, μ, T) is AT then its entropy is zero.

Proof. First, note that no Bernoulli dynamical system is AT. Indeed, apply the proposition 4.4 to get the weakly mixing 2-step compact group extension $(T_{\phi})_{\psi}$ of T which is not AT. But, by [33] $(T_{\phi})_{\psi}$ is again Bernoulli with same entropy as T . It follows from the Ornstein isomorphism theorem that T is not AT.

Assume that the entropy of T is strictly positive. By the Kolmogorov-Sinai theorem there exists a Bernoulli factor with the same entropy. But the factor of the system with the AT property is AT. We get that T is not AT. \square

4.3. Absolutely continuous cocycles over irrational rotations without AT property. Denote $\mathbb{T} = [0, 1)$ and let $Tx = x + \alpha$ be an irrational rotation. Consider

$$F(x, y) = \chi_{\mathbb{T} \times [0, \frac{1}{2})}(x, y) - \chi_{\mathbb{T} \times [\frac{1}{2}, 1)}(x, y) = 2\chi_{\mathbb{T} \times [0, \frac{1}{2})}(x, y) - 1 = f(y),$$

where $f(y) = 2\chi_{[0, \frac{1}{2})}(y) - 1$. For $m \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned} \hat{f}(m) &= \int_0^1 f(y)e^{-2\pi imy} dy \\ &= 2 \int_0^1 \chi_{[0, \frac{1}{2})}(y)e^{-2\pi imy} dy - \int_0^1 e^{-2\pi imy} dy = 2 \int_0^{1/2} e^{-2\pi imy} dy \\ &= \frac{1}{-\pi im} e^{-2\pi imy} \Big|_0^{1/2} = \frac{-1}{\pi im} (e^{-\pi im} - 1) = \begin{cases} 0 & \text{if } m = 2k \\ \frac{2}{\pi im} & \text{if } m = 2k + 1. \end{cases} \end{aligned}$$

Moreover

$$F(x, y) = \sum_{m=-\infty}^{\infty} \hat{f}(m)e^{2\pi imy} = \sum_{k=-\infty}^{\infty} \frac{2}{(2k+1)\pi i} e^{2\pi i(2k+1)y}.$$

Notice that functions

$$e^{2\pi imy} =: \xi_m(x, y) \in L^2(\mathbb{T}) \otimes e^{2\pi imy}$$

where the latter subspace is U_{T_ϕ} -invariant for each cocycle $\phi : \mathbb{T} \rightarrow \mathbb{T}$. In what follows we assume that

$$\phi(x) = e^{2\pi i(x+g(x))}$$

where $g : \mathbb{T} \rightarrow \mathbb{R}$ is a ‘‘smooth’’ function: we will precise conditions imposed on g later; it is however at least that g is absolutely continuous and its derivative is a.e. equal to a function of bounded variation. It follows that the spectral measure of F is equal to the sum of spectral measures of its orthogonal projections on subspaces $L^2(\mathbb{T}) \otimes e^{2\pi imy}$. Let us compute the spectral measure of ξ_m :

$$\int \xi_m \circ T_\phi^n \cdot \overline{\xi_m} dx dy = \int_0^1 e^{2\pi im(nx + \frac{n(n-1)}{2}\alpha + g^{(n)}(x))} dx.$$

For $n \neq 0$ we have (using integration by parts for Fourier-Stieltjes integrals as in [22]):

$$\begin{aligned} \int_0^1 e^{2\pi im(nx + g^{(n)}(x))} dx &= \frac{1}{2\pi im} \int_0^1 \frac{1}{n + g'^{(n)}(x)} de^{2\pi im(nx + g^{(n)}(x))} \\ &= \frac{1}{2\pi im} \cdot (-1) \int_0^1 e^{2\pi im(nx + g^{(n)}(x))} d\left(\frac{1}{n + g'^{(n)}(x)}\right). \end{aligned}$$

We will now assume additionally that $g' > -1 + \delta_0$ ($0 < \delta_0 < 1$) so that we can pass to a well-known estimation (see again [22]):

$$\begin{aligned} \left| \int \xi_m \circ T_\phi^n \cdot \overline{\xi_m} dx dy \right| &\leq \frac{1}{2\pi|m|} \text{Var} \left(\frac{1}{n + g'^{(n)}(x)} \right) \\ &\leq \frac{1}{2\pi|m|} \frac{\text{Var} g'^{(n)}}{(1 - \delta_0)^2 n^2} \leq \frac{1}{2\pi|m|} \frac{\text{Var} g'}{(1 - \delta_0)^2 |n|}. \end{aligned}$$

Hence (for $n \neq 0$)

$$\begin{aligned} |\hat{\sigma}_F(n)| &= \left| \int F \circ T_\phi^n \cdot \overline{F} dx dy \right| = \left| \sum_{m \neq 0} \hat{f}(m) \int \xi_m \circ T_\phi^n \cdot \overline{\xi_m} dx dy \right| \\ &\leq \sum_{m \neq 0} |\hat{f}(m)| \frac{1}{2\pi|m|} \frac{\text{Var}(g')}{(1 - \delta_0)^2 |n|} = \frac{\text{Var}(g')}{2\pi(1 - \delta_0)^2 |n|} \sum_{m \neq 0} \frac{|\hat{f}(m)|}{|m|} = O\left(\frac{1}{|n|}\right). \end{aligned}$$

Notice however that the constant (in the expression $O\left(\frac{1}{|n|}\right)$) which appears can be made as small as we need by assuming that $\text{Var}(g')$ is small. It follows that σ_F is an absolutely continuous measure whose density d is given by $d(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, where $a_0 = 1$ and for $n \neq 0$, $a_n = \widehat{\sigma}_F(n)$; moreover $|a_n| \leq C \frac{1}{|n|}$ with $C > 0$ as small as we need.

Proposition 4.7. Assume that $Tx = x + \alpha$, $Sx' = x' + \beta$ where $\alpha, \beta, 1$ rationally independent. Let $\phi(x) = e^{2\pi i(x+g(x))}$, $\psi(x') = e^{2\pi i(x'+h(x))}$ where $g, h : \mathbb{T} \rightarrow \mathbb{R}$ are absolutely continuous, with the derivatives (a.e.) of bounded variation and bounded away from $-1 + \delta_0$ for some $0 < \delta_0 < 1$. If the variations of g' and h' are sufficiently small then $T_\phi \times S_\psi$ is not AT.

Proof. It is enough to show that the factor $(T \times S)_{e^{2\pi i(x+x'+g(x)+h(x'))}}$ is not AT. We consider

$$F(x, x', y) = \chi_{\mathbb{T} \times \mathbb{T} \times [0, \frac{1}{2})} - \chi_{\mathbb{T} \times \mathbb{T} \times [\frac{1}{2}, 1)}.$$

By repeating all above calculations we end up with

$$|\widehat{\sigma}_F(n)| \leq O\left(\frac{1}{|n|^2}\right),$$

where the bounding constant is as small as we need. It follows that the density $d(z) = 1 + \sum_{n \neq 0} \widehat{\sigma}_F(n) z^n$ of σ_F is as close to 1 as we need and therefore σ_F is an SBH measure. \square

4.4. Nil-rotations without AT property. We consider nil-rotations S in dimension 3 only. Hence S is defined on $(\mathbb{R}^3, *) /_* \mathbb{Z}^3$ where we recall that the multiplication $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy')$ on \mathbb{R}^3 is the same as the multiplication in the Heisenberg group of upper triangle matrices. Moreover

$$S((x, y, z) * \mathbb{Z}^3) = (\alpha, \beta, 0) * (x, y, z) * \mathbb{Z}^3.$$

It is well-known (see e.g. [11]) that each such nil-rotation is isomorphic to a skew product transformation T_ϕ on $\mathbb{T}^2 \times \mathbb{S}^1$ where $T(x, y) = (x + \alpha, y + \beta)$, $\phi = e^{2\pi i \varphi}$ and

$$\varphi(x, y) = \alpha\{y\} - (\{x\} + \alpha)[\{y\} + \beta] + \gamma$$

(the nil-rotation is hence ergodic if $\alpha, \beta, 1$ are rationally independent; it follows that if a nil-rotation is ergodic so are all its non-zero powers). It is classical (Parry) that nil-rotations have countable Lebesgue spectrum in the orthocomplement of the subspace of eigenfunctions.

Proposition 4.8. For every ergodic nil-rotation S on $(\mathbb{R}^3, *) /_* \mathbb{Z}^3$ there exists $q \geq 1$ such that S^q is not AT. Moreover if in the above representation of S as a skew product T_ϕ , $\frac{1}{2} < \beta < 1$ is sufficiently close to 1 then S is not AT.

Proof. The method for spectral calculations which we apply below comes from [11].

As before we consider the function

$$F(x, y, z) = \chi_{\mathbb{T} \times \mathbb{T} \times [0, \frac{1}{2})}(x, y, z) - \chi_{\mathbb{T} \times \mathbb{T} \times [\frac{1}{2}, 1)}(x, y, z) = f(z),$$

where $f(z) = 2\chi_{[0, \frac{1}{2})}(z) - 1$, we look at its Fourier decomposition for $L^2(X) \otimes e^{2\pi i m z}$ where $m \in \mathbb{Z} \setminus \{0\}$ and we have

$$\hat{f}(m) = \begin{cases} 0 & \text{if } m = 2k \\ \frac{2}{\pi i m} & \text{if } m = 2k + 1. \end{cases}$$

It follows that

$$\begin{aligned} \langle F \circ T_\phi^n, F \rangle &= \sum_{m \neq 0} \hat{f}(m) \int \xi_m \circ T_\phi^n \cdot \bar{\xi}_m \, dx dy dz \\ &= \sum_{m \neq 0} \hat{f}(m) \int e^{2\pi i m \varphi^{(n)}(x,y)} \, dx dy. \end{aligned}$$

As F is real valued we only need to consider $n > 0$. We have

$$\int e^{2\pi i m \varphi^{(n)}(x,y)} \, dx dy = \int c_n(y) \left(\int e^{2\pi i m (x \cdot \sum_{j=0}^{n-1} [\{y+j\beta\} + \beta])} \, dx \right) dy.$$

Notice that if $q \geq 1$ is so that

$$(12) \quad q\beta > 1 \text{ then } \sum_{j=0}^{q-1} [\{y+j\beta\} + \beta] \geq 1 \text{ for each } y \in [0, 1).$$

It follows that for each $n \neq 0$

$$\langle F \circ T_\phi^{nq}, F \rangle = 0$$

and the spectral measure F for T_ϕ^q is purely Lebesgue (cf. Corollary 3.5), which completes the first part of the proposition.

If $\frac{1}{2} < \beta < 1$ in view of (12) for $n = 2$ (and for all $n \geq 2$) we have $\langle F \circ T_\phi^n, F \rangle = 0$. Hence we have only to control the case $n = 1$. We have

$$\begin{aligned} \langle F \circ T_\phi, F \rangle &= \sum_{m \neq 0} \hat{f}(m) \int e^{2\pi i m \varphi(x,y)} \, dx dy \\ &= \sum_{m \neq 0} \hat{f}(m) \int e^{2\pi i m (\alpha\{y\} - (\{x\} + \alpha)\{y\} + \beta)} \, dx dy \\ &= \sum_{m \neq 0} \hat{f}(m) \left(\int_0^{1-\beta} b_m(y) \left(\int e^{2\pi i m (x \cdot [\{y\} + \beta])} \, dx \right) dy \right. \\ &\quad \left. + \int_{1-\beta}^1 b_m(y) \left(\int e^{2\pi i m (x \cdot [\{y\} + \beta])} \, dx \right) dy \right) \\ &= \sum_{m \neq 0} \hat{f}(m) \int_0^{1-\beta} b_m(y) \left(\int e^{2\pi i m (x \cdot [\{y\} + \beta])} \, dx \right) dy = \sum_{m \neq 0} \hat{f}(m) \int_0^{1-\beta} b_m(y) \, dy \\ &= \sum_{m \neq 0} \hat{f}(m) e^{2\pi i m \alpha} \int_0^{1-\beta} e^{2\pi i m \alpha \{y\}} \, dy = \sum_{m \neq 0} \hat{f}(m) e^{2\pi i m \alpha} \frac{1}{2\pi i m \alpha} \left(e^{2\pi i m \alpha \{1-\beta\}} - 1 \right). \end{aligned}$$

Now, the series $\sum_{m \neq 0} \hat{f}(m) e^{2\pi i m \alpha} \frac{1}{2\pi i m \alpha}$ is absolutely summable and the functions $\beta \mapsto e^{2\pi i m \alpha \{1-\beta\}} - 1$ are continuous. It follows that if $1/2 < \beta < 1$ is sufficiently close to 1 then the density of σ_F (which is a trigonometric polynomial of degree 1) is as close to 1 as we want, and therefore σ_F is SBH, so the corresponding nil-rotation is not AT. \square

4.5. Gaussian systems and the AT property. In this section we will study the non AT property in the class of zero entropy Gaussian dynamical systems. We will show the existence of mixing zero entropy Gaussian systems for which the 4th Cartesian product is not AT. Let us recall the definition of Gaussian systems.

Definition 4.9. Given a symmetric Borel probability measure σ on the circle $[0, 1)$ we call (real) Gaussian system of spectral measure σ the dynamical system $(\Omega, \mathcal{A}, T_\sigma, \mu)$ where:

- Ω is $\mathbb{R}^{\mathbb{Z}}$.
- \mathcal{A} is the borelian σ -algebra.
- S is the shift : $(T_\sigma(\omega))_n = \omega_{n+1}$.
- μ is defined on the cylinder by $\mu(\omega_{j_1} \in A_1, \dots, \omega_{j_n} \in A_n)$ is the probability of visiting the set $A_1 \times \dots \times A_n$ for a Gaussian vector $(X_{j_1}, \dots, X_{j_n})$ of zero mean and covariances

$$\text{Cov}(X_{j_s}, X_{j_t}) = \int_{\mathbb{T}} z^{j_s - j_t} d\sigma(z).$$

Such a system is then generated by real stationary (centered) Gaussian process, namely

$$X_n = X_0 \circ T_\sigma^n, \quad n \in \mathbb{Z}, \quad \text{where } X_0(\omega) = \omega_0.$$

The basic account on the spectral analysis of Gaussian dynamical systems may be found in [5]. We recall that the maximal spectral multiplicity of every Gaussian dynamical systems is $+\infty$ or 1.

Let us point out that the transformation $S : (\omega_n) \mapsto (-\omega_n)$ preserves the Gaussian measure and commute with any Gaussian system. In addition, for the partition $\mathcal{P} = \{P_0, P_1\}$, given by $P_0 = \{X_0 > 0\}$, we have $SP_0 = P_1$.

With this remark in mind we will compute the Fourier coefficients of the spectral measure $\sigma_{\chi_{P_0} - \chi_{P_1}}$.

Lemma 4.10. Let $X = (X_n)_{n \in \mathbb{Z}}$ be a stationary centered Gaussian process with spectral measure σ satisfying $\hat{\sigma}(n) \in (-1, 1)$ for $n \neq 0$. Then

$$\mu\{X_0 > 0, X_n > 0\} = \frac{1}{4} + \frac{1}{2\pi} \arcsin(\hat{\sigma}(n)).$$

Proof. Put $Z_n = X_n - \hat{\sigma}(n)X_0$. It follows that Z_n and X_0 are independent, the distribution of Z_n is Gaussian with variance $1 - \hat{\sigma}(n)^2$ and for any Borel function $\phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\mathbb{E}(\phi(X_0, Z_n) | X_0) = h(Z_n), \quad \text{where } h(z) = \mathbb{E}(\phi(X_0, z)).$$

Now, by taking $\phi(x, z) = 1$ if $x > 0$ and $z > -\hat{\sigma}(n)x$, and 0 otherwise we obtain that

$$\begin{aligned} \mu\{X_0 > 0, X_n > 0\} &= \mathbb{E} \left(\mathbb{E}(\chi_{\{X_0 > 0\}} \chi_{\{Z_n > -\hat{\sigma}(n)X_0\}} | X_0) \right) \\ &= \mathbb{E} \left(\chi_{\{X_0 > 0\}} (\mathbb{E}(\chi_{\{Z_n > -\hat{\sigma}(n)X_0\}} | X_0)) \right) \\ &= \mathbb{E} \left(\chi_{\{X_0 > 0\}} \mu \left(\frac{Z_n}{\text{Var } Z_n} > \frac{-\hat{\sigma}(n)X_0}{\sqrt{1 - \hat{\sigma}(n)^2}} \mid X_0 \right) \right) \\ &= \mathbb{E} \left(\chi_{\{X_0 > 0\}} \left\{ 1 - G \left(\frac{-\hat{\sigma}(n)X_0}{\sqrt{1 - \hat{\sigma}(n)^2}} \right) \right\} \right) \end{aligned}$$

where $G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du$. But, since $1 - G(-x) = G(x)$, we get

$$\begin{aligned} \mu\{X_0 > 0, X_n > 0\} &= \mathbb{E} \left(\chi_{\{X_0 > 0\}} \left\{ G \left(\frac{(\hat{\sigma}(n))X_0}{\sqrt{1 - \hat{\sigma}(n)^2}} \right) \right\} \right) \\ &= \int_0^\infty \left\{ G \left(\frac{\hat{\sigma}(n)u}{\sqrt{1 - \hat{\sigma}(n)^2}} \right) \right\} G'(u) du. \end{aligned}$$

An easy calculation by taking the derivatives of the following functions yields

$$\int_0^{+\infty} G(au)G'(u)du = \frac{1}{2\pi} \arctan a + \frac{1}{4}, \quad \text{for any } a \in \mathbb{R},$$

and by taking $a = \frac{\hat{\sigma}(n)}{\sqrt{1 - \hat{\sigma}(n)^2}}$ we conclude. \square

It follows from the lemma above that we have

$$\hat{\sigma}_{\chi_{P_0} - \chi_{P_1}}(n) = \frac{2}{\pi} \arcsin(\hat{\sigma}(n)).$$

We now pass to a similar problem but in the Cartesian product $(X \times X, \mathcal{B}, \mu \otimes \mu, T \times T)$ in which we consider the process (Y_n) where $Y_n = Y_0 \circ (T \times T)^n$ and $Y_0(\omega_1, \omega_2) = X_0(\omega_1)X_0(\omega_2)$. We first must extend Lemma 4.10.

Lemma 4.11. Let $X = (X_n)_{n \in \mathbb{Z}}$ be a stationary centered Gaussian process with spectral measure σ such that $\hat{\sigma}(n) \in (-1, 1)$ for $n \neq 0$ and let $Y_n(\omega_1, \omega_2) = X_n(\omega_1)X_n(\omega_2)$. Then

$$\mu(\{Y_0 > 0, Y_n > 0\}) = \frac{1}{4} + \frac{1}{\pi^2} \arcsin^2(\hat{\sigma}(n)).$$

Proof. It is easy to see that

$$\begin{aligned} \mu \otimes \mu\{Y_0 > 0, Y_n > 0\} &= (\mu\{X_0 > 0, X_n > 0\})^2 + (\mu\{X_0 < 0, X_n > 0\})^2 \\ &\quad + (\mu\{X_0 > 0, X_n < 0\})^2 + (\mu\{X_0 < 0, X_n < 0\})^2. \end{aligned}$$

Using Lemma 4.10 and the fact that $S \in C(T)$ we obtain that

$$\begin{aligned} \mu(\{X_0 < 0, X_n < 0\}) &= \mu(\{S\{X_0 < 0, X_n < 0\}\}) \\ &= \mu(\{X_0 > 0, X_n > 0\}) = \frac{1}{4} + \frac{1}{2\pi} \arcsin(\hat{\sigma}(n)), \end{aligned}$$

and then

$$\begin{aligned} \mu\{X_0 < 0, X_n > 0\} &= \mu\{X_0 < 0\} - \mu\{X_0 < 0, X_n < 0\} = \frac{1}{4} - \frac{1}{2\pi} \arcsin(\hat{\sigma}(n)), \\ \mu\{X_0 < 0, X_n > 0\} &= \frac{1}{4} - \frac{1}{2\pi} \arcsin(\hat{\sigma}(n)). \end{aligned}$$

and the proof of the lemma is complete. \square

Since $X_n(\omega) = \omega_n$ ($\omega = (\omega_n)_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$) and $S \times Id$ is in the centralizer of $T \times T$,

$$\begin{aligned} \mu \otimes \mu(\{Y_0 < 0, Y_n < 0\}) &= \mu \otimes \mu(S \times Id(\{Y_0 < 0, Y_n < 0\})) \\ &= \mu \otimes \mu(\{Y_0 > 0, Y_n > 0\}) = \frac{1}{4} + \frac{1}{\pi^2} \arcsin(\hat{\sigma}(n))^2 \end{aligned}$$

and since $\mu \otimes \mu(\{Y_0 < 0\}) = \frac{1}{2}$,

$$\mu \otimes \mu(\{Y_0 < 0, Y_n > 0\}) = \mu \otimes \mu(\{Y_0 > 0, Y_n < 0\}) = \frac{1}{4} - \frac{1}{\pi^2} \arcsin(\hat{\sigma}(n))^2.$$

Consider $\bar{T} = (T \times T) \times (T \times T)$ with measure $\bar{\mu} = (\mu \otimes \mu) \otimes (\mu \otimes \mu)$ and the process $(Z_n)_{n \in \mathbb{Z}}$ where $Z_n = Z_0 \otimes \bar{T}^n$ and

$$Z_0(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \omega^{(4)}) = Y_0(\omega^{(1)}, \omega^{(2)})Y_0(\omega^{(3)}, \omega^{(4)}).$$

By the argument from the beginning of the proof of Lemma 4.11 it follows that

$$\bar{\mu}(\{Z_0 > 0, Z_n > 0\}) = \frac{1}{4} + \frac{4}{\pi^4} \arcsin^4(\hat{\sigma}(n)).$$

Therefore if we put $P'_0 = \{Z_0 < 0\}$ and $P'_1 = \{Z_0 \geq 0\}$ then for any $n \in \mathbb{Z}$ we have

$$\hat{\sigma}_{\chi_{P'_0} - \chi_{P'_1}}(n) = \frac{16}{\pi^4} (\arcsin(\hat{\sigma}(n)))^4.$$

Finally notice that $P'_1 = S \times Id \times Id \times Id(P'_0)$, where $S \times Id \times Id \times Id \in C(\bar{T})$.

Before we formulate the main result of this section we recall the following result of Körner [24] (which is a strengthening of a result of Ivašëv-Musatov [3]).

Theorem 4.12. Assume that $\phi : [1, \infty) \rightarrow [1, \infty)$ is a continuous positive function such that

- (i) $\int_1^\infty \phi(x)^2 dx = +\infty$,
- (ii) $(\exists K > 1) \quad K\phi(x) \geq \phi(y) \geq \phi(x)/K$ for $2x \geq y \geq x \geq 1$.

Then there exists a singular probability measure σ on \mathbb{T} such that for each $n \neq 0$,

$$|\hat{\sigma}(n)| \leq \phi(|n|).$$

We easily verify that for each $c > 0$ the function $\phi(x) = c/\sqrt{x}$ satisfies the assumptions of Theorem 4.12.

Notice that if σ satisfies the assertion of Theorem 4.12 then so does the measure $\tilde{\sigma}(A) = \sigma(\bar{A})$ (since $|\hat{\tilde{\sigma}}(-n)| = |\hat{\sigma}(n)|$) and also

$$(13) \quad \left| \left(\frac{1}{2}(\sigma + \tilde{\sigma}) \right) \hat{\gamma}(n) \right| \leq \phi(|n|).$$

In other words the assertion of Theorem 4.12 holds in the class of symmetric measures.

Proposition 4.13. There exists a mixing Gaussian zero entropy dynamical system (X, \mathcal{B}, T, μ) such that $T \times T \times T \times T$ is not AT.

Proof. Using Theorem 4.12 (and (13)) we can find σ a probability symmetric singular measure on the circle \mathbb{T} such that for $n \geq 1$

$$(14) \quad |\hat{\sigma}(n)| \leq \frac{c}{\sqrt{n}}.$$

The constant $c > 0$ has to be small enough so that for $|x| \leq \frac{c}{\log 2}$ we have $|\arcsin x| \leq 2|x|$ and we assume that

$$c \leq \pi^{1/2} \left(\frac{1 + \varepsilon_0}{86} \right)^{1/4}.$$

We now have

$$\sum_{k \neq 0} |\hat{\sigma}_{\chi_{P'_0} - \chi_{P'_1}}(k)| \leq \frac{32}{\pi^4} \sum_{k \geq 1} (\arcsin(\hat{\sigma}(k)))^4 \leq \frac{512}{\pi^4} \sum_{k \geq 1} (\hat{\sigma}(k))^4$$

$$\leq \frac{512c^4}{\pi^4} \sum_{k \geq 1} \frac{1}{k^2} \leq 1 + \varepsilon_0.$$

It follows that $\sigma_{\chi_{P'_0} - \chi_{P'_1}}$ is an SBH measure and the proof of the proposition is complete. \square

Question. Is it true that for every automorphism T its Cartesian square $T \times T$ does not have the AT property?

Remark 4.14. In the recent paper [25], Körner shows that given $\alpha \in (\frac{1}{2}, 1)$ there is a singular measure μ on the circle such that $\mu * \mu$ is absolutely continuous with density of Lipschitz class $\alpha - \frac{1}{2}$. However the measure μ is not symmetric. If Körner's construction can be "symmetrized" then we would obtain a zero entropy Gaussian automorphism whose Cartesian square $T \times T$ is not AT.

4.6. Gaussian cocycles and non AT property. Following [28] in this section we consider extensions of Gaussian systems via Gaussian cocycles. So we assume that $(X_n)_{n \in \mathbb{Z}}$ is a stationary centered real Gaussian process inducing the dynamical system $T = T_\sigma$ acting on (Ω, μ_σ) , where $\Omega = \mathbb{R}^\mathbb{Z}$ and the probability measure σ , always assumed to be continuous, is the spectral measure of the process (and we can assume that $X_n(\omega) = \omega_n$). We then consider the skew product $T_{e^{2\pi i X_0}}$ acting on $(\Omega \times \mathbb{S}^1, \mu_\sigma \otimes \lambda)$.

It has been proved in [28] that if $\hat{\sigma}(n) \geq 0$ for each $n \in \mathbb{Z}$ then $T_{\exp(2\pi i X_0)}$ has countable Lebesgue spectrum in the orthocomplement of $L^2(\Omega, \mu_\sigma) \otimes 1$. Based on the proof of this result we will now show the following.

Proposition 4.15. Assume that $\hat{\sigma}(n) \geq 0$ for each $n \in \mathbb{Z}$. Then there exists m_0 such that for each $m \geq m_0$, $T_{\exp(2\pi i X_0)}^m$ is not AT.

Proof. Take the partition (A, A^c) of \mathbb{S}^1 into the upper and the lower semicircle and consider $(\Omega \times A, \Omega \times A^c)$. As in Section 4.3 we notice that the Fourier decomposition of $F(\omega, z) = (\chi_{\Omega \times A} - \chi_{\Omega \times A^c})(\omega, z)$ is of the form

$$F(\omega, z) = f(z) = \sum_{k=-\infty}^{\infty} \frac{2}{(2k+1)\pi i} z^{2k+1}.$$

Proceeding as in Section 4.3 we obtain that

$$(15) \quad |\hat{\sigma}_F(n)| = \left| \sum_{m \neq 0} \hat{f}(m) \int_{\Omega} e^{2\pi i m X_0^{(n)}} d\mu_\sigma \right|.$$

An elementary calculation using the fact that the Fourier transform of σ is positive (see [28]) shows that

$$\|X_0^{(n)}\|_2^2 \geq |n| \hat{\sigma}(0) = |n|$$

and therefore

$$\left| \int_{\Omega} e^{2\pi i m X_0^{(n)}} d\mu \right| = e^{-2\pi^2 m^2 \|X_0^{(n)}\|_2^2} \leq e^{-C m^2 |n|}.$$

In view of (15) it follows that the Fourier transform of σ_F still decreases exponentially and the assertion follows from Corollary 3.5. \square

Remark 4.16. It follows that for each Gaussian system T_σ where $\sigma = \eta * \eta$ we have a Gaussian cocycle such that the corresponding skew product has a power which is not AT.

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