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UNIFORM EXPONENTIAL GROWTH FOR SOME $SL(2, \mathbb{R})$ MATRIX PRODUCTS

ARTUR AVILA AND THOMAS ROBLIN

Abstract. Given a hyperbolic matrix $H \in SL(2, \mathbb{R})$, we prove that for almost every $R \in SL(2, \mathbb{R})$, any product of length $n$ of $H$ and $R$ grows exponentially fast with $n$ provided the matrix $R$ occurs less than $o\left(\sqrt[3]{n \log n \log \log n}\right)$ times.

1. Introduction

For $t, \theta \in \mathbb{R}$, let $H = H(t)$ be the hyperbolic matrix $\begin{pmatrix} \exp \frac{1}{2}t & 0 \\ 0 & \exp -\frac{1}{2}t \end{pmatrix}$ and let $R = R(\theta)$ be the rotation matrix $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$. For a finite word $w = w_n \ldots w_1$ on the symbols $H$ and $R$, we let $|w|$ denote its length and we let $m(w)$ denote the number of occurrences of $R$ in $w$. For any such word, and for any choice of parameters $t$ and $\theta$, we let $A_w(t, \theta)$ denote the corresponding matrix product in $SL(2, \mathbb{R})$, and denote by $\|A_w(t, \theta)\|$ its norm.

By the Oseledets Theorem, for a typical large word $w$ on $H$ and $R$, the size of the matrix product is given up to subexponential error, by $e^{L(t, \theta)|w|}$, where $L(t, \theta)$ is the Lyapunov exponent of the Bernoulli product giving equal probabilities for $H$ and $R$. By Furstenberg’s Theorem (cf [3]), $L(t, \theta) > 0$ unless $t = 0$ or $\theta = \pi/2 \mod \pi$, thus hyperbolic behavior prevails under a very mild “transversality condition” on the pair $(H, R)$.

Here we are interested in the following subtler question: Assuming some stronger transversality condition on the pair $(H, R)$, can one ensure hyperbolic behavior just by limiting the frequency of rotation elements in the word? A basic question in this direction, raised by Bochi and Fayad in [1], is whether for almost every $t$ and $\theta$, a condition of the type $C(t, \theta)m(w) \leq |w|$ implies that $\|A_w(t, \theta)\|$ grows exponentially. While this question is still open, in [2], Fayad and Krikorian showed that for almost every $t$ and $\theta$, one has exponential growth provided $m(w) \leq |w|^\alpha$ with $0 < \alpha < 1/2$. Our goal in this paper will be to show that the weaker condition $C(t, \theta)m(w) \log m(w) \log \log m(w) \leq |w|$ suffices.

Theorem 1. For every $t > 0$, $0 < \gamma < \frac{1}{2}$ and almost every $\theta \in \mathbb{R}$, there exists $\epsilon > 0$ such that for any word $w$ on $H$ and $R$, if $m(w) \leq \epsilon |w|(|\log |w| \log \log |w|)^{-1}$, then the spectral radius of $A_w(t, \theta)$ is at least $e^{\epsilon|w|\gamma}$.

In fact, our proof allows us to take for $R$ a general matrix of $SL(2, \mathbb{R})$, presented in its Cartan decomposition form, as follows.

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Theorem 2. For every $t > 0$, $s > 0$, $\alpha \in \mathbb{R}$, $0 < \gamma < \frac{t}{4}$ and almost every $\theta \in \mathbb{R}$, there exists $\epsilon > 0$ such that for any word $w$ on $H = H(t)$ and $R = R(\theta)H(s)R(\alpha)$, if $m(w) \leq \epsilon |w|(\log |w| \log \log |w|)^{-1}$, then the spectral radius of $A_w$ is at least $e^{|w|\gamma}$.

Corollary. For every $t > 0$, $0 < \gamma < \frac{t}{4}$ and almost every $R \in SL(2, \mathbb{R})$ with respect to the Haar measure, there exists $\epsilon > 0$ such that for any word $w$ on $H = H(t)$ and $R$, if $m(w) \leq \epsilon |w|(\log |w| \log \log |w|)^{-1}$, then the spectral radius of $A_w$ is at least $e^{|w|\gamma}$.

2. Proof of the theorems

We now give a detailed proof of theorem 1. Then we shall indicate how theorem 2 is obtained following the same lines.

From now on we fix $t > 0$, and drop the dependence on $t$ from the notation.

For a given word $w$ we shall use the notations $w_{[i,j]} = w_i \ldots w_j$ for $1 \leq i \leq j \leq |w|$. We also let $a_w, b_w, c_w, d_w : \mathbb{R} \to \mathbb{R}$ be defined so that $A_w(\theta) = \begin{pmatrix} a_w(\theta) & b_w(\theta) \\ c_w(\theta) & d_w(\theta) \end{pmatrix}$.

Let us say that a function $\psi, \gamma \in \mathbb{Z}_{+}$ is good if

\begin{equation}
\forall k, l \geq 1, \, \psi(k) + \psi(l) \leq \psi(k + l) - \log 2.
\end{equation}

We will mostly work with multiples (by reals greater than 1) of the functions $\psi_1(m) = m(1 + \log^2 m)$ and $\psi_2(m) = m(1 + \log m)(1 + \log \max\{e, m\})$ (with $0 \log 0 = 0$). Both $\psi_1$ and $\psi_2$ are easily seen to be good.

Given a good function $\psi$ and $0 < \gamma \leq \frac{t}{2}$, for any word $w$ of length $n$, we let $F_w(\psi, \gamma) = F_w$ be the set of all $\theta \in [0, \pi)$ such that

\begin{equation}
\log |a_{w[k+1]}| - \psi(m(w_{[k+1]})) \geq n-k \gamma - \psi(m(w_{[k+1]}))
\end{equation}

for all $0 < k < n$, but $\log |a_w| < n \gamma - \psi(m(w))$.

Notice that if $F_w$ is not empty, necessarily $w_1 = w_n = R$. In view of (1), it follows that on the set $F_w$,

\begin{equation}
|a_w| \leq \frac{1}{2} |a_{w[k+1]} a_{w[k+1]}|, \quad \forall 0 < k < n.
\end{equation}

Lemma 1. For every $w$ we have, writing $|w| = n$ and $m(w) = m$:

\begin{equation}
|F_w| \leq 8n^2 e^{\psi(m) (1 + \log (\frac{n}{2})) - \psi(m)}.
\end{equation}

Proof. Since $a_w$ is in general a polynomial of degree $m(\omega)$ in $\cos \theta$, as is easily checked, the set $F_w$ is the union of at most $4nm$ intervals. Now, in order to bound the size of such an interval, we show that the derivative of $a_w$ with respect to $\theta$ at any (fixed) point of $F_w$ is not too small.

Since the derivative of $R(\theta)$ is $R(\pi)R(\theta)$, using the product rule, it is easy to derive the following formula for the derivative of $a_w$:

\begin{equation}
a'_w = \sum_{k, w_k = R} c_{w[k]} a_{w[k+1]} - a_{w[k]} b_{w[k+1]}.
\end{equation}

On the one hand, we have, for all $0 < k < n$,

\begin{equation}
a_w = a_{w[k]} a_{w[k+1]} + c_{w[k]} b_{w[k+1]}.
\end{equation}

In view of (3), this shows that

\begin{equation}
\frac{1}{2} \leq \frac{c_{w[k]} b_{w[k+1]}}{a_{w[k]} a_{w[k+1]}} \leq \frac{3}{2}.
\end{equation}
In particular, for each \(0 < k < n\), \(c_{w[1:k]} a_{w[k+1:n]}\) and \(-a_{w[1:k]} b_{w[k+1:n]}\) have the same sign.

On the other hand, one easily sees that \(\forall 1 < k < n\), the upper left entry of the matrix \(A_{w[1:k]} R(\frac{\pi}{2}) A_{w[1:k]}\) is \(c_{w[1:k]} a_{w[k+1:n]} - a_{w[1:k]} b_{w[k+1:n]} = c_{w[1:k]} a_{w[k]} - a_{w[1:k-1]} b_{w[k]}\) if \(w_k = R\) and \(c_{w[1:k]} b_{w[k+1:n]} - a_{w[1:k]} b_{w[k+1:n]} = e^{-\pi} c_{w[1:k-1]} a_{w[k]} - e^{\pi} a_{w[1:k-1]} b_{w[k]}\) if \(w_k = H\) (indeed, \(R(\frac{\pi}{2}) H(t) = H(-t) R(\frac{\pi}{2})\)).

After finite iteration, we deduce from these observations that the quantities \(c_{w[1:k]} a_{w[k+1:n]}\) and \(-a_{w[1:k]} b_{w[k+1:n]}\) for \(k\) varying from 1 to \(n-1\) have all the same sign; among them, the summands in (5). Therefore, taking \(k\) with \(w_k = R\) so that \(m(w[1:k]) = \lfloor \frac{m}{2} \rfloor\) where \(m = m(w)\), we have

\[
|a'_w| \geq |c_{w[1:k]} a_{w[k+1:n]}| + |a_{w[1:k]} b_{w[k+1:n]}| \geq 2|c_{w[1:k]} a_{w[k+1:n]} c_{w[1:k]} b_{w[k+1:n]}|^\frac{1}{2}.
\]

From (7) and (2), we get (at any point \(\theta \in F_w\)):

\[
|a'_w| \geq |c_{w[1:k]} a_{w[k+1:n]}| \geq e^{\frac{n}{2} - \psi(\frac{\pi}{2}) - \psi(m - \lfloor \frac{m}{2} \rfloor)}
\]

From the above minoration, we deduce that any interval in \(F_w\) as defined by (2) is of length less than \(2e^{\psi(\frac{\pi}{2}) + \psi(m - \lfloor \frac{m}{2} \rfloor)}\). Since \(F_w\) is the union of at most \(4m\) such intervals, the result follows.

\[\square\]

Lemma 2. If \(F_w \neq \emptyset\) then

\[n \leq m(1 + \frac{1}{\lambda} \psi(m)),\]

where \(n = |w|\) and \(m = m(w)\).

Proof. Let us fix some \(\theta \in F_w\), and write \(w = w[k+r+1:n] H^r w[1:k]\) with \(r\) maximal. Since \(w_1 = w_n = R\), as we have already observed, one has \(0 < k < n - r\), \(m(w[1:k]), m(w[k+r+1:n]) \geq 1\), and

\[r \geq \frac{n - m}{m - 1}.
\]

We have

\[a_w = e^{\frac{\pi}{2}} a_{w[1:k]} a_{w[k+r+1:n]} + e^{-\frac{\pi}{2}} c_{w[1:k]} b_{w[k+r+1:n]}^*.
\]

Observe that in general \(\max(a_w^2 + b_w^2, c_w^2 + d_w^2) \leq |w|\), so that here

\[|c_{w[1:k]} b_{w[k+r+1:n]}| \leq e^{(n-r)\frac{\pi}{2}}.
\]

From (1),(2) and (9), we get

\[2e^{n\frac{\pi}{2} - \psi(m)} \leq e^{\frac{n}{2} - \psi(m)} e^{\frac{\pi}{2} - \psi(m)} e^{(n-r)\frac{\pi}{2}}.
\]

Hence \(rt < \psi(m)\), which combined with (8) gives the result.

From now on, let \(E(\psi, \gamma)\) denote the set of all \(\theta \in [0, \pi]\) such that

\[\log|a_w(\theta)| < |w|\gamma - \psi(m(w))\]

for some word \(w\).

Lemma 3. There exists some constant \(c > 0\) such that \(|E(\lambda(\psi/2), \gamma)| = O_{\lambda \geq 1} (e^{-\lambda})\).
We first notice that if \( m \) converges. By the Borel-Cantelli lemma, we conclude that for almost every \( \theta \) (12)

On the other hand, by the use of Stirling’s formula, we find that

Let \( \lambda \)

Proof. \( \square \)

Here \( \forall \gamma \)

Using again (12), we obtain

So, summing over \( n \) in (11) and then reversing the order of summation yields

For large \( \lambda \), this sum is finite and less than \( e^{-\lambda} \).

Lemma 4. Let \( 0 < \gamma < \frac{\lambda}{2} \). There exists some constant \( c > 0 \) such that \( |E(\lambda/2, \gamma) \setminus E(\gamma, \lambda/2)| = O_{\lambda \geq 1}(e^{-c\lambda}) \).

Proof. We first notice that if \( F_w(\lambda/2, \gamma) \setminus E(\gamma, \lambda/2) \neq \emptyset \), then \( \lambda w_i(\gamma, \lambda/2) \).

Here \( \forall m \geq 2, \psi_1(m - \lfloor m/2 \rfloor) + \psi_2(\lfloor m/2 \rfloor) - \psi_2(m) \leq -C_0m(1 + \log \max(\{e, m\})).

Using again (12), we obtain

We conclude as before. \( \square \)

The lemmata 3 and 4 show that for \( 0 < \gamma < \frac{\lambda}{2} \), the sum \( \sum_{\lambda \in \mathbb{N}} |E(\lambda/2, \gamma)| \)

It follows that for almost every \( \theta \), if \( |w| \) is large and \( m(w) \) is much smaller than \( |w|(\log |w| \log \log |w|)^{-1} \), then \( \frac{1}{|w|} \log \| A_w(\theta) \| \) is close to \( \frac{\lambda}{2} \), as well as \( \frac{1}{|w|^2} \log \| A_{ww}(\theta) \| \). But

\[ A_{ww}(\theta) - A_w \text{tr} A_w + id = A_w(\theta)^2 - A_w \text{tr} A_w + id = 0, \]
since $A_w \in \text{SL}(2, \mathbb{R})$, which shows that $\frac{1}{|w|} \log |\text{tr}A_w|$ is close to $\frac{1}{2}$, yielding the estimate on the spectral radius in theorem 1.

In order to prove theorem 2 by the same method, we consider, instead of the words on $H$ and $R$, words $w = w_n \ldots w_1$ on $H(t), R(\theta), H(s)$ and $R(\alpha)$ such that the last three ones always appear consecutively, except maybe at the ends of the word, and $m(w)$ is now the number of these occurring in $w$. Then the proof goes the same way, notably the considerations of sign in lemma 1.

References

