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# UNIFORM EXPONENTIAL GROWTH FOR SOME $SL(2, \mathbb{R})$ MATRIX PRODUCTS

ARTUR AVILA AND THOMAS ROBLIN

ABSTRACT. Given a hyperbolic matrix  $H \in SL(2, \mathbb{R})$ , we prove that for almost every  $R \in SL(2, \mathbb{R})$ , any product of length  $n$  of  $H$  and  $R$  grows exponentially fast with  $n$  provided the matrix  $R$  occurs less than  $o(\frac{n}{\log n \log \log n})$  times.

## 1. INTRODUCTION

For  $t, \theta \in \mathbb{R}$ , let  $H = H(t)$  be the hyperbolic matrix  $\begin{pmatrix} \exp \frac{1}{2}t & 0 \\ 0 & \exp -\frac{1}{2}t \end{pmatrix}$  and let  $R = R(\theta)$  be the rotation matrix  $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ . For a finite word  $w = w_n \dots w_1$  on the symbols  $H$  and  $R$ , we let  $|w|$  denote its length and we let  $m(w)$  denote the number of occurrences of  $R$  in  $w$ . For any such word, and for any choice of parameters  $t$  and  $\theta$ , we let  $A_w(t, \theta)$  denote the corresponding matrix product in  $SL(2, \mathbb{R})$ , and denote by  $\|A_w(t, \theta)\|$  its norm.

By the Oseledec's Theorem, for a typical large word  $w$  on  $H$  and  $R$ , the size of the matrix product is given up to subexponential error, by  $e^{L(t, \theta)|w|}$ , where  $L(t, \theta)$  is the Lyapunov exponent of the Bernoulli product giving equal probabilities for  $H$  and  $R$ . By Furstenberg's Theorem (cf [3]),  $L(t, \theta) > 0$  unless  $t = 0$  or  $\theta = \pi/2 \pmod{\pi}$ , thus hyperbolic behavior prevails under a very mild "transversality condition" on the pair  $(H, R)$ .

Here we are interested in the following subtler question: Assuming some stronger transversality condition on the pair  $(H, R)$ , can one ensure hyperbolic behavior just by limiting the frequency of rotation elements in the word? A basic question in this direction, raised by Bochi and Fayad in [1], is whether for almost every  $t$  and  $\theta$ , a condition of the type  $C(t, \theta)m(w) \leq |w|$  implies that  $\|A_w(t, \theta)\|$  grows exponentially. While this question is still open, in [2], Fayad and Krikorian showed that for almost every  $t$  and  $\theta$ , one has exponential growth provided  $m(w) \leq |w|^\alpha$  with  $0 < \alpha < 1/2$ . Our goal in this paper will be to show that the weaker condition  $C(t, \theta)m(w) \log m(w) \log \log m(w) \leq |w|$  suffices.

**Theorem 1.** *For every  $t > 0$ ,  $0 < \gamma < \frac{t}{2}$  and almost every  $\theta \in \mathbb{R}$ , there exists  $\epsilon > 0$  such that for any word  $w$  on  $H$  and  $R$ , if  $m(w) \leq \epsilon|w|(\log |w| \log \log |w|)^{-1}$ , then the spectral radius of  $A_w(t, \theta)$  is at least  $e^{|w|^\gamma}$ .*

In fact, our proof allows us to take for  $R$  a general matrix of  $SL(2, \mathbb{R})$ , presented in its Cartan decomposition form, as follows.

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**Theorem 2.** For every  $t > 0$ ,  $s > 0$ ,  $\alpha \in \mathbb{R}$ ,  $0 < \gamma < \frac{t}{2}$  and almost every  $\theta \in \mathbb{R}$ , there exists  $\epsilon > 0$  such that for any word  $w$  on  $H = H(t)$  and  $R = R(\theta)H(s)R(\alpha)$ , if  $m(w) \leq \epsilon|w|(\log|w|\log\log|w|)^{-1}$ , then the spectral radius of  $A_w$  is at least  $e^{|w|\gamma}$ .

**Corollary.** For every  $t > 0$ ,  $0 < \gamma < \frac{t}{2}$  and almost every  $R \in SL(2, \mathbb{R})$  with respect to the Haar measure, there exists  $\epsilon > 0$  such that for any word  $w$  on  $H = H(t)$  and  $R$ , if  $m(w) \leq \epsilon|w|(\log|w|\log\log|w|)^{-1}$ , then the spectral radius of  $A_w$  is at least  $e^{|w|\gamma}$ .

## 2. PROOF OF THE THEOREMS

We now give a detailed proof of theorem 1. Then we shall indicate how theorem 2 is obtained following the same lines.

From now on we fix  $t > 0$ , and drop the dependence on  $t$  from the notation.

For a given word  $w$  we shall use the notations  $w_{[i,j]} = w_j \dots w_i$  for  $1 \leq i \leq j \leq |w|$ . We also let  $a_w, b_w, c_w, d_w : \mathbb{R} \rightarrow \mathbb{R}$  be defined so that  $A_w(\theta) = \begin{pmatrix} a_w(\theta) & b_w(\theta) \\ c_w(\theta) & d_w(\theta) \end{pmatrix}$ .

Let us say that a function  $\psi : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$  is good if

$$(1) \quad \forall k, l \geq 1, \quad \psi(k) + \psi(l) \leq \psi(k+l) - \log 2.$$

We will mostly work with multiples (by reals greater than 1) of the functions  $\psi_1(m) = m(1 + \log^2 m)$  and  $\psi_2(m) = m(1 + \log m)(1 + \log \log \max\{e, m\})$  (with  $0 \log 0 = 0$ ). Both  $\psi_1$  and  $\psi_2$  are easily seen to be good.

Given a good function  $\psi$  and  $0 < \gamma \leq \frac{t}{2}$ , for any word  $w$  of length  $n$ , we let  $F_w(\psi, \gamma) = F_w$  be the set of all  $\theta \in [0, \pi)$  such that

$$(2) \quad \begin{aligned} & \log |a_{w_{[1,k]}}| \geq k\gamma - \psi(m(w_{[1,k]})) \text{ and } \log |a_{w_{[k+1,n]}}| \geq (n-k)\gamma - \psi(m(w_{[k+1,n]})) \\ & \text{for all } 0 < k < n, \text{ but } \log |a_w| < n\gamma - \psi(m(w)). \end{aligned}$$

Notice that if  $F_w$  is not empty, necessarily  $w_1 = w_n = R$ . In view of (1), it follows that on the set  $F_w$ ,

$$(3) \quad |a_w| \leq \frac{1}{2} |a_{w_{[1,k]}} a_{w_{[k+1,n]}}|, \quad \forall 0 < k < n.$$

**Lemma 1.** For every  $w$  we have, writing  $|w| = n$  and  $m(w) = m$ :

$$(4) \quad |F_w| \leq 8n^2 e^{\psi(\lfloor \frac{n}{2} \rfloor) + \psi(m - \lfloor \frac{n}{2} \rfloor) - \psi(m)}.$$

*Proof.* Since  $a_w$  is in general a polynomial of degree  $m(w)$  in  $\cos \theta$ , as is easily checked, the set  $F_w$  is the union of at most  $4nm$  intervals. Now, in order to bound the size of such an interval, we show that the derivative of  $a_w$  with respect to  $\theta$  at any (fixed) point of  $F_w$  is not too small.

Since the derivative of  $R(\theta)$  is  $R(\frac{\pi}{2})R'(\theta)$ , using the product rule, it is easy to derive the following formula for the derivative of  $a_w$ :

$$(5) \quad a'_w = \sum_{k, w_k=R} c_{w_{[1,k]}} a_{w_{[k+1,n]}} - a_{w_{[1,k]}} b_{w_{[k+1,n]}}.$$

On the one hand, we have, for all  $0 < k < n$ ,

$$(6) \quad a_w = a_{w_{[1,k]}} a_{w_{[k+1,n]}} + c_{w_{[1,k]}} b_{w_{[k+1,n]}}.$$

In view of (3), this shows that

$$(7) \quad \frac{1}{2} \leq -\frac{c_{w_{[1,k]}} b_{w_{[k+1,n]}}}{a_{w_{[1,k]}} a_{w_{[k+1,n]}}} \leq \frac{3}{2}.$$

In particular, for each  $0 < k < n$ ,  $c_{w_{[1\ k]}} a_{w_{[k+1\ n]}}$  and  $-a_{w_{[1\ k]}} b_{w_{[k+1\ n]}}$  have the same sign.

On the other hand, one easily sees that  $\forall 1 < k < n$ , the upper left entry of the matrix  $A_{w_{[k+1\ n]}} R(\frac{\pi}{2}) A_{w_{[1\ k]}}$  is  $c_{w_{[1\ k]}} a_{w_{[k+1\ n]}} - a_{w_{[1\ k]}} b_{w_{[k+1\ n]}} = c_{w_{[1\ k-1]}} a_{w_{[k\ n]}} - a_{w_{[1\ k-1]}} b_{w_{[k\ n]}}$  if  $w_k = R$  and  $c_{w_{[1\ k]}} a_{w_{[k+1\ n]}} - a_{w_{[1\ k]}} b_{w_{[k+1\ n]}} = e^{-t} c_{w_{[1\ k-1]}} a_{w_{[k\ n]}} - e^t a_{w_{[1\ k-1]}} b_{w_{[k\ n]}}$  if  $w_k = H$  (indeed,  $R(\frac{\pi}{2})H(t) = H(-t)R(\frac{\pi}{2}) = H(t)H(-2t)R(\frac{\pi}{2})$ ).

After finite iteration, we deduce from these observations that the quantities  $c_{w_{[1\ k]}} a_{w_{[k+1\ n]}}$  and  $-a_{w_{[1\ k]}} b_{w_{[k+1\ n]}}$  for  $k$  varying from 1 to  $n-1$  have all the same sign; among them, the summands in (5). Therefore, taking  $k$  with  $w_k = R$  so that  $m(w_{[1\ k]}) = \lfloor \frac{m}{2} \rfloor$  where  $m = m(w)$ , we have

$$|a'_w| \geq |c_{w_{[1\ k]}} a_{w_{[k+1\ n]}}| + |a_{w_{[1\ k]}} b_{w_{[k+1\ n]}}| \geq 2|a_{w_{[1\ k]}} a_{w_{[k+1\ n]}} c_{w_{[1\ k]}} b_{w_{[k+1\ n]}}|^{\frac{1}{2}}.$$

From (7) and (2), we get (at any point  $\theta \in F_w$ ):

$$\begin{aligned} |a'_w| &\geq |a_{w_{[1\ k]}} a_{w_{[k+1\ n]}}| \\ &\geq e^{n\frac{t}{2} - \psi(\lfloor \frac{m}{2} \rfloor) - \psi(m - \lfloor \frac{m}{2} \rfloor)} \end{aligned}$$

From the above minoration, we deduce that any interval in  $F_w$  as defined by (2) is of length less than  $2e^{\psi(\lfloor \frac{m}{2} \rfloor) + \psi(m - \lfloor \frac{m}{2} \rfloor) - \psi(m)}$ . Since  $F_w$  is the union of at most  $4nm$  such intervals, the result follows.  $\square$

**Lemma 2.** *If  $F_w \neq \emptyset$  then*

$$n \leq m(1 + \frac{1}{t}\psi(m)),$$

where  $n = |w|$  and  $m = m(w)$ .

*Proof.* Let us fix some  $\theta \in F_w$ , and write  $w = w_{[k+r+1\ n]} H^r w_{[1\ k]}$  with  $r$  maximal. Since  $w_1 = w_n = R$ , as we have already observed, one has  $0 < k < n-r$ ,  $m(w_{[1\ k]}), m(w_{[k+r+1\ n]}) \geq 1$ , and

$$(8) \quad r \geq \frac{n-m}{m-1}.$$

We have

$$(9) \quad a_w = e^{r\frac{t}{2}} a_{w_{[1\ k]}} a_{w_{[k+r+1\ n]}} + e^{-r\frac{t}{2}} c_{w_{[1\ k]}} b_{w_{[k+r+1\ n]}}.$$

Observe that in general  $\max(a_\omega^2 + c_\omega^2, b_\omega^2 + d_\omega^2) \leq e^{|\omega|t}$ , so that here

$$|c_{w_{[1\ k]}} b_{w_{[k+r+1\ n]}}| \leq e^{(n-r)\frac{t}{2}}.$$

From (1),(2) and (9), we get

$$\begin{aligned} 2e^{n\frac{t}{2} - \psi(m)} &\leq e^{n\frac{t}{2} - \psi(m(w_{[1\ k]})) - \psi(m(w_{[k+r+1\ n]}))} \\ &\leq e^{r\frac{t}{2}} |a_{w_{[1\ k]}} a_{w_{[k+r+1\ n]}}| \\ &< e^{n\frac{t}{2} - \psi(m)} + e^{(n-2r)\frac{t}{2}}. \end{aligned}$$

Hence  $rt < \psi(m)$ , which combined with (8) gives the result.  $\square$

From now on, let  $E(\psi, \gamma)$  denote the set of all  $\theta \in [0, \pi)$  such that

$$(10) \quad \log |a_w(\theta)| < |w|\gamma - \psi(m(w)) \text{ for some word } w.$$

**Lemma 3.** *There exists some constant  $c > 0$  such that  $|E(\lambda\psi_1, \frac{t}{2})| = O_{\lambda \geq 1}(e^{-c\lambda})$ .*

*Proof.* Let  $E_n = E_n(\lambda\psi_1, \frac{t}{2}) \subset E = E(\lambda\psi_1, \frac{t}{2})$  be the set of  $\theta$  such that  $n$  is the minimal length of a word  $w$  such that (10) holds. Clearly  $E$  is the disjoint union of the  $E_n$ 's and each  $E_n$  is covered by the  $F_w$ 's with  $|w| = n$ .

We then apply lemmas 1 and 2 to estimate  $|E_n|$  for  $n \geq 2$  as follows:

$$(11) \quad |E_n| \leq \sum_{|w|=n} |F_w| \leq 8n^2 \sum_m \binom{n}{m} e^{\lambda(\psi_1(m - [\frac{m}{2}]) + \psi_1([\frac{m}{2}]) - \psi_1(m))},$$

where the sum runs over the  $2 \leq m \leq n$  such that  $n \leq m(1 + \frac{1}{t}\lambda\psi_1(m))$ , which implies  $n \leq C_0\lambda m^2 \log^2 m$ . Here and in the sequel,  $C_0, C_1, \dots$  stand for positive constants independent of  $m, n$  or  $\lambda$ .

For  $n = 1$ , notice that  $E_1 = \{\theta \mid |\cos \theta| < e^{\frac{t}{2} - \lambda}\}$ .

It is readily seen that  $\forall m \geq 2$ ,  $\psi_1(m - [\frac{m}{2}]) + \psi_1([\frac{m}{2}]) - \psi_1(m) \leq -C_1 m \log m$ . On the other hand, by the use of Stirling's formula, we find that

$$(12) \quad \binom{n}{m} \leq e^{m \log n - m \log m + C_2 m}.$$

So, summing over  $n$  in (11) and then reversing the order of summation yields

$$\begin{aligned} |E| &\leq |E_1| + \sum_{m \geq 2} e^{(C_3 - C_1\lambda)m \log m} \sum_{n \leq C_0\lambda m^2 \log^2 m} n^{(m+2)} \\ &\leq C_4 e^{-\lambda} + \sum_{m \geq 2} e^{(C_5 - C_1\lambda)m \log m + (m+3) \log \lambda}. \end{aligned}$$

For large  $\lambda$ , this sum is finite and less than  $e^{-c\lambda}$ .  $\square$

**Lemma 4.** *Let  $0 < \gamma < \frac{t}{2}$ . There exists some constant  $c > 0$  such that  $|E(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2})| = O_{\lambda \geq 1}(e^{-c\lambda})$ .*

*Proof.* We first notice that if  $F_w(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2}) \neq \emptyset$ , then  $\lambda\psi_1(m(w)) \geq (\frac{t}{2} - \gamma)|w|$ . Thus, proceeding as in the previous lemma, we get (even for  $n = 1$ )

$$|E_n(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2})| \leq 8n^2 \sum_{\substack{\lambda\psi_1(m) \geq (\frac{t}{2} - \gamma)n \\ m \geq 2}} \binom{n}{m} e^{\lambda(\psi_2(m - [\frac{m}{2}]) + \psi_2([\frac{m}{2}]) - \psi_2(m))}.$$

Here  $\forall m \geq 2$ ,  $\psi_2(m - [\frac{m}{2}]) + \psi_2([\frac{m}{2}]) - \psi_2(m) \leq -C_6 m(1 + \log \log \max\{e, m\})$ . Using again (12), we obtain

$$\begin{aligned} |E(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2})| &\leq \sum_{m \geq 2} e^{(C_7 - C_6\lambda)m(1 + \log \log \max\{e, m\}) - m \log m} \sum_{n \leq C_8\lambda m \log^2 m} n^{(m+2)} \\ &\leq \sum_{m \geq 2} e^{(C_9 - C_6\lambda)m(1 + \log \log \max\{e, m\}) + (m+3) \log \lambda}. \end{aligned}$$

We conclude as before.  $\square$

The lemmata 3 and 4 show that for  $0 < \gamma < \frac{t}{2}$ , the sum  $\sum_{\lambda \in \mathbb{N}^*} |E(\lambda\psi_2, \gamma)|$  converges. By the Borel-Cantelli lemma, we conclude that for almost every  $\theta$ , there exists  $\lambda \geq 1$  such that for all word  $w$ ,  $\log |a_w(\theta)| \geq |w|\gamma - \lambda\psi_2(m(w))$ .

It follows that for almost every  $\theta$ , if  $|w|$  is large and  $m(w)$  is much smaller than  $|w|(\log |w| \log \log |w|)^{-1}$ , then  $\frac{1}{|w|} \log \|A_w(\theta)\|$  is close to  $\frac{t}{2}$ , as well as  $\frac{1}{|w|^2} \log \|A_{ww}(\theta)\|$ . But

$$A_{ww}(\theta) - A_w \text{tr} A_w + \text{id} = A_w(\theta)^2 - A_w \text{tr} A_w + \text{id} = 0,$$

since  $A_w \in SL(2, \mathbb{R})$ , which shows that  $\frac{1}{|w|} \log |\text{tr} A_w|$  is close to  $\frac{t}{2}$ , yielding the estimate on the spectral radius in theorem 1.

In order to prove theorem 2 by the same method, we consider, instead of the words on  $H$  and  $R$ , words  $w = w_n \dots w_1$  on  $H(t)$ ,  $R(\theta)$ ,  $H(s)$  and  $R(\alpha)$  such that the last three ones always appear consecutively, except maybe at the ends of the word, and  $m(w)$  is now the number of these occurring in  $w$ . Then the proof goes the same way, notably the considerations of sign in lemma 1.

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