

UNIFORM EXPONENTIAL GROWTH FOR SOME $SL(2, \mathbb{R})$ MATRIX PRODUCTS

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ABSTRACT. Given a hyperbolic matrix $H \in SL(2, \mathbb{R})$, we prove that for almost every $R \in SL(2, \mathbb{R})$, any product of length n of H and R grows exponentially fast with n provided the matrix R occurs less than $o(\frac{n}{\log n \log \log n})$ times.

1. INTRODUCTION

For $t, \theta \in \mathbb{R}$, let $H = H(t)$ be the hyperbolic matrix $\begin{pmatrix} \exp \frac{1}{2}t & 0 \\ 0 & \exp -\frac{1}{2}t \end{pmatrix}$ and let $R = R(\theta)$ be the rotation matrix $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$. For a finite word $w = w_n \dots w_1$ on the symbols H and R , we let $|w|$ denote its length and we let $m(w)$ denote the number of occurrences of R in w . For any such word, and for any choice of parameters t and θ , we let $A_w(t, \theta)$ denote the corresponding matrix product in $SL(2, \mathbb{R})$, and denote by $\|A_w(t, \theta)\|$ its norm.

By the Oseledec's Theorem, for a typical large word w on H and R , the size of the matrix product is given up to subexponential error, by $e^{L(t, \theta)|w|}$, where $L(t, \theta)$ is the Lyapunov exponent of the Bernoulli product giving equal probabilities for H and R . By Furstenberg's Theorem (cf [3]), $L(t, \theta) > 0$ unless $t = 0$ or $\theta = \pi/2 \pmod{\pi}$, thus hyperbolic behavior prevails under a very mild "transversality condition" on the pair (H, R) .

Here we are interested in the following subtler question: Assuming some stronger transversality condition on the pair (H, R) , can one ensure hyperbolic behavior just by limiting the frequency of rotation elements in the word? A basic question in this direction, raised by Bochi and Fayad in [1], is whether for almost every t and θ , a condition of the type $C(t, \theta)m(w) \leq |w|$ implies that $\|A_w(t, \theta)\|$ grows exponentially. While this question is still open, in [2], Fayad and Krikorian showed that for almost every t and θ , one has exponential growth provided $m(w) \leq |w|^\alpha$ with $0 < \alpha < 1/2$. Our goal in this paper will be to show that the weaker condition $C(t, \theta)m(w) \log m(w) \log \log m(w) \leq |w|$ suffices.

Theorem 1. *For every $t > 0$, $0 < \gamma < \frac{t}{2}$ and almost every $\theta \in \mathbb{R}$, there exists $\epsilon > 0$ such that for any word w on H and R , if $m(w) \leq \epsilon|w|(\log |w| \log \log |w|)^{-1}$, then the spectral radius of $A_w(t, \theta)$ is at least $e^{|w|^\gamma}$.*

In fact, our proof allows us to take for R a general matrix of $SL(2, \mathbb{R})$, presented in its Cartan decomposition form, as follows.

This research was partially conducted during the period A.A. served as a Clay Research Fellow.

Theorem 2. For every $t > 0$, $s > 0$, $\alpha \in \mathbb{R}$, $0 < \gamma < \frac{t}{2}$ and almost every $\theta \in \mathbb{R}$, there exists $\epsilon > 0$ such that for any word w on $H = H(t)$ and $R = R(\theta)H(s)R(\alpha)$, if $m(w) \leq \epsilon|w|(\log|w|\log\log|w|)^{-1}$, then the spectral radius of A_w is at least $e^{|w|\gamma}$.

Corollary. For every $t > 0$, $0 < \gamma < \frac{t}{2}$ and almost every $R \in SL(2, \mathbb{R})$ with respect to the Haar measure, there exists $\epsilon > 0$ such that for any word w on $H = H(t)$ and R , if $m(w) \leq \epsilon|w|(\log|w|\log\log|w|)^{-1}$, then the spectral radius of A_w is at least $e^{|w|\gamma}$.

2. PROOF OF THE THEOREMS

We now give a detailed proof of theorem 1. Then we shall indicate how theorem 2 is obtained following the same lines.

From now on we fix $t > 0$, and drop the dependence on t from the notation.

For a given word w we shall use the notations $w_{[i,j]} = w_j \dots w_i$ for $1 \leq i \leq j \leq |w|$. We also let $a_w, b_w, c_w, d_w : \mathbb{R} \rightarrow \mathbb{R}$ be defined so that $A_w(\theta) = \begin{pmatrix} a_w(\theta) & b_w(\theta) \\ c_w(\theta) & d_w(\theta) \end{pmatrix}$.

Let us say that a function $\psi : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ is good if

$$(1) \quad \forall k, l \geq 1, \quad \psi(k) + \psi(l) \leq \psi(k+l) - \log 2.$$

We will mostly work with multiples (by reals greater than 1) of the functions $\psi_1(m) = m(1 + \log^2 m)$ and $\psi_2(m) = m(1 + \log m)(1 + \log \log \max\{e, m\})$ (with $0 \log 0 = 0$). Both ψ_1 and ψ_2 are easily seen to be good.

Given a good function ψ and $0 < \gamma \leq \frac{t}{2}$, for any word w of length n , we let $F_w(\psi, \gamma) = F_w$ be the set of all $\theta \in [0, \pi)$ such that

$$(2) \quad \begin{aligned} & \log |a_{w_{[1,k]}}| \geq k\gamma - \psi(m(w_{[1,k]})) \text{ and } \log |a_{w_{[k+1,n]}}| \geq (n-k)\gamma - \psi(m(w_{[k+1,n]})) \\ & \text{for all } 0 < k < n, \text{ but } \log |a_w| < n\gamma - \psi(m(w)). \end{aligned}$$

Notice that if F_w is not empty, necessarily $w_1 = w_n = R$. In view of (1), it follows that on the set F_w ,

$$(3) \quad |a_w| \leq \frac{1}{2} |a_{w_{[1,k]}} a_{w_{[k+1,n]}}|, \quad \forall 0 < k < n.$$

Lemma 1. For every w we have, writing $|w| = n$ and $m(w) = m$:

$$(4) \quad |F_w| \leq 8n^2 e^{\psi(\lfloor \frac{n}{2} \rfloor) + \psi(m - \lfloor \frac{n}{2} \rfloor) - \psi(m)}.$$

Proof. Since a_w is in general a polynomial of degree $m(w)$ in $\cos \theta$, as is easily checked, the set F_w is the union of at most $4nm$ intervals. Now, in order to bound the size of such an interval, we show that the derivative of a_w with respect to θ at any (fixed) point of F_w is not too small.

Since the derivative of $R(\theta)$ is $R(\frac{\pi}{2})R'(\theta)$, using the product rule, it is easy to derive the following formula for the derivative of a_w :

$$(5) \quad a'_w = \sum_{k, w_k=R} c_{w_{[1,k]}} a_{w_{[k+1,n]}} - a_{w_{[1,k]}} b_{w_{[k+1,n]}}.$$

On the one hand, we have, for all $0 < k < n$,

$$(6) \quad a_w = a_{w_{[1,k]}} a_{w_{[k+1,n]}} + c_{w_{[1,k]}} b_{w_{[k+1,n]}}.$$

In view of (3), this shows that

$$(7) \quad \frac{1}{2} \leq -\frac{c_{w_{[1,k]}} b_{w_{[k+1,n]}}}{a_{w_{[1,k]}} a_{w_{[k+1,n]}}} \leq \frac{3}{2}.$$

In particular, for each $0 < k < n$, $c_{w_{[1\ k]}} a_{w_{[k+1\ n]}}$ and $-a_{w_{[1\ k]}} b_{w_{[k+1\ n]}}$ have the same sign.

On the other hand, one easily sees that $\forall 1 < k < n$, the upper left entry of the matrix $A_{w_{[k+1\ n]}} R(\frac{\pi}{2}) A_{w_{[1\ k]}}$ is $c_{w_{[1\ k]}} a_{w_{[k+1\ n]}} - a_{w_{[1\ k]}} b_{w_{[k+1\ n]}} = c_{w_{[1\ k-1]}} a_{w_{[k\ n]}} - a_{w_{[1\ k-1]}} b_{w_{[k\ n]}}$ if $w_k = R$ and $c_{w_{[1\ k]}} a_{w_{[k+1\ n]}} - a_{w_{[1\ k]}} b_{w_{[k+1\ n]}} = e^{-t} c_{w_{[1\ k-1]}} a_{w_{[k\ n]}} - e^t a_{w_{[1\ k-1]}} b_{w_{[k\ n]}}$ if $w_k = H$ (indeed, $R(\frac{\pi}{2})H(t) = H(-t)R(\frac{\pi}{2}) = H(t)H(-2t)R(\frac{\pi}{2})$).

After finite iteration, we deduce from these observations that the quantities $c_{w_{[1\ k]}} a_{w_{[k+1\ n]}}$ and $-a_{w_{[1\ k]}} b_{w_{[k+1\ n]}}$ for k varying from 1 to $n-1$ have all the same sign; among them, the summands in (5). Therefore, taking k with $w_k = R$ so that $m(w_{[1\ k]}) = \lfloor \frac{m}{2} \rfloor$ where $m = m(w)$, we have

$$|a'_w| \geq |c_{w_{[1\ k]}} a_{w_{[k+1\ n]}}| + |a_{w_{[1\ k]}} b_{w_{[k+1\ n]}}| \geq 2|a_{w_{[1\ k]}} a_{w_{[k+1\ n]}} c_{w_{[1\ k]}} b_{w_{[k+1\ n]}}|^{\frac{1}{2}}.$$

From (7) and (2), we get (at any point $\theta \in F_w$):

$$\begin{aligned} |a'_w| &\geq |a_{w_{[1\ k]}} a_{w_{[k+1\ n]}}| \\ &\geq e^{n\frac{t}{2} - \psi(\lfloor \frac{m}{2} \rfloor) - \psi(m - \lfloor \frac{m}{2} \rfloor)} \end{aligned}$$

From the above minoration, we deduce that any interval in F_w as defined by (2) is of length less than $2e^{\psi(\lfloor \frac{m}{2} \rfloor) + \psi(m - \lfloor \frac{m}{2} \rfloor) - \psi(m)}$. Since F_w is the union of at most $4nm$ such intervals, the result follows. \square

Lemma 2. *If $F_w \neq \emptyset$ then*

$$n \leq m(1 + \frac{1}{t}\psi(m)),$$

where $n = |w|$ and $m = m(w)$.

Proof. Let us fix some $\theta \in F_w$, and write $w = w_{[k+r+1\ n]} H^r w_{[1\ k]}$ with r maximal. Since $w_1 = w_n = R$, as we have already observed, one has $0 < k < n-r$, $m(w_{[1\ k]}), m(w_{[k+r+1\ n]}) \geq 1$, and

$$(8) \quad r \geq \frac{n-m}{m-1}.$$

We have

$$(9) \quad a_w = e^{r\frac{t}{2}} a_{w_{[1\ k]}} a_{w_{[k+r+1\ n]}} + e^{-r\frac{t}{2}} c_{w_{[1\ k]}} b_{w_{[k+r+1\ n]}}.$$

Observe that in general $\max(a_\omega^2 + c_\omega^2, b_\omega^2 + d_\omega^2) \leq e^{|\omega|t}$, so that here

$$|c_{w_{[1\ k]}} b_{w_{[k+r+1\ n]}}| \leq e^{(n-r)\frac{t}{2}}.$$

From (1), (2) and (9), we get

$$\begin{aligned} 2e^{n\frac{t}{2} - \psi(m)} &\leq e^{n\frac{t}{2} - \psi(m(w_{[1\ k]})) - \psi(m(w_{[k+r+1\ n]}))} \\ &\leq e^{r\frac{t}{2}} |a_{w_{[1\ k]}} a_{w_{[k+r+1\ n]}}| \\ &< e^{n\frac{t}{2} - \psi(m)} + e^{(n-2r)\frac{t}{2}}. \end{aligned}$$

Hence $rt < \psi(m)$, which combined with (8) gives the result. \square

From now on, let $E(\psi, \gamma)$ denote the set of all $\theta \in [0, \pi)$ such that

$$(10) \quad \log |a_w(\theta)| < |w|\gamma - \psi(m(w)) \text{ for some word } w.$$

Lemma 3. *There exists some constant $c > 0$ such that $|E(\lambda\psi_1, \frac{t}{2})| = O_{\lambda \geq 1}(e^{-c\lambda})$.*

Proof. Let $E_n = E_n(\lambda\psi_1, \frac{t}{2}) \subset E = E(\lambda\psi_1, \frac{t}{2})$ be the set of θ such that n is the minimal length of a word w such that (10) holds. Clearly E is the disjoint union of the E_n 's and each E_n is covered by the F_w 's with $|w| = n$.

We then apply lemmas 1 and 2 to estimate $|E_n|$ for $n \geq 2$ as follows:

$$(11) \quad |E_n| \leq \sum_{|w|=n} |F_w| \leq 8n^2 \sum_m \binom{n}{m} e^{\lambda(\psi_1(m - [\frac{m}{2}]) + \psi_1([\frac{m}{2}]) - \psi_1(m))},$$

where the sum runs over the $2 \leq m \leq n$ such that $n \leq m(1 + \frac{1}{t}\lambda\psi_1(m))$, which implies $n \leq C_0\lambda m^2 \log^2 m$. Here and in the sequel, C_0, C_1, \dots stand for positive constants independent of m, n or λ .

For $n = 1$, notice that $E_1 = \{\theta \mid |\cos \theta| < e^{\frac{t}{2} - \lambda}\}$.

It is readily seen that $\forall m \geq 2$, $\psi_1(m - [\frac{m}{2}]) + \psi_1([\frac{m}{2}]) - \psi_1(m) \leq -C_1 m \log m$. On the other hand, by the use of Stirling's formula, we find that

$$(12) \quad \binom{n}{m} \leq e^{m \log n - m \log m + C_2 m}.$$

So, summing over n in (11) and then reversing the order of summation yields

$$\begin{aligned} |E| &\leq |E_1| + \sum_{m \geq 2} e^{(C_3 - C_1\lambda)m \log m} \sum_{n \leq C_0\lambda m^2 \log^2 m} n^{(m+2)} \\ &\leq C_4 e^{-\lambda} + \sum_{m \geq 2} e^{(C_5 - C_1\lambda)m \log m + (m+3) \log \lambda}. \end{aligned}$$

For large λ , this sum is finite and less than $e^{-c\lambda}$. \square

Lemma 4. *Let $0 < \gamma < \frac{t}{2}$. There exists some constant $c > 0$ such that $|E(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2})| = O_{\lambda \geq 1}(e^{-c\lambda})$.*

Proof. We first notice that if $F_w(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2}) \neq \emptyset$, then $\lambda\psi_1(m(w)) \geq (\frac{t}{2} - \gamma)|w|$. Thus, proceeding as in the previous lemma, we get (even for $n = 1$)

$$|E_n(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2})| \leq 8n^2 \sum_{\substack{\lambda\psi_1(m) \geq (\frac{t}{2} - \gamma)n \\ m \geq 2}} \binom{n}{m} e^{\lambda(\psi_2(m - [\frac{m}{2}]) + \psi_2([\frac{m}{2}]) - \psi_2(m))}.$$

Here $\forall m \geq 2$, $\psi_2(m - [\frac{m}{2}]) + \psi_2([\frac{m}{2}]) - \psi_2(m) \leq -C_6 m(1 + \log \log \max\{e, m\})$. Using again (12), we obtain

$$\begin{aligned} |E(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2})| &\leq \sum_{m \geq 2} e^{(C_7 - C_6\lambda)m(1 + \log \log \max\{e, m\}) - m \log m} \sum_{n \leq C_8\lambda m \log^2 m} n^{(m+2)} \\ &\leq \sum_{m \geq 2} e^{(C_9 - C_6\lambda)m(1 + \log \log \max\{e, m\}) + (m+3) \log \lambda}. \end{aligned}$$

We conclude as before. \square

The lemmata 3 and 4 show that for $0 < \gamma < \frac{t}{2}$, the sum $\sum_{\lambda \in \mathbb{N}^*} |E(\lambda\psi_2, \gamma)|$ converges. By the Borel-Cantelli lemma, we conclude that for almost every θ , there exists $\lambda \geq 1$ such that for all word w , $\log |a_w(\theta)| \geq |w|\gamma - \lambda\psi_2(m(w))$.

It follows that for almost every θ , if $|w|$ is large and $m(w)$ is much smaller than $|w|(\log |w| \log \log |w|)^{-1}$, then $\frac{1}{|w|} \log \|A_w(\theta)\|$ is close to $\frac{t}{2}$, as well as $\frac{1}{|w|^2} \log \|A_{ww}(\theta)\|$. But

$$A_{ww}(\theta) - A_w \text{tr} A_w + \text{id} = A_w(\theta)^2 - A_w \text{tr} A_w + \text{id} = 0,$$

since $A_w \in SL(2, \mathbb{R})$, which shows that $\frac{1}{|w|} \log |\text{tr} A_w|$ is close to $\frac{t}{2}$, yielding the estimate on the spectral radius in theorem 1.

In order to prove theorem 2 by the same method, we consider, instead of the words on H and R , words $w = w_n \dots w_1$ on $H(t)$, $R(\theta)$, $H(s)$ and $R(\alpha)$ such that the last three ones always appear consecutively, except maybe at the ends of the word, and $m(w)$ is now the number of these occurring in w . Then the proof goes the same way, notably the considerations of sign in lemma 1.

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