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Counterexamples to Strichartz inequalities for the wave equation in domains II

1 Introduction

Let Ω be a smooth manifold of dimension $d \geq 2$ with C^∞ boundary $\partial\Omega$, equipped with a Riemannian metric g . Let Δ_g be the Laplace-Beltrami operator associated to g on Ω , acting on $L^2(\Omega)$ with Dirichlet boundary condition. Let $0 < T < \infty$ and consider the wave equation with Dirichlet boundary conditions:

$$\begin{cases} (\partial_t^2 - \Delta_g)u = 0 \text{ on } \Omega \times [0, T], \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \\ u|_{\partial\Omega} = 0. \end{cases} \quad (1.1)$$

Strichartz estimates are a family of dispersive estimates on solutions $u : \Omega \times [0, T] \rightarrow \mathbb{C}$ to the wave equation (1.1). In their most general form, local Strichartz estimates state that

$$\|u\|_{L^q([0, T], L^r(\Omega))} \leq C(\|u_0\|_{\dot{H}^\gamma(\Omega)} + \|u_1\|_{\dot{H}^{\gamma-1}}), \quad (1.2)$$

where $\dot{H}^\gamma(\Omega)$ denotes the homogeneous Sobolev space over Ω and where the pair (q, r) is wave admissible in dimension d , i.e. it satisfies $2 \leq q \leq \infty$, $2 \leq r < \infty$ and moreover

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma, \quad \frac{2}{q} + \frac{d-1}{r} \leq \frac{d-1}{2}. \quad (1.3)$$

When equality holds in (1.3) the pair (q, r) is called sharp wave admissible in dimension d . Estimates involving $r = \infty$ hold when $(q, r, d) \neq (2, \infty, 3)$, but typically require the use of Besov spaces.

In \mathbb{R}^d and for $g_{ij} = \delta_{ij}$, Strichartz estimates in the context of the wave and Schrödinger equations have a long history, beginning with Strichartz pioneering work [31], where he proved the particular case $q = r$ for the wave and (classical) Schrödinger equation. This was later generalized to mixed $L^q((-T, T), L^r(\Omega))$ norms by Ginibre and Velo [9] for Schrödinger equation, where (q, r) is sharp admissible and $q > 2$; the wave estimates were obtained independently by Ginibre-Velo [11] and Lindblad-Sogge [21], following earlier work by Kapitanski [16]. The remaining endpoints for both equations were finally settled by Keel and Tao [19].

In that case $\gamma = \frac{(d+1)}{2}(\frac{1}{2} - \frac{1}{r})$ and one can obtain a global estimate with $T = \infty$; (see also Kato [18], Cazenave-Weissler [7]).

However, for general manifolds phenomena such as trapped geodesics or finiteness of volume can preclude the development of global estimates, leading us to consider local in time estimates.

In the variable coefficients case, even without boundary, the situation is much more complicated: we simply recall here the pioneering work of Staffilani and Tataru [30], dealing with compact, non trapping perturbations of the flat metric and recent work of Bouclet and Tzvetkov [4] in the context of Schrodinger equation, which considerably weakens the decay of the perturbation (retaining the non trapping character at spatial infinity). On compact manifolds without boundary, due to the finite speed of propagation, it is enough to work in coordinate charts and to establish local Strichartz estimates for variable coefficients wave operators in \mathbb{R}^d : we recall here the works by Kapitanski [17] and Mockenhaupt, Seeger and Sogge [25] in the case of smooth coefficients when one can use the Lax parametrix construction to obtain the appropriate dispersive estimates. In the case of $C^{1,1}$ coefficients, Strichartz estimates were shown in the works by Smith [26] and by Tataru [32], the latter work establishing the full range of local estimates; here the lack of smoothness prevents the use of Fourier integral operators and instead wave packets and coherent state methods are used to construct parametrices for the wave operator.

Let us recall a result for the flat space: if we denote by Δ the Euclidian Laplace operator, then the Strichartz estimates for the wave equation posed on \mathbb{R}^d read as follows (see [19]):

Proposition 1.1. *Let (q, r) be a wave admissible pair in dimension $d \geq 2$. If u satisfies*

$$(\partial_t^2 - \Delta)u = 0, \quad [0, T] \times \mathbb{R}^d, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \quad (1.4)$$

for some $0 < T < \infty$, $u_0, u_1 \in C^\infty(\mathbb{R}^d)$, then there is a constant $C = C_T$ such that

$$\|u\|_{L^q([0,T], L^r(\mathbb{R}^d))} \leq C(\|u_0\|_{\dot{H}^{\frac{(d+1)}{2}(\frac{1}{2}-\frac{1}{r})}(\mathbb{R}^d)} + \|u_1\|_{\dot{H}^{\frac{(d+1)}{2}(\frac{1}{2}-\frac{1}{r})-1}(\mathbb{R}^d)}). \quad (1.5)$$

In this paper we prove that Strichartz estimates for the wave equation inside the domain Ω suffer losses when compared to the usual case \mathbb{R}^d , at least for a subset of the usual range of indices, under the assumption that there exists a point in $T^*\partial\Omega$ where the second fundamental form on the boundary of the manifold has a strictly positive eigenfunction.

Assumption 1. We assume that there exists a point $(\rho_0, \vartheta_0) \in T^*(\partial\Omega \times \mathbb{R})$ and a bicharacteristic which is tangential to $\partial\Omega \times \mathbb{R}$ at (ρ_0, ϑ_0) having exactly second order contact with the boundary. We call such a point a *gliding point*.

Our main result reads as follows:

Theorem 1.2. *Let (q, r) be a sharp wave admissible pair in dimension $d \in \{2, 3, 4\}$ with $r > 4$. Under the Assumption 1, for every small $\epsilon > 0$ there exist sequences $V_{h,j,\epsilon} \in C^\infty(\bar{\Omega})$,*

$j = \overline{0,1}$ such that the solution $V_{h,\epsilon}$ to the wave equation with Dirichlet boundary conditions

$$\begin{cases} (\partial_t^2 - \Delta_g)V_{h,\epsilon} = 0, \\ V_{h,\epsilon}|_{t=0} = V_{h,0,\epsilon}, \quad \partial_t V_{h,\epsilon}|_{t=0} = V_{h,1,\epsilon}, \\ V_{h,\epsilon}|_{\partial\Omega \times [0,T]} = 0, \end{cases} \quad (1.6)$$

satisfies

$$\sup_{\epsilon > 0, h \in (0,1], j} h^{-\frac{(d+1)}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{6}(\frac{1}{4}-\frac{1}{r})+2\epsilon+j} \|V_{h,j,\epsilon}\|_{L^2(\Omega)} \leq 1 \quad (1.7)$$

and

$$\lim_{h \rightarrow 0} \|V_{h,\epsilon}\|_{L_t^q([0,T], L^r(\Omega))} = \infty. \quad (1.8)$$

Moreover $V_{h,\epsilon}$ has compact support for the normal variable in $(0, h^{\frac{1-\epsilon}{2}}]$ and is well localized at spatial frequency $\frac{1}{h}$ in the tangential variable.

Remark 1.3. The proof of Theorem 1.2 will show that the restriction on the dimension comes only from the fact that for $d \geq 5$ all admissible pairs (q, r) satisfy $r \leq 4$.

For a manifold with smooth, strictly geodesically concave boundary (i.e. for which the second fundamental form is strictly negative definite), the Melrose and Taylor parametrix yields the Strichartz estimates for the wave equation with Dirichlet boundary condition for the range of exponents in (1.3) (not including the endpoints) as shown in the paper of Smith and Sogge [27]. If the concavity assumption is removed, however, the presence of multiply reflecting geodesic and their limits, the gliding rays, prevent the construction of a similar parametrix!

Note that on an exterior domain a source point does not generate caustics and that the presence of caustics generated in small time near a source point is the one which makes things difficult inside a strictly convex set.

Recently, Burq, Lebeau and Planchon [5], [6] established Strichartz type inequalities on a manifold with boundary using the $L^r(\Omega)$ estimates for the spectral projectors obtained by Smith and Sogge [28]. The range of triples (q, r, γ) that can be obtained in this manner, however, is restricted by the allowed range of r in the square function estimate for the wave equation, which controls the norm of u in the space $L^r(\Omega, L^2(-T, T))$ (see [28]). In dimension 3, for example, this restricts the indices to $q, r \geq 5$. The work of Blair, Smith and Sogge [3] expands the range of indices q and r obtained in [5]: specifically, they show that if Ω is a compact manifold with boundary (or without boundary but with Lipschitz metric g) and (q, r, γ) is a triple satisfying the first condition in (1.3) together with the restriction

$$\begin{cases} \frac{3}{q} + \frac{d-1}{r} \leq \frac{d-1}{2}, & d \leq 4 \\ \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}, & d \geq 4, \end{cases}$$

then the Strichartz estimates (1.2) hold true for solutions u to (1.1) satisfying Dirichlet or Neumann homogeneous boundary conditions, with a constant C depending on Ω and T .

Remark 1.4. Notice that Theorem 1.2 states for instance that the scale-invariant Strichartz estimates fail for $\frac{3}{q} + \frac{1}{r} > \frac{15}{24}$, whereas the result of Blair, Smith and Sogge states that such estimates hold if $\frac{3}{q} + \frac{1}{r} \leq \frac{1}{2}$. Of course, the counterexample places a lower bound on the loss for such indices (q, r) , and the work [3] would place some upper bounds, but this concise statement shows one explicit gap in our knowledge that remains to be filled.

A very interesting and natural question would be to determine the sharp range of exponents for (1.2) in any dimension $d \geq 2$!

A classical way to prove Strichartz inequalities is to use dispersive estimates: the fact that weakened dispersive estimates can *still* imply optimal (and scale invariant) Strichartz estimates for the solution of the wave equation was first noticed by Lebeau: in [20] he proved that a loss of derivatives is unavoidable for the wave equation inside a strictly convex domain, and this appears because of swallowtail type caustics in the wave front set of u :

$$|\chi(hD_t)u(t, x)| \lesssim h^{-d} \min(1, (h/t)^{\frac{d-2}{2} + \frac{1}{4}}).$$

However, these estimates, although optimal for the dispersion, imply Strichartz type inequalities without losses, but with indices (q, r, d) satisfying

$$\frac{1}{q} \leq \frac{(d-1)}{4} \left(\frac{1}{2} - \frac{1}{r} \right).$$

A natural strategy for proving Theorem 1.2 would be to use the Rayleigh whispering gallery modes which accumulate their energy near the boundary contributing to large $L^r(\Omega)$ norms. Applying the semi-classical Schrödinger evolution shows that a loss of derivatives is necessary for the Strichartz estimates. However, when dealing with the wave operator this strategy fails as the gallery modes satisfy the Strichartz estimates of the free space, as it is shown in [15].

In the proof of Theorem 1.2 we shall proceed in a different manner, using co-normal waves with multiply reflected cusps at the boundary, together with Melrose's Theorem of glancing rays to reduce the study of the iterated boundary operators to the Friedlander case, in which case all the computations are explicit. We only recall here the main ingredients of the proof and show how this can be used to construct a counterexample under the much more general assumptions of Theorem 1.2. The reduction to the model case relies essentially on Melrose's Theorem [24] of glancing surfaces.

The organization of the paper is as follows: in Section 2 we show that in order to prove Theorem 1.2 it is enough to consider the two-dimensional case. In Section 3 we deal with a strictly convex domain of dimension two and use the model construction to determine an approximate solution of (1.6) which satisfies Theorem 1.2. In the Appendix we compute the L^r norms of a cusp.

2 Reduction to the two dimensional case

Let Ω satisfy the assumptions of Theorem 1.2. Write local coordinates on Ω as (x, y_1, \dots, y_{d-1}) with $x > 0$ on Ω , $\partial\Omega = \{(0, y) | y = (y_1, \dots, y_{d-1}) \in \mathbb{R}^{d-1}\}$ and local coordinates induced by the product $X = \Omega \times \mathbb{R}_t$, as (x, y, t) .

Local coordinates on the base induce local coordinates on the cotangent bundle, namely $(\rho, \vartheta) = (x, y, t, \xi, \eta, \tau)$ on T^*X near $\pi^{-1}(q)$, $q \in T^*\partial X$, where $\pi : T^*X \rightarrow T^*X$ is the canonical inclusion from the cotangent bundle into the b -cotangent bundle defined by ${}^bT^*X = T^*\overset{\circ}{X} \cup T^*\partial X$. The corresponding local coordinates on the boundary are denoted (y, t, η, τ) (on a neighborhood of a point q in $T^*\partial X$). The metric function in $T^*\Omega$ has the form

$$g(x, y, \xi, \eta) = A(x, y)\xi^2 + 2 \sum_{j=1}^{d-1} C_j(x, y)\xi\eta_j + \sum_{j,k=1}^{d-1} B_{j,k}(x, y)\eta_j\eta_k,$$

with $A, B_{j,k}, C_j$ smooth. Moreover, these coordinates can be chosen so that $A(x, y) = 1$ and $C_j(x, y) = 0$ (see [14, Appendix C]). Thus, in this coordinates chart the metric on the boundary writes

$$g(0, y, \xi, \eta) = \xi^2 + \sum_{j,k=1}^{d-1} B_{j,k}(0, y)\eta_j\eta_k.$$

On $T^*\partial\Omega$ the metric g takes even a simpler form, since introducing geodesic coordinates we can assume moreover that, locally,

$$B_{1,1}(0, y) = 1, \quad B_{1,j}(0, y) = 0 \quad \forall j \in \{2, \dots, d-1\}.$$

Hence, if $R(x, y, \eta) := \sum_{j,k=1}^{d-1} B_{j,k}(x, y)\eta_j\eta_k$, then for small x we have

$$\begin{aligned} R(x, y, \eta) &= (1 + x\partial_x B_{1,1}(0, y_1, 0) + O(x|y'|))\eta_1^2 + \\ &\quad + \sum_{j=1}^{d-1} (x\partial_x B_{1,j}(0, y) + O(x^2))\eta_1\eta_j + \sum_{j,k=2}^{d-1} B_{j,k}(x, y)\eta_j\eta_k. \end{aligned} \quad (2.1)$$

The Assumption 1 on the domain Ω is equivalent to saying that there exists a point $(0, y_0, \xi_0, \eta_0)$ on $T^*\Omega$ where the boundary is microlocally strictly convex, i.e. that there exists a bicharacteristic passing through this point that intersects $\partial\Omega$ tangentially having exactly second order contact with the boundary and remaining in the complement of $\partial\bar{\Omega}$. If $p \in C^\infty(T^*X \setminus \{o\})$ (where we write o for the "zero" section) denotes the principal symbol of the wave operator $\partial_t^2 - \Delta_g$, this last condition translates into

$$\tau^2 = R(0, y_0, \eta_0), \quad \{p, x\} = \frac{\partial p}{\partial \xi} = 2\xi_0 = 0, \quad (2.2)$$

$$\{\{p, x\}, p\} = \left\{ \frac{\partial p}{\partial \xi}, p \right\} = 2\partial_x R(0, y_0, \eta_0) > 0, \quad (2.3)$$

where $\{f_1, f_2\}$ denotes the Poisson bracket

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial \vartheta} \frac{\partial f_2}{\partial \rho} - \frac{\partial f_1}{\partial \rho} \frac{\partial f_2}{\partial \vartheta}.$$

Denote the gliding point (in $T^*\Omega \times \mathbb{R}$) of the Assumption 1 by

$$(\rho_0, \vartheta_0) = (0, y_0, 0, 0, \eta_0, \tau_0 = -\sqrt{R(0, y_0, \eta_0)}).$$

We start the proof of Theorem 1.2 by reducing the problem to the study of the two dimensional case. Consider the following assumptions:

Assumption 2. Let $\tilde{\Omega}$ be a smooth manifold of dimension 2 with C^∞ boundary and with a Riemannian metric \tilde{g} . Suppose that in a chart of local coordinates $\tilde{\Omega} = \{(x, \tilde{y}) | x > 0, \tilde{y} \in \mathbb{R}\}$ and that the Laplace-Beltrami operator associated to \tilde{g} is given by

$$\partial_x^2 + (1 + xb(\tilde{y}))\partial_{\tilde{y}}^2,$$

where $b(\tilde{y})$ is a smooth function. Suppose in addition that there exists a point $(0, \tilde{y}_0, \tilde{\xi}_0, \tilde{\eta}_0) \in T^*\tilde{\Omega}$ and a bicharacteristic intersecting the boundary tangentially at this point and having exactly second order contact with the boundary. This is equivalent to saying that at $(0, \tilde{y}_0, \tilde{\xi}_0, \tilde{\eta}_0)$ the following holds

$$\tilde{\xi}_0 = 0, \quad 2b(\tilde{y}_0) > 0.$$

We suppose $b(\tilde{y}_0) = 1$ and that there exists a neighborhood \mathcal{N} of y_0 and $0 < b_0 < 1/4$ small enough such that for $\tilde{y} \in \mathcal{N}$ we have $|b(\tilde{y}) - 1| \leq b_0$.

Theorem 2.1. *Under the Assumption 2, given $T > 0$, for every $\epsilon > 0$ small enough there exist sequences $\tilde{V}_{h,j,\epsilon}$, $j \in \{0, 1\}$ and approximate solutions $\tilde{V}_{h,\epsilon}$ to the wave equation on $\tilde{\Omega}$ with Dirichlet boundary condition*

$$\begin{cases} \partial_t^2 V - \partial_x^2 V - (1 + xb(\tilde{y}))\partial_{\tilde{y}}^2 V = 0, & \text{on } \tilde{\Omega} \times \mathbb{R} \\ V|_{t=0} = \tilde{V}_{h,0,\epsilon}, \quad \partial_t V|_{t=0} = \tilde{V}_{h,1,\epsilon}, \\ V|_{\partial\Omega \times [0,T]} = 0, \end{cases} \quad (2.4)$$

which satisfy the following conditions:

- First, $\tilde{V}_{h,\epsilon}$ is an approximate solution to (2.4) in the sense that

$$\partial_t^2 \tilde{V}_{h,\epsilon} - \partial_x^2 \tilde{V}_{h,\epsilon} - (1 + xb(\tilde{y}))\partial_{\tilde{y}}^2 \tilde{V}_{h,\epsilon} = O_{L^2(\tilde{\Omega})}(1/h), \quad \|\tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \leq 1. \quad (2.5)$$

- Secondly, $\tilde{V}_{h,\epsilon}$ writes as a sum

$$\tilde{V}_{h,\epsilon}(x, \tilde{y}, t) = \sum_{n=0}^N v_{h,\epsilon}^n(x, \tilde{y}, t), \quad (2.6)$$

where the functions $v_{h,\epsilon}^n(x, \tilde{y}, t)$ satisfy the following conditions:

– for $4 < r < \infty$:

$$\begin{cases} \|v_{h,\epsilon}^n(\cdot, t)\|_{L^r(\tilde{\Omega})} \geq Ch^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{6}(\frac{1}{4}-\frac{1}{r})+2\epsilon}, \\ \sup_{\epsilon>0} (\|v_{h,\epsilon}^n(\cdot, t)\|_{L^2(\tilde{\Omega})} + h\|\partial_t v_{h,\epsilon}^n(\cdot, t)\|_{L^2(\tilde{\Omega})}) \leq 1, \end{cases} \quad (2.7)$$

where the constants $C > 0$ are independent of h and n ;

- $v_{h,\epsilon}^n(x, \tilde{y}, t)$ are essentially supported for the time variable t in almost disjoint intervals of time and for the tangential variable \tilde{y} in almost disjoint intervals.
- $\tilde{V}_{h,\epsilon}$ are supported for the normal variable $x \in [0, \tilde{C}_\epsilon h^{(1-\epsilon)/2}]$ with $\tilde{C}_\epsilon > 0$ independent of h and localized at spatial frequency $\frac{1}{h}$ in the tangential variable \tilde{y} . Moreover,

$$\sup_{\epsilon>0} \|\tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \lesssim 1, \quad \sup_{\epsilon>0} \|\partial_{\tilde{y}} \tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \lesssim \frac{1}{h}, \quad \sup_{\epsilon>0} \|\partial_{\tilde{y}}^2 \tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \lesssim \frac{1}{h^2}; \quad (2.8)$$

In the rest of this section we show how Theorem 2.1 implies Theorem 1.2. Suppose we have proved Theorem 2.1. Let (Ω, g) be a Riemannian manifold of dimension $d > 2$ satisfying the assumptions of Theorem 1.2 and let $(0, y_0, \xi_0, \eta_0) \in T^*\Omega$ be a point satisfying (2.2), (2.3). If e_1 is the eigenfunction corresponding to the strictly positive eigenvalue of the second fundamental form associated to the metric g then from (2.1) it follows that local coordinates can be chosen such that $y_0 = 0 \in \mathbb{R}^{d-1}$, $\eta_0 = (1, 0, \dots, 0) \in \mathbb{R}^{d-1}$ and such that the Laplace-Beltrami operator Δ_g be given by

$$\Delta_g = \partial_x^2 + \sum_{j,k=1}^{d-1} B_{j,k}(x, y) \partial_j \partial_k, \quad (2.9)$$

where for x and $|y'|$ close to zero

$$B_{1,1}(x, y) = 1 + x \partial_x B_{1,1}(0, y_1, 0) + O(x|y'|) + O(x^2),$$

and for $j \in \{2, \dots, d-1\}$

$$B_{1,j}(x, y) = x \partial_x B_{1,j}(0, y) + O(x^2).$$

Define $\tilde{\Omega} = \{(x, y_1) | x > 0, y_1 \in \mathbb{R}\}$ the two dimensional manifold equipped with the metric

$$\tilde{g}(x, y_1, \xi, \eta_1) = \xi^2 + (1 + xb(y_1))\eta_1^2, \quad b(y_1) := \partial_x B_{1,1}(0, y_1, 0), \quad b(0) = 1.$$

Applying Theorem 2.1 near $(0, y_1 = 0, 0, \eta_1 = 1) \in T^*\tilde{\Omega}$ we obtain, for $\epsilon > 0$ small enough, sequences $\tilde{V}_{h,\epsilon,j}$, $j \in \{0, 1\}$ such that the solution $\tilde{V}_{h,\epsilon}$ to (2.4) satisfies (2.6), (2.7) and (2.8). Let $\chi \in C_0^\infty(\mathbb{R}^{d-2})$ be a cut-off function supported in the coordinate chart such that $\chi = 1$ in a neighborhood of $0 \in \mathbb{R}^{d-2}$ and for $j \in \{0, 1\}$ set

$$V_{h,\epsilon,j}(x, y_1, y') := h^{-(d-2)/4} \tilde{V}_{h,\epsilon/3,j}(x, y_1) e^{-\frac{|y'|^2}{2h}} \chi(y'). \quad (2.10)$$

Proposition 2.2. *The solution $V_{h,\epsilon}$ to the wave equation (1.6) with Dirichlet boundary condition where Δ_g is given by (2.9) and with initial data $(V_{h,\epsilon,0}, V_{h,\epsilon,1})$ defined in (2.10) satisfies (1.7), (1.8).*

Remark 2.3. Notice that Proposition 2.2 implies immediately Theorem 1.2.

Proof. We proceed by contradiction. Let (q, r) be a sharp wave admissible pair in dimension $d > 2$ with $r > 4$ and set

$$\beta(r, d) = \frac{(d+1)}{2} \left(\frac{1}{2} - \frac{1}{r} \right) + \frac{1}{6} \left(\frac{1}{4} - \frac{1}{r} \right).$$

Let us suppose to the contrary that the operator

$$\sin t \sqrt{-\Delta_g} : L^2(\Omega) \rightarrow L^q([0, T], L^r(\Omega))$$

is bounded by $h^{-\beta(r,d)+2\epsilon}$. Let $\tilde{V}_{h,\epsilon/3}$ be the approximate solution to (2.4) with initial data $(\tilde{V}_{h,\epsilon/3,j})_{j=0,1}$ satisfying all the conditions in Theorem 2.1. We define

$$W_{h,\epsilon}(x, y, t) := h^{-(d-2)/4} \tilde{V}_{h,\epsilon/3}(x, y_1, t) e^{-\frac{|y'|^2}{2h}} \chi(y').$$

Lemma 2.4. *There exists constants c_j , $j = 0, 1$, independent of h such that $W_{h,\epsilon}$ satisfies*

$$\|W_{h,\epsilon}\|_{L^q([0,T], L^r(\Omega))} \geq c_0 h^{-\beta(r,d)+2\epsilon/3}, \quad (2.11)$$

$$\|W_{h,\epsilon}|_{t=0}\|_{L^2(\Omega)} + h \|\partial_t W_{h,\epsilon}|_{t=0}\|_{L^2(\Omega)} \leq c_1. \quad (2.12)$$

Proof. Indeed, using the special form of $\tilde{V}_{h,\epsilon}$ provided by Theorem 2.1 we can estimate

$$\|W_{h,\epsilon}\|_{L^q([0,T], L^r(\Omega))}^q = \int_0^T \|W_{h,\epsilon}\|_{L^r(\Omega)}^q dt \quad (2.13)$$

$$= \left(\int_0^T \left\| \sum_{n=0}^N v_{h,\epsilon/3}^n \right\|_{L^r(\tilde{\Omega})}^q dt \right) \times \|h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \chi(y')\|_{L^r(\mathbb{R}^{d-2})}^q \quad (2.14)$$

$$\geq c_\epsilon h^{-\frac{q(d-2)}{2}(\frac{1}{2}-\frac{1}{r})} \sum_{k \leq N/5} \int_{t \in I_k} \left\| \sum_{n=0}^N v_{h,\epsilon/3}^n \right\|_{L^r(\tilde{\Omega})}^q dt + O(h^\infty) \quad (2.15)$$

$$\simeq c_\epsilon h^{-\frac{q(d-2)}{2}(\frac{1}{2}-\frac{1}{r})} \sum_{k \leq N/5} |I_k| \|v_{h,\epsilon/3}^0\|_{L^r(\tilde{\Omega})}^q + O(h^\infty) \quad (2.16)$$

$$\simeq c_\epsilon h^{-\frac{q(d-2)}{2}(\frac{1}{2}-\frac{1}{r})} \|v_{h,\epsilon/3}^0\|_{L^r(\tilde{\Omega})}^q + O(h^\infty) \quad (2.17)$$

$$\geq c_{0,\epsilon} h^{(-\beta(r,d)+2\epsilon/3)q}. \quad (2.18)$$

To estimate the $L^2(\Omega)$ norm we use again the fact that $v_{h,\epsilon}^n$ and its time derivative have disjoint essential supports in the tangential variable y_1 . For $W_{h,\epsilon}(\cdot, 0)$ we have, for instance

$$\|W_{h,\epsilon}|_{t=0}\|_{L^2(\Omega)} = \|\tilde{V}_{h,\epsilon/3,0}\|_{L^2(x,y_1)} \|h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \chi(y')\|_{L^2(\mathbb{R}^{d-2})} \lesssim 1.$$

□

Let $V_{h,\epsilon}$ be the solution to the wave equation (1.6) with initial data $(V_{h,\epsilon,j})_{j=0,1}$ and write

$$V_{h,\epsilon} = W_{h,\epsilon} + w_{h,\epsilon,err}.$$

If we denote $\Delta_{\tilde{g}} = \partial_x^2 + (1 + xb(y_1))\partial_{y_1}^2$, $\square_{\tilde{g}} = \partial_t^2 - \Delta_{\tilde{g}}$, then $W_{h,\epsilon}$ solves

$$\begin{cases} \square_{\tilde{g}} W_{h,\epsilon} &= \square_{\tilde{g}} \tilde{V}_{h,\epsilon/3} h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \chi(y'), \\ W_{h,\epsilon}|_{t=0} &= V_{h,\epsilon,0}, \quad \partial_t W_{h,\epsilon}|_{t=0} = V_{h,\epsilon,1}, \\ W_{h,\epsilon}|_{\partial\Omega \times [0,T]} &= 0. \end{cases} \quad (2.19)$$

Since $V_{h,\epsilon}$ is a solution to (1.6), $w_{h,\epsilon,err}$ must satisfy the following equation

$$\begin{cases} \square_g w_{h,\epsilon,err} &= -\square_{\tilde{g}} \tilde{V}_{h,\epsilon/3} h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \chi(y') - (1 + xb(y_1))\partial_{y_1}^2 W_{h,\epsilon} + \\ &+ \sum_{j,k=1}^{d-1} B_{j,k}(x,y)\partial_{y_j,y_k}^2 W_{h,\epsilon}, \\ w_{h,\epsilon,err}|_{t=0} &= 0, \quad \partial_t w_{h,\epsilon,err}|_{t=0} = 0, \\ w_{h,\epsilon}|_{\partial\Omega \times [0,T]} &= 0, \end{cases} \quad (2.20)$$

where we set $\square_g := \partial_t^2 - \Delta_g$ and we used that

$$\Delta_g - \Delta_{\tilde{g}} = -(1 + xb(y_1))\partial_{y_1}^2 + \sum_{j,k=1}^{d-1} B_{j,k}(x,y)\partial_{y_j,y_k}^2.$$

Lemma 2.5. *The solution $w_{h,\epsilon,err}$ to the wave equation (2.20) satisfies*

$$\|(\partial_t^2 - \Delta_g)w_{h,\epsilon,err}\|_{L^2(\Omega)} \simeq O(h^{-2(1-(1-\epsilon/3)/2)})\|w_{h,\epsilon,err}\|_{L^2(\Omega)} \quad (2.21)$$

$$\geq O(h^{-\epsilon/3})\|w_{h,\epsilon,err}\|_{\dot{H}^{-1}(\Omega)}. \quad (2.22)$$

Moreover,

$$\|w_{h,\epsilon,err}\|_{L^q([0,T],L^r(\Omega))} \leq C_\epsilon h^{-\beta(r,d)+2\epsilon-\epsilon/3}. \quad (2.23)$$

Proof. We start with (2.23). Assume we have already proved (2.21). The Duhamel formula for $w_{h,\epsilon,err}$ writes

$$w_{h,\epsilon,err}(x,y,t) = \int_0^t \frac{\sin(t-s)\sqrt{-\Delta_g}}{\sqrt{-\Delta_g}} ((\partial_t^2 - \Delta_g)w_{h,\epsilon,err}(x,y,s)) ds. \quad (2.24)$$

Using the Minkowski inequality together with (2.21) we find

$$\|w_{h,\epsilon,err}(\cdot, t)\|_{L^r(\Omega)} = \left\| \int_0^t \frac{\sin(t-s)\sqrt{-\Delta_g}}{\sqrt{-\Delta_g}} ((\partial_t^2 - \Delta_g)w_{h,\epsilon,err}(\cdot, s)) ds \right\|_{L^r(\Omega)} \quad (2.25)$$

$$\lesssim \int_0^t \left\| \frac{\sin(t-s)\sqrt{-\Delta_g}}{\sqrt{-\Delta_g}} ((\partial_t^2 - \Delta_g)w_{h,\epsilon,err}(\cdot, s)) \right\|_{L^r(\Omega)} ds \quad (2.26)$$

$$\lesssim h^{-\beta(r,d)+2\epsilon} \|(\sqrt{-\Delta_g})^{-1}(\partial_t^2 - \Delta_g)w_{h,\epsilon,err}\|_{L^1([0,T],L^2(\Omega))} \quad (2.27)$$

$$\simeq h^{-\beta(r,d)+2\epsilon} \|(\partial_t^2 - \Delta_g)w_{h,\epsilon,err}\|_{L^1([0,T],\dot{H}^{-1}(\Omega))} \quad (2.28)$$

$$\lesssim h^{-\beta(r,d)+2\epsilon-\epsilon/3}, \quad (2.29)$$

where in the third line we used that the wave operator $\sin t\sqrt{-\Delta_g}$ was supposed to be bounded by $h^{-\beta(r,d)+2\epsilon}$ and where in the last line we used (2.21).

We now proceed with (2.21). In order to do this we use the special form of Δ_g and the fact that $\tilde{V}_{h,\epsilon/3}(x, y_1, t)$ (and therefore $V_{h,\epsilon}$) is supported for $x \in [0, \tilde{C}_{\epsilon/3}h^{(1-\epsilon/3)/2}]$. The inhomogeneous part of the equation (2.20) writes

$$\square \tilde{V}_{h,\epsilon/3} h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \chi(y') + (1 + xb(y_1)) \partial_{y_1}^2 W_{h,\epsilon} - \sum_{j,k=1}^{d-1} B_{j,k}(x, y) \partial_{y_j, y_k}^2 W_{h,\epsilon}. \quad (2.30)$$

The $L^2(\Omega)$ norm of $\square \tilde{V}_{h,\epsilon/3} h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \chi(y')$ is estimated using the last condition in Theorem 2.1 and its contribution in the norm of the non-linear term of (2.20) is $O_{L^2(\Omega)}(1/h)$. The last two terms in (2.30) write

$$\begin{aligned} & - (1 + xb(y_1)) \partial_{y_1}^2 W_{h,\epsilon} + \sum_{j,k=1}^{d-1} B_{j,k}(x, y) \partial_{y_j, y_k}^2 W_{h,\epsilon} = h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \times \\ & \times \left[\left(B_{1,1}(x, y) - 1 - b(y_1) \right) \chi(y') \partial_{y_1}^2 \tilde{V}_{h,\epsilon/3} - \frac{1}{h} \sum_{j=2}^{d-1} B_{1,j}(x, y) \left(y_j \chi(y') + h \partial_{y_j} \chi(y') \right) \partial_{y_1} \tilde{V}_{h,\epsilon/3} \right. \\ & \left. + \frac{1}{h^2} \sum_{j,k=2}^{d-1} \left(y_j y_k \chi(y') - h(y_j \partial_{y_k} \chi(y') + y_k \partial_{y_j} \chi(y') + \delta_{j=k}) + h^2 \partial_{y_j, y_k}^2 \chi(y') \right) B_{j,k}(x, y) \tilde{V}_{h,\epsilon/3} \right]. \end{aligned} \quad (2.31)$$

If $|y'| \geq h^{(1-\epsilon')/2}$ for some $\epsilon' > 0$, then $e^{-\frac{|y'|^2}{2h}} \leq C_M h^M$, for all $M \geq 0$, thus taking $\epsilon' = \epsilon/3$ we can estimate the $L^2(\Omega)$ norm of (2.31) as follows

$$\begin{aligned} & \left\| - (1 + xb(y_1)) \partial_{y_1}^2 W_{h,\epsilon} + \sum_{j,k=1}^{d-1} B_{j,k}(x, y) \partial_{y_j} \partial_{y_k} W_{h,\epsilon} \right\|_{L^2(\Omega)} \\ & \lesssim h^{-2+(1-\epsilon/3)} \|\tilde{V}_{h,\epsilon/3}\|_{L^2(\tilde{\Omega})} \\ & \lesssim h^{-1-\epsilon/3}, \end{aligned} \quad (2.32)$$

where we used that

$$\sup_{\epsilon>0} \|\tilde{V}_{h,\epsilon/3}\|_{L^2(\tilde{\Omega})} \lesssim 1, \quad \sup_{\epsilon>0} \|\partial_{y_1} \tilde{V}_{h,\epsilon/3}\|_{L^2(\tilde{\Omega})} \lesssim \frac{1}{h}, \quad \sup_{\epsilon>0} \|\partial_{y_1}^2 \tilde{V}_{h,\epsilon/3}\|_{L^2(\tilde{\Omega})} \lesssim \frac{1}{h^2}.$$

In the same way we can estimate

$$\begin{aligned}
& \left\| -(1 + xb(y_1))\partial_{y_1}^2 W_{h,\epsilon} + \sum_{j,k=1}^{d-1} B_{j,k}(x, y)\partial_{y_j, y_k}^2 W_{h,\epsilon} \right\|_{\dot{H}^{-1}(\Omega)} \\
& \lesssim h \left\| -(1 + xb(y_1))\partial_{y_1}^2 W_{h,\epsilon} + \sum_{j,k=1}^{d-1} B_{j,k}(x, y)\partial_{y_j, y_k}^2 W_{h,\epsilon} \right\|_{L^2(\Omega)} \\
& \lesssim h^{-\epsilon/3}. \quad (2.33)
\end{aligned}$$

For the last inequality we used the following lemma

Lemma 2.6. *Let $f(x, y) : \Omega \rightarrow \mathbb{R}$ be localized at frequency $1/h$ in the $y \in \mathbb{R}^{d-1}$ variable, i.e. such that there exists $\psi \in C_0^\infty(\mathbb{R}^{d-1} \setminus \{0\})$ with $\psi(hD_y)f = f$. Then there exists a constant $C > 0$ independent of h such that one has*

$$\|f\|_{\dot{H}^{-1}(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}.$$

Proof. (of Lemma 2.6:) Since $\psi(hD_y)f = f$ we have

$$\begin{aligned}
\|f\|_{\dot{H}^{-1}(\Omega)} &= \sup_{\|g\|_{\dot{H}^1(\Omega)} \leq 1} \int \psi f \bar{g} \leq \|f\|_{L^2(\Omega)} \times \sup_{\|g\|_{\dot{H}^1(\Omega)} \leq 1} \|\psi(hD_y)g\|_{L^2(\Omega)} \\
&\leq h\|f\|_{L^2(\Omega)} \|\tilde{\psi}(hD_y)\nabla_y g\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)},
\end{aligned}$$

where we set $\tilde{\psi}(\eta) = |\eta|^{-1}\psi(\eta)$. Hence Lemma 2.6 is proved. \square

\square

\square

End of the proof of Proposition 2.2:

Recall that we have assumed that the operator

$$\sin t\sqrt{-\Delta_g} : L^2(\Omega) \rightarrow L^q([0, T], L^r(\Omega))$$

is bounded by $h^{-\beta(r,d)+2\epsilon}$. This assumption implies

$$\|V_{h,\epsilon}\|_{L^q([0,T], L^r(\Omega))} \lesssim C_{0,\epsilon} h^{-\beta(r,d)+2\epsilon} (\|V_{h,\epsilon,0}\|_{L^2(\Omega)} + \|V_{h,\epsilon,1}\|_{\dot{H}^{-1}(\Omega)}) \quad (2.34)$$

$$\lesssim C_{1,\epsilon} h^{-\beta(r,d)+2\epsilon}, \quad (2.35)$$

where $C_{j,\epsilon} > 0$ are independent of h . If (2.34) were true, together with (2.11) it would yield

$$h^{-\beta(r,d)+2\epsilon/3} \lesssim \|W_{h,\epsilon}\|_{L^q([0,T], L^r(\Omega))} \quad (2.36)$$

$$\lesssim (\|V_{h,\epsilon}\|_{L^q([0,T], L^r(\Omega))} + \|w_{h,\epsilon, err}\|_{L^q([0,T], L^r(\Omega))}). \quad (2.37)$$

The last estimate together with (2.23) and (2.34) gives a contradiction, since it would imply

$$h^{-\beta(r,d)+2\epsilon/3} \lesssim h^{-\beta(r,d)+2\epsilon} + h^{-\beta(r,d)+2\epsilon-\epsilon/3}$$

which is obviously not true. The proof is complete. \square

3 Form of an approximate solution in $2D$

We are reduced to prove Theorem 2.1. We may suppose $T = 1$. In what follows we fix $\epsilon > 0$ small enough and we do not mention anymore the dependence on ϵ of the solution of the wave equation (1.1) we shall construct. We keep the notations of Theorem 1.2 in the two-dimensional case. Let therefore Ω be a Riemannian manifold of dimension $d = 2$ with smooth boundary $\partial\Omega$ satisfying the assumptions of Theorem 2.1 and let g denote its Riemannian metric. Let local coordinates be chosen such that Ω be given by

$$\Omega = \{(x, y) | x > 0, y \in \mathbb{R}\}, \quad (3.1)$$

and the Laplace-Beltrami operator Δ_g associated to the metric g be given by

$$\Delta_g = \partial_x^2 + (1 + xb(y))\partial_y^2, \quad (3.2)$$

where b is a smooth function. Set $X = \Omega \times \mathbb{R}_t$, let $\square_g = \partial_t^2 - \Delta_g$ denote the wave operator on X and let $p \in C^\infty(T^*X \setminus o)$ be the principal symbol of \square_g , which is homogeneous of degree 2 in $T^*X \setminus o$ (where we write o for the "zero section" of T^*X),

$$p(x, y, t, \xi, \eta, \tau) = \xi^2 + (1 + xb(y))\eta^2 - \tau^2. \quad (3.3)$$

The characteristic set $P := \text{Char}(p) \subset T^*X \setminus o$ of \square_g is defined by $p^{-1}(\{0\})$. If we denote $N^*\partial\Omega$ the conormal bundle of ∂X we notice that $\text{Char}(p) \cap N^*\partial\Omega = \emptyset$, meaning that the boundary is non-characteristic for \square_g .

We briefly recall some definitions we shall use in the rest of the paper (for details see [14] or [34], for example). Let us consider the Dirichlet problem for \square_g :

$$\square_g u = 0, \quad u|_{\partial X} = 0. \quad (3.4)$$

The statement of the propagation of singularities of solutions to (3.4) has two main ingredients: locating singularities of a distribution, as captured by the wave front set, and describing the curves along which they propagate, namely the bicharacteristics. Both of these are closely related to an appropriate notion of "phase space", in which both the wave front set and the bicharacteristics are located. On manifolds without boundary, this phase space is the standard cotangent bundle T^*X . In presence of boundaries the phase space is the b -cotangent bundle, ${}^bT^*X$. Let o denote the zero section of ${}^bT^*X$. Then ${}^bT^*X \setminus o$ is equipped with an \mathbb{R}^+ -action (fiberwise multiplication) which has no fixed points. There is a natural non-injective "inclusion" $\pi : T^*X \rightarrow {}^bT^*X$. We define the elliptic, glancing and hyperbolic sets in $T^*\partial X$ as follows:

$$\begin{aligned} \mathcal{E} &= \{q \in \pi(T^*X) \setminus o | \pi^{-1}(q) \cap \text{Char}(p) = \emptyset\}, \\ \mathcal{G} &= \{q \in \pi(T^*X) \setminus o | \text{Card}(\pi^{-1}(q) \cap \text{Char}(p)) = 1\}, \\ \mathcal{H} &= \{q \in \pi(T^*X) \setminus o | \text{Card}(\pi^{-1}(q) \cap \text{Char}(p)) \geq 2\}, \end{aligned}$$

with Card denoting the cardinality of a set; each of these is a conic subset of $\pi(T^*X) \setminus o$. Note that in $T^*\overset{\circ}{X}$, π is the identity map, so every point $q \in T^*\overset{\circ}{X}$ is either elliptic or glancing, depending on whether $q \notin \text{Char}(p)$ or $q \in \text{Char}(p)$.

The canonical local coordinates on T^*X will be denoted $(x, y, t, \xi, \eta, \tau)$, so one forms are $\alpha = \xi dx + \eta dy + \tau dt$. Let $(\rho, \vartheta) = (x, y, t, \xi, \eta, \tau)$ on T^*X near $\pi^{-1}(q)$, $q \in T^*\partial X$, and corresponding coordinates (y, t, η, τ) on a neighborhood \mathcal{U} of q in $T^*\partial X$. Consequently,

$$\mathcal{E} \cap \mathcal{U} = \{(y, t, \eta, \tau) | \tau^2 < \eta^2\},$$

$$\mathcal{G} \cap \mathcal{U} = \{(y, t, \eta, \tau) | \tau^2 = \eta^2\},$$

$$\mathcal{H} \cap \mathcal{U} = \{(y, t, \eta, \tau) | \tau^2 > \eta^2\}.$$

Let $\rho = \rho(s) = (x, y, t)(s)$, $\vartheta = \vartheta(s) = (\xi, \eta, \tau)(s)$ be a bicharacteristic of $p(\rho, \vartheta)$, i.e. such that (ρ, ϑ) satisfies

$$\frac{d\rho}{ds} = \frac{\partial p}{\partial \vartheta}, \quad \frac{d\vartheta}{ds} = -\frac{\partial p}{\partial \rho}, \quad p(\rho(0), \vartheta(0)) = 0. \quad (3.5)$$

We say that $(\rho(s), \vartheta(s))|_{s=0}$ on the boundary ∂X is a gliding point if it satisfies

$$x(\rho(0)) = 0, \quad \frac{d}{ds}x(\rho(0)) = 0, \quad \frac{d^2}{ds^2}x(\rho(0)) < 0. \quad (3.6)$$

This is equivalent to saying that $(\rho, \vartheta) \in T^*X \setminus o$ is a gliding point if

$$p(\rho, \vartheta) = 0, \quad \{p, x\}|_{(\rho, \vartheta)} = 0, \quad \{\{p, x\}, p\}|_{(\rho, \vartheta)} > 0. \quad (3.7)$$

The assumption on the domain Ω is equivalent to saying that there exists a point $(0, y_0, \xi_0, \eta_0)$ on $T^*\Omega$ through which there exists a bicharacteristic passing tangentially and having exactly second order contact with $\partial\Omega$. From (3.7) we see that this last condition writes

$$\tau^2 = (1 + xb(y))\eta^2|_{x=0}, \quad \{p, x\} = \frac{\partial p}{\partial \xi} = 2\xi_0 = 0, \quad (3.8)$$

$$\{\{p, x\}, p\} = \left\{ \frac{\partial p}{\partial \xi}, p \right\} = 2b(y_0)\eta_0^2 > 0. \quad (3.9)$$

We can suppose that $b(y_0) = 1$ and that for some small $0 < b_0 < 1/4$ we have $|b(y) - 1| \leq b_0$ for y in a fixed neighborhood \mathcal{N} of y_0 . Denote the gliding point (in $T^*\partial X$) by

$$\pi(\rho_0, \vartheta_0) = (y_0, 0, \eta_0, \tau_0 = -\eta_0). \quad (3.10)$$

Suppose without loss of generality that $y_0 = 0$, $\eta_0 = 1$, thus $\pi(\rho_0, \vartheta_0) = (0, 0, 1, -1) \in \mathcal{G}$. We define the semi-classical wave front set $WF_h(u)$ of a distribution u on \mathbb{R}^3 to be the complement of the set of points $(\rho = (x, y, t), \zeta = (\xi, \eta, \tau)) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus 0)$ for which there exists a symbol $a(\rho, \zeta) \in \mathcal{S}(\mathbb{R}^6)$ such that $a(\rho, \zeta) \neq 0$ and for all integers $m \geq 0$ the following holds

$$\|a(\rho, hD_\rho)u\|_{L^2} \leq c_m h^m.$$

3.1 Choice of an approximate solution

We look for an approximate solution to the equation (1.6) of the form

$$u_h(x, y, t) = \int_{\xi, \eta, \tau} e^{\frac{i}{h}(\theta + \zeta\xi + \frac{\xi^3}{3})} g_h d\xi d\eta d\tau, \quad (3.11)$$

where the phase functions $\theta(x, y, t, \eta, \tau)$, $\zeta(x, y, \eta, \tau)$ are real valued and homogeneous in (η, τ) of degree 1 and 2/3, respectively, and where we have, moreover,

$$\zeta_0(\eta, \tau) := \zeta(0, y, \eta, \tau) = -\frac{(\tau^2 - \eta^2)}{\eta^2} \eta^{2/3}. \quad (3.12)$$

Here g_h is a symbol to be determined in the next sections. The functions θ , ζ must solve an eikonal equation that we derive in what follows. We denote by $\langle \cdot, \cdot \rangle$ the symmetric bilinear form obtained by polarization of the second order homogeneous principal symbol p of the wave operator \square_g ,

$$\langle da, db \rangle = \partial_x a \partial_x b + (1 + xb(y)) \partial_y a \partial_y b - \partial_t a \partial_t b. \quad (3.13)$$

Applying the wave operator $h^2 \square_g$ to u_h , the main contribution becomes

$$\begin{aligned} (\partial_x \theta + \xi \partial_x \zeta)^2 + (1 + xb(y)) (\partial_y \theta + \xi \partial_y \zeta)^2 - (\partial_t \theta + \xi \partial_t \zeta)^2 = \\ = \langle d\theta, d\theta \rangle - 2\xi \langle d\theta, d\zeta \rangle + \xi^2 \langle d\zeta, d\zeta \rangle. \end{aligned} \quad (3.14)$$

In order to eliminate this term after integrations by parts in ξ we ask that the right hand side of (3.14) to be a nontrivial multiple of $\partial_\xi \Phi$, where we set

$$\Phi = \theta + \zeta\xi + \frac{\xi^3}{3}. \quad (3.15)$$

This is equivalent to determine θ , ζ solutions to

$$\begin{cases} \langle d\theta, d\theta \rangle - \zeta \langle d\zeta, d\zeta \rangle = 0, \\ \langle d\theta, d\zeta \rangle = 0. \end{cases} \quad (3.16)$$

The system (3.16) is a nonlinear system of partial differential equations, which is elliptic where $\zeta > 0$ (shadow region), hyperbolic where $\zeta < 0$ (illuminated region) and parabolic where $\zeta = 0$ (caustic curve or surface). It is crucial that there is a solution of the form

$$\phi^\pm = \theta \mp \frac{2}{3}(-\zeta)^{3/2} \quad (3.17)$$

with θ , ζ smooth. In terms of (3.17), the eikonal equation takes the form

$$p(x, y, t, d\phi^\pm) = 0 \quad (3.18)$$

by taking the sum and the difference of the equations (3.18). It is easy, by Hamilton-Jacobi theory, to find many smooth solutions to the eikonal equation (3.18). Solutions with the singularity (3.17) arise from solving the initial value problem for (3.18) off an initial surface which does not have the usual transversality condition, corresponding to the fact that there are bicharacteristics tangent to the boundary.

3.1.1 Geometric reduction

Let $X = \Omega \times \mathbb{R}$ as before. Let p and q be functions on T^*X with independent differentials at a point $(\rho, \vartheta) \in T^*X \setminus o$. We denote by P and Q the hypersurfaces defined by p and q , respectively.

Definition 3.1. We say that the hypersurfaces P, Q in the symplectic manifold T^*X are glancing surfaces at (ρ, ϑ) if

1. $\{p, q\}((\rho, \vartheta)) = 0$,
2. $\{p, \{p, q\}\}((\rho, \vartheta)) \neq 0$ and $\{q, \{q, p\}\}((\rho, \vartheta)) \neq 0$.

In our case we take q to be the defining function of the boundary $\partial\Omega$, therefor $q = x$, and p the symbol of the wave operator \square_g defined in (3.3). Precisely,

$$Q = \{q(x, y, t, \xi, \eta, \tau) = x = 0\}, \quad P = \{p = \xi^2 + (1 + xb(y))\eta^2 - \tau^2 = 0\}, \quad (3.19)$$

which are glancing at (ρ_0, ϑ_0) defined in (3.10). The nondegeneracy conditions in Definition 3.1 hold at a point (ρ, ϑ) with $\{p, q\} = 0$ if and only if $\partial\Omega$ is strictly convex at (ρ, ϑ) .

Remark 3.2. A model case of a pair of glancing surfaces is given by

$$Q_F = \{q_F(x, y, \xi, \eta, \tau) = x = 0\}, \quad P_F = \{p_F = \xi^2 + (1 + x)\eta^2 - \tau^2 = 0\}, \quad (3.20)$$

which have a second order intersection at the point

$$(\bar{\rho}_0, \bar{\vartheta}_0) := (0, y_0 = 0, t_0 = 0, 0, \eta_0 = 1, \tau_0 = -1) \in T^*X_F \setminus o.$$

This model case was studied in [15]. There is a deep geometrical reason underlying the similarity of the general gliding ray parametrice for (3.19) and the one for the model example (3.20), which will facilitate solution to the eikonal equation.

Theorem 3.3. *Let P and Q be two hypersurfaces in $T^*X \setminus o$ satisfying the glancing conditions in Definition 3.1 at $(\rho_0, \vartheta_0) \in P \cap Q \subset T^*X \setminus o$. Then there exist real functions θ and ζ which are C^∞ in a conic neighborhood \mathcal{U} of $(\rho_0, 1, -1) \in X \times \mathbb{R}^2$, are homogeneous of degrees one and two-thirds, respectively, and have the following properties*

- $\zeta_0 := \zeta|_{x=0} = -(\tau^2 - \eta^2)\eta^{-4/3}$ and $\partial\zeta|_{\partial X} > 0$ on $\mathcal{U} \cap \partial X \times \mathbb{R}^2$,
- $d_{y,t}(\partial_\eta\theta, \partial_\tau\theta)$ are linearly independent on \mathcal{U} ,
- the system (3.16) holds in $\zeta \leq 0$ and in Taylor series on ∂X .

Moreover, ζ is a defining function for the fold set denoted Σ . By translation invariance in time ζ is independent of t while the phase function θ is linear in the time variable.

Remark 3.4. Theorem 3.3 determines the phase functions θ , ζ , solutions to the eikonal equations (3.16). In what follows we shall use the construction of the model case (3.20) in order to determine a parametrice for the general case (3.19). This will be possible using the symplectomorphism generated by the restriction of the phase function θ to ∂X .

Remark 3.5. Notice in fact that if P and Q are the hypersurfaces in the symplectic space $T^*X \setminus o$ defined in (3.19) and glancing at $(\rho_0, \vartheta_0) \in T^*X \setminus o$, then there exists a canonical transformation

$$\chi : \Gamma \subset T^*X_F \setminus o \rightarrow T^*X \setminus o, \quad (3.21)$$

defined in a conic neighborhood Γ of $(\bar{\rho}_0, \bar{\vartheta}_0)$ and taking $(\bar{\rho}_0, \bar{\vartheta}_0)$ to (ρ_0, ϑ_0) and the model pair P_F and Q_F to P and Q . The fact that χ , which is symplectic, maps Q_F onto Q means that it defines a local canonical transformation from the quotient space of Q_F , modulo its Hamilton fibration, to the corresponding quotient space of Q , which is naturally identified as the cotangent space of the hypersurface

$$Q/\mathbb{R}H_q \simeq T^*\partial X.$$

Now, as we just said, on Q (and similarly on Q_F) the symplectic form gives a Hamilton foliation. Let this determine an equivalence relation \sim . Then $Q \cap P/\sim$ has the structure of a symplectic manifold with boundary, and it is naturally isomorphic to the closure of the "hyperbolic" set in $T^*\partial X$, the region over which real rays pass, and similarly $Q_F \cap P_F/\sim$.

Therefor, the restriction of χ to $T^*\partial X_F$, that we denote χ_∂ , is also a canonical transformation from a neighborhood $\gamma \subset T^*\partial X_F \setminus o$ of $\pi(\bar{\rho}_0, \bar{\vartheta}_0)$ to a neighborhood of $\pi(\rho_0, \vartheta_0) \in T^*\partial X \setminus o$,

$$\begin{aligned} \chi_\partial : \gamma \rightarrow T^*\partial X \setminus o, \quad \gamma \subset T^*\partial X_F \setminus o, \\ \gamma = \{(y, t, \eta, \tau) \in T^*\partial X_F \mid \exists \xi, (0, y, t, \xi, \eta, \tau) \in \Gamma\} \end{aligned}$$

defined in the hyperbolic region by

$$\chi_\partial^{-1} : (y, t, d_y\theta_0, d_t\theta_0) \rightarrow (d_\eta\theta_0, d_\tau\theta_0, \eta, \tau), \quad \chi_\partial^{-1}(\pi(\rho_0, \vartheta_0)) = \pi(\bar{\rho}_0, \bar{\vartheta}_0), \quad (3.22)$$

where $\theta_0 := \theta|_{\partial X}$ is the restriction to ∂X of the phase function θ introduced in Theorem 3.3.

Remark 3.6. The map χ_∂ has the important property that near $\pi(\bar{\rho}_0, \bar{\vartheta}_0)$, it conjugates the billiard ball map $\delta^\pm \subset (T^*\partial X \setminus o) \times (T^*\partial X \setminus o)$ to the normal form δ_F^\pm introduced in (3.32) in Section 3.2. Roughly speaking, the billiard ball maps are defined as follows: if (y, t, η, τ) is a hyperbolic point and if $\xi_+ > 0$ denotes the positive solution to $p(0, y, t, \xi, \eta, \tau) = 0$ we consider the integral curve $(\rho(s), \vartheta(s)) = \exp(sH_p)(0, y, t, \xi_+, \eta, \tau)$ of the Hamiltonian vector field of p starting at $(0, y, t, \xi_+, \eta, \tau)$; if it intersects transversally $T^*X|_{\partial X}$ at a time $s_1 > 0$ and lies entirely in T^*X for $s \in (0, s_1)$ we set $(0, y', t', \xi'_-, \eta', \tau') = \exp(s_1 H_p)(0, y, t, \xi_+, \eta, \tau)$ and define $\delta^+(y, t, \eta, \tau) := (y', t', \eta', \tau')$. Its local inverse is denoted δ^- . An interpolating Hamiltonian for the billiard ball maps δ^\pm is ζ_0 and we have $\delta^\pm(y, t, \eta, \tau) = \exp(\pm \frac{4}{3} H_{(-\zeta_0)^{3/2}})$.

3.1.2 Phase functions in the Friedlander's model

In the Friedlander's model of the half space $\Omega_F := \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}\}$ with Laplace operator defined by $\Delta_F := \partial_x^2 + (1+x)\partial_y^2$ studied in [15], the equation (3.18) has the solution

$$\phi_F^\pm = \theta_F \mp \frac{2}{3}(-\zeta_F)^{3/2}, \quad (3.23)$$

where

$$\theta_F(x, y, t, \eta, \tau) = y\eta + t\tau, \quad \zeta_F(x, y, \eta, \tau) = \left(x - \frac{\tau^2 - \eta^2}{\eta^2}\right)\eta^{2/3}, \quad (3.24)$$

as can be seen by direct computation. This solution serves very much as a guide to the general construction as we shall see in the next sections.

3.1.3 Phase functions associated to the special operator \square_g

Notice that in Section 2 we reduced the problem to the construction of a solution to the wave equation (1.6) in the special two-dimensional case of the half-space Ω defined in (3.1) together with the Laplace operator Δ_g given in (3.2). Using the simple form of the symbol of the associated wave operator \square_g we easily find

$$\zeta(x, y, \eta, \tau) = \left(xb(y) - \frac{\tau^2 - \eta^2}{\eta^2}\right)\eta^{2/3}, \quad (3.25)$$

where we recall that $b(0) = 1$ and that for y in a fixed neighborhood \mathcal{N} of 0 we have $|b(y) - 1| \leq b_0$ for some $0 < b_0 < 1/4$ small enough.

We do not determine θ explicitly, but we can manage to estimate its derivatives which will be useful later. Using the second eikonal equation in (3.16) and the fact that $\partial_x \zeta|_{x=0} = b(y)\eta^{2/3}$ with $b(y)$ close to 1 for y close to 0 and η on the support of Ψ we have

$$\partial_x \theta = -(1 + xb(y)) \frac{\partial_y \zeta}{\partial_x \zeta} \partial_y \theta = -(1 + xb(y)) \frac{x \partial_y b(y)}{b(y)} \partial_y \theta,$$

and introducing this in the first eikonal equation in (3.16) gives

$$(\partial_y \theta)^2 = \frac{\tau^2 + (\eta^2(1 + xb(y)) - \tau^2)(b^2(y) + (1 + xb(y))(x \partial_y b(y))^2)}{(1 + xb(y))(1 + (1 + xb(y))(x \partial_y b(y))^2/b^2(y))},$$

and we deduce

$$\partial_{\eta, y}^2 \theta_0 = \frac{\eta b^2(y)}{\sqrt{\tau^2 + (\eta^2 - \tau^2)b^2(y)}}, \quad \partial_{\eta, x}^2 \theta_0 = 0, \quad (3.26)$$

$$\partial_{\tau, y}^2 \theta_0 = \frac{\tau(1 - b^2(y))}{\sqrt{\tau^2 + (\eta^2 - \tau^2)b^2(y)}}, \quad \partial_{\tau, x}^2 \theta_0 = 0. \quad (3.27)$$

From (3.27) it follows that for $y \in \mathcal{N}$ and $0 \leq x \lesssim a$ we have $|\partial_{\tau, y}^2 \theta| \leq b_0$.

3.2 A model operator

In [15] we proved Theorem 1.2 in the case of a two-dimensional, strictly convex domain $\Omega_F = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}\}$ with Laplace operator given by

$$\square_F = \partial_t^2 - \partial_x^2 - (1+x)\partial_y^2. \quad (3.28)$$

Remark 3.7. In this paper we want to construct examples for general manifolds with a gliding ray, but the heart of the matter is well illustrated by the particular example studied in [15] which will generalize using Melrose's equivalence of glancing hypersurfaces theorem. Therefor we start by recalling the main steps in the construction of [15] and then use the particular solution of the model case to define an approximate solution for the more general domain described in Section 3.

Let $X_F = \Omega_F \times \mathbb{R}$ and let $p_F \in C^\infty(T^*X_F \setminus o)$ denote the homogeneous symbol of the model wave operator \square_F , $p_F(x, y, t, \xi, \eta, \tau) = \xi^2 + (1+x)\eta^2 - \tau^2$. Consider the wave equation

$$\begin{cases} \partial_t^2 v - \partial_x^2 v - (1+x)\partial_y^2 v = 0 \\ v|_{\partial\Omega_F \times [0,1]} = 0. \end{cases} \quad (3.29)$$

We have chosen in [15] an approximate solution to the equation (3.29) of the form

$$\begin{aligned} u_{F,h}(x, y, t) = \frac{1}{h} \int_{\xi, \eta, \tau} e^{\frac{i}{h}(y\eta + t\tau + (x+1 - \frac{\tau^2}{\eta^2})\eta^{2/3}\xi + \frac{\xi^3}{3})} g_F(t, \xi, \eta, h) \times \\ \times \Psi(\eta) \delta\left(\frac{\tau}{h} = -\frac{\eta}{h}(1+a)^{1/2}\right) d\xi d\eta d\tau \end{aligned} \quad (3.30)$$

where the symbol g_F is a smooth function independent of x, y , δ denotes the Dirac and $\Psi \in C_0^\infty(\mathbb{R}^*)$ is supported for η in a small neighborhood of 1, $0 \leq \Psi(\eta) \leq 1$, $\Psi(\eta) = 1$ for η near 1 and where $a = h^\alpha$ for some $\alpha \in (0, \frac{2}{3})$.

Remark 3.8. Notice that the case $\alpha = 0$ would correspond to a data localized away from the boundary: in this situation the finite speed of propagation of the wave flow yields the same Strichartz estimates as in the flat case. Sharp Strichartz estimates follow even if the data is very close to the boundary but the wave is transversal, in which case there is only one reflection at the boundary. On the other hand, the case $\alpha = 2/3$ corresponds to considering highly-multiply reflected waves localized in a $h^{2/3}$ -neighborhood of the boundary which propagate along the boundary (an example of such waves is given by the gallery modes), when one should manage to prove the same result for general operators using the parametrix introduced by Eskin [8] together with the approach in [15, Theorem 1.8]. Notice that in [15, Thm.1.8] we proved sharp Strichartz estimates for data in the space of the gallery modes, which are eigenfunctions of the Laplace operator which concentrate their energy near the boundary and contribute in this way to large $L^r(\Omega)$ norms.

Remark 3.9. This specific choice of the solution to (3.29) is motivated by the following: if $v(t, x, y)$ satisfies (3.29), then taking the Fourier transform \hat{v} in time and tangential variable it follows that \hat{v} can be expressed using Airy's function $Ai((x+1 - \frac{\tau^2}{\eta^2})\eta^{2/3})$ and its derivative.

Remark 3.10. The particular manifold studied in [15] is one for which the eigenmodes are explicitly expressed in terms of Airy's function and can be written as

$$e^{iy\eta} Ai(|\eta|^{2/3}x - \omega_k),$$

where the domain is Ω_F and the Dirichlet condition dictates that $-\omega_k$ be the zeroes of the Airy function $Ai(-\omega_k) = 0$. Rewriting the mode in the form

$$e^{iy\eta} Ai(|\eta|^{2/3}(x - a)),$$

the eigenvalue is $|\eta|(1+a)^{1/2}$, which means that such a wave moves with velocity $(1+a)^{1/2}$ in the y direction. To construct wave-fronts that do not disperse requires superimposing waves with the same value of a . If one ignores the boundary condition for the moment, the superposition of such waves over a range $\eta \simeq 1/h$ would give, as can be seen by the asymptotic expansions of the Airy function, a solution living in an h -depending neighborhood of the cusp

$$y - (1+a)^{1/2}t = \pm|a - x|^{3/2}, \quad x \in [0, a].$$

The goal is to construct a similar solution that satisfies boundary conditions at $x = 0$, while taking a as small as possible depending on h . Rather than attempt to deal with the zeros of the Airy function, the boundary conditions are met by taking a superposition of localized cusp solutions, each term in the sum being chosen to cancel off the boundary values of the previous one. The relation between amplitudes in the sum is dictated by the billiard ball maps.

We now concentrate on the approximate solution (3.30) and describe its properties:

After the change of variables $\xi = \eta^{1/3}s$, the phase function Φ_F of (3.30) becomes homogeneous of degree one in (η, τ) and its associated Lagrangian manifold writes

$$\Lambda_{\Phi_F} := \{(x, y, t, s, \eta, \tau) | s^2 + (x + 1 - \frac{\tau^2}{\eta^2}) = 0, \partial_\eta \Phi_F = 0, \partial_\tau \Phi_F = 0\} \subset T^*X_F \setminus o. \quad (3.31)$$

Let $\text{pr}_F : \Lambda_{\Phi_F} \rightarrow X_F$ denote the natural projection and let Σ_F be the set of singular points of pr_F . The points where the Jacobian of $d(\text{pr}_F)$ vanishes lie over the caustic set, thus the fold set is given by $\Sigma_F = \{s = 0\}$ and the caustic is defined by $\text{pr}_F(\Sigma_F) = \{x + (1 - \frac{\tau^2}{\eta^2}) = 0\}$.

If the symbol is chosen such that on the boundary to be localized away from the caustic set $\text{pr}_F(\Sigma_F)$, $\Lambda_{\Phi_F|_{\partial X_F}}$ is the graph of a pair of canonical transformations, the billiard ball maps δ_F^\pm . Roughly speaking, the billiard ball maps $\delta_F^\pm : T^*\partial X_F \rightarrow T^*\partial X_F$, defined on the hyperbolic region \mathcal{H} , continuous up to the boundary, smooth in the interior, are defined at a point of $T^*\partial X_F$ by taking the two rays that lie over this point, in the variety $\text{Char}(p_F)$, and following the null bicharacteristic through these points until you pass over ∂X_F again, projecting such a point onto $T^*\partial X_F$ (a gliding point being "a diffractive point viewed from the other side of the boundary", there is no bicharacteristic in $T^*\partial X_F$ through it, but in any neighborhood of a gliding point there are hyperbolic points).

In the model case the analysis is simplified by the presence of a large *commutative* group of symmetries, the translations in (y, t) , and the billiard ball maps have specific formulas

$$\delta_F^\pm(y, t, \eta, \tau) = \left(y \pm 4\left(\frac{\tau^2}{\eta^2} - 1\right)^{1/2} \pm \frac{8}{3}\left(\frac{\tau^2}{\eta^2} - 1\right)^{3/2}, t \mp 4\left(\frac{\tau^2}{\eta^2} - 1\right)^{1/2}\frac{\tau}{\eta}, \eta, \tau \right). \quad (3.32)$$

Away from $\text{pr}_F(\Sigma_F)$ these maps have no recurrent points, since under iteration $t((\delta_F^\pm)^n) \rightarrow \pm\infty$ as $n \rightarrow \infty$. The composite relation with n factors $\Lambda_{\Phi_F|_{x=0}} \circ \dots \circ \Lambda_{\Phi_F|_{x=0}}$ has, always away from $\text{pr}_F(\Sigma_F)$, $n + 1$ components, obtained namely using the graphs of the iterates $(\delta_F^\pm)^n$,

$$(\delta_F^\pm)^n(y, t, \eta, \tau) = \left(y \pm 4n\left(\frac{\tau^2}{\eta^2} - 1\right)^{1/2} \pm \frac{8}{3}n\left(\frac{\tau^2}{\eta^2} - 1\right)^{3/2}, t \mp 4n\left(\frac{\tau^2}{\eta^2} - 1\right)^{1/2}\frac{\tau}{\eta}, \eta, \tau \right). \quad (3.33)$$

All these graphs, of the powers of δ_F^\pm , are disjoint away from $\text{pr}_F(\Sigma_F)$ and locally finite, in the sense that only a finite number of components meet any compact subset of $\{\frac{\tau^2}{\eta^2} - 1 > 0\}$. Since $(\delta_F^\pm)^n$ are all immersed canonical relations, it is necessary to find a parametrization of each to get at least microlocal representations of the associated Fourier integral operators. We see that a parametrization of $\Lambda_{\Phi_F|_{x=0}}$ is

$$y\eta + t\tau + \frac{4}{3}\eta\left(\frac{\tau^2}{\eta^2} - 1\right)^{3/2},$$

thus the iterated Lagrangians $(\Lambda_{\Phi_F|_{x=0}})^{on}$ are parametrized by

$$y\eta + t\tau + \frac{4}{3}n\eta\left(\frac{\tau^2}{\eta^2} - 1\right)^{3/2},$$

and the corresponding phase functions associated to $(\Lambda_{\Phi_F})^{on}$ will be given by

$$\Phi_F^n = \Phi_F + \frac{4}{3}n\eta\left(\frac{\tau^2}{\eta^2} - 1\right)^{3/2}. \quad (3.34)$$

The domain Ω_F being strictly convex, at each point on the boundary there is a bicharacteristic that intersects the boundary $\partial\Omega_F$ tangentially having exactly second order contact with the boundary and remaining in the complement of $\bar{\Omega}_F$. In [15] we considered $(\bar{\rho}_0, \bar{\vartheta}_0) \in T^*X_F$ to be such a point and assumed without loss of generality that

$$\pi(\bar{\rho}_0, \bar{\vartheta}_0) = (0, 0, 1, -1) \in T^*\partial X_F,$$

where $\pi : T^*X_F \rightarrow T^*X_F$ is the natural, non-injective inclusion. In a neighborhood of $(\bar{\rho}_0, \bar{\vartheta}_0)$ we then constructed a sequence $u_{F,h}^n$ of approximate solutions to (3.29) of the form

$$u_{F,h}^n(x, y, t) = \frac{1}{h} \int_{\xi, \eta, \tau} e^{i\Phi_F^n} g_F^n(t, \xi, \eta, h) \Psi(\eta) \delta\left(\frac{\tau}{h} = -\frac{\eta}{h}(1+a)^{1/2}\right) d\xi d\eta d\tau,$$

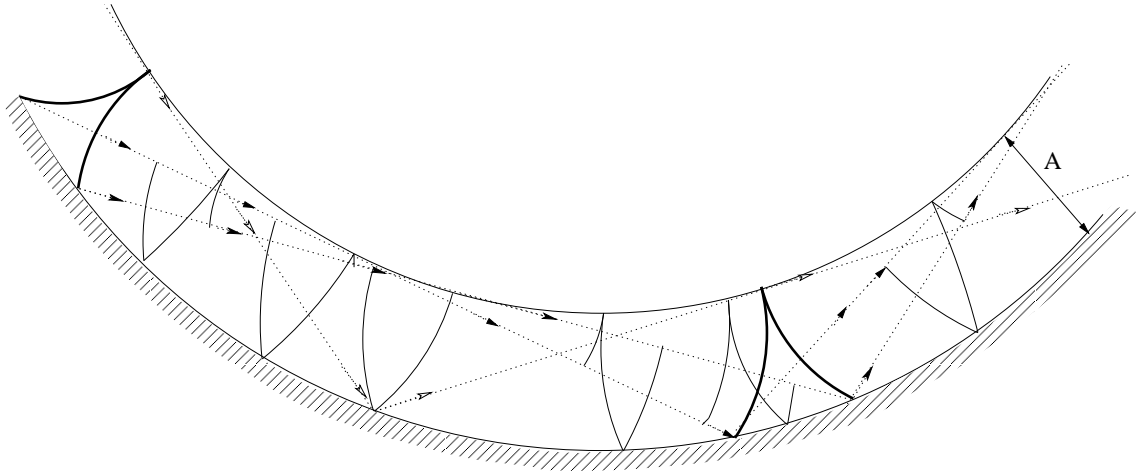


Figure 1: Propagation of the cusp.

where Φ_F^n were the phase functions defined in (3.34) such that $\Lambda_{\Phi_F^n} = (\Lambda_{\Phi_F})^{on}$ and where the symbols g_F^n have been chosen such that on the boundary the Dirichlet condition to be satisfied by

$$U_{F,h}(x, y, t) = \sum_{n=0}^N u_{F,h}^n(x, y, t).$$

In fact on the boundary $\{x = 0\}$ the phases have two critical, non-degenerate points, thus each $u_{F,h}^n$ writes as a sum of two trace operators, $\text{Tr}_{\pm}(u_{F,h}^n)$, localized for $y - (1+a)^{1/2}t$ near $\pm \frac{2}{3}na^{3/2}$, respectively, and in order to obtain a contribution $O_{L^2}(h^\infty)$ on the boundary we had to define the symbol g_F^{n+1} such that $\text{Tr}_-(g_F^n) + \text{Tr}_+(g_F^{n+1}) = O_{L^2}(h^\infty)$. This was possible by Egorov theorem, as long as $N \ll a^{3/2}/h$. This last condition, together with the assumption of finite time $T = 1$, allowed to estimate the number of iterations N and the "distance to the gliding region" a . We finally found $a = h^{\frac{1-\epsilon}{2}}$ for $\epsilon > 0$ as small as we want.

The motivation of this construction came from the fact that near the caustic set $\text{pr}_F(\Sigma_F)$ we noticed a singularity of cusp type for which we were able to estimate the $L^r(\Omega)$ norm. Localizing the symbols at $t = 0$ in a small, fixed, neighborhood of the caustic set, we then proved that the respective "pieces of cusps" propagate until they reach the boundary but that short after that their contribution becomes $O_{L^2}(h^\infty)$, since as t increases, one quickly quits a neighborhood of the Lagrangian $\Lambda_{\Phi_F^n}$ which contains the semi-classical wave front set $WF_h(u_{F,h}^n)$ of $u_{F,h}^n$. This argument is valid for all $u_{F,h}^n$, thus the approximate solutions $u_{F,h}^n$ have almost disjoint essential supports and the $L^q([0, 1], L^r(\Omega))$ norms of the sum U_h can be bounded from below by the sum of the $L^r(\Omega)$ norms of each $u_{F,h}^n$ on small intervals of time of size \sqrt{a} .

3.2.1 Construction of an approximate solution in the model case

In this section we recall the construction of the symbols for the model case of the wave operator \square_F defined by (3.28). In Section 3.2 we introduced the approximate solution used in [15]. Using the Dirac distribution we easily get

$$u_{F,h}(x, y, t) = \int_{\xi, \eta} e^{\frac{i}{h}(\theta_F + \zeta_F \xi + \frac{\xi^3}{3})} g_F(t, \xi, \eta, h) d\xi d\eta, \quad (3.35)$$

where the symbol g_F is a smooth function supported for η in a small neighborhood of 1 and where $a = h^\delta$, $0 < \delta < 2/3$ will be chosen later. Recall from (3.24) that the model phase functions are given by

$$\theta_F(x, y, t, \eta, \tau) = y\eta + t\tau, \quad \zeta_F(x, y, \eta, \tau) = \left(x - \frac{\tau^2 - \eta^2}{\eta^2}\right)\eta^{2/3},$$

and on the support of the symbol of (3.30) we have $\tau = -(1+a)^{1/2}\eta$. Applying the wave operator \square_F to $u_{F,h}$ and integrating by parts with respect to ξ using the eikonal system (3.16) gives

$$\begin{aligned} \square_F u_{F,h}(x, y, t) &= \int e^{\frac{i}{h}(\theta_F + \zeta_F \xi + \frac{\xi^3}{3})} \left(\partial_t^2 g_F + \frac{i}{h} 2\tau \partial_t g_F + \frac{1}{h^2} \eta^{4/3} ((x-A)\eta^{2/3} + \xi^2) g_F \right) d\xi d\eta \\ &= \int e^{\frac{i}{h}(\theta_F + \zeta_F \xi + \frac{\xi^3}{3})} \left(\partial_t^2 g_F + \frac{i}{h} (\eta^{4/3} \partial_\xi g_F + 2\tau \partial_t g_F) \right) d\xi d\eta. \end{aligned} \quad (3.36)$$

Let $\Phi_F = \theta_F + \zeta_F \xi + \frac{\xi^3}{3}$. After the change of variables $\xi = \eta^{1/3} s$, Φ_F becomes homogeneous of degree 1 in η and the transport equation in (3.36) becomes independent of the variable η .

3.2.2 Choice of the symbols and main properties

Definition 3.11. Let $\lambda \geq 1$. For a given compact $K \subset \mathbb{R}$ we define the space $\mathcal{S}_K(\lambda)$, consisting of functions $\varrho(z, \lambda) \in C^\infty(\mathbb{R})$ which satisfy

1. $\sup_{z \in \mathbb{R}, \lambda \geq 1} |\partial_z^\alpha \varrho(z, \lambda)| \leq C_\alpha$, where C_α are constants independent of λ ,
2. If $\psi(z) \in C_0^\infty$ is a smooth function equal to 1 in a neighborhood of K , $0 \leq \psi \leq 1$ then $(1 - \psi)\varrho \in O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty})$.

Here $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decreasing functions.

An example of function $\varrho(z, \lambda) \in \mathcal{S}_K(\lambda)$, $K \subset \mathbb{R}$ is the following: let $k(z)$ be the smooth function on \mathbb{R} defined by

$$k(z) = \begin{cases} C \exp(-1/(1-|z|^2)), & \text{if } |z| < 1, \\ 0, & \text{if } |z| \geq 1, \end{cases}$$

where C is a constant chosen such that $\int_{\mathbb{R}} k(z) dz = 1$. Define a mollifier $k_\lambda(z) := \lambda k(\lambda z)$ and let $\tilde{\varrho} \in C_0^\infty(K)$ be a smooth function with compact support included in K . If we set $\varrho(z, \lambda) = (\tilde{\varrho} * k_\lambda)(z)$, then one can easily check that ϱ belongs to $\mathcal{S}_K(\lambda)$.

Let $\epsilon > 0$ be fixed, small enough and let $a = h^{(1-\epsilon)/2}$. In what follows we continue to write a instead of replacing it by the respective power of h to keep the notations in [15].

We also let $\lambda = \lambda(h) = a^{3/2}/h$ and define $K_0 = [-c_0, c_0]$ for some small $0 < c_0 < 1$ and let $\varrho(\cdot, \lambda) \in \mathcal{S}_{K_0}(\lambda)$ be the smooth function defined in Definition 3.11. We define

$$g_F(t, \xi, \eta, h) = \varrho\left(\frac{t + 2\xi(1+a)^{1/2}\eta^{-1/3}}{2(1+a)^{1/2}a^{1/2}}, \lambda\right)\Psi(\eta), \quad (3.37)$$

where $\Psi \in C_0^\infty(\mathbb{R}^*)$ is supported for η in a small neighborhood of 1 and $0 \leq \Psi(\eta) \leq 1$.

Proposition 3.12. ([15, Prop.6]) *On the boundary $u_{F,h}$ writes (modulo $O_{L^2}(\lambda^{-\infty})$) as a sum of two trace operators,*

$$u_{F,h}(0, y, t) = \sum_{\pm} \text{Tr}_{\pm}(u_{F,h})(y, t), \quad (3.38)$$

where

$$\begin{aligned} \text{Tr}_{\pm}(u_{F,h})(y, t) := & h^{1/3} \int_{\eta} e^{\frac{i}{h}(y\eta - t(1+a)^{1/2}\eta \mp \frac{2}{3}a^{3/2}\eta)} \Psi(\eta) (\eta\lambda)^{-1/6} \times \\ & \times I_{\pm}(\varrho(\cdot, \lambda))_{\eta}\left(\frac{t}{2(1+a)^{1/2}a^{1/2}}, \lambda\right) d\eta, \end{aligned} \quad (3.39)$$

where $I_{\pm}(\varrho(\cdot, \lambda))_{\eta}(z, \lambda)$ are given by

$$I_{\pm}(\varrho(\cdot, \lambda))_{\eta}(z, \lambda) = e^{\pm i\pi/2 - i\pi/4} \frac{\eta\lambda}{2\pi} \int_w e^{i\eta\lambda(w(z-z') \mp \frac{2}{3}((1-w)^{3/2}-1))} \kappa(w) a_{\pm}(w, \eta\lambda) \varrho(z', \lambda) dw. \quad (3.40)$$

Here κ is a smooth function supported for w as close as we want to 0 and

$$a_{\pm}(w, \eta, \lambda) \simeq (1-w)^{-1/4} \sum_{j \geq 0} a_{\pm, j} (-1)^{-j/2} (1-w)^{-3j/2} (\eta\lambda)^{-j}$$

are the asymptotic expansions of the symbols of the Airy functions. Moreover, the symbols $k(w)a_{\pm}(w, \eta\lambda)$ are elliptic at $w = 0$.

Proposition 3.13. ([15, Lemma 4]) *Let $p \in \mathbb{Z}$ and $K_p = [-c_0 + p, c_0 + p]$. Then for some small $0 < c_0 < 1$ and η belonging to the support of Ψ we have*

$$I_{\pm, \eta} : \mathcal{S}_{K_p}(\lambda) \rightarrow \mathcal{S}_{K_{p \mp 1}}(\lambda).$$

Proposition 3.14. ([15, Chp.3.3]) *For η belonging to the support of Ψ the operators $J_{\pm, \eta}$ defined for some $\tilde{\lambda} \geq 1$ and $\check{\varrho} \in \mathcal{S}_{K_{\mp 1}}(\tilde{\lambda})$ by*

$$\begin{aligned} J_{\pm}(\check{\varrho}(\cdot, \tilde{\lambda}))_{\eta}(z', \lambda) := \\ e^{\mp i\pi/2 + i\pi/4} \frac{\eta\lambda}{2\pi} \int e^{i\eta\lambda((z'-z)w \pm \frac{2}{3}((1-w)^{3/2}-1))} b_{\pm}(w, \eta\lambda) \check{\varrho}(z, \tilde{\lambda}) dz dw \end{aligned} \quad (3.41)$$

where $b_{\pm}(w, \eta\lambda) = \frac{k(w)}{a_{\pm}(w, \eta\lambda)}$ are asymptotic expansions in $(\eta\lambda)^{-1}$ satisfy

$$\begin{aligned}\check{\varrho}(\cdot, \tilde{\lambda}) &= I_{\pm}(J_{\pm}(\check{\varrho}(\cdot, \tilde{\lambda}))_{\eta}(\cdot, \lambda))_{\eta}(\cdot, \lambda) + O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}) + O_{\mathcal{S}(\mathbb{R})}(\tilde{\lambda}^{-\infty}), \\ \varrho(\cdot, \tilde{\lambda}) &= J_{\pm}(I_{\pm}(\varrho(\cdot, \tilde{\lambda}))_{\eta}(\cdot, \lambda))_{\eta}(\cdot, \lambda) + O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}) + O_{\mathcal{S}(\mathbb{R})}(\tilde{\lambda}^{-\infty}).\end{aligned}$$

3.2.3 Iteration

In this section we iterate the preceding construction a sufficiently large number of times, such that the sum of the iterates satisfies the Dirichlet boundary condition in finite time. Here we just recall the main results of [15, Section 3.3.1].

Proposition 3.15. ([15, Prop.7]) *Let $N \simeq \lambda h^{\epsilon}$ and $1 \leq n \leq N$. For $\eta \in \text{supp}(\Psi)$ we have*

$$(J_{+}(\cdot)_{\eta} \circ I_{-}(\cdot)_{\eta})^{\circ n} : \mathcal{S}_{K_0}(\lambda) \rightarrow \mathcal{S}_{K_{2n}}(\lambda/n) \quad \text{uniformly in } n. \quad (3.42)$$

Notice that since $\lambda/n \geq h^{-\epsilon} \gg 1$, then $O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}) = O_{\mathcal{S}(\mathbb{R})}((\lambda/n)^{-\infty}) = O_{\mathcal{S}(\mathbb{R})}(h^{\infty})$.

Moreover, if T_k denotes the translation operator which to a given function $\varrho(z)$ associates $\varrho(z+k)$ then the operator defined above writes as a convolution

$$(T_1 \circ J_{+}(\cdot)_{\eta} \circ I_{-}(\cdot)_{\eta} \circ T_1)^{\circ n} = (F_{\eta\lambda})^{*n},$$

where

$$(F_{\eta\lambda})^{*n}(z) = \frac{\eta\lambda}{2\pi} \int_w e^{i\eta\lambda(wz+n(2w+\frac{4}{3}((1-w)^{3/2}-1)))} \left(\kappa(w)a_{+}(w, \eta\lambda)b_{-}(w, \eta\lambda) \right)^n dw. \quad (3.43)$$

Definition 3.16. Let $\varrho(\cdot, \lambda) \in \mathcal{S}_{K_0}(\lambda)$ and $\eta \in \text{supp}(\Psi)$. For $1 \leq n \leq N$, $N \simeq \lambda h^{\epsilon}$ set

$$\varrho^n(z, \eta, \lambda) := (-1)^n (T_1 \circ J_{+}(\cdot)_{\eta} \circ I_{-}(\cdot)_{\eta} \circ T_1)^n(\varrho(\cdot, \lambda))(z), \quad \varrho^0(z, \eta, \lambda) = \varrho(z, \lambda).$$

From Proposition 3.15 it follows that $\varrho^n(z, \eta, \lambda) \in \mathcal{S}_{K_0}(\lambda/n)$.

Definition 3.17. For $0 \leq n \leq N$ with $N \simeq \lambda h^{\epsilon}$ also define

$$\begin{aligned}u_{F,h}^n(x, y, t) &:= \int_{s,\eta} e^{\frac{i}{h}\Phi_F^n(x,y,t,\eta^{1/3}s,\eta, -(1+a)^{1/2}\eta)} \times \\ &\quad \times \varrho^n\left(\frac{t+2\eta^{1/3}s(1+a)^{1/2}}{2(1+a)^{1/2}a^{1/2}} - 2n, \eta, \lambda\right) \Psi(\eta) \eta^{1/3} ds d\eta, \quad (3.44)\end{aligned}$$

where

$$\Phi_F^n(x, y, t, \xi, \eta, \tau) = \theta_F(x, y, t, \eta, \tau) + \xi \zeta_F(x, y, \eta, \tau) + \frac{\xi^3}{3} + \frac{4}{3}n(-\zeta_{F,0})^{3/2}(\eta, \tau).$$

Proposition 3.18. ([15, Prop.8]) *This choice of the symbols gives for all $0 \leq n \leq N-1$*

$$\text{Tr}_{-}(u_{F,h}^n)(y, t) + \text{Tr}_{+}(u_{F,h}^{n+1})(y, t) = O_{L^2}(\lambda^{-\infty}). \quad (3.45)$$

Proof. The equality (3.45) follows from the relation

$$e^{i\pi/2} I_-(T_1(\varrho^n(\cdot, \eta, \lambda)))_\eta + e^{-i\pi/2} \circ I_+(T_{-1}(\varrho^{n+1}(\cdot, \eta, \lambda)))_\eta = O_{S(\mathbb{R})}(\lambda^{-\infty}),$$

together with the fact that the operators $I_{\pm, \eta}$ are of convolution type so they commute with translations. \square

Proposition 3.19. ([15, Lemma 4.3]) *If $0 \leq n \leq N$, $u_{F, h}^n(\cdot, t)$ is essentially supported for t in the interval*

$$[4na^{1/2} - 2a^{1/2}(1 + c_0), 4na^{1/2} + 2a^{1/2}(1 + c_0)](1 + a)^{1/2},$$

meaning that for t outside this interval its contribution is $O_{L^2}(h^\infty)$.

3.3 Construction of an approximate solution to (1.6)

In this section we construct an approximate solution to (1.6) satisfying Theorem 2.1. It will be essentially based on the model construction recalled in Section 3.2 together with the Melrose's equivalence of glancing surfaces Theorem 3.3.

Recall the form of the approximate solution u_h we considered in (3.11), where the functions θ and ζ are those given in Theorem 3.3 and where the symbol has to be determined. Away from the caustic set defined by the locus where $\xi = \zeta = 0$, there are two main contributions in u_h denoted $u_{h, \pm}$ with phase functions $\phi^\pm = \theta \mp \frac{2}{3}(-\zeta)^{3/2}$ given in (3.17). These are the phases corresponding to the Airy functions $A_\pm(\zeta)$ and one can think (at least away from the boundary $x = 0$) of the part $u_{h, -}$ corresponding to $A_-(\zeta)$ as a free wave or the "incoming piece": after hitting the boundary it gives rise to the outgoing one which corresponds to $A_+(\zeta) \frac{A_-(\zeta_0)}{A_+(\zeta_0)}$ with phase $-\frac{2}{3}(-\zeta)^{3/2} + \frac{4}{3}(-\zeta_0)^{3/2}$. The oscillatory part $\frac{4}{3}(-\zeta_0)^{3/2}$ corresponds to the billiard ball map shift corresponding to reflection.

Therefore, inspired from the Friedlander's case, in order to obtain a solution to (1.6) satisfying the Dirichlet boundary condition we construct a superposition u_h^n of localized cusp solutions to (1.6), where u_h^0 is of the form (3.11), each term in the sum being chosen to cancel off the boundary values of the previous one. Precisely, we take u^n of the form

$$u_h^n(x, y, t) = \int_{\xi, \eta, \tau} e^{\frac{i}{h}\Phi^n(x, y, t, \xi, \eta, \tau)} g_h^n d\xi d\eta d\tau, \quad (3.46)$$

for some symbols g_h^n suitably chosen and where the choice of the phase functions

$$\Phi^n(x, y, t, \xi, \eta, \tau) := \theta(x, y, t, \eta, \tau) + \xi\zeta(x, y, \eta, \tau) + \frac{\xi^3}{3} + \frac{4}{3}n(-\zeta_0)^{3/2}(\eta, \tau) \quad (3.47)$$

is dictated by the billiard ball maps, so that on the boundary the following holds:

$$u_{h, -}^n + u_{h, +}^{n+1} = O_{L^2}(h^\infty). \quad (3.48)$$

3.3.1 The boundary condition

We have to determine the symbols g_h^n in (3.46) such that u_h^n be approximate solution to (1.6) in a sense to be precise. We start by defining their restriction to the boundary, using the condition (3.48).

We consider an operator J

$$J(f)(y, t) := \frac{1}{(2\pi h)^2} \int_{\eta, \tau} e^{\frac{i}{h}\theta_0(y, t, \eta, \tau)} d_h(y, \eta, \tau) \widehat{f}(\eta/h, \tau/h) d\eta d\tau, \quad (3.49)$$

where $d_h(y, \eta, \tau) = d(y, \eta/h, \tau/h)$ is an elliptic symbol of order 0 and type $(1, 0)$, compactly supported in a conic neighborhood of the glancing point $\pi(\rho_0, \vartheta_0)$ and where θ_0 is the restriction to the boundary of the phase function θ introduced in Theorem 3.3.

This is an elliptic Fourier integral operator in a neighborhood of $(\pi(\bar{\rho}_0, \bar{\vartheta}_0), \pi(\rho_0, \vartheta_0))$, with canonical relation χ_∂ given by the symplectomorphism generated by θ_0 and such that $\chi_\partial(\pi(\bar{\rho}_0, \bar{\vartheta}_0)) = \pi(\rho_0, \vartheta_0)$ (see the remarks following Theorem 3.3).

Remark 3.20. Using Proposition 3.18 it is clear that if we define $u_{h,\pm}^n$ on the boundary as the composition $J \circ Tr_\pm(u_{F,h}^n)$ of the elliptic FIO with the trace operators $Tr_\pm(u_{F,h}^n)$, the condition (3.48) will be fulfilled. We first compute $J \circ Tr_\pm(u_{F,h}^n)$. We keep the notations of Section 3.2.

Lemma 3.21. *On the boundary $J \circ Tr_\pm(u_{F,h}^n)$ writes*

$$J \circ Tr_\pm(u_{F,h}^n)(y, t) = h^{1/3} \int e^{\frac{i}{h}(\theta_0(y, t, \eta, -\eta(1+a)^{1/2}) + \frac{2}{3}(2n\mp 1)(-\zeta_0)^{3/2}(\eta, -\eta(1+a)^{1/2}))} \lambda^{-1/6} \eta^{-1/2} \\ \times I_\pm(g_h^n(\cdot, y, \eta))_\eta \left(\frac{\partial_\tau \theta_0}{\partial_\tau \zeta_0 (-\zeta_0)^{1/2}}(y, t, \eta, -\eta(1+a)^{1/2}) - 2n, \lambda \right) d\eta, \quad (3.50)$$

where

$$g_h^n(z, y, \eta) \simeq \Psi(\eta) \eta^{1/3} \left(\sum_{k \geq 0} h^{k/2} a^{-k/2} \mu_k(y, \eta, h) \partial^k \varrho^n(z, \eta, \lambda) \right), \quad (3.51)$$

where $\mu_k(y, \eta, h)$ are symbols of order 0 and type $(1, 0)$ independent of n . Moreover, if $\eta \in \text{supp}(\Psi)$ and $1 \leq n \leq N \simeq \lambda h^\epsilon$ for some small $\epsilon > 0$ then $g_h^n(\cdot, y, \eta) \in \mathcal{S}_{K_0}(\lambda/n)$.

We recall that $\lambda = h^{3\alpha/2-1}$. Recall also

$$\partial_\tau \zeta_0(\eta, -(1+a)^{1/2}\eta) = 2(1+a)^{1/2}\eta^{-1/3}, \quad (-\zeta_0)^{1/2}(\eta, -(1+a)^{1/2}\eta) = a^{1/2}\eta^{1/3}.$$

Proof. We compute explicitly the restriction of each $u_{F,h}^n$ to $x = 0$, for $0 \leq n \leq N$.

$$u_{F,h}^n(0, \bar{y}, \bar{t}) = \int_{\xi, \bar{\eta}} e^{\frac{i}{h}(\bar{y}\bar{\eta} - \bar{t}(1+a)^{1/2}\bar{\eta} + \xi(x-a) + \frac{\xi^3}{3} + \frac{4}{3}na^{3/2}\bar{\eta})} \times \\ \times \Psi(\bar{\eta}) \varrho^n \left(\frac{\bar{t} + 2\xi(1+a)^{1/2}\bar{\eta}^{-1/3}}{2(1+a)^{1/2}a^{1/2}} - 2n, \bar{\eta}, \lambda \right) d\xi d\bar{\eta}. \quad (3.52)$$

Taking $\xi = a^{1/2}\eta^{1/3}v$, the integral in ξ in (3.52) becomes, modulo $O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty})$

$$\begin{aligned} & \Psi(\bar{\eta})a^{1/2}\bar{\eta}^{1/3}\frac{\bar{\eta}\lambda}{2\pi}\int_{z,w}\int_v e^{i\bar{\eta}\lambda(\frac{v^3}{3}-v(1-w))}dve^{i\bar{\eta}\lambda w(\frac{\bar{t}}{2(1+a)^{1/2}a^{1/2}}-2n-z)}\varrho^n(z,\bar{\eta},\lambda)dwdz = \\ & \Psi(\bar{\eta})h^{1/3}\frac{\bar{\eta}\lambda}{2\pi}\int_{z,w} Ai(-(\bar{\eta}\lambda)^{2/3}(1-w))e^{i\bar{\eta}\lambda w(\frac{\bar{t}}{2(1+a)^{1/2}a^{1/2}}-2n-z)}\kappa(w)\varrho^n(z,\bar{\eta},\lambda)dwdz, \end{aligned} \quad (3.53)$$

where $\kappa(w) \in C_0^\infty(\mathbb{R})$ is supported for w as close as we want to 0 and where Ai is the Airy function. We shall use the decomposition $Ai(z) = A^+(z) + A^-(z)$, where

$$A^\pm(-(\bar{\eta}\lambda)^{2/3}(1-w)) = e^{\pm\frac{i\pi}{2}-\frac{i\pi}{4}}e^{\mp\frac{2}{3}i\bar{\eta}\lambda}(\bar{\eta}\lambda)^{-1/6}a_\pm(w,\bar{\eta},\lambda), \quad (3.54)$$

where $a_\pm(w,\bar{\eta},\lambda)$ are defined in Proposition 3.12. For details see the proof of Proposition 3.12 from [15, Prop.6]. Finally, (3.52) becomes

$$\begin{aligned} u_{F,h}^n(0,\bar{y},\bar{t}) &= h^{1/3}\sum_{\pm}\int_{\bar{\eta}} e^{\frac{i}{h}(\bar{y}\bar{\eta}-\bar{t}(1+a)^{1/2}\bar{\eta}\mp\frac{2}{3}a^{3/2}\bar{\eta}+\frac{4}{3}na^{3/2}\bar{\eta})}\times \\ & \quad \times \Psi(\bar{\eta})(\bar{\eta}\lambda)^{-1/6}I_{\pm}(\varrho^n(\cdot,\bar{\eta},\lambda))_{\bar{\eta}}\left(\frac{\bar{t}}{2(1+a)^{1/2}a^{1/2}}-2n,\lambda\right)d\bar{\eta}, \end{aligned} \quad (3.55)$$

where $I_{\pm}(\varrho^n(\cdot,\bar{\eta},\lambda))_{\bar{\eta}}(z,\lambda)$ are defined in (3.40). Therefor, $Tr_{\pm}(u_{F,h}^n)(\bar{y},\bar{t})$ are the contributions corresponding to the \pm signs, respectively, in the right hand side of (3.55).

We can proceed to compute $J \circ Tr_{\pm}(u_{F,h}^n)(\bar{y},\bar{t})$:

$$\begin{aligned} J \circ Tr_{\pm}(u_{F,h}^n)(y,t) &= \frac{h^{1/3}}{(2\pi h)^2}\int e^{\frac{i}{h}(\theta_0(y,t,\eta,\tau)-\bar{y}(\eta-\bar{\eta})-\bar{t}(\tau+\bar{\eta}(1+a)^{1/2})\mp\frac{2}{3}a^{3/2}\bar{\eta}+\frac{4}{3}na^{3/2}\bar{\eta})}\times \\ & \quad \times d_h(y,\eta,\tau)\Psi(\bar{\eta})(\bar{\eta}\lambda)^{-1/6}I_{\pm}(\varrho^n(\cdot,\bar{\eta},\lambda))_{\bar{\eta}}\left(\frac{\bar{t}}{2(1+a)^{1/2}a^{1/2}}-2n,\lambda\right)d\bar{\eta}d\bar{y}d\bar{t}d\eta d\tau. \end{aligned} \quad (3.56)$$

Since the symbol is independent of \bar{y} , the integration in \bar{y} gives $\eta = \bar{\eta}$. Now we are in a situation where the stationary phase theorem can be applied in the variables (\bar{t},τ) . To do this, we use the next proposition:

Proposition 3.22. ([14, Thm.7.7.7]) *Let $f(z,\tau)$ be a real valued function in $C^\infty(\mathbb{R}^{m+1})$, $\tau_0 \in \mathbb{R}$. If K is a compact subset of \mathbb{R}^{2+m} and $\sigma \in C_0^\infty(K)$, then*

$$\begin{aligned} & \int_{\tau,\bar{t}} e^{\frac{i}{h}(f(z,\tau)-\bar{t}(\tau-\tau_0))}\sigma(\tau,\bar{t},z)d\tau d\bar{t} \\ & \quad \simeq (2\pi i)e^{\frac{i}{h}f(z,\tau_0)}\sum_{\nu \geq 0}(ih)^\nu(\partial_\tau \partial_{\bar{t}})^\nu \left(e^{\frac{i}{h}r(z,\tau)}\sigma(\tau,\bar{t},z) \right) \Big|_{\tau=\tau_0,\bar{t}=\partial_\tau f(z,\tau_0)}. \end{aligned} \quad (3.57)$$

Here $r(z,\tau) = f(z,\tau) - f(z,\tau_0) - (\tau - \tau_0)\partial_\tau f(z,\tau_0)$.

Remark 3.23. In the sum (3.57) the \bar{t} derivative must act on σ , and τ derivatives acting on $e^{\frac{i}{h}r(\tau,z)}$ bring out with h^{-1} a derivative of r vanishing at $\tau = \tau_0$. Another τ derivative must act on it to give a non-zero contribution. This shows that the terms in the sum are $O(h^{\nu/2})$, for at most $\nu/2$ derivations bring out a factor h^{-1} .

We apply Proposition 3.22 with $z = (y, t, \eta) \in \mathbb{R}^3$, $f(z, \tau) = \theta_0(z, \tau)$, $\tau_0 = -\eta(1+a)^{1/2}$ and with symbol

$$\sigma(\tau, \bar{t}, z) := d_h(y, \eta, \tau) I_{\pm}(\varrho^n(\cdot, \eta, \lambda))_{\eta} \left(\frac{\bar{t}}{2(1+a)^{1/2}a^{1/2}} - 2n, \lambda \right).$$

Consequently, $J \circ Tr_{\pm}(u_{E,h}^n)$ admits the asymptotic expansion

$$\begin{aligned} J \circ Tr_{\pm}(u_{E,h}^n)(y, t) &\simeq h^{1/3} \int e^{\frac{i}{h}(\theta_0(y,t,\eta,-\eta(1+a)^{1/2}) + \frac{2}{3}(2n \mp 1)(-\zeta_0)^{3/2}(\eta, -\eta(1+a)^{1/2}))} (\eta\lambda)^{-1/6} \Psi(\eta) \\ &\times \left(\sum_{k \geq 0} (ih)^k a^{-k/2} 2^{-k} (1+a)^{-k/2} \partial^k \left(e^{\frac{i}{h}r(y,t,\eta,\tau)} d_h(y, \eta, \tau) \right) \right) \Big|_{\left\{ \begin{array}{l} \tau = -\eta(1+a)^{1/2}, \\ \bar{t} = \partial_{\tau} \theta_0(y, t, \eta, -\eta(1+a)^{1/2}) \end{array} \right.} \\ &\times \partial^k I_{\pm}(\varrho^n(\cdot, \eta, \lambda))_{\eta} \left(\frac{\partial_{\tau} \theta_0(y, t, \eta, -\eta(1+a)^{1/2})}{2(1+a)^{1/2}a^{1/2}} - 2n, \lambda \right) d\eta. \quad (3.58) \end{aligned}$$

According to the Remark 3.23 it follows that each τ derivative of order k writes

$$i^k 2^{-k} (1+a)^{-k/2} \partial^k \left(e^{\frac{i}{h}r(y,t,\eta,\tau)} d_h(y, \eta, \tau) \right) \Big|_{\left\{ \begin{array}{l} \tau = -\eta(1+a)^{1/2}, \\ \bar{t} = \partial_{\tau} \theta_0(y, t, \eta, -\eta(1+a)^{1/2}) \end{array} \right.} =: h^{-k/2} \mu_k(y, \eta, h),$$

where the main contribution of $\mu_{2\nu}$ is $(\partial_{\tau}^2 r)^{\nu} d_h e^{\frac{i}{h}r}$ and that of $\mu_{2\nu-1}$ is $h(\partial_{\tau}^2 r)^{\nu} \partial_{\tau} d_h e^{\frac{i}{h}r}$, all the other terms in the sum defining μ_k being positive powers of h ; since d_h is a symbol of order 0 and type $(1, 0)$, we deduce that μ_k is also a symbol of order 0 and type $(1, 0)$.

Notice, moreover, that $I_{\pm}(\varrho^n)_{\eta}$ is a convolution product and consequently $\partial^k(I_{\pm}(\varrho^n)_{\eta}) = I_{\pm}(\partial^k \varrho^n)_{\eta}$. Since from Proposition 3.15 and Definition 3.16 the symbols $\varrho^n(\cdot, \eta, \lambda)$ belong to $\mathcal{S}_{K_0}(\lambda/n)$, where $K_0 = [-c_0, c_0]$, it follows that the sum

$$\Psi(\eta) \eta^{1/3} \left(\sum_{k \geq 0} h^{k/2} a^{-k/2} \mu_k(y, \eta, h) \partial^k \varrho^n(z, \eta, \lambda) \right),$$

denoted $g_h^n(z, y, \eta)$ in the Lemma, belongs also to $\mathcal{S}_{K_0}(\lambda/n)$. This achieves the proof of Lemma 3.21. \square

3.3.2 Transport equations

The symbol g_h^n defined in Lemma 3.21 will be the one we were looking for to define u_h^n in (3.46). However, notice that since we want u_h^n to solve (1.6), at least at first order, the symbols we choose should satisfy at least the eikonal (3.16) and the first transport equations

associated to the wave operator \square_g . Since there is no reason for which g_h^n defined in (3.51) would satisfy such transport equations and since we need to use their precise form in order to have the condition (3.48) satisfied on the boundary, we shall take the composition of g_h^n with an integral curve of the vector field $\langle 2d\Phi^n, d. \rangle - \eta^{-1/3} \langle d\zeta, d\zeta \rangle \partial_s$ associated to the first transport equation associated to the wave operator. We first determine this integral curve.

Lemma 3.24. *The functions*

$$\eta^{-2/3}\zeta + s^2, \quad \partial_\tau\theta + \eta^{1/3}s\partial_\tau\zeta \quad (3.59)$$

are integral curves of the vector field $\langle 2d\Phi^n, d. \rangle - \eta^{-1/3} \langle d\zeta, d\zeta \rangle \partial_s$, where we recall that Φ^n is the homogeneous phase function

$$\Phi^n(x, y, t, \eta^{1/3}s, \eta, \tau) = \theta(x, y, t, \eta, \tau) + \eta^{1/3}s\zeta(x, y, \eta, \tau) + \eta\frac{s^3}{3} + \frac{4}{3}n(-\zeta_0)^{3/2}(\eta, \tau).$$

Proof. The Hamiltonian system writes

$$\begin{cases} \dot{x} = 2(\partial_x\theta + \eta^{1/3}s\partial_x\zeta), \\ \dot{y} = 2(1 + xb(y))(\partial_y\theta + \eta^{1/3}s\partial_y\zeta), \\ \dot{t} = -2\tau, \\ \dot{s} = -\eta^{-1/3} \langle d\zeta, d\zeta \rangle \end{cases} \quad (3.60)$$

and we can compute the derivative of the first integral curve in (3.59)

$$\begin{aligned} \overbrace{(s^2 + \eta^{-2/3}\zeta)}^{\dot{}} &= 2\dot{s}s + \eta^{-2/3}\dot{\zeta} = 2\eta^{-1/3} \langle d\zeta, d\zeta \rangle s + \eta^{-2/3}(\dot{x}\partial_x\zeta + \dot{y}\partial_y\zeta) \\ &= 2\eta^{-1/3} \langle d\zeta, d\zeta \rangle s - 2\eta^{-4/3} \langle d\theta, d\zeta \rangle + 2\eta^{-1/3} \langle d\zeta, d\zeta \rangle s \\ &= 0, \end{aligned}$$

where we used the eikonal equations (3.16). For the second one we have

$$\begin{aligned} \overbrace{(\partial_\tau\theta + \eta^{1/3}\partial_\tau\zeta)}^{\dot{}} &= \dot{x}\partial_{\tau,x}^2\theta + \dot{y}\partial_{\tau,y}^2\theta + \dot{t}\partial_{\tau,t}^2\theta + \eta^{1/3}s(\dot{x}\partial_{\tau,x}^2\zeta + \dot{y}\partial_{\tau,y}^2\zeta) + \eta^{1/3}\dot{s}\partial_\tau\zeta \\ &= \partial_\tau(\langle d\theta, d\theta \rangle - \zeta \langle d\zeta, d\zeta \rangle) + 2\eta^{-2/3}s\partial_\tau \langle d\theta, d\zeta \rangle \\ &\quad + \partial_\tau(\zeta \langle d\zeta, d\zeta \rangle) + \eta^{-4/3}s^2\partial_\tau \langle d\zeta, d\zeta \rangle \\ &= \langle d\zeta, d\zeta \rangle \partial_\tau\zeta = \eta^{2/3}(s^2 + \eta^{-2/3}\zeta)\partial_\tau \langle d\zeta, d\zeta \rangle \\ &= 0 \end{aligned}$$

on the Lagrangian Λ_{Φ^n} which contains the semi-classical wave front set $WF_h(u_h^n)$,

$$\Lambda_{\Phi^n} := \{(x, y, t, \eta^{1/3}s, \eta, \tau) | \partial_s\Phi^n = \eta^{-2/3}\zeta + s^2 = 0, \partial_\eta\Phi^n = 0, \tau = -\eta(1 + a)^{1/2}\}.$$

□

Now we are in the situation when we can define u_h^n everywhere:

Definition 3.25. Let g_h^n be the one defined in (3.51) and for $0 \leq n \leq N \simeq \lambda h^\epsilon$ let

$$u_h^n(x, y, t) := \int_{s, \eta} e^{\frac{i}{h} \Phi^n(x, y, t, \eta^{1/3} s, \eta, -\eta(1+a)^{1/2})} \times g_h^n \left(\frac{\partial_\tau \theta + \eta^{1/3} s \partial_\tau \zeta}{\partial_\tau \zeta_0 (-\zeta_0)^{1/2}}(x, y, t, \eta, -\eta(1+a)^{1/2}) - 2n, y, \eta \right) ds d\eta. \quad (3.61)$$

Remark 3.26. Notice that since $\partial_\tau \theta$, $\eta^{1/3} \partial_\tau \zeta$ are homogeneous of degree 0 in (η, τ) , for $\tau = -\eta(1+a)^{1/2}$ they are independent of η , so the term in the first variable in g_h^n depends only of (x, y, t) and a . Recall also that $\partial_\tau \zeta_0 (-\zeta_0)^{1/2}(\eta, -\eta(1+a)^{1/2}) = 2(1+a)^{1/2} a^{1/2}$.

We shall prove that the restriction to u_h^n defined in (3.61) to the boundary coincides with the sum of the two terms in (3.58). This will imply (3.48).

Proposition 3.27. *On the boundary $\partial\Omega$ we have*

$$u_h^n(0, y, t) = \sum_{\pm} J(\text{Tr}_{\pm}(u_{F,h}^n))(y, t, h). \quad (3.62)$$

Moreover,

$$J(\text{Tr}_-(u_{F,h}^n))(y, t) + J(\text{Tr}_+(u_{F,h}^{n+1}))(y, t) = O_{L^2}(h^\infty). \quad (3.63)$$

Proof. We first proceed with (3.63). Since J is an elliptic Fourier integral operator the proof follows from Proposition 3.18, since

$$\text{Tr}_-(u_{F,h}^n)(y, t, h) + \text{Tr}_+(u_{F,h}^{n+1})(y, t, h) = O_{L^2}(h^\infty).$$

We now prove (3.62). At $x = 0$ the integral in s in (3.61) writes

$$\begin{aligned} & \Psi(\eta) \eta^{1/3} \int e^{\frac{i}{h} (\eta^{1/3} s \zeta_0(\eta, -\eta(1+a)^{1/2}) + \eta \frac{s^3}{3})} \\ & \quad \times \partial^k \varrho^n \left(\frac{\partial_\tau \theta_0 + \eta^{1/3} s \partial_\tau \zeta_0}{\partial_\tau \zeta_0 (-\zeta_0)^{1/2}}(y, t, \eta, -\eta(1+a)^{1/2}) - 2n, \eta, \lambda \right) ds \\ & = h^{1/3} \Psi(\eta) \frac{\eta \lambda}{2\pi} \int e^{i\eta \lambda w} \left(\frac{\partial_\tau \theta_0(y, t, \eta, -\eta(1+a)^{1/2})}{2(1+a)^{1/2} a^{1/2}} - 2n - z \right) \\ & \quad \times \text{Ai}(-(\eta \lambda)^{2/3} (1-w)) \times \kappa(w) \partial^k \varrho(z, \lambda) dz dw + O_{\mathcal{S}(\mathbb{R})}((\eta \lambda)^{-\infty}), \end{aligned} \quad (3.64)$$

where κ is a smooth function supported for w as close as we want to 0. We distinguish two contributions in (3.64), obtained using the decomposition $\text{Ai}(z) = A^+(z) + A^-(z)$, where $A_{\pm}(z)$ are recalled in (3.54). Consequently, (3.64) becomes

$$h^{1/3} \Psi(\eta) e^{\mp \frac{2}{3} i \eta \lambda} (\eta \lambda)^{-1/6} I_{\pm}(\partial^k \varrho^n(\cdot, \eta, \lambda))_{\eta} \left(\frac{\partial_\tau \theta_0(y, t, \eta, -\eta(1+a)^{1/2})}{2(1+a)^{1/2} a^{1/2}} - 2n, \lambda \right).$$

Using again the fact that $I_{\pm}(\varrho^n)_{\eta}$ is a convolution product we obtain $I_{\pm}(\partial^k \varrho^n)_{\eta} = \partial^k (I_{\pm}(\varrho^n)_{\eta})$, which allows achieve the proof of (3.62). \square

Let u_h^n be defined by (3.61) above. We now show that it is an approximate solution to (1.6) and we obtain L^2 bounds for $\square_g u_h^n$. Applying the wave operator \square_g to u_h^n and using the eikonal equations (3.16) yields

$$\begin{aligned} \square_g u_h^n(x, y, t) &= \int_{s, \eta, \tau} e^{\frac{i}{h} \Phi^n(x, y, t, \eta^{1/3} s, \eta, -\eta(1+a)^{1/2})} \times \\ &\times \left(\frac{i}{h} \langle 2d\Phi^n, dg_h^n \rangle + \eta^{-1/3} \langle d\zeta, d\zeta \rangle \partial_s g_h^n + (\square_g \Phi^n) g_h^n + \square_g g_h^n \right) ds d\eta. \end{aligned} \quad (3.65)$$

Using Lemma 3.24 we obtain the following:

Proposition 3.28.

$$\|\square_g u_h^n(\cdot, t)\|_{L^2(\Omega)} = O(h^{-1}) \|u_h^n(\cdot, t)\|_{L^2(\Omega)}. \quad (3.66)$$

Remark 3.29. This result is useful since in order to estimate the error between the approximate solution we are constructing and the exact solution to (1.6) we are going to use the same approach as in Lemma 2.5 where an error of at least

Proof. Using the eikonal equations (3.16) and we integrating by parts with respect to s we find

$$\begin{aligned} \square_g u_h^n(x, y, t) &= \int_{s, \eta} e^{\frac{i}{h} \Phi^n(x, y, t, \eta^{1/3} s, \eta, -\eta(1+a)^{1/2})} \left(\frac{i}{h} ((\square_g \Phi^n) g_h^n + 2(1 + xb(y)) \partial_y \Phi_h \partial_2 g_h^n) \right. \\ &+ \frac{1}{4(1+a)a} \langle d\partial_\tau \Phi, d\partial_\tau \Phi \rangle - \partial_\tau \zeta \partial_\tau \langle d\zeta, d\zeta \rangle \partial_1^2 g_h^n + \frac{1}{2(1+a)^{1/2} a^{1/2}} \square_g (\partial_\tau \Phi) \partial_1 g_h^n \\ &\left. - (1 + xb(y)) \partial_2^2 g_h^n \right) \left(\frac{\partial_\tau \theta + \eta^{1/3} s \partial_\tau \zeta}{2(1+a)^{1/2} a^{1/2}}(x, y, t, \eta, -\eta(1+a)^{1/2}) - 2n, y, \eta, h \right) ds d\eta, \end{aligned} \quad (3.67)$$

Here $\partial_1 g_h^n, \partial_2 g_h^n$ denote the derivatives of $g_h^n(z, y, \eta)$ with respect to z and y , respectively.

We estimate the $L^2(\Omega)$ norm of the terms in the integral above: since $\square_g \Phi^n$ is bounded in (x, y) , uniformly in $0 \leq n \leq N \simeq \lambda h^\epsilon$, independent of t then

$$\left\| \int_{s, \eta} e^{\frac{i}{h} \Phi^n(x, y, t, \eta^{1/3} s, \eta, -\eta(1+a)^{1/2})} (\square_g \Phi^n) g_h^n \right\|_{L^2(\Omega)} \lesssim \|\square_g \Phi^n\|_{L^\infty(\Omega)} \|u_h^n\|_{L^2(\Omega)}.$$

The remaining terms have coefficients $\lesssim a^{-1}$ and their symbols are uniformly bounded functions in (x, y) as well. In order to estimate the $L^2(\Omega)$ norms of these terms involving derivatives of g_h^n with respect to its first variable we use the fact that $g_h^n(\cdot, y, \eta) \in \mathcal{S}_{K_0}(\lambda/n)$ so that for every $k \geq 0$ there exist constants $C_k > 0$ independent of h such that $\sup_z |\partial_1^k g_h^n(z, \eta, \lambda)| \leq C_k$ and the essential support of $\partial_1^k g_h^n$ is included in $K_0 = [-c_0, c_0]$. To handle the terms involving derivatives of g_h^n with respect to y we use the form (3.51) and the fact that μ_k are symbols of order 0 and type $(1, 0)$. \square

3.4 Main properties of the parametrix

In this section we state the main properties of the sum

$$U_h(x, y, t) := \sum_{n=0}^N u_h^n(x, y, t), \quad (3.68)$$

where $u_h^n(x, y, t)$ are introduced in (3.61) and where $N \simeq \lambda h^\epsilon$ for some small $\epsilon > 0$. We first prove that each u_h^n is essentially supported for t in an interval of time of size $a^{1/2}$ and that they have almost disjoint supports. Because of this reason, in order to construct U_h on an interval of time of size one we shall take $a = h^\alpha$ with to satisfy

$$Na^{1/2} \simeq \lambda h^\epsilon h^{\alpha/2} \simeq 1, \quad \text{where} \quad \lambda = h^{3\alpha/2-1}. \quad (3.69)$$

This condition gives immediately $\alpha = \frac{1-\epsilon}{2}$, $\epsilon > 0$ being the one fixed from the beginning. From now on, a and N will be those determined by (3.69),

$$a = h^{(1-\epsilon)/2} \quad \text{and} \quad N \simeq h^{-(1-\epsilon)/4}.$$

Remark 3.30. Recall that the assumption that the number N should be "much" smaller than the parameter λ was necessary in the construction of the model solution (see Section 3.2). Precisely, the construction of the operator $(J_{+, \eta} \circ I_{-, \eta})^{on}$ in the proof of Proposition 3.15 involves a stationary phase argument with (large) parameter λ/n ; although this parameter was large, but not a fixed, positive power of h^{-1} , the construction would have failed since the contribution coming from the error terms wouldn't have belonged to $O_{\mathcal{S}(\mathbb{R})}(h^\infty)$, which is possible only taking $n \leq N \leq \lambda h^\epsilon$. Since the loss of derivatives in the $L_t^q L^r(\Omega)$ norms becomes larger when ϵ goes to zero, we choose to take $N \simeq \lambda h^\epsilon$ (for details see [15, Prop.3.6]).

3.4.1 Wave front set

Lemma 3.31. *Let u_h^n be given in (3.61). Then the wave front set $WF_h(u_h^n)$ of u_h^n is contained in the Lagrangian set Λ_{Φ^n} defined as follows*

$$\begin{aligned} \Lambda_{\Phi^n} := \{ & (x, y, t, \eta^{1/3}s, \eta, -\eta(1+a)^{1/2}) | \zeta(x, y, \eta, -\eta(1+a)^{1/2}) + \eta^{2/3}s^2 = 0, \\ & (\partial_\eta \theta - (1+a)^{1/2} \partial_\tau \theta + s\zeta)(x, y, t, 1, -(1+a)^{1/2}) + \frac{s^3}{3} + \frac{4}{3}na^{3/2} = 0 \}. \end{aligned} \quad (3.70)$$

In other words, outside any neighborhood of Λ_{Φ^n} the contribution of u_h^n is $O_{L^2}(h^\infty)$. Notice that to define Λ_{Φ^n} we used here the homogeneity of the phases θ and ζ in (η, τ) of degrees 1 and 2/3, respectively.

Proof. Let (x, y) be such that $|\partial_s \Phi^n| \geq c$ for some $c > 0$ and let L_1 be the operator $L_1 := \frac{h}{i} \frac{1}{|s^2 + \eta^{-2/3}\zeta|} \partial_s$. After each integration by parts with respect to s we gain a factor $h^{1-\alpha/2}$ and the contribution of u_h^n in this case is obviously $O_{L^2}(h^\infty)$.

Let now $|\partial_\eta \Phi^n| \geq c > 0$ for some positive constant c : we shall perform (repeated) integrations by parts using this time the operator $L_2 := \frac{h}{i} \frac{\partial_\eta \Phi^n}{|\partial_\eta \Phi^n|^2} \partial_\eta$. We need however to estimate the derivatives with respect to η of g_h^n , precisely we have to estimate

$$L_2^{*m} \left(g_h^n \left(\frac{\partial_\tau \theta + \eta^{1/3} s \partial_\tau \zeta}{2(1+a)^{1/2} a^{1/2}} (x, y, t, \eta, -\eta(1+a)^{1/2}) - 2n, y, \eta \right) \right).$$

Since $\partial_\tau \theta$ and $\eta^{1/3} \partial_\tau \zeta$ are homogeneous of degree 0 with respect to (η, τ) , the first variable in the symbol g_h^n is independent of η .

The symbol $g_h^n(z, y, \eta)$ is an asymptotic sum whose general term is of the form

$$h^{k/2} a^{-k/2} \Psi(\eta) \eta^{1/3} \mu_k(y, \eta, h) \partial^k \varrho^n(z, \eta, \lambda),$$

where we recall from Definition 3.16 and Proposition 3.15 of Section 3.2 that $\partial^k \varrho^n$ writes as a convolution

$$\partial^k \varrho^n(z, \eta, \lambda) = (F_{\eta\lambda})^{*n} * \partial^k \varrho^0(\cdot, \lambda)(z), \quad \forall k \geq 0,$$

where $\varrho^0 \in \mathcal{S}_{K_0}(\lambda)$ is independent of η and where $(F_{\eta\lambda})^{*n}$ is defined in (3.43). Since μ_k are symbols of order 0 and type $(1, 0)$ it will be enough to estimate the contribution of the terms involving the derivatives on $(F_{\eta\lambda})^{*n}$. We recall that $(F_{\eta\lambda})^{*n}$ has the explicit form

$$(F_{\eta\lambda})^{*n}(z) = \frac{\eta\lambda}{2\pi} \int_w e^{i\eta\lambda(wz + n(2w + \frac{4}{3}((1-w)^{3/2} - 1)))} (c(w, \eta\lambda))^n dw,$$

where $c(w, \eta\lambda) := \kappa(w) a_+(w, \eta\lambda) b_-(w, \eta\lambda) \simeq \kappa^2(w) \sum_{j \geq 0} c_j (1-w)^{-3j/2} (\eta\lambda)^{-j}$ with $c_0 = 1$. Making the change of variables $\tilde{w} = nw$ and setting $\tilde{\lambda} = \lambda/n \geq h^{-\epsilon} \gg 1$, we can write

$$(F_{\eta\lambda})^{*n}(z) = \frac{\eta\tilde{\lambda}}{2\pi} \int_{\tilde{w}} e^{i\eta\tilde{\lambda}(z\tilde{w} + n^2 f(\frac{\tilde{w}}{n}))} c^n\left(\frac{\tilde{w}}{\lambda}, \eta n \tilde{\lambda}\right) d\tilde{w},$$

where we set $f(w) := 2w + \frac{4}{3}((1-w)^{3/2} - 1)$. Hence one η -derivative yields

$$\begin{aligned} \partial_\eta (F_{\eta\lambda})^{*n}(z) &= \frac{1}{\eta} (F_{\eta\lambda})^{*n}(z) + \frac{\eta\tilde{\lambda}}{2\pi} \int_{\tilde{w}} e^{i\eta\tilde{\lambda}(z\tilde{w} + n^2 f(\frac{\tilde{w}}{n}))} i\tilde{\lambda} (z\tilde{w} + n^2 f(\frac{\tilde{w}}{n})) c^n\left(\frac{\tilde{w}}{\lambda}, \eta n \tilde{\lambda}\right) d\tilde{w} \\ &\quad + \frac{\eta\tilde{\lambda}}{2\pi} \int_{\tilde{w}} e^{i\eta\tilde{\lambda}(z\tilde{w} + n^2 f(\frac{\tilde{w}}{n}))} n \partial_\eta c\left(\frac{\tilde{w}}{\lambda}, \eta n \tilde{\lambda}\right) c^{n-1}\left(\frac{\tilde{w}}{\lambda}, \eta n \tilde{\lambda}\right) d\tilde{w}. \end{aligned} \quad (3.71)$$

The symbol of the third term in the right hand side of (3.71) is $n \partial_\eta c(w, \eta\lambda) c^{n-1}(w, \eta\lambda)$ and we have

$$\partial_\eta c(w, \eta\lambda) = -\eta^{-2} \lambda^{-1} \sum_{j \geq 1} j c_j (1-w)^{-3j/2} (\eta\lambda)^{-(j-1)},$$

and since $n \ll \lambda$, the contribution from this term is easily handled with.

The symbol in the second term in the right hand side of (3.71) equals the symbol of $(F_{\eta\lambda})^{*n}$ multiplied by the factor $i\tilde{\lambda}(z\tilde{w} + \lambda n f(\frac{\tilde{w}}{n}))$. Recall that on the support of $c(w, \eta\lambda)$

we have $w = \tilde{w}/n \in \text{supp}(\kappa)$ is as close to zero as we want and there $f(w) = w^2/2 + O(w^3)$, hence $n^2 f(\frac{\tilde{w}}{n}) = \tilde{w}^2/2 + O(\tilde{w}^3/n)$. On the other hand, when we take the convolution product of the second term in (3.71) with $\varrho^0(\cdot, \lambda)$ we find that the critical points of the phase in the oscillatory integral obtained in this way,

$$\frac{\eta\tilde{\lambda}}{2\pi} \int_{\tilde{w}, z'} e^{i\eta\tilde{\lambda}((z-z')\tilde{w} + n^2 f(\frac{\tilde{w}}{n}))} i\tilde{\lambda}((z-z')\tilde{w} + n^2 f(\frac{\tilde{w}}{n})) c^n(\frac{\tilde{w}}{n}, \eta n\tilde{\lambda}) \varrho^0(z', \lambda) d\tilde{w} dz',$$

are given by $\tilde{w} = 0$ and $z = z'$. The phase function which we denoted by $\phi_n(z, z', \tilde{w})$ satisfies $\phi_n(z, z, 0) = 0$. Applying the stationary phase theorem in \tilde{w} and z' , the first term in the asymptotic expansion obtained in this way vanishes, and the next ones are multiplied by strictly negative, integer powers of $\tilde{\lambda}$, hence the contribution from this term will be also bounded.

Notice that when we take higher order derivatives in η of ϱ^n , we obtain symbols which are products of $\tilde{\lambda}^j (\phi_n)^j \partial_\eta^{k-j} (c^n(\tilde{w}/n, \eta n\tilde{\lambda}))$ and can be dealt with in the same way, taking into account this time that the first j terms in the asymptotic expansion obtained after applying the stationary phase vanish. As a consequence, after each integration by parts in η using the operator L_2 we gain a factor h , meaning that the contribution of u_h^n is $O_{L^2}(h^\infty)$. \square

3.4.2 Localization of the supports

In the rest of this section we shall localize the essential supports of u_h^n . Recall from Proposition 3.19 that in the Friedlander's case the solutions $u_{F,h}^n$ to (3.29) were essentially supported in time in the intervals, respectively

$$[4na^{1/2} - 2a^{1/2}(1 + c_0), 4na^{1/2} + 2a^{1/2}(1 + c_0)](1 + a)^{1/2}, \quad (3.72)$$

where c_0 is a constant sufficiently small and such that $\varrho \in \mathcal{S}_{K_0}(\lambda)$, where $K_0 = [-c_0, c_0]$.

We prove a similar property for u_h^n : precisely, their essential supports in the time variables will be contained in almost disjoint intervals obtained as the image of (3.72) by the symplectomorphisme χ_∂ . For, we define

$$\bar{t}_{0,\pm} = \pm 2(1 + a)^{1/2} a^{1/2}, \quad \bar{y}_0 = \pm 2(1 + a)a^{1/2} \mp \frac{2}{3}a^{3/2},$$

and for $n \geq 1$ we set

$$\bar{t}_{n,\pm} = \bar{t}_{0,\pm} + 4n(1 + a)^{1/2} a^{1/2}, \quad \bar{y}_{n,\pm} = \bar{y}_{0,\pm} + 4na^{1/2} + \frac{8}{3}na^{3/2},$$

so that $(\bar{y}_{n,\pm}, \bar{t}_{n,\pm}, 1, -(1 + a)^{1/2}) := (\delta_F^+)^n(\bar{y}_{0,\pm}, \bar{t}_{0,\pm}, 1, -(1 + a)^{1/2})$, where we recall that δ_F^+ is the billiard ball map of the model case defined in (3.32). Notice that for $n \geq 0$ we have

$$\bar{t}_{n,+} = \bar{t}_{n+1,-}, \quad \bar{y}_{n,+} = \bar{y}_{n+1,-}, \quad \bar{y}_{n,\pm} - \bar{t}_{n,\pm}(1 + a)^{1/2} + \frac{4}{3}na^{3/2} = \mp \frac{2}{3}a^{3/2}.$$

Since $(\bar{y}_{n,\pm}, t_{n,\pm}) \in \Lambda_{\Phi_F^n|_{x=0}}$ and using Proposition 3.19 it follows that each u_F^n is also essentially supported for the tangential variable in a small neighborhood of $[\bar{y}_{n,-}, \bar{y}_{n,+}]$.

For $0 \leq n \leq N$, we define

$$I_n(c) := [\bar{t}_{n,-} - 2c(1+a)^{1/2}a^{1/2}, \bar{t}_{n,+} + 2c(1+a)^{1/2}a^{1/2}] \quad (3.73)$$

and also

$$D_n(c) := \{(\bar{y}, \bar{t}) | \bar{t} \in I_n(c), \bar{y} - (1+a)^{1/2}\bar{t} - \frac{4}{3}na^{3/2} \in [-\frac{2}{3}a^{3/2}, \frac{2}{3}a^{3/2}]\}. \quad (3.74)$$

Remark 3.32. Notice that with these notations $u_{F,h}^n$ is essentially supported in the set

$$\{(x, y, t) | 0 \leq x \leq a, (y, t) \in D_n(c_0)\}.$$

Notice moreover that since $na^{1/2} \lesssim 1$ for $0 \leq n \leq N$, the essential support in the tangential variable is contained in an interval of same size as the time interval (3.72).

For small $x \geq 0$ we define

$$\chi_x(y, t) := (\partial_\eta \theta, \partial_\tau \theta)(x, y, t, 1, -(1+a)^{1/2}).$$

Theorem 3.3 implies that if x is sufficiently small, then the application χ_x is invertible.

We prove the following:

Proposition 3.33. *If $c_0 > 0$ is sufficiently small, then u_h^n have almost disjoint supports in the time and tangential variables (y, t) . Precisely, it is sufficient to show that*

$$\text{ess-supp}(u_h^n) \subset \{(x, y, t) | 0 \leq x \lesssim a, (y, t) \in \chi_x^{-1}(D_n(c_0))\}. \quad (3.75)$$

By $\text{ess-supp}(u_h^n)$ we denote the closure of the set of points outside which u_h^n is $O_{L^2}(h^\infty)$.

Proof. Since according to Lemma 3.21 we have $g_h^n(\cdot, y, \eta) \in \mathcal{S}_{[-c_0, c_0]}(\lambda/n)$, on the essential support (in the first variable) of the symbol g_h^n the following holds

$$\begin{aligned} |(\partial_\tau \theta + \eta^{1/3} s \partial_\tau \zeta)(x, y, t, \eta, -\eta(1+a)^{1/2}) - 4n(1+a)^{1/2}a^{1/2}| \\ \leq 2c_0(1+a)^{1/2}a^{1/2}. \end{aligned} \quad (3.76)$$

Let $c \in (0, 1)$ be such that

$$|\partial_\tau \theta(x, y, t, \eta, -\eta(1+a)^{1/2}) - 4n(1+a)^{1/2}a^{1/2}| \geq 2(1+c)(1+a)^{1/2}a^{1/2}. \quad (3.77)$$

We show that on the essential support of u_h^n we must have $c \leq c_0$, therefor the contribution of u_h^n will be $O_{L^2}(h^\infty)$ for (x, y, t) outside a set on which

$$|\partial_\tau \theta(x, y, t, \eta, -\eta(1+a)^{1/2}) - 4n(1+a)^{1/2}a^{1/2}| \leq 2(1+c_0)(1+a)^{1/2}a^{1/2}. \quad (3.78)$$

The inequalities (3.76) and (3.77) yield

$$\begin{aligned} |\eta^{1/3} s \partial_\tau \zeta(x, y, \eta, -\eta(1+a)^{1/2})| &\geq |\partial_\tau \theta(x, y, t, \eta, -\eta(1+a)^{1/2}) - 4n(1+a)^{1/2} a^{1/2}| \\ &\quad - |(\partial_\tau \theta + \eta^{1/3} s \partial_\tau \zeta)(x, y, t, \eta, -\eta(1+a)^{1/2}) - 4n(1+a)^{1/2} a^{1/2}| \\ &\geq 2(1+c-c_0)(1+a)^{1/2} a^{1/2}. \end{aligned}$$

On the other hand, on the Lagrangian Λ_{Φ^n} introduced in (3.70) we have

$$|\eta^{1/3} s| = (-\zeta)^{1/2}(x, y, \eta, -\eta(1+a)^{1/2}), \quad \zeta \leq 0. \quad (3.79)$$

From the properties of the phase function ζ stated in Theorem 3.3 we deduce that $\zeta \leq 0$ and from its explicit form (3.25) we obtain

$$0 \leq -\zeta(x, y, \eta, -\eta(1+a)^{1/2}) = (a - xb(y))\eta^{2/3} \leq a\eta^{2/3}, \quad (3.80)$$

and also

$$\partial_\tau \zeta(x, y, \eta, -\eta(1+a)^{1/2}) = 2(1+a)^{1/2} \eta^{-1/3}. \quad (3.81)$$

Since, according to Lemma 3.31, outside any neighborhood of the Lagrangian set Λ_{Φ^n} the contribution of u_h^n equals $O_{L^2}(h^\infty)$, it follows from (3.79), (3.80), (3.81) together with the consequence of the inequalities (3.76) and (3.77) stated above that if $(x, y, t) \in \text{ess-supp}(u_h^n)$ and c is such that (3.77) holds, then

$$(1+c-c_0)^2 \leq 1 \quad \text{which yields} \quad c \leq c_0. \quad (3.82)$$

Till now we proved that if $(x, y, t) \in \text{ess-supp}(u_h^n)$, then

$$\partial_\tau \theta(x, y, t, 1, -(1+a)^{1/2}) \in I_n(c_0), \quad (3.83)$$

(which follows using the homogeneity of degree 0 of $\partial_\tau \theta$ in (η, τ)) and also that

$$0 \leq x \leq a/b(y) \leq a/(1-b_0), \quad (3.84)$$

where we recall that $b_0 \in (0, 1/4)$ is such that on a fixed, sufficiently small neighborhood \mathcal{N} of $y_0 = 0$ we have $|b(y) - 1| \leq b_0$. To conclude that $\chi_x(y, t) \in D_n(c_0)$ it remains to show that

$$\begin{aligned} &|\partial_\eta \theta(x, y, t, \eta, -\eta(1+a)^{1/2}) \\ &\quad - (1+a)^{1/2} \partial_\tau \theta(x, y, t, \eta, -\eta(1+a)^{1/2}) - \frac{4}{3} n a^{3/2}| \leq \frac{2}{3} a^{3/2}, \end{aligned} \quad (3.85)$$

but this follows immediately from Lemma 3.31 together with (3.80). \square

In what follows we show that u_h^n are essentially supported in time in (almost disjoint) intervals of size $\simeq a^{1/2}$. We also prove that if we consider shrunk enough intervals of time J_n , the respective pieces of cusps do not "live" enough to reach the boundary: this property will be useful to estimate from below the L^q in time norm of U_h defined in (3.68), which will

be bounded from below by a sum of integrals over J_n on which only the cusp u_h^n will have a nontrivial contribution. This bound will be estimated in Section 3.4.3 as the sum over n of the $L^q(J_n)$ norms of u_h^n .

We set

$$J_n := \text{proj}_t \left(\text{ess-supp}(u_h^n) \cap \{(x, \chi_x^{-1}(D_n(c_0/3 - 1))), 0 \leq x \leq a/(1 - b_0)\} \right), \quad (3.86)$$

where proj_t denotes the projection on the time variable.

Lemma 3.34. *If $|J_n|$ denotes the size of J_n , then*

$$|J_n| \geq \frac{1}{2} c_0 a^{1/2}. \quad (3.87)$$

Moreover, if c_0 is chosen sufficiently small and if (x, y, t) is such that $t \in J_n$, then

$$a/2 \leq x \leq a/(1 - b_0). \quad (3.88)$$

Proof. We start with (3.88): let $(x, y, t) \in \text{ess-supp}(u_h^n)$, $0 \leq x \leq a/(1 - b_0)$, so that

$$|\partial_\tau \theta(x, y, t, 1, -(1 + a)^{1/2}) - 4n(1 + a)^{1/2} a^{1/2}| \leq 2 \frac{c_0}{3} (1 + a)^{1/2} a^{1/2}. \quad (3.89)$$

Combining the inequality (3.76) which holds on the support of the symbol g_h^n with the condition (3.79) which assures the localization on the Lagrangian Λ_{Φ_n} and using the explicit form (3.25) of ζ , we obtain

$$\begin{aligned} |\partial_\tau \theta(x, y, t, 1, -(1 + a)^{1/2}) - 4n(1 + a)^{1/2} a^{1/2} + 2(a - xb(y))^{1/2} (1 + a)^{1/2}| \\ \leq 2c_0 (1 + a)^{1/2} a^{1/2}. \end{aligned} \quad (3.90)$$

The last two inequalities yield

$$xb(y) \geq a \left(1 - \frac{16}{9} c_0^2\right),$$

and if we take $c_0 \leq 3/8$, using $3/4 < b(y) \leq 1 + b_0 < 5/4$ for $y \in \mathcal{N}$, we get (3.88).

Now we proceed with (3.87): we write $I_n(c_0/3 - 1) = [\bar{t}_-, \bar{t}_+]$, where

$$\bar{t}_\pm := (4n(1 + a)^{1/2} a^{1/2} \pm 2 \frac{c_0}{3} (1 + a)^{1/2} a^{1/2}).$$

Since θ is linear in time, the derivative $\partial_\eta \theta(x, y, t, \eta, \tau)$ is independent of t and we write it $\partial_\eta \theta(x, y, \cdot, \eta, \tau)$. The application $y \rightarrow \partial_\eta \theta(a/b(y), y, \cdot, 1, -(1 + a)^{1/2})$ is a diffeomorphism in a neighborhood of $y_0 = 0$ since from the estimates (3.26) in Section 3.1.3 it follows that $\partial_{y,\eta}^2 \theta \in [(1 - b_0)^2, (1 + b_0)^2]$, hence for small $b_0 \in (0, 1/2)$ this derivative doesn't vanish. This implies that there exist unique points y_\pm such that

$$\partial_\eta \theta(a/b(y_\pm), y_\pm, \cdot, 1, -(1 + a)^{1/2}) = (1 + a)^{1/2} \bar{t}_\pm + \frac{4}{3} n a^{3/2}.$$

Since $\partial_{t,\tau}^2 \theta = 1$, we can also determine unique points t_{\pm} such that

$$\partial_{\tau} \theta(a/b(y_{\pm}), y_{\pm}, t_{\pm}, 1, -(1+a)^{1/2}) = \bar{t}_{\pm}.$$

Moreover, t_{\pm} belong to J_n , since from the choice of y_{\pm} we easily see that

$$(a/b(y_{\pm}), y_{\pm}, t_{\pm}) \in \text{ess-supp}(u_h^n) \cap \{(x, \alpha_x^{-1}(D_n(c_0/3 - 1))), 0 \leq x \leq a/(1 - b_0)\}.$$

Therefore we estimate $|J_n| \geq |t_+ - t_-|$ and it remains to show that $c_0 a^{1/2} \lesssim |t_+ - t_-|$. Write

$$\begin{aligned} \bar{t}_+ - \bar{t}_- &= (t_+ - t_-) + (y_+ - y_-) \int_0^1 \partial_{y,\tau}^2 \theta|_{((1-o)y_+ + oy_-)} do \\ &\quad + \left(\frac{a}{b(y_+)} - \frac{a}{b(y_-)} \right) \int_0^1 \partial_{x,\tau}^2 \theta|_{((1-o)a/b(y_+) + oa/b(y_-))} do, \end{aligned} \quad (3.91)$$

and, on the other hand,

$$\begin{aligned} \bar{t}_+ - \bar{t}_- &= (1+a)^{-1/2} \left((y_+ - y_-) \int_0^1 \partial_{y,\eta}^2 \theta|_{((1-o)y_+ + oy_-)} do \right. \\ &\quad \left. + \left(\frac{a}{b(y_+)} - \frac{a}{b(y_-)} \right) \int_0^1 \partial_{x,\eta}^2 \theta|_{((1-o)a/b(y_+) + oa/b(y_-))} do \right). \end{aligned} \quad (3.92)$$

Since $\partial_{x,\eta} \theta_0 = \partial_{x,\tau} \theta_0 = 0$ and $x \leq a/(1 - b_0)$, the last terms in the right hand sides of (3.91), (3.92), respectively, are $O(a^2)$. Since $\partial_{y,\eta}^2 \theta \in [(1 - b_0)^2, (1 + b_0)^2]$, using (3.92) we get $|y_+ - y_-| \leq \frac{25}{16} |\bar{t}_+ - \bar{t}_-|$. Since from (3.27) in Section 3.1.3 we have $\partial_{y,\tau}^2 \theta \leq b_0(1 + 2b_0) \leq 3/8$ if $b_0 < 1/4$, using (3.91) we finally get $|t_+ - t_-| \geq \frac{2}{5} |\bar{t}_+ - \bar{t}_-| > \frac{1}{2} c_0 a^{1/2}$. \square

In the next lemma we prove that for $(y, t) \in \chi_x^{-1}(D_k(c_0/3 - 1))$, in the sum $U_h(x, y, t)$ defined in (3.68) there is at most one cusp to consider, $u_h^k(x, y, t)$, the contribution from all the others $u_h^n(x, y, t)$ with $n \neq k$ being $O_{L^2}(h^\infty)$.

Lemma 3.35. *If c_0 is chosen small enough and $k \geq 0$, then if*

$$(x, y, t) \in \{(x, \chi_x^{-1}(D_k(c_0/3 - 1))), 0 \leq x \leq a/(1 - b_0)\} \quad (3.93)$$

the following holds

$$U_h(x, y, t) = u_h^k(x, y, t) + O_{L^2}(h^\infty). \quad (3.94)$$

Proof. Suppose that $(x, y, t) \in \text{ess-supp}(u_h^n)$ for some $0 \leq n \leq N$. We show that $n = k$. First we notice that we must have $x \leq a/(1 - b_0)$, otherwise being localized outside a neighborhood of Λ_{Φ^n} . From (3.93) we have

$$\begin{aligned} \partial_{\tau} \theta(x, y, t, 1, -(1+a)^{1/2}) &\in I_k(c_0/3 - 1) \\ &= [4k(1+a)^{1/2} a^{1/2} - 2\frac{c_0}{3}(1+a)^{1/2} a^{1/2}, 4k(1+a)^{1/2} a^{1/2} + 2\frac{c_0}{3}(1+a)^{1/2} a^{1/2}]. \end{aligned} \quad (3.95)$$

Suppose that $n \neq k$. We have to show that the contribution from u_h^n is $O_{L^2}(h^\infty)$. It is enough to prove that if c_0 is chosen small enough, then the inequality (3.76) is not satisfied. We write

$$\begin{aligned} 2c_0(1+a)^{1/2}a^{1/2} &\geq |\partial_\tau \Phi^n| = |\partial_\tau \Phi^k - 4(n-k)(1+a)^{1/2}a^{1/2}| \\ &\geq 4|n-k|(1+a)^{1/2}a^{1/2} - |\partial_\tau \theta + 4k(1+a)^{1/2}a^{1/2}| - \eta^{1/3}|s|\partial_\tau \zeta \\ &\geq 4(1+a)^{1/2}a^{1/2} - 2\frac{c_0}{3}(1+a)^{1/2}a^{1/2} - \eta^{1/3}|s|\partial_\tau \zeta, \end{aligned}$$

where $\theta = \theta(x, y, t, \eta, -\eta(1+a)^{1/2})$, $\zeta = \zeta(x, y, \eta, -\eta(1+a)^{1/2})$. Together with (3.79), (3.80) and (3.81), the last inequality yields

$$(a - xb(y))^{1/2} \geq (2 - \frac{4}{3}c_0)a^{1/2} \geq \frac{3}{2}a^{1/2}, \quad (3.96)$$

if we take $c_0 \leq \frac{3}{8}$ for example (since this values are suitable, c_0 will be fixed in $(0, 3/8]$). This is obviously a contradiction, showing that we can only have $n = k$. \square

Remark 3.36. Lemma 3.35 shows that u_h^n have almost disjoint essential supports in the variables (y, t) . Therefor, Proposition 3.28 applies and shows that U_h defined in (3.68) is also an approximate solution to (1.6) in the sense that

$$\|\square_g U_h(\cdot, t)\|_{L^2(\Omega)} \leq O(h^{-1})\|U_h(\cdot, t)\|_{L^2(\Omega)}. \quad (3.97)$$

In the rest of this section we prove that U_h satisfies the Dirichlet boundary condition

$$U_h|_{\partial\Omega \times [0,1]} = O(h^\infty).$$

Moreover, using the results in the Appendix, we obtain a bound from below of the $L_t^q L^r$ norms of U_h in terms of its L^2 norm. If we had dealt with an exact solution, this bound would have indicated that a loss of derivatives would be necessary in the Strichartz estimates for the wave operator \square_g . The problem is that there is no reason for which U_h would solve (1.6). However, since U_h satisfies (3.97), we are able to handle the error between the parametrix we constructed and the precise solution to (1.6) using the techniques in Section 2.

We proceed with the details:

Proposition 3.37. *(Dirichlet boundary condition for U_h) The approximate solution U_h to (1.6) defined in (3.68) satisfies the Dirichlet boundary condition*

$$U_h|_{\partial\Omega \times [0,1]} = O(h^\infty). \quad (3.98)$$

Proof. Recall from Proposition 3.27 that we have

$$J(Tr_-(u_{F,h}^n))(y, t) + J(Tr_+(u_{F,h}^{n+1}))(y, t) = O_{L^2}(h^\infty),$$

from which we deduce

$$U_h(0, y, t) = J(Tr_+(u_{F,h}^0))(y, t) + J(Tr_-(u_{F,h}^N))(y, t). \quad (3.99)$$

We have to prove that the term in the last line is $O(h^\infty)$ for $t \in [0, 1]$. Notice that the first term in the right hand side of (3.99) is essentially supported for t in a small interval that doesn't meet $[0, 1]$, hence its contribution is clearly trivial for $t \in [0, 1]$. We now proceed with the second term in (3.99): if its essential support in time meets the interval $[0, 1]$, then we construct u_h^{N+1} like in Section 3.3 and we take $U_h + u_h^{N+1}$ instead of U_h as an approximate solution to (1.6); we can repeat this finitely many times, say M times, until $u_{h,-}^{N+M} := Tr_-(u_{F,h}^{N+M})$ will be supported in a small interval outside a neighborhood of $[0, 1]$. Using again Proposition 3.27, the contribution of $u_{h,+}^{N+M-k} := Tr_-(u_{F,h}^{N+M-k})$ will be cancelled by $u_-^{N+M-k-1}$ for all $0 \leq k \leq N+M$ and this allows to conclude. \square

3.4.3 Strichartz estimates for the approximate solution U_h

Proposition 3.38. *Let $r > 4$ and $\epsilon > 0$ be the one fixed in Section 3. We define*

$$\beta(r) = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{1}{6}\left(\frac{1}{4} - \frac{1}{r}\right) \quad (3.100)$$

and let $\beta \leq \beta(r) - \epsilon$. Then the approximate solution U_h of the wave equation (1.6) satisfies

$$h^\beta \|U_h\|_{L^q([0,1], L^r(\Omega))} \geq h^{-7\epsilon/8} \|U_h|_{t=0}\|_{L^2(\Omega)} \gg \|U_h|_{t=0}\|_{L^2(\Omega)}. \quad (3.101)$$

Remark 3.39. Notice that the condition $\beta < \beta(r)$ shows that U_h can't satisfy the Strichartz inequalities of the free case, a loss of at least $\frac{1}{6}(\frac{1}{4} - \frac{1}{r})$ derivatives being unavoidable.

Proof. We estimate from below the $L^q([0, 1], L^r(\Omega))$ norm of U_h using Proposition 4.1 from the Appendix. The key point here is that u_h^n have almost disjoint supports in time and in the tangential variable, hence we can bound from below the $L^q([0, 1])$ norm by a sum of integrals over small intervals of time J_k on which there will be only one cusp, u_h^k to consider, the contribution from all the others being trivial. The intervals J_k will be the ones defined in (3.86) for which Lemma 3.34 applies.

$$\begin{aligned} \|U_h\|_{L^q([0,1], L^r(\Omega))}^q &= \int_0^1 \|U_h\|_{L^r(\Omega)}^q dt = \int_0^1 \left\| \sum_{n=0}^N u_h^n \right\|_{L^r(\Omega)}^q dt \\ &\geq \sum_{k \leq N/5} \int_{t \in J_k} \left\| \sum_{n=0}^N u_h^n \right\|_{L^r(\Omega)}^q dt + O(h^\infty) \\ &\simeq \sum_{k \leq N/5} |J_k| \|u_h^0\|_{L^r(\Omega)}^q + O(h^\infty) \\ &\simeq \|u_h^0\|_{L^r(\Omega)}^q + O(h^\infty). \end{aligned} \quad (3.102)$$

Indeed, we have shown in Lemma 3.35 that for t belonging to small intervals of time J_k there is only u_h^k to be considered in the sum since the contribution from each u_h^n with $n \neq k$ is $O_{L^2}(h^\infty)$. In the last line of (3.102) we have used Lemma 3.34 to estimate from below $|J_k|$, uniformly in k , by $\frac{1}{2}c_0a^{1/2}$, where $c_0 \in (0, 3/8)$ is fixed (as shown in (3.87)), and the fact that $N \simeq a^{-1/2}$.

On the other hand, for $t \in J_k$, the "piece" of cusp $u_h^k(\cdot, t)$ does not "live" enough to reach the boundary, as it is shown in the last part of Lemma 3.34. Precisely, from (3.88) it follows that if $t \in J_k$ then $x \geq a/2$, therefor on the essential support of $u_h^k(\cdot, t)$ the normal variable doesn't approach the boundary; this means that the restrictions of u_h^k to J_k have *disjoint* supports.

Moreover, we see from Proposition 4.1 that for $t \in J_k$ the $L^r(\Omega)$ norms of $u_h^k(\cdot, t)$ are all equivalent to the $L^r(\Omega)$ norm of $u_h^0(\cdot, 0)$. Using Proposition 4.1 we deduce that there are constants C independent of h such that for $r = 2$

$$\|U_h|_{t=0}\|_{L^2(\Omega)} \simeq h\|\partial_t U_h|_{t=0}\|_{L^2(\Omega)} \simeq \|u_h^0(\cdot, 0)\|_{L^2(\Omega)} \simeq h^{1+\frac{\delta}{4}}, \quad (3.103)$$

while for $r > 4$

$$\|U_h\|_{L^q([0,1], L^r(\Omega))} \geq Ch^{\frac{1}{3}+\frac{5}{3r}}. \quad (3.104)$$

Since $\alpha = (1 - \epsilon)/2$ we deduce that for $\beta \leq \beta(r) - \epsilon$ the following holds

$$\begin{aligned} h^\beta \|U_h\|_{L^q([0,1], L^r(\Omega))} &\geq Ch^{\beta(r)-\epsilon} h^{\frac{1}{3}+\frac{5}{3r}} = Ch^{-7\epsilon/8+1+(1-\epsilon)/8} \simeq h^{-7\epsilon/8} \|u_h(\cdot, 0)\|_{L^2} \\ &\gg (\|U_h|_{t=0}\|_{L^2(\Omega)} + h\|\partial_t U_h|_{t=0}\|_{L^2}). \end{aligned} \quad (3.105)$$

Remark 3.40. Using Proposition 4.1 we can also estimate the L^r norms for $2 \leq r < 4$,

$$\|U_h\|_{L^q([0,1], L^r(\Omega))} \geq Ch^{\frac{1}{r}+\frac{1}{2}+\delta(\frac{1}{r}-\frac{1}{4})}. \quad (3.106)$$

Notice however that in this case there is no contradiction when comparing (3.106) to the usual Strichartz inequalities of the free case (1.5) recalled in Proposition 1.1.

□

3.4.4 End of the proof of Theorem 2.1

We can now achieve the proof of Theorem 2.1. Let $\epsilon > 0$ be the one fixed in Section 3 above and N be given by (3.69). Consider the L^2 normalized approximate solution to (1.6)

$$v_{h,\epsilon}^n(x, y, t) := \frac{1}{\|U_h(\cdot, 0)\|_{L^2(\Omega)}} u_h^n(x, y, t),$$

and set

$$\tilde{V}_{h,\epsilon}(x, y, t) := \sum_{n=0}^N v_h^n(x, y, t) = \frac{1}{\|U_h(\cdot, 0)\|_{L^2(\Omega)}} U_h(x, y, t).$$

We claim that $\tilde{V}_{h,\epsilon} v_h^n$ satisfy the conditions of Theorem 2.1. Notice that this would achieve the proof of Theorem 1.2, since in Section 2 we showed that matters can be reduced to proving Theorem 2.1. Indeed, it follows from Proposition 4.1 that for $4 < r < \infty$, $v_{h,\epsilon}^n$ satisfy

$$\begin{cases} \|v_{h,\epsilon}^n(\cdot, t)\|_{L^r(\Omega)} \geq Ch^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{6}(\frac{1}{4}-\frac{1}{r})+2\epsilon}, & \text{for } t \in J_n \\ \sup_{\epsilon>0} \|v_{h,\epsilon}^n(\cdot, t)\|_{L^2(\Omega)} \leq 1, \end{cases} \quad (3.107)$$

where in order to bound uniformly the L^2 norms we use the fact that for $t \in J_n$ and $0 \leq n \leq N$

$$\|u_h^n(\cdot, t)\|_{L^2(\Omega)} \simeq \|u_h^0(\cdot, t)\|_{L^2(\Omega)} \simeq \|u_h^0(\cdot, 0)\|_{L^2(\Omega)} = \|U_h|_{t=0}\|_{L^2(\Omega)}. \quad (3.108)$$

From Proposition 3.33, the cusps $v_{h,\epsilon}^n$ have almost disjoint essential supports in the time and tangential variable and for the normal variable in an interval of size $a = h^\alpha$. Moreover, the approximate solution $\tilde{V}_{h,\epsilon}$ is localized at spatial frequency $1/h$ and satisfies

$$\|\tilde{V}_{h,\epsilon}\|_{L^2(\Omega)} \lesssim 1, \quad \|\partial_y \tilde{V}_{h,\epsilon}\|_{L^2(\Omega)} \lesssim \frac{1}{h}, \quad \|\partial_y^2 \tilde{V}_{h,\epsilon}\|_{L^2(\Omega)} \lesssim \frac{1}{h^2}, \quad (3.109)$$

with constants independent of ϵ , which follows from the spectral localization together with the uniform bounds of the derivatives of g_h^n with respect to y .

From Proposition 3.28 and the almost orthogonality property of the supports in y we also obtain

$$\square_g \tilde{V}_{h,\epsilon} = O_{L^2(\Omega)}(1/h). \quad (3.110)$$

Finally, Proposition 3.37 assures that the Dirichlet boundary condition is satisfied by the restriction of $\tilde{V}_{h,\epsilon}$ to the interval of time $[0, 1]$:

$$\tilde{V}_{h,\epsilon}|_{\partial\Omega \times [0,1]} = O(h^\infty). \quad (3.111)$$

4 Appendix

In this section we compute the L^r norms of the phase integrals associated to a cusp type Lagrangian. We prove the following:

Proposition 4.1. *For $t \in J_n$ defined in (3.86), the $L^r(\Omega)$ norms of a cusp $u_h^n(\cdot, t)$ of the form (3.61) satisfy*

- for $2 \leq r < 4$

$$\|u_h^n(\cdot, t)\|_{L^r(\Omega)} \simeq h^{\frac{1}{r} + \frac{1}{2} + \delta(\frac{1}{r} - \frac{1}{4})}, \quad (4.1)$$

$$\|u_h^n(\cdot, 0)\|_{L^2(\Omega)} \simeq h^{1 + \frac{\delta}{4}}; \quad (4.2)$$

- for $r > 4$

$$\|u_h^n(\cdot, t)\|_{L^r(\Omega)} \simeq h^{\frac{1}{3} + \frac{5}{3r}}. \quad (4.3)$$

Proof. Recall that $\lambda = h^{3\alpha/2-1} \gg 1$, $a = h^\alpha$, $N \simeq \lambda h^\epsilon$, $\epsilon > 0$ and let $n \in \{0, \dots, N\}$ be fixed. Recall the form of the cusp u_h^n ,

$$u_h^n(x, y, t) := \int_{s, \eta} e^{\frac{i}{h}\Phi^n(x, y, t, \eta^{1/3}s, \eta, -\eta(1+a)^{1/2})} g_h^n\left(\frac{\partial_\tau \theta + \eta^{1/3}s\partial_\tau \zeta}{\partial_\tau \zeta_0(-\zeta_0)^{1/2}} - 2n, \eta, \lambda\right) ds d\eta, \quad (4.4)$$

where

$$\Phi^n(x, y, t, \xi, \eta, \tau) := \theta(x, y, t, \eta, \tau) + \xi\zeta(x, y, \eta, \tau) + \frac{\xi^3}{3} + \frac{4}{3}n(-\zeta_0)^{3/2}(\eta, \tau) \quad (4.5)$$

and where, according to Lemma 3.21, the symbol satisfies $g_h^n(\cdot, y, \eta) \in \mathcal{S}_{K_0}(\lambda/(n+1))$. Recall also from (3.25) that

$$\zeta(x, y, \eta, -\eta(1+a)^{1/2}) = (xb(y) - a)\eta^{2/3}.$$

To estimate the $L^r(\Omega)$ norms of u_h^n , we must distinguish three cases:

- If $|xb(y) - a| \leq Mh^{2/3}$ for some constant $M \geq 1$, we make the change of variables $x = x(X, y) := (a - h^{2/3}X)/b(y)$ which is allowed since for $t \in J_n$, on the essential support of u_h^n the factor $b(y)$ remains close to 1 by Proposition 3.33.

In order to estimate the L^r norm with respect to y we also need to make the change of variables $y \rightarrow \partial_\eta \theta(x(X, y), y, t, \eta, -\eta(1+a)^{1/2}) + \frac{4}{3}na^{3/2} =: Y(X, y, t)$. We verify that the jacobian of this application doesn't vanish. Since

$$\begin{aligned} \partial_\eta(\theta(x, y, t, \eta, -\eta(1+a)^{1/2})) &= \partial_\eta \theta(x, y, t, 1, -(1+a)^{1/2}) \\ &\quad - (1+a)^{1/2} \partial_\tau \theta(x, y, t, 1, -(1+a)^{1/2}), \end{aligned} \quad (4.6)$$

we compute

$$\begin{aligned} \frac{dY}{dy}(X, y, t) &= \left(\partial_{y, \eta}^2 \theta + \frac{dx(X, y)}{dy} \partial_{x, \eta}^2 \theta - (1+a)^{1/2} \frac{dx(X, y)}{dy} \partial_{x, \tau}^2 \theta \right. \\ &\quad \left. - (1+a)^{1/2} \partial_{y, \tau}^2 \theta \right) \Big|_{(x(X, y), y, t, 1, -(1+a)^{1/2})}. \end{aligned} \quad (4.7)$$

From (3.26), (3.27) of Section 3.1.3 we have

$$\begin{aligned} \partial_{\eta, y}^2 \theta_0(y, t, 1, -(1+a)^{1/2}) &= \frac{b^2(y)}{\sqrt{1+a(1-b^2(y))}}, & \partial_{\eta, x}^2 \theta_0(y, t, 1, -(1+a)^{1/2}) &= 0, \\ \partial_{\tau, y}^2 \theta_0(y, t, 1, -(1+a)^{1/2}) &= \frac{(1+a)^{1/2}(1-b^2(y))}{\sqrt{1+a(1-b^2(y))}}, & \partial_{\tau, x}^2 \theta_0(y, t, 1, -(1+a)^{1/2}) &= 0, \end{aligned}$$

and since from the curvature assumption it follows that in a neighborhood of 0 we have $|b(y) - 1| \leq b_0$ for some $0 < b_0 < 1/4$, we deduce that $\frac{dY}{dy}(X, y, t) \neq 0$ in a neighborhood

of (ρ_0, ϑ_0) (here we also used that if $t \in J_n$ then $x \lesssim a$ on the essential support of $u_h^n(\cdot, t)$). We write $y = y(X, Y, t)$ and set $s = h^{1/3}\xi$. We set $Q(X, u) = \frac{\xi^3}{3} - X\xi$ and for $\beta : \mathbb{R} \rightarrow [0, 1]$ we define

$$\begin{aligned} f_\beta^n(X, Y, t, \eta, h) &:= \int e^{i\eta Q(X, \xi)} \beta(\xi) \\ &\times g_h^n\left(\frac{\partial_\tau \theta(x(X, y), y(X, Y, t), t, 1, -(1+a)^{1/2})}{2a^{1/2}(1+a)^{1/2}} + h^{1/3-\alpha/2}\xi - 2n, y(X, Y, t), \eta, \lambda\right) d\xi. \end{aligned} \quad (4.8)$$

We introduce

$$F_\beta^n(X, Y, t, h) := \int e^{i\eta Y/h} f_\beta^n(X, Y, t, \eta, h) d\eta, \quad (4.9)$$

and we make integrations by parts with respect to η in order to compute

$$Y^p F_\beta^n(X, Y, t, h) = (ih)^p \int e^{i\eta Y/h} \partial_\eta^p f_\beta^n(X, Y, t, \eta, h) d\eta. \quad (4.10)$$

The derivatives of f_β^n are estimated using the precise form of the symbol g_h^n : recall that

$$g_h^n(z, y, \eta) \simeq \Psi(\eta) \eta^{1/3} \left(\sum_{k \geq 0} h^{k/2} a^{-k/2} \mu_k(y, \eta, h) \partial_z^k \varrho^n(z, \eta, \lambda) \right), \quad (4.11)$$

where

$$\partial_z^k \varrho^n(z, \eta, \lambda) = (F_{\eta\lambda})^{*n} * \partial_z^k \varrho^0(\cdot, \lambda)(z), \quad \forall k \geq 0, \quad (4.12)$$

with $\varrho^0(\cdot, \lambda) \in \mathcal{S}_{K_0}(\lambda)$ independent of η and where $(F_{\eta\lambda})^{*n}$ is defined in (3.43). The derivatives of $(F_{\eta\lambda})^{*n}$ with respect to η were estimated in the proof of Lemma 3.31 and we saw there that the convolution product of those derivatives with ϱ^0 is bounded uniformly in n . On the other hand, μ_k are symbols of order 0 and type $(1, 0)$ hence all the derivatives $\partial_\eta^{p-l}(\eta^{1/3}\Psi(\eta)\mu_k)(y, \eta, h)$ are bounded (on the support of $\Psi(\eta)$) by constants C_{p-l} . We now estimate $\partial_\eta^p f_\beta^n$:

$$\begin{aligned} \partial_\eta^p f_\beta^n(X, Y, t, \eta, h) &= \int e^{i\eta Q(X, \xi)} \beta(\xi) \sum_{l=0}^p C_p^l (iQ(X, \xi))^{p-l} \\ &\times \partial_\eta^l g_h^n\left(\frac{\partial_\tau \theta(x(X, y), y(X, Y, t), t, 1, -(1+a)^{1/2})}{2a^{1/2}(1+a)^{1/2}} + h^{1/3-\alpha/2}\xi - 2n, \eta, \lambda\right) d\xi. \end{aligned} \quad (4.13)$$

To estimate $\partial_\eta^l g_h^n$ we use (4.11). We first take $\beta(u) := 1_{|u| \leq \sqrt{1+M}}$ and estimate the L^∞ norms of $w^p F_\beta^n$ in this case, in order to use the multiplier's theorem to bound the L^r

norms of u_h^n . We have

$$\begin{aligned} \left\| \left(\frac{Y}{h} \right)^p F_\beta^n(X, Y, t, h) \right\|_{L_Y^\infty} &\leq \sum_{k \geq 0} h^{k/2} a^{-k/2} \sum_{l=0}^p C(p, l) \sup_{|u| \leq \sqrt{1+M}} |Q(X, u)|^{l-j} \\ &\quad \times \sum_{j=0}^l \int_\eta |\partial_\eta^{l-j} (\Psi(\eta) \eta^{1/3} \mu_k(y, \eta, h))| \\ &\quad \times |\partial_\eta^j \partial_z^k \varrho^n \left(\frac{\partial_\tau \theta(x(X, y), y(X, Y, t), t, 1, -(1+a)^{1/2})}{2a^{1/2}(1+a)^{1/2}} + h^{1/3-\delta/2} u - 2n, \eta, \lambda) \right)| d\eta \\ &\leq C_{p,M}, \end{aligned} \quad (4.14)$$

for some constants $C_{p,M}$ depending only of p and M . The derivatives of $\partial_z^k \varrho^n$ with respect of η are estimated using the formula (4.12) as in the proof of Lemma 3.31. Following the computations in that proof, we obtain the bounds (4.14) uniformly in n .

However, we have not finished yet since we need an uniform bound of $\|W^p F_1^n(X, hW, t, h)\|_{L^\infty}$ and therefor we have to consider also the case $1-\beta$. On the support of $(1-\beta)$ we have $\sqrt{1+M} \leq |u| \lesssim h^{\frac{\alpha}{2}-\frac{1}{3}}$, and the contribution of the integral (4.4) is $O_{L^2}(h^\infty)$. Indeed, in that case $|s^2 + \eta^{-2/3} \zeta| \geq h^{2/3}$ and using the operator $L_1 = \frac{h}{i} \frac{1}{|s^2 + \eta^{-2/3} \zeta|} \partial_s$ like in the proof of Lemma 3.31 we obtain at each integration by parts a factor $h^{1/3-\alpha/2}$.

As a result we obtain

$$\begin{aligned} \|u_h^n(\cdot, t)\|_{L^r(|xb(y)-a| \leq Mh^{2/3}, y)}^r &= h^{r/3} \int_{|X| \leq h^{2/3}} |F_1^n(X, Y, t, h)|^r \frac{dx(X, y)}{dX} \frac{dy(X, Y, t)}{dY} dY dX \\ &\stackrel{Y=hW}{\simeq} h^{5/3+r/3} \|F_1^n(X, hW, t, h)\|_{L^r(|X| \leq M, W)}^r \\ &\simeq h^{5/3+r/3}, \end{aligned} \quad (4.15)$$

where in the second line we used the estimates (4.14), while in the last line in (4.15) we used the fact that μ_0 is elliptic.

- If $|xb(y) - a| \in (Mh^{2/3}, a]$ with $M \gg 1$ big enough we apply the stationary phase:

Proposition 4.2. ([14, Thm.7.7.5]) *Let $K \subset \mathbb{R}$ be a compact set, $f \in C_0^\infty(K)$, $\phi \in C^\infty(\overset{\circ}{K})$ such that $\phi(0) = \phi'(0) = 0$, $\phi''(0) \neq 0$, $\phi' \neq 0$ in $\overset{\circ}{K} \setminus \{0\}$. Let $\omega \gg 1$, then for every $k \geq 1$ we have*

$$\int e^{i\omega\phi(u)} f(u) du \simeq \frac{(2\pi i)^{1/2} e^{i\omega\phi(0)}}{(\omega\phi''(0))^{1/2}} \sum_{j \geq 0} \omega^{-j} L_j f. \quad (4.16)$$

Here C is bounded when ϕ stays in a bounded set in $C^\infty(\overset{\circ}{K})$, $|u|/|\phi'(u)|$ has a uniform bound and

$$L_j f = \sum_{\nu-\mu=j} \sum_{2\nu \geq 3\mu} \frac{i^{-j} 2^{-\nu}}{\mu! \nu!} (\phi''(0))^{-\nu} \partial^{2\nu} (\kappa^\mu f)(0). \quad (4.17)$$

where $\kappa(u) = \phi(u) - \phi(0) - \frac{\phi''(0)}{2} u^2$ vanishes of third order at 0.

We make the change of variable $s = (a - xb(y))^{1/2}(\pm 1 + u)$ to compute the integral in s in (4.4). Recall that it writes

$$\sum_{k \geq 0} h^{k/2} a^{-k/2} \Psi(\eta) \eta^{1/3} \mu_k(y, \eta, h) \int_s e^{\frac{i}{h} \eta (\frac{s^3}{3} - s(xb(y) - a))} \partial_z^k \varrho^n(z + s/a^{1/2} - 2n, \eta, \lambda) ds, \quad (4.18)$$

where $z := \frac{\partial_\tau \theta(x, y, t, 1, -(1+a)^{1/2})}{2a^{1/2}(1+a)^{1/2}}$. Using Proposition 4.2 with $\phi_\pm(u) = \frac{u^3}{3} \pm u^2$ and large parameter $\omega = (a - xb(y))^{3/2}/h \gg 1$, $\kappa_\pm(u) = u^3/3$, the integral in s in (4.18) becomes

$$\begin{aligned} & (h\pi)^{1/2} \eta^{-2/3} (a - xb(y))^{-1/4} e^{\mp \frac{2}{3} i \eta (a - xb(y))^{3/2} / h \pm \frac{i\pi}{4}} \sum_{j \geq 0} h^j (a - xb(y))^{-3j/2} \eta^{-j} \\ & \times L_j \left(\partial_z^k \varrho^n \left(\frac{\partial_\tau \theta(x, y, t, 1, -(1+a)^{1/2})}{2a^{1/2}(1+a)^{1/2}} + (a - xb(y))^{1/2}(\pm 1 + u)/a^{1/2} - 2n, \eta, \lambda \right) \right) \Big|_{u=0}. \end{aligned} \quad (4.19)$$

Since $\partial_z^k \varrho^n$ writes as the convolution product (4.12), we have, for z as above,

$$\begin{aligned} u_h^n(x, y, t) &= (\pi h)^{1/2} \sum_{k \geq 0} h^{k/2} a^{-k/2} \sum_{j \geq 0} h^j (a - xb(y))^{-3j/2 - 1/4} L_j(\partial_z^k \varrho^0)(\cdot) * \\ & * \int_\eta e^{\frac{i}{h} \eta (\theta(x, y, t, 1, -(1+a)^{1/2}) + \frac{4}{3} n a^{3/2} \mp \frac{2}{3} (a - xb(y))^{3/2})} \frac{\Psi(\eta)}{\eta^{j+1}} \mu_k(y, \eta, h) (F_{\eta\lambda})^{*n} d\eta \Big|_{(z \pm (a - xb(y))^{1/2} / a^{1/2} - 2n)}. \end{aligned} \quad (4.20)$$

We set

$$F^{n, k, j}(\cdot, y, \eta, h) := \frac{\Psi(\eta)}{\eta^{j+1}} \mu_k(y, \eta, h) (F_{\eta\lambda})^{*n}(\cdot).$$

Since $\Psi(\eta)$ is compactly supported for η in a neighborhood of 1, the Fourier transform $\widehat{F^{n, k, j}}$ with respect to η of each $F^{n, k, j}$ is rapidly decreasing and the integral in η in (4.20) becomes

$$\widehat{F^{n, k, j}}(\cdot, y, \frac{(\theta(x, y, t, 1, -(1+a)^{1/2}) + \frac{4}{3} n a^{3/2} \mp \frac{2}{3} (a - xb(y))^{3/2})}{h}, h)$$

We make again the changes of variables $x = x(X, y) := (a - h^{2/3} X)/b(y)$, then $y = y(X, Y, t)$ where $Y = \theta(x(X, y), y, t, 1, -(1+a)^{1/2}) + \frac{4}{3} n a^{3/2}$ and finally $Y = hW$. Then (4.20) becomes

$$u_h^n(x, y, t) := \pi^{1/2} \sum_{k, j \geq 0} h^{k/2} a^{-k/2} u_h^{n, k, j}(x, y, t), \quad (4.21)$$

where we define, for $z = \frac{\partial_\tau \theta(x(X, y), y(X, hW, t), t, 1, -(1+a)^{1/2})}{2a^{1/2}(1+a)^{1/2}}$,

$$\begin{aligned} u_h^{n, k, j}(x, y, t) &:= h^{1/3} X^{-1/4 - 3j/2} L_j(\partial_z^k \varrho^0(\cdot, \lambda)) * \\ & * \widehat{F^{n, k, j}}(\cdot, y(X, hW, t), W \mp \frac{2}{3} X^{3/2}, h) \Big|_{(z + h^{1/3} - \alpha/2 X^{1/2} - 2n)}. \end{aligned} \quad (4.22)$$

- If $r > 4$ then a simple computation shows that for $k \geq 0$ the L^r norms of each $u^{n,k,j}$ can be estimated from above as follows

$$\begin{aligned} \|u_h^{n,k,j}(\cdot, t)\|_{L^r((a-xb(y)) \in (Mh^{2/3}, a], y)}^r &\lesssim h^{r(1/2+j+5/3r-1/6-j)} \int_M^{Ah^{-2/3}} X^{-r(1/4+3j/2)} dX \\ &\lesssim h^{r/3+5/3} \frac{M^{1-r(1/4+3j/2)}}{(r(1/4+3j/2)-1)}, \end{aligned} \quad (4.23)$$

and since the operators L_j are of order $2j$, for each j there will be $2j$ terms in the sum $\sum_j u_h^{n,k,j}$: summing up over $j \geq 0$ (taking $M \geq 2$ for example) and using that $\varrho^n \in \mathcal{S}_{K_0}(\lambda/(n+1))$ (which assures uniform bounds for the derivatives $\partial_z^k \varrho^n$ for each $n, k \geq 0$), yields

$$\left\| \sum_{j \geq 0} u_h^{n,k,j}(\cdot, t) \right\|_{L^r((a-xb(y)) \in (Mh^{2/3}, a], y)} \lesssim C(r) h^{r/3+5/3}, \quad C(r) = \frac{1}{r/4-1}.$$

On the other hand

$$\|u_h^n(\cdot, t)\|_{L^r((a-xb(y)) \in (Mh^{2/3}, a], y)} \lesssim \sum_{k \geq 0} h^{k(1-\delta/2)} \left\| \sum_{j \geq 0} u_h^{n,k,j}(\cdot, t) \right\|_{L^r((a-xb(y)) \in (Mh^{2/3}, a], y)}.$$

For $k = 0$, due to the ellipticity of the symbol $\mu_0(y, \eta, h) = a_h(y, \eta, -\eta(1+a)^{1/2})$ we can estimate also from below the L^r norm of $u_h^{n,0,j}(\cdot, t)$ by $C(r)h^{1/3+5/3r}$ and consequently we deduce

$$\|u_h^{n,0}(\cdot, t)\|_{L^r((a-xb(y)) \in (Mh^{2/3}, a], y)} \simeq C(r)h^{r/3+5/3}.$$

Hence (4.2) follows.

- We deal now with the case $r \in [2, 4)$: since in this case we do not expect any contradiction to the usual Strichartz estimates, we shall concentrate on the L^2 norms of $\sum_{j \geq 0} u_h^{n,k,j}(\cdot, t)$ only, and prove (4.2). We let (4.1) as an exercise, since in the proof of Theorem 1.2 we do not use this estimate.

For $j = 0$ we compute, as before

$$\int_M^{ah^{-2/3}} X^{-1/2} dX \simeq 2(ah^{-2/3})^{1/2},$$

while for $j \geq 1$ we have $2(1/4+3j/2)-1 > 0$ and

$$\int_M^{ah^{-2/3}} X^{-2(1/4+3j/2)} dX = -\frac{X^{1-2(1/4+3j/2)}|_M^{ah^{-2/3}}}{2(1/4+3j/2)-1} \simeq \frac{M^{1/2-3j}}{3j-1/2}.$$

For $M \geq 2$ the sum of $\|u_h^{n,k,j}(\cdot, y, t)\|_{L^2((a-xb(y)) \in (Mh^{2/3}, a], y)}^2$ over $j \geq 1$ (where for each j we count $2j$ terms which appear in the expression of $L_j(\partial_z^k \varrho^n)$) is small

enough compared to $\|u_h^{n,k,0}(\cdot, y, t)\|_{L^2((a-xb(y)) \in (Mh^{2/3}, a], y)}$, while for $k = 0$ we can estimate also from below, as before

$$\left\| \sum_{j \geq 0} u_h^{n,0,j}(\cdot, t) \right\|_{L^2((a-xb(y)) \in (Mh^{2/3}, a], y)} \simeq h^{1/3+5/6} (ah^{-2/3})^{1/4} = h^{1+\delta/8}.$$

We have proved (4.1) for $r = 2$; for $r \in (2, 4)$, as already mentioned, we do not give the proof since it is not used in the proof of Theorem 2.1 and since it follows exactly in the same way as for $r = 2$.

- We say a few words about the last regime, $(a - xb(y)) \geq Ma$ for some $M > 1$: in this case we use Lemma 3.31 we obtain that the contribution in each $u_h^n(\cdot, t)$ is $O_{L^2}(h^\infty)$, since in this case we are localized away from a neighborhood of the Lagrangian Λ_{Φ^n} .

□

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