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# Bisets as categories, and tensor product of induced bimodules

Serge Bouc

**Abstract :** Bisets can be considered as categories. This note uses this point of view to give a simple proof of a Mackey-like formula expressing the tensor product of two induced bimodules.

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**Keywords :** Biset, induced, bimodule, tensor product.

## 1. Introduction

Let  $R$  be a commutative ring, let  $G$  and  $H$  be finite groups, let  $X$  be a subgroup of  $H \times G$ , and  $M$  be an  $RX$ -module. If  $m \in M$  and  $(h, g) \in X$ , set  $h \cdot m \cdot g^{-1} = (h, g) \cdot m$  : this is a slight extension of the usual correspondence between  $R(H \times G)$ -modules and  $(RH, RG)$ -bimodules.

The object of this note is to give a simple proof of the following result :

**1.1. Theorem :** *Let  $R$  be a commutative ring, let  $G$ ,  $H$ , and  $K$  be finite groups, let  $X$  be a subgroup of  $H \times G$  and  $Y$  be a subgroup of  $K \times H$ . Let  $M$  be an  $RX$ -module, and  $N$  be an  $RY$ -module. Then there is an isomorphism of  $(RK, RG)$ -bimodules*

$$(\text{Ind}_Y^{K \times H} N) \otimes_{RH} (\text{Ind}_X^{H \times G} M) \cong \bigoplus_{t \in [p_2(Y) \setminus H/p_1(X)]} \text{Ind}_{Y * {}^{(t,1)}X}^{K \times G} (N \otimes_{k_2(Y) \cap {}^t k_1(X)} {}^{(t,1)}M) ,$$

where the notation is as follows (cf. [1]) :

$$p_1(X) = \{h \in H \mid \exists g \in G, (h, g) \in X\}, \quad k_1(X) = \{h \in H \mid (h, 1) \in X\}$$

$$p_2(Y) = \{h \in H \mid \exists k \in K, (k, h) \in Y\}, \quad k_2(Y) = \{h \in H \mid (1, h) \in Y\}$$

$$Y * {}^{(t,1)}X = \{(k, g) \in K \times G \mid \exists h \in H, (k, h) \in Y, (h^t, g) \in X\} .$$

The action of  $(k, g) \in Y * {}^{(t,1)}X$  on  $N \otimes_{k_2(Y) \cap {}^t k_1(X)} {}^{(t,1)}M$  is given by

$$k \cdot (n \otimes m) \cdot g^{-1} = (k \cdot n \cdot h^{-1}) \otimes (h^t \cdot m \cdot g^{-1}) ,$$

if  $h \in H$  is chosen such that  $(k, h) \in Y$  and  $(h^t, g) \in X$ .

## 2. Functors over bisets

Recall that when  $G$  and  $H$  are groups, an  $(H, G)$ -biset  $U$  is a set equipped with a left action of  $H$  and a right action of  $G$  which commute, i.e. such that  $(hu)g = h(ug)$  for any  $h \in H$ ,  $u \in U$ , and  $g \in G$ .

**2.1. Notation :** Let  $G$  and  $H$  be groups. When  $U$  is an  $(H, G)$ -biset, let  $\langle U \rangle$  denote the following category :

- The objects of  $\langle U \rangle$  are the elements of  $U$ .
- If  $u, v \in U$ , then

$$\text{Hom}_{\langle U \rangle}(u, v) = \{(h, g) \in H \times G \mid hu = vg\} .$$

- If  $u, v, w \in U$ , the composition of the morphisms  $(h, g) : u \rightarrow v$  and  $(h', g') : v \rightarrow w$  is the morphism  $(h'h, g'g) : u \rightarrow w$ .
- If  $u \in U$ , the identity morphism  $\text{Id}_u : u \rightarrow u$  is the pair  $(1, 1) \in G \times G$ .

Note that the category  $\langle U \rangle$  is a groupoid (any morphism is an isomorphism), and that for any  $u \in U$ , the group

$$A(u) = \text{Hom}_{\langle U \rangle}(u, u) = \{(h, g) \in H \times G \mid hu = ug\}$$

is a subgroup of  $H \times G$ .

A functor  $M$  from  $\langle U \rangle$  to a category  $\mathcal{C}$  consists of a collection of objects  $M(u)$  of  $\mathcal{C}$ , for  $u \in U$ , together with morphisms

$$M(h, g) : M(u) \rightarrow M(hug^{-1})$$

in the category  $\mathcal{C}$ , for  $(h, g) \in H \times G$ , fulfilling the usual functorial conditions. In particular, for each  $u \in U$ , there is a group homomorphism  $A(u) \rightarrow \text{Aut}_{\mathcal{C}} M(u)$ .

Functors from  $\langle U \rangle$  to  $\mathcal{C}$  are the objects of a category  $\text{Fun}(\langle U \rangle, \mathcal{C})$ , in which the morphisms are natural transformation of functors.

**2.2. Notation :** When  $\mathcal{C}$  is a subcategory of the category **Sets** of sets, and  $M$  is a functor  $\langle U \rangle \rightarrow \mathcal{C}$ , the image of  $m \in M(u)$  by the map  $M(h, g) : M(u) \rightarrow M(hug^{-1})$ , for  $(h, g) \in H \times G$ , will be denoted by  $hmg^{-1}$ .

In this case, a functor  $M : \langle U \rangle \rightarrow \mathcal{C}$  is a collection of objects  $M(u)$  of  $\mathcal{C}$ , for  $u \in U$ , together with morphisms  $m \mapsto hmg^{-1} : M(u) \rightarrow M(hug^{-1})$  in  $\mathcal{C}$ ,

for  $(h, g) \in H \times G$ , such that  $h'(hmg^{-1})g'^{-1} = (h'h)m(g'g)^{-1}$  and  $1m1 = m$ , for any  $(h, g), (h', g')$  in  $H \times G$ , any  $u \in U$ , and any  $m \in M(u)$ .

**2.3. Example :** Suppose that  $\mathcal{C} = \mathbf{Sets}$ . Then the disjoint union  $\bigsqcup M = \bigsqcup_{u \in U} M(u)$  becomes an  $(H, G)$ -biset, and the map  $\bigsqcup M \rightarrow U$  sending elements of  $M(u)$  to  $u$ , for  $u \in U$ , is a map of  $(H, G)$ -bisets. Conversely, if  $\pi : S \rightarrow U$  is a map of  $(H, G)$ -bisets, then the assignment  $u \mapsto \pi^{-1}(u)$  is a functor from  $\langle U \rangle$  to  $\mathbf{Sets}$ .

In other words, a functor  $\langle U \rangle \rightarrow \mathbf{Sets}$  is just an  $(H, G)$ -biset over  $U$ . More precisely, the category  $\mathbf{Fun}(\langle U \rangle, \mathbf{Sets})$  of such functors is equivalent to the category of  $(H, G)$ -bisets over  $U$ .

**2.4. Example :** Let  $R$  be a commutative ring. In the remainder of this note, the category  $\mathcal{C}$  will be the category  $R\text{-Mod}$  of (left)  $R$ -modules. If  $M$  is functor from  $\langle U \rangle$  to  $R\text{-Mod}$ , then for each  $u \in U$ , the  $R$ -module  $M(u)$  has a natural structure of  $RA(u)$ -module.

Conversely, let  $[H \backslash U / G]$  be a set of representatives of  $(H, G)$ -orbits on  $U$ . Equivalently  $[H \backslash U / G]$  is a set of representatives of isomorphism classes in the category  $\langle U \rangle$ . Since  $\langle U \rangle$  is a groupoid, it is equivalent to its full subcategory  $[H \backslash U / G]$ . In particular, this yields an equivalence of categories

$$(2.5) \quad \mathbf{Fun}(\langle U \rangle, R\text{-Mod}) \cong \prod_{u \in [H \backslash U / G]} RA(u)\text{-Mod} .$$

**2.6. Remark :** In the situation of Example 2.4, the direct sum

$$\Sigma(M) = \bigoplus_{u \in U} M(u)$$

has a natural structure of  $(RH, RG)$ -bimodule, i.e. using the usual group isomorphism  $(h, g) \mapsto (h, g^{-1})$  from  $H \times G^{op}$  to  $H \times G$ , of left  $R(H \times G)$ -module.

Moreover, it is easy to see that there is an isomorphism of  $(RH, RG)$ -bimodules

$$\Sigma(M) \cong \bigoplus_{u \in [H \backslash U / G]} \text{Ind}_{A(u)}^{H \times G} M(u) .$$

### 3. Product of bisets, and product of functors

Let  $G, H$  and  $K$  be groups. If  $U$  is an  $(H, G)$ -biset and  $V$  is a  $(K, H)$ -biset, recall that the product (or *composition*) of  $V$  and  $U$  is the set

$$V \times_H U = (V \times U) / H ,$$

where the right action of  $H$  on  $(V \times U)$  is defined by  $(v, u) \cdot h = (vh, h^{-1}u)$ , for  $v \in V$ ,  $u \in U$ , and  $h \in H$ . The set  $V \times_H U$  is a  $(K, G)$ -biset for the following action

$$\forall z \in K, \forall x \in G, \forall v \in V, \forall u \in U, \quad z \cdot (v, {}_H u) \cdot x = (zv, {}_H ux) \quad ,$$

where  $(v, {}_H u)$  denotes the  $H$ -orbit of  $(v, u)$ .

**3.1. Definition :** *Let  $G$ ,  $H$ , and  $K$  be finite groups. Let  $U$  be a finite  $(H, G)$ -biset, and  $V$  be a finite  $(K, H)$ -biset. If  $M$  is a functor  $\langle U \rangle \rightarrow R\text{-Mod}$  and  $N$  is a functor  $\langle V \rangle \rightarrow R\text{-Mod}$ , the tensor product  $N \otimes_H M$  is the functor  $\langle V \times_H U \rangle \rightarrow R\text{-Mod}$  defined by*

$$(N \otimes_H M)(v, {}_H u) = \left( \bigoplus_{h \in H} N(vh) \otimes_R M(h^{-1}u) \right) / \mathcal{I}_{v,u} \quad ,$$

where  $\mathcal{I}_{v,u}$  is the  $R$ -submodule generated by the elements of the form

$$[ny \otimes y^{-1}m]_{hy} - [n \otimes m]_h \quad ,$$

where  $y \in H$ , and where  $[n \otimes m]_h$  denotes the element  $n \otimes m$  of the component indexed by  $h \in H$  in the direct sum, for  $n \in N(vh)$ , and  $m \in M(h^{-1}u)$ .

If  $(k, g) \in K \times G$ , then by definition

$$k [n \otimes m]_h g = [kn \otimes mg]_h \quad .$$

**3.2. Remark :** It follows from this definition that

$$(N \otimes_H M)((v, {}_H u)) \cong N(v) \otimes_{RH_{v,u}} M(u) \quad ,$$

where  $H_{v,u}$  is the set of elements  $h \in H$  such that  $vh = v$  and  $hu = u$ .

**3.3. Lemma :** *There is an isomorphism of  $(RK, RG)$ -bimodules*

$$\Sigma(N) \otimes_{RH} \Sigma(M) \cong \Sigma(N \otimes_H M) \quad ,$$

sending (from right to left) the element  $[n \otimes m]_h$  to  $n \otimes_{RH} m$ .

**Proof :** To be more precise, the map  $\alpha$  from

$$\Sigma(N \otimes_H M) = \bigoplus_{(v, {}_H u) \in V \times_H U} \left( \bigoplus_{h \in H} N(vh) \otimes_R M(h^{-1}u) \right) / \mathcal{I}_{v,u}$$

sending the element  $[n \otimes m]_h$  in the component indexed by  $(v, {}_H u)$  to the element  $n \otimes m$  of the tensor product

$$\Sigma(N) \otimes_{RH} \Sigma(M) = \left( \bigoplus_{v \in V} N(v) \right) \otimes_{RH} \left( \bigoplus_{u \in U} M(u) \right)$$

is well defined. To show that it is an isomorphism, define a map

$$\beta : \Sigma(N) \otimes_{RH} \Sigma(M) \rightarrow \Sigma(N \otimes_H M)$$

in the following way : choose a set  $S$  of representatives of the classes  $(v, {}_H u)$ . Now map the element  $n \otimes_{RH} m \in N(v) \otimes M(u) \subseteq \Sigma(N) \otimes_{RH} \Sigma(M)$  to  $[n \otimes m]_h$ , where  $h \in H$  is chosen such that  $(vh^{-1}, hu) \in S$ . Again, it is easy to see that this map is well defined, and that the maps  $\alpha$  and  $\beta$  are mutual inverse isomorphisms of  $(RK, RG)$ -bimodules.  $\square$

**3.4. Corollary :** *Let  $G, H,$  and  $K$  be finite groups. Let  $X$  be a subgroup of  $H \times G$  and  $Y$  be a subgroup of  $K \times H$ . Let  $M$  be an  $RX$ -module, and  $N$  be an  $RY$ -module. Then there is an isomorphism of  $(RK, RG)$ -bimodules*

$$(\text{Ind}_Y^{K \times H} N) \otimes_{RH} (\text{Ind}_X^{H \times G} M) \cong \bigoplus_{t \in [p_2(Y) \setminus H/p_1(X)]} \text{Ind}_{Y^{*(t,1)} X}^{K \times G} (N \otimes_{k_2(Y) \cap {}^t k_1(X)} ({}^{t,1} M)) .$$

**Proof :** Set  $U = (H \times G)/X$ . Then  $U$  is an  $(H, G)$ -biset by  $h \cdot (t, s)X \cdot g = (ht, g^{-1}s)X$ , and this biset is transitive. If  $u$  is the point  $X$  of  $U$ , then  $A(u) = X$ , and the equivalence of categories 2.5 reads

$$\text{Fun}(\langle U \rangle, R\text{-Mod}) \cong RX\text{-Mod} .$$

More precisely, for an  $RX$ -module  $M$ , this equivalence yields a functor  $\tilde{M} : \langle U \rangle \rightarrow R\text{-Mod}$  in the following way : for any  $(h, g) \in H \times G$ , set

$$\tilde{M}((h, g)X) = M .$$

Next, fix a set  $S$  of representatives of elements of  $U$ , i.e.  $X$ -cosets in  $H \times G$ . For  $(t, s) \in S$ , and  $(h, g) \in H \times G$ , define a map

$$\tilde{M}(h, g) : \tilde{M}((t, s)X) = M \rightarrow \tilde{M}((ht, gs)X) = M$$

by  $\tilde{M}(h, g)(m) = (y, x)m$ , where  $(y, x)$  is the unique element of  $X$  such that  $(ht, gs)(y, x)^{-1} \in S$ .

Then it is easy to check that  $\tilde{M}$  is indeed a functor, and that there is an isomorphism of  $(RH, RG)$ -bimodules

$$\Sigma(\tilde{M}) \cong \text{Ind}_X^{H \times G} M .$$

Similarly, set  $V = (K \times H)/Y$ , and define a functor  $\tilde{N}\langle V \rangle \rightarrow R\text{-Mod}$ , using the  $RY$ -module  $N$ . Then the corollary is a straightforward consequence of the lemma, applied to the functors  $\tilde{M}$  and  $\tilde{N}$ , using Remark 2.6 and Remark 3.2.  $\square$

## References

- [1] S. Bouc. Foncteurs d'ensembles munis d'une double action. *J. of Algebra*, 183(0238):664–736, 1996.

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