

The CRT is the scaling limit of unordered binary trees

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We prove that a uniform, rooted unordered binary tree with n vertices has the Brownian continuum random tree as its scaling limit for the Gromov-Hausdorff topology. The limit is thus, up to a constant factor, the same as that of uniform plane trees or labeled trees. Our analysis rests on a combinatorial and probabilistic study of appropriate trimming procedures of trees.

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The Brownian Continuum Random Tree (CRT), introduced by Aldous [2], is a natural object that arises in various situations in Probability Theory. It is known to be the universal scaling limit for conditioned critical Galton-Watson trees with finite variance offspring distribution [4, 21, 12], or of random labeled trees on n vertices (Cayley trees) [2, 10, 1].

Several distinct proofs for the convergence of discrete trees towards the CRT exist in the literature, taking advantage of the specific aspects of the considered models, which in passing yield various equivalent constructions of the CRT. For instance, a specific¹ proof of the convergence of Cayley trees rests on the fact that a uniform Cayley tree is a uniform spanning tree of the complete graph, which can be constructed *via* the Aldous-Broder algorithm and leads to a limiting “stick-breaking construction” of the CRT. On the other hand, the convergence of conditioned Galton-Watson trees to the CRT can be obtained by appropriate encodings of trees by random walks. In [3], Aldous conjectures that many other models of trees, for which it is harder to have a good probabilistic understanding, also have the CRT as a continuum limit.

In this article, we will focus on one of these models, namely, the family of rooted binary unordered trees, considered as graph-theoretic trees without planar or labeled structure. The main goal of this paper is to prove that a uniformly chosen rooted binary unordered tree with n leaves converges, after renormalization of distances by \sqrt{n} , to a constant multiple of the CRT.

From a combinatorial perspective, models of plane trees are very close to the case studied here: their generating functions' singularities are of the same type (a square-root singularity), and heuristically, combinatorial families with generating functions bearing the same singularity type should have a similar continuous limit. Nevertheless, the probabilistic methods developed for plane or labeled trees are not valid anymore for unordered trees, and another method has to be developed to derive their scaling limits. The present work is a step in this direction, for the particular case of binary, rooted, unordered trees.

We mention that in a very recent paper, another model of rooted unlabeled, unordered trees has been investigated by Drmota & Gittenberger [11]: this is the model of Pólya trees where

¹This can also be obtained in the framework of Galton-Watson trees by choosing the particular Poisson offspring distribution

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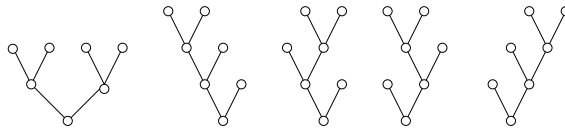


Figure 1: Representation of the 5 rooted binary plane trees with 4 leaves. There are 2 (rooted) binary trees with 4 leaves (the 4 last ones on the picture representing the same unordered tree).

there is no restriction on the authorized degrees. They compute the limiting height and limiting profile of these trees (with n nodes, when $n \rightarrow +\infty$) and show that, up to constant factors, they respectively converge to the maximum of the Brownian excursion, and the local time of the Brownian excursion (which is similar to the case of binary plane trees). Even if this result is of a different kind from ours, and does not imply the convergence of the rescaled tree to the CRT for the Gromov-Hausdorff topology, it gives an important indication that it should be the case.

1.1 Plane trees

Let us recall the standard definition of plane trees (see Figures 1, 2 and 3). Let $\mathcal{U} = \bigsqcup_{n \geq 0} \mathbb{N}^n$ be the set of words with integer letters, where $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^0 = \{\emptyset\}$. For $u \in \mathcal{U}$, let $|u|$ be its length, and we generally note $u = u_1 \dots u_n$ with $n = |u|$. The concatenation of the words u and v is written uv , and we write $u \preceq v$ if u is a prefix of v , meaning that there exists $w \in \mathcal{U}$ such that $uw = v$. This defines a partial order on \mathcal{U} . If $A \subset \mathcal{U}$ and $u \in \mathcal{U}$, we let

$$uA = \{uv : v \in A\}.$$

If $u, v \in \mathcal{U}$, we let $u \wedge v$ be the longest common prefix to u and v . The set \mathcal{U} is endowed with the lexicographical order \leq : we have $u \leq v$ if $u \preceq v$ or if $u \wedge v$ is a strict prefix of u and v such that $u_{|u \wedge v|+1} < v_{|u \wedge v|+1}$ (on \mathbb{N}). The order (\mathcal{U}, \leq) is total.

Definition 1 *A rooted plane tree is a finite subset $\mathfrak{t} \subset \mathcal{U}$ containing \emptyset , such that if $ui \in \mathfrak{t}$ with $u \in \mathcal{U}$ and $i \in \mathbb{N}$, then*

- $uj \in \mathfrak{t}$ for $1 \leq j \leq i$, and
- $u \in \mathfrak{t}$.

(See Figure 2). The elements of \mathfrak{t} are called vertices, and \emptyset is called the root. A vertex $ui \in \mathfrak{t}$, with $u \in \mathcal{U}$ and $i \in \mathbb{N}$, is called a child of u . Their number is denoted by $c_u(\mathfrak{t}) = \sup\{i \geq 1 : ui \in \mathfrak{t}\} \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$. The notion of brothers, ancestors, descendants are induced by that of child as in the standard life. The length $|u|$ is equally called height of u .

In a plane tree \mathfrak{t} , the subtree of \mathfrak{t} rooted at $u \in \mathfrak{t}$ is the plane tree $\mathfrak{t}_u = \{v \in \mathcal{U} : uv \in \mathfrak{t}\}$.

A plane tree has a representation as a plane graph, where each vertex is linked by an edge to its children, which are ordered from left to right in lexicographical order (as done on Figure 2).

We say that a rooted plane tree \mathfrak{t} is binary if $c_u(\mathfrak{t}) \in \{0, 2\}$ for every $u \in \mathfrak{t}$. The vertices having no children are called the leaves the set of which is denoted by $L(\mathfrak{t})$, while $I(\mathfrak{t}) = \mathfrak{t} \setminus L(\mathfrak{t})$.

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denotes the set of *internal vertices*. In turn, the internal vertices can be partitioned into three sets

$$I(\mathfrak{t}) = I_0(\mathfrak{t}) \sqcup I_1(\mathfrak{t}) \sqcup I_2(\mathfrak{t}),$$

where for $i \in \{0, 1, 2\}$, $I_i(\mathfrak{t})$ is the set of internal vertices having i children being themselves internal vertices (see Figure 2). We let

$$S(\mathfrak{t}) = L(\mathfrak{t}) \setminus \{u_1, u_2 : u \in I_0(\mathfrak{t})\},$$

the set of leaves that are children of the vertices of $I_1(\mathfrak{t})$. These leaves will be called “skeleton leaves” of \mathfrak{t} .

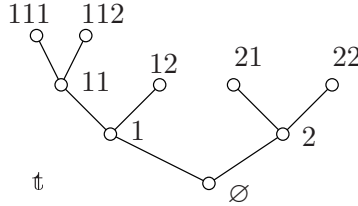


Figure 2: A plane tree \mathfrak{t} such that $L(\mathfrak{t}) = \{111, 112, 12, 21, 22\}$, $I_2(\mathfrak{t}) = \{\emptyset\}$, $I_0(\mathfrak{t}) = \{11, 2\}$, $I_1(\mathfrak{t}) = \{1\}$, $S(\mathfrak{t}) = \{12\}$.

We let $|\mathfrak{t}|$ denote the number of leaves of \mathfrak{t} . We let \mathbb{T} be the set of binary rooted plane trees, and $\mathbb{T}_n = \{\mathfrak{t} : |\mathfrak{t}| = n\} \subset \mathbb{T}$ be the subset of those having n leaves (hence $2n - 1$ vertices and $2n - 2$ edges). It is a simple exercise to enumerate rooted binary plane trees via their generating functions. Namely, we have

$$\#\mathbb{T}_n = \frac{1}{n} \binom{2n-2}{n-1},$$

the $(n - 1)$ -th Catalan number, which also counts numerous families of combinatorial objects. Note the equivalent

$$\#\mathbb{T}_n \underset{n \rightarrow \infty}{\sim} \frac{4^{n-1}}{\sqrt{\pi n^{3/2}}}. \quad (1)$$

1.2 Unordered trees

A rooted, binary unordered tree is a rooted, binary plane tree in which the planar order has been “forgotten” (but not the root). Two rooted binary plane trees \mathfrak{t} and \mathfrak{t}' are equivalent if \mathfrak{t}' can be obtained from \mathfrak{t} by exchanging the order of the various children of \mathfrak{t} , while preserving the rooted genealogical structure: ancestors and brothers remain as such in the process. Formally, consider $\mathfrak{t} \in \mathbb{T}$ and a family of permutations $\sigma = (\sigma_u, u \in I(\mathfrak{t}))$ of the set $\{1, 2\}$, indexed by the internal vertices of \mathfrak{t} . For $u = u_1 \dots u_n \in \mathfrak{t}$, we let

$$\sigma(u) = \sigma_{\emptyset}(u_1)\sigma_{u_1}(u_2) \dots \sigma_{u_1 \dots u_{n-1}}(u_n),$$

and $\sigma(\emptyset) = \emptyset$. Now for $\mathfrak{t}' \in \mathbb{T}$ we say that $\mathfrak{t} \sim \mathfrak{t}'$ if there exists such a σ with

$$\mathfrak{t}' = \sigma(\mathfrak{t}).$$

Note that there are at most $2^{|\mathfrak{t}|-1}$ elements in the class of \mathfrak{t} , since $\#I(\mathfrak{t}) = |\mathfrak{t}| - 1$.

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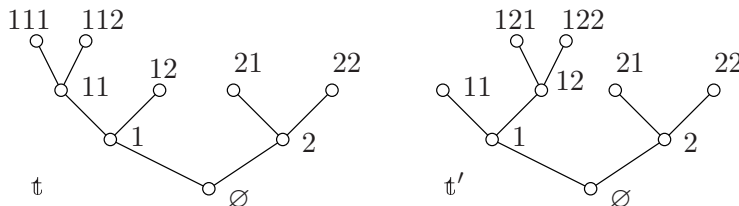


Figure 3: Two plane trees \mathfrak{t} and \mathfrak{t}' that are equal as trees.

Definition 2 *The quotient set $\mathbf{T} = \mathbb{T} / \sim$ is called the set of (binary, rooted, unordered) trees. projection. We also let $\mathbf{T}_n = \mathbb{T}_n / \sim$ be those trees which have n leaves, and set $|\mathfrak{t}| = n$ for $\mathfrak{t} \in \mathbf{T}_n$.*

It will be convenient to fix once and for all a section $s : \mathbf{T} \rightarrow \mathbb{T}$ of the canonical projection $\mathbb{T} \rightarrow \mathbf{T}$, so $s(\mathfrak{t})$ is a choice among all the planar representatives of \mathfrak{t} . In this way, there will be a non-ambiguous way to consider a vertex of an unordered tree, the number of its children, one of its edges, and so on.

The numbers $(\#\mathbf{T}_n, n \geq 1)$ are the *Etherington-Wedderburn numbers*, the first ones are 1, 1, 1, 2, 3, 6, 11, 23, 46, 98, 207, 451, ... (referred to as A001190 in Sloane's On-Line Encyclopedia of Integer Sequences). No closed formula is available, but the asymptotic enumeration of elements of \mathbf{T} has been studied by Otter [22], using the properties of the generating function for trees². Let $\mathfrak{T}(z) = \sum_{n \geq 1} \#\mathbf{T}_n z^n$ be the generating function for rooted binary trees, counted according to their number of leaves. A decomposition of a tree at its root into a multiset of two subtrees yields the formula

$$\mathfrak{T}(z) = z + \frac{\mathfrak{T}(z^2) + \mathfrak{T}(z)^2}{2}. \quad (2)$$

One shows that the radius of convergence ϱ of T satisfies

$$\mathfrak{T}(\varrho) = 1. \quad (3)$$

A detailed analysis of the recursive equation (2) entails

$$\#\mathbf{T}_n \underset{n \rightarrow \infty}{\sim} \frac{\mathbf{c}}{2\sqrt{\pi n^{3/2}} \varrho^n}, \quad (4)$$

where

$$\mathbf{c} := \sqrt{2\varrho + 2\varrho^2 \mathfrak{T}'(\varrho^2)} = 1.1300337 \dots, \quad (5)$$

see [18, Note VII.22, p.477]. The equivalent (4) is similar to (1), which hints at the similarity of structure of large trees with large planar trees.

Important notice. From this section onwards, we will only be interested in rooted, binary, plane or unordered trees, and the words “rooted, binary”, will *always* be implicit when dealing with the two combinatorial families of trees we have just defined. To avoid confusion, from now on,

²There is a slight difference in the article of Otter, as he considers trees with vertex-degrees either equal to 1, 2 or 3. It is easy to see that this is actually equivalent to the family of binary unordered trees, by adding “ghost” edges at every vertex with only one child.

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- a *plane tree* stands for a rooted, binary plane tree,
- while a *tree* is a rooted, binary, unordered tree.

1.3 Gromov-Hausdorff topology

It is natural to consider trees (plane or not) as finite metric spaces, by endowing the set of their vertices with the usual graph distance. Formally, for $\mathfrak{t} \in \mathbb{T}$ and $u, v \in \mathfrak{t}$, we let

$$d_{\mathfrak{t}}(u, v) = |u| + |v| - 2|u \wedge v|.$$

In the sequel, we will often improperly identify \mathfrak{t} with the metric space $(\mathfrak{t}, d_{\mathfrak{t}})$, or even with its isometry class, i.e. the collection of metric spaces that are isometric to it. It is straightforward that $(\mathfrak{t}, d_{\mathfrak{t}})$ and $(\mathfrak{t}', d_{\mathfrak{t}'})$ are isometric spaces whenever $\mathfrak{t} \sim \mathfrak{t}'$, so that a tree \mathfrak{t} will also denote the isometry class of the metric space $(s(\mathfrak{t}), d_{s(\mathfrak{t})})$. We will adopt the notation $a(X, d) = (X, ad)$ for (X, d) a metric space (identified with its isometry class) and $a > 0$. Therefore, the notation $a\mathfrak{t}, at$ will stand for the metric spaces $\mathfrak{t}, \mathfrak{t}$ with distances multiplied by a .

There is a common way to define a topology (even a metric) on spaces of metric spaces. Such topologies have been developed in Geometry for the last 30 years, following the ideas of Gromov [19]. Their use in Probability in the context of random real trees has been popularized by Evans and his coauthors [15, 16, 17], and has been applied successfully in various situations, for instance by Duquesne & Le Gall [13], Le Gall [20]. We let \mathcal{M} be the set of isometry classes of compact metric spaces, which we endow with the *Gromov-Hausdorff distance* defined by

$$d_{\text{GH}}((X, d), (X', d')) = \inf_{\phi, \phi'} \delta_H(\phi(X), \phi'(X')), \quad (6)$$

the infimum being taken over the set of isometric embeddings ϕ, ϕ' from $(X, d), (X', d')$ into a common metric space (Z, δ) , and where δ_H is the usual Hausdorff distance between compact subsets of Z . The reader can consult [9, Chapter 7] for basic properties of this distance, which turns \mathcal{M} into a Polish metric space (this is a straightforward extension of the proof of [16, Theorem 1]).

Of particular importance for our purposes is the subset $\mathcal{T} \subset \mathcal{M}$ of \mathbb{R} -trees, i.e. of compact spaces (\mathcal{T}, d) such that for every pair of points $x, y \in \mathcal{T}$,

- there exists an isometry $\varphi_{x,y} : [0, d(x, y)] \rightarrow \mathcal{T}$ with $\varphi_{x,y}(0) = x$ and $\varphi_{x,y}(d(x, y)) = y$.
- for every continuous injective $q : [0, 1] \rightarrow \mathcal{T}$ with $q(0) = x$ and $q(1) = y$, it holds that $q([0, 1]) = \varphi_{x,y}([0, d(x, y)])$.

The first property says that (\mathcal{T}, d) is a *geodesic space*, and the second is a “tree property” that there is a unique way to travel between two points without backtracking. The set \mathcal{T} is closed in \mathcal{M} .

We should mention that when dealing with the convergence of plane trees, a natural topology is inherited from the uniform distance between (rescaled) contour processes, as defined later in Section 4.4. This topology is stronger than the the Gromov-Hausdorff topology, since contour processes encode also the lexicographical order of the vertices. For unordered trees, the Gromov-Hausdorff distance is very natural since it is by essence a distance on metric spaces, which ignores any extra structure, like planarity.

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Finally, we stress that there is a variant of the Gromov-Hausdorff distance on pointed metric spaces [16], which would be natural in our context since we are interested in rooted trees. It is straightforward to check that the results to come remain valid in this setting, by taking the root vertex as the distinguished point.

1.4 The Brownian continuum random tree

In order to state our main result, it remains to briefly describe the CRT, which arises as the scaling limit of plane trees, as we now recall.

Let \mathcal{E} be the set of continuous functions f defined on an interval $[0, \sigma_f]$, that are non-negative, and satisfy $f(0) = f(\sigma_f) = 0$. It is a complete space when endowed with the distance

$$D(f, g) = \sup_{t \geq 0} |f(t \wedge \sigma_f) - g(t \wedge \sigma_g)|.$$

With every $f \in \mathcal{E}$, we can associate an \mathbb{R} -tree, following [13]. Define a function d_f on $[0, \sigma_f]^2$ by

$$d_f(s, t) = f_s + f_t - 2\check{f}_{s,t},$$

where by definition

$$\check{f}_{s,t} = \inf_{s \wedge t \leq u \leq s \vee t} f_u.$$

It is easy to see that d_f is a pseudo-distance on $[0, \sigma_f]$. It is not a distance as it does not separate points, so we let $s \equiv_f t$ if $d_f(s, t) = 0$, defining an equivalence relation on $[0, \sigma_f]$ (see Figure 4).

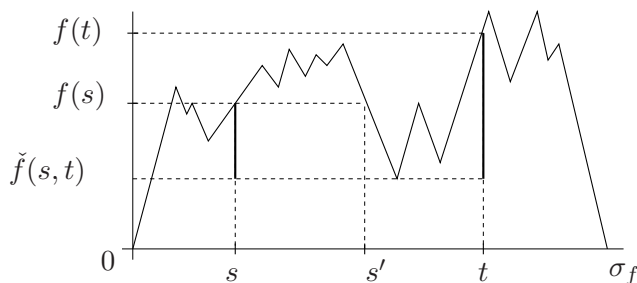


Figure 4: Graph of a function f from \mathcal{E} . In this example $s \equiv_f s'$ and the distance $d_f(s, t) = d_f(s', t) = f(s) + f(t) - 2\check{f}(s, t)$.

The quotient space $\mathcal{T}_f := [0, \sigma_f] / \equiv_f$ is endowed with the distance induced by d_f , which we still call d_f . The canonical projection $[0, \sigma_f] \rightarrow \mathcal{T}_f$ being obviously continuous, the target space is compact. The (isometry class of the) space \mathcal{T}_f is thus an element of \mathcal{M} , and turns out to be an element of \mathcal{S} . The space \mathcal{T}_f is naturally *rooted*, i.e. comes with a distinguished point ρ , which is the \equiv_f -equivalence class of 0. Finally, the mapping $f \mapsto \mathcal{T}_f$ is continuous from (\mathcal{E}, D) to $(\mathcal{S}, d_{\text{GH}})$.

Now, let $(e_t, 0 \leq t \leq 1)$ denote the standard normalized Brownian excursion ([23, Chapter XII]). The CRT is the random isometry class \mathcal{T}_{2e} , defining a random variable in $(\mathcal{S}, d_{\text{GH}})$. Note the convention to use **twice** the distance d_e . This is only a matter of convenience, and some references call ‘‘CRT’’ the metric space \mathcal{T}_e .

1.5 Main result

We are now ready to state our main theorem. As a motivation, we start with a well-known result. Let \mathbb{P}_n be the uniform distribution on \mathbb{T}_n . The following result is a re-interpretation in terms of \mathbb{R} -trees of a result of Aldous, see [4] and [13].

Proposition 1 *Let \mathcal{T}_n be a random variable with distribution \mathbb{P}_n . Then*

$$\frac{1}{\sqrt{2n}}\mathcal{T}_n \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{T}_{2e},$$

where the convergence in distribution holds with respect to the Gromov-Hausdorff topology on \mathcal{M} .

Our main result is an analog of the previous statement for trees. We let \mathbf{P}_n be the uniform distribution on \mathbf{T}_n .

Theorem 1 *Let \mathcal{T}_n be a random variable with distribution \mathbf{P}_n . Then*

$$\frac{\mathbf{c}}{\sqrt{2n}}\mathcal{T}_n \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{T}_{2e},$$

where the convergence in distribution holds with respect to the Gromov-Hausdorff topology on \mathcal{M} , and where \mathbf{c} is given in (5).

In particular, one will notice that a tree with law \mathbf{P}_n is typically about \mathbf{c} times smaller (indeed \mathbf{c} is greater than 1) than a typical \mathbb{P}_n -distributed plane tree (in passing, a quick computation based on Figure 1 shows that the mean height of a random variable with distribution \mathbb{P}_4 is $14/5$, while it equals $5/2$ for a random variable with distribution \mathbf{P}_4).

1.6 Contents of the paper and strategy of the proof of Theorem 1

In Section 2.1 we present a trimming operation for plane and unordered trees: the idea is to keep in a large plane tree with n leaves, only the vertices having at least $\lfloor \epsilon n \rfloor$ descendants, which in turn can be encoded as a plane tree with edge-lengths, called the $\lfloor \epsilon n \rfloor$ -skeleton. A similar operation is defined on (unordered) trees in Section 2.2.

This operation is the main tool in our study. Asymptotically, the $\lfloor \epsilon n \rfloor$ -skeletons of \mathbb{P}_n and \mathbf{P}_n -distributed random variables will turn out to be directly comparable thanks to two very similar “local limit theorems” (Propositions 2 and 3), which give the scaling limits for the densities of these $\lfloor \epsilon n \rfloor$ -skeletons. These are obtained via combinatorial arguments developed in Sections 2.1 and 2.2.

These local limit theorems form the cornerstone of our study, and they are, at the intuitive level, the main explanation of the similar asymptotic behaviour of both families of trees. However, these results are not sufficient by themselves to entail Theorem 1. First, because the pointwise convergence of the density of a random variable is not sufficient to get convergence in distribution. We need to check that the limiting formulas in Propositions 2 and 3 define indeed probability distributions, i.e. to ensure that no mass disappears, or goes into a singular part. A second problem is that the convergence of the $\lfloor \epsilon n \rfloor$ skeleton does not imply immediately the convergence of the non-trimmed trees under \mathbf{P}_n : we have to rule out the possibility that the

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parts of the trees that have been removed in the trimming procedure are thin “hair” of very large diameter although they contain a small amount of leaves (at most $\lfloor \epsilon n \rfloor$). The identification of the limit is then a consequence of Proposition 1.

Let us describe more precisely these steps. The local limit result, Proposition 2, concerning the $\lfloor \epsilon n \rfloor$ -skeleton under \mathbb{P}_n , is shown to imply a convergence in distribution in Section 4. To this end, we make a strong use of the convergence of the contour process of a \mathbb{P}_n -distributed plane tree toward the Brownian excursion (Proposition 4). This involves a careful translation of our trimming operations in terms of operations on contour processes, and more generally on excursion functions seen as encoding \mathbb{R} -trees as explained in Section 1.4. This is done in Sections 4.1 and 4.2.

As a matter of fact, in the plane case, the convergence of contour processes under \mathbb{P}_n is sufficiently robust to entail that of the $\lfloor \epsilon n \rfloor$ -skeleton to an ϵ -trimmed version of the CRT, described in term of the Brownian excursion (Corollary 1 in Section 4.3, and Sections 4.5 and 4.6). This gives the wanted convergence in distribution for $\lfloor \epsilon n \rfloor$ -trimming of a \mathbb{P}_n -distributed plane tree, and a simple comparison argument (thanks to Propositions 2 and 3), allows to prove that $\lfloor \epsilon n \rfloor$ -skeletons under \mathbf{P}_n converges also in distribution toward the same limit, up to a constant factor (Section 5.1).

Finally, we show in Section 5.2 that the trimmed versions are not too far from the whole tree with high probability: this part, which amounts to controlling the maximal height of all the subtrees appearing in the difference between a tree and its $\lfloor \epsilon n \rfloor$ -skeleton relies on moment estimates for the height of a \mathbf{P}_n -distributed tree, obtained by Broutin & Flajolet [8].

Acknowledgments. We are grateful to Mathilde Weill, whose help and insights in the early stage of this work have been very important. Grateful thanks are due to Nicolas Broutin and Philippe Flajolet for the interest they have taken in the problem of estimating the total height of a random \mathbf{P}_n -distributed tree, and for keeping us informed of the progress of their work, an extended abstract of which can be found in [8]. These results are indeed crucial in obtaining a tightness argument of the kind of Lemma 22.

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Our main tool to prove Theorem 1 will be to use a mass-trimming of the tree from the leaves. The underlying idea is that the combinatorics of the resulting tree are a lot more tractable, and show quite easily the universal aspects of different tree models, at least for binary trees. In this perspective, we are going to compare the trimming of plane trees and trees with respective distributions $\mathbb{P}_n, \mathbf{P}_n$.

2.1 The case of plane trees

2.1.1 a -trimmed tree

Let \mathfrak{t} be a plane tree, and for $a \geq 0$, the a -trimmed tree is defined to be

$$\mathfrak{t}[a] = \{u \in \mathfrak{t} : |\mathfrak{t}_v| > a \quad \forall v \prec u\}.$$

Of course, to check that $u \in \mathfrak{t}[a]$ it suffices to verify that $|\mathfrak{t}_v| > a$ where v is the parent of u . However, with our definition, $\mathfrak{t}[a]$ always contains \emptyset , and obviously defines a plane tree. For instance, $\mathfrak{t}[0] = \mathfrak{t}[1] = \mathfrak{t}$ (see also Figure 5). From now on we assume that $a \geq 2$.

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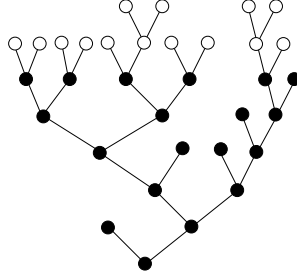


Figure 5: A plane tree \mathfrak{t} ; the tree $\mathfrak{t}[a]$ with $a = 3$, i.e. those vertices whose fathers have 4 leaves at least among their descendants, is represented by the black vertices

Let us discuss the reconstruction of a plane tree \mathfrak{t} , starting from its a -trimming $\mathfrak{t}[a]$. By definition of $\mathfrak{t}[a]$ it must hold that $|\mathfrak{t}_u| \leq a$ for every $u \in L(\mathfrak{t}[a])$ (otherwise the children of u would belong to $\mathfrak{t}[a]$ as well), and moreover, if u is such that u_1, u_2 are both in $L(\mathfrak{t}[a])$, then it must hold that $|\mathfrak{t}_{u_1}| + |\mathfrak{t}_{u_2}| > a$ (otherwise $|\mathfrak{t}_u| \leq a$ so u_1 could not be in $\mathfrak{t}[a]$). The following lemma is a (straightforward) kind of converse to this observation.

Lemma 1 *Let $\mathfrak{t}_0 \in \mathbb{T}$, and $(\mathfrak{t}_{(u)}, u \in L(\mathfrak{t}_0)) \in \mathbb{T}^{L(\mathfrak{t}_0)}$ be a family of plane trees indexed by the leaves of \mathfrak{t}_0 , such that*

- $|\mathfrak{t}_{(u)}| \leq a$ for every $u \in \mathfrak{t}$,
- $|\mathfrak{t}_{(u_1)}| + |\mathfrak{t}_{(u_2)}| > a$ whenever $u_1, u_2 \in L(\mathfrak{t})$,

therefore $\mathfrak{t}_0 = \mathfrak{t}[a]$, where

$$\mathfrak{t} = \mathfrak{t}_0 \cup \bigcup_{u \in L(\mathfrak{t}_0)} u\mathfrak{t}_{(u)}. \quad (7)$$

Consequently, a plane tree can be recovered in a one-to-one fashion from its a -trimming $\mathfrak{t}[a]$ and an appropriate family of plane trees indexed by the leaves of $\mathfrak{t}[a]$. Define

$$\mathbb{T}_{\leq a} = \{\mathfrak{t} \in \mathbb{T} : |\mathfrak{t}| \leq a\},$$

the set of trees having at most a leaves, and

$$\mathbb{F}_a = \{\mathfrak{f} = (\mathfrak{t}, \mathfrak{t}') : \mathfrak{t}, \mathfrak{t}' \in \mathbb{T}, |\mathfrak{t}| \vee |\mathfrak{t}'| \leq a, |\mathfrak{t}| + |\mathfrak{t}'| > a\}.$$

A pair of the form $\mathfrak{f} = (\mathfrak{t}, \mathfrak{t}')$ is called a forest with two tree components, namely $\mathfrak{f}_1 = \mathfrak{t}, \mathfrak{f}_2 = \mathfrak{t}'$ with natural notations. The size $|\mathfrak{f}|$ is then defined as $|\mathfrak{f}| = |\mathfrak{t}| + |\mathfrak{t}'|$, and by convention, for $u \in \mathcal{U}$ and $\mathfrak{f} = (\mathfrak{t}, \mathfrak{t}')$, we let $u\mathfrak{f}$ be the set $\{u\} \cup u_1\mathfrak{t} \cup u_2\mathfrak{t}'$.

With this notation, we can rewrite (7) as

$$\mathfrak{t} = \mathfrak{t}_0 \cup \bigcup_{u \in S(\mathfrak{t}_0)} u\mathfrak{t}_{(u)} \cup \bigcup_{u \in I_0(\mathfrak{t}_0)} u\mathfrak{f}_{(u)},$$

where the $\mathfrak{t}_{(u)}$ are in $\mathbb{T}_{\leq a}$ and the $\mathfrak{f}_{(u)}$ are in \mathbb{F}_a . In the forthcoming Lemma 3, we will make one further adjustment by ordering the elements of $S(\mathfrak{t}_0)$ and $I_0(\mathfrak{t}_0)$ in lexicographical order, allowing us to label the families $\mathfrak{t}_{(u)}$ and $\mathfrak{f}_{(u)}$ by integers rather than elements of \mathfrak{t} .

2 Trimming trees

In order to study the shape of $\mathfrak{t}[a]$ we introduce two operations on plane trees. Their aim is to encode in some sense $\mathfrak{t}[a]$: when a is large, the tree $\mathfrak{t}[a]$ will have few true branching points ($\#I_2(\mathfrak{t}[a])$ is small), and a lot of skeleton leaves. The operations defined below in the context of general binary plane trees will then be applied on $\mathfrak{t}[a]$ in Section 3.

2.1.2 Contraction of a plane tree

We first define an operation of concatenation. Assume given two plane trees $\mathfrak{t}, \mathfrak{t}'$, together with labels $\mathbf{x} = (x_u, u \in \mathfrak{t}), \mathbf{x}' = (x'_u, u \in \mathfrak{t}')$ taking their values in some set E (typically, E will be the set \mathbb{Z}_+). For $x \in E$, we define the *concatenation*

$$\Xi(x, (\mathfrak{t}, \mathbf{x}), (\mathfrak{t}', \mathbf{x}')) = (\mathfrak{t}'', \mathbf{x}''),$$

to be the tree having its root \emptyset marked by x , and having the two marked tree \mathfrak{t} and \mathfrak{t}' as subtrees rooted at the children 1 and 2 of \emptyset . Formally

$$\mathfrak{t}'' = \{\emptyset\} \cup 1\mathfrak{t} \cup 2\mathfrak{t}' \in \mathbb{T},$$

and $\mathbf{x}'' = (x''_u, u \in \mathfrak{t}'')$ is defined by

$$x''_{\emptyset} = x, \quad x''_{1u} = x_u, \quad u \in \mathfrak{t}, \quad x''_{2u} = x'_u, \quad u \in \mathfrak{t}'.$$

Let us introduce the contraction application Π . The action of Π amounts to suppressing all the leaves of \mathfrak{t} , and then to contracting the maximal chains of vertices with degree 2 in the resulting graph (while keeping its planar structure). This provides a new labeled binary tree \mathfrak{t}' , the label of a vertex being simply the size of the chain that has been contracted underneath it (see Figure 6).

Let us be formal: for every $\mathfrak{t} \in \mathbb{T} \setminus \{\{\emptyset\}\}$ (so that $|\mathfrak{t}| \geq 2$), we define the *contraction* of \mathfrak{t} as follows. Let $U(\mathfrak{t})$ be the vertex with minimal height having degree 2 in \mathfrak{t} satisfying

$$\forall v \prec u, \quad |\mathfrak{t}_{v1}| = 1 \text{ xor } |\mathfrak{t}_{v2}| = 1,$$

where xor denotes exclusive or. It must be understood that if no such v as in the definition exists, then $U(\mathfrak{t}) = \emptyset$. Then the contraction of $\mathfrak{t} \in \mathbb{T}$ is defined in a recursive way as a plane tree with integer labels $\Pi(\mathfrak{t}) = (\mathfrak{s}, \mathbf{x})$ as follows:

- If $|\mathfrak{t}_{U(\mathfrak{t})1}| = |\mathfrak{t}_{U(\mathfrak{t})2}| = 1$ then $\mathfrak{s} = \{\emptyset\}$ and $x_{\emptyset} = |U(\mathfrak{t})|$,
- otherwise, it must hold that $|\mathfrak{t}_{U(\mathfrak{t})1}| \wedge |\mathfrak{t}_{U(\mathfrak{t})2}| \geq 2$, and we set

$$\Pi(\mathfrak{t}) = \Xi(|U(\mathfrak{t})|, \Pi(\mathfrak{t}_{U(\mathfrak{t})1}), \Pi(\mathfrak{t}_{U(\mathfrak{t})2})). \quad (8)$$

This branching contraction procedure, together with the interpretation given at the beginning of Section 2.1.2, entails that the vertices of $\Pi(\mathfrak{t})$ are naturally associated with a subset of the vertices of \mathfrak{t} via an application $\pi_{\mathfrak{t}}$. As one may observe on Figure 6, the branching structure of $\Pi(\mathfrak{t})$ is given by the relative positions of the vertices in $I_2(\mathfrak{t}) \cup I_0(\mathfrak{t})$. More precisely, we let $\pi_{\mathfrak{t}}(\emptyset) = U(\mathfrak{t})$, and recursively, for $u \in I(\Pi(\mathfrak{t}))$, we set

$$\pi_{\mathfrak{t}}(u1) = U(\mathfrak{t}_{\pi_{\mathfrak{t}}(u)1}), \quad \pi_{\mathfrak{t}}(u2) = U(\mathfrak{t}_{\pi_{\mathfrak{t}}(u)2}). \quad (9)$$

2 Trimming trees

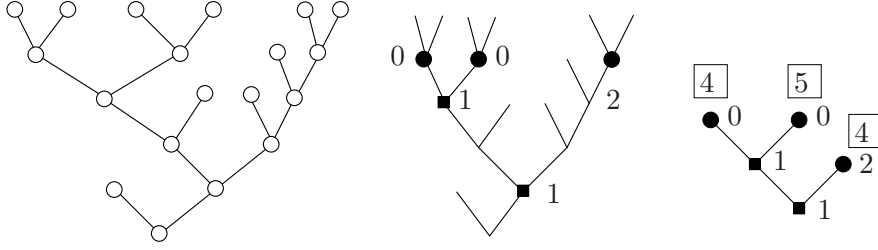


Figure 6: The plane tree $\mathfrak{t}' = \mathfrak{t}[3]$ (where \mathfrak{t} is given in Figure 5). The elements of $I_2(\mathfrak{t}')$ (represented with squares), and elements of $I_0(\mathfrak{t}')$ (represented by small black discs) are the vertices giving the structure of the projection $\Pi(\mathfrak{t}') = (\mathfrak{s}, \mathbf{x})$, represented on the third picture.

It is easy to check that $\pi_{\mathfrak{t}}$ is injective, with image $I_2(\mathfrak{t}) \cup I_0(\mathfrak{t})$. More precisely, if $\mathfrak{s} = \Pi(\mathfrak{t})$, then $\pi_{\mathfrak{t}}(I(\mathfrak{s})) = I_2(\mathfrak{t})$ and $\pi_{\mathfrak{t}}(L(\mathfrak{s})) = I_0(\mathfrak{t})$. Moreover, $\pi_{\mathfrak{t}}$ is increasing for the lexicographical order.

Noticing that the plane trees contracted to the tree $(\{\emptyset\}, x)$ are exactly those of the form $\{v, v1, v2 : v \preceq u\}$ for a word $u \in \{1, 2\}^x$, and using an induction argument, one may prove straightforwardly the following Lemma:

Lemma 2 *Let \mathfrak{s} be a binary plane tree and $\mathbf{x} = (x_u, u \in \mathfrak{s})$. Then*

$$\#\Pi^{-1}(\mathfrak{s}, \mathbf{x}) = 2^{|\mathbf{x}|},$$

where $|\mathbf{x}| = \sum_{u \in \mathfrak{s}} x_u$.

2.1.3 Encoding of the a -trimming of \mathfrak{t} : the a -skeleton of \mathfrak{t}

We now go one step further in our simplification of trees. Let $(\mathfrak{s}, \mathbf{x}) = \Pi(\mathfrak{t}[a])$ be the contracted tree associated with the a -trimming of \mathfrak{t} . It is implicit that $\Pi(\mathfrak{t}[a])$ is indeed well-defined, meaning that $|\mathfrak{t}[a]| \geq 2$, and this is equivalent to the fact that $|\mathfrak{t}| > a$, which is thus assumed.

For our purposes, it will be useful to assign to the leaves of \mathfrak{s} a *second integer label*, giving the total number of descendant leaves they had in the initial tree (corresponding to the size of the corresponding forest of pairs of trees). Formally, for $u \in L(\mathfrak{s})$,

$$y_u = |\mathfrak{t}_{\pi_{\mathfrak{t}[a]}(u)}|,$$

where the mapping $\pi_{\mathfrak{t}[a]}$ is as defined in (9), and note that if $u \in L(\mathfrak{s})$, it must hold that $\pi_{\mathfrak{t}[a]}(u)$ is an element of $I_0(\mathfrak{t}[a])$. Also, it holds that $y_v > a$ for every $v \in L(\mathfrak{s})$.

Definition 3 *The triple $(\mathfrak{s}, \mathbf{x}, \mathbf{y})$ is called the a -skeleton of \mathfrak{t} , and is denoted by $\text{Sk}_a(\mathfrak{t})$.*

The general a -skeleton is thus a triple $(\mathfrak{s}, \mathbf{x}, \mathbf{y})$ such that

- $\mathfrak{s} \in \mathbb{T}$,
- $\mathbf{x} = (x_u, u \in \mathfrak{s})$ is a vector of non-negative integers,
- $\mathbf{y} = (y_u, u \in L(\mathfrak{s}))$ is a vector of integers in $(a, 2a]$.

2 Trimming trees

Summing up our study so far, we obtain the following result.

Lemma 3 *Any (binary) plane tree \mathfrak{t} such that $|\mathfrak{t}| > a$ can be obtained from the following data, in a one-to-one way:*

1. an a -skeleton $(\mathfrak{s}, \mathbf{x}, \mathbf{y})$
2. one of the $2^{|\mathbf{x}|}$ elements $\mathfrak{t}_0 \in \Pi^{-1}(\mathfrak{s}, \mathbf{x})$,
3. two sequences $(\mathfrak{t}(i), 1 \leq i \leq |\mathbf{x}|) \in \mathbb{T}_{\leq a}^{|\mathbf{x}|}$ and $(\mathfrak{f}(i), 1 \leq i \leq |\mathfrak{s}|) \in \mathbb{F}_a^{|\mathfrak{s}|}$ such that

$$|\mathfrak{f}(j)| = y_{v_j}, \quad 1 \leq j \leq |\mathfrak{s}|,$$

where $v_1 \leq \dots \leq v_{|\mathfrak{s}|}$ is the ordered list of leaves of $|\mathfrak{s}|$

by the formula:

$$\mathfrak{t} = \mathfrak{t}_0 \cup \bigcup_{i=1}^{|\mathbf{x}|} u_i \mathfrak{t}(i) \cup \bigcup_{j=1}^{|\mathfrak{s}|} l_j \mathfrak{f}(j),$$

where $u_1 \leq \dots \leq u_{|\mathbf{x}|}$ and $l_1 \leq \dots \leq l_{|\mathfrak{s}|}$ respectively denote the ordered lists of elements of $S(\mathfrak{t}_0)$ and $I_0(\mathfrak{t}_0)$, so in particular, $l_i = \pi_{\mathfrak{t}_0}(v_j), 1 \leq j \leq |\mathfrak{s}|$.

2.2 The case of trees

The procedure we adopt to trim a tree is similar to the one for plane trees. The reconstruction properties of a tree from its trimming are not as smooth as in the plane case though, as we will now see. First, we focus on what we call a labeled tree. A labeled plane tree is a pair $(\mathfrak{t}, \mathbf{g})$ where \mathfrak{t} is a plane tree and $\mathbf{g} : \mathfrak{t} \rightarrow E$ is a function from \mathfrak{t} to some set E . We define a new equivalence relation \approx for labeled plane trees $(\mathfrak{t}, \mathbf{g})$, by $(\mathfrak{t}, \mathbf{g}) \approx (\mathfrak{t}', \mathbf{g}')$ if there exist permutations $\sigma = (\sigma_u, u \in \mathfrak{t})$ as in Sect. 1.2 such that

$$\mathfrak{t}' = \sigma(\mathfrak{t}), \quad g'_{\sigma(u)} = g_u, \quad u \in \mathfrak{t}.$$

A labeled tree is just an equivalence class for \approx . Such an equivalence class has at most $2^{\#I(\mathfrak{t})} = 2^{|\mathfrak{t}|-1}$ representatives. When this upper-bound is attained, we say that the associated (rooted) labeled tree (or any of its plane representatives) has *no symmetries*. A simple criterion for having no symmetry is given by the following intuitive result. We say that a labeled plane tree $(\mathfrak{t}, \mathbf{g})$ (or its \approx -equivalence class) is *good* if the components of \mathbf{g} are pairwise distinct.

Lemma 4 *A good labeled plane tree $(\mathfrak{t}, \mathbf{g})$ has no symmetries.*

Proof. Endow the label set E with any particular linear order. With any good labeled tree, one associates a particular planar representative $(\mathfrak{t}, \mathbf{g})$ in which for every pair of vertices born from the same parent, the one with larger label is always the first child. The other planar representatives are obtained by deciding the order in which the two children of an interior vertex should be displayed, knowing that they can always be distinguished, as they have different labels, so there are exactly $2^{|\mathfrak{t}|-1}$ of them. \square

Next, the reader will easily be convinced that Π and $\mathfrak{t} \mapsto \mathfrak{t}[a]$ are class operations, meaning that if $\mathfrak{t} \sim \mathfrak{t}'$ then $\mathfrak{t}[a] \sim \mathfrak{t}'[a]$, and $\Pi(\mathfrak{t}) \approx \Pi(\mathfrak{t}')$. This validates the following

2 Trimming trees

Definition 4 We say that a plane tree \mathfrak{t} with $|\mathfrak{t}| > a$ is a -good if it satisfies the following properties

- the labeled tree $\Pi(\mathfrak{t}[a])$ is good, and
- for every $u \in I_0(\mathfrak{t}[a])$, it holds that $|\mathfrak{t}_{u1}| \neq |\mathfrak{t}_{u2}|$.

We say that a tree \mathfrak{t} is a -good if one (hence any) of its planar representatives is.

If \mathfrak{t} is an a -good tree, we can choose in some non-ambiguous way a plane \approx -representative of its skeleton. To be specific, we will set

$$\text{Sk}_a(\mathfrak{t}) := (\mathfrak{s}, \mathbf{x}, \mathbf{y})$$

to be the representative such that $x_{u1} > x_{u2}$ for every $u \in I(\mathfrak{s})$, and call it the a -skeleton of \mathfrak{t} . The general a -skeleton of an a -good tree (we naturally call such objects a -good skeletons) is thus a triple $(\mathfrak{s}, \mathbf{x}, \mathbf{y})$ such that

- \mathfrak{s} is a plane tree,
- $\mathbf{x} = (x_u, u \in \mathfrak{s})$ is such that $(\mathfrak{s}, \mathbf{x})$ is good and $x_{u1} > x_{u2}$ whenever $u \in I(\mathfrak{s})$, and
- $\mathbf{y} = (y_u, u \in L(\mathfrak{s}))$ is such that $y_u \in (a, 2a]$ for $u \in L(\mathfrak{s})$.

The key observation is the following fact.

Lemma 5 Let $\mathfrak{t} \sim \mathfrak{t}'$ be a -good plane trees such that $\mathfrak{t}[a] = \mathfrak{t}'[a]$, and such that for every $u \in I_0(\mathfrak{t}[a])$, one has

$$|\mathfrak{t}_{u1}| > |\mathfrak{t}_{u2}| \quad \text{and} \quad |\mathfrak{t}'_{u1}| > |\mathfrak{t}'_{u2}|.$$

Then for every $u \in L(\mathfrak{t}[a])$, it holds that $\mathfrak{t}_u \sim \mathfrak{t}'_u$.

Proof. Let $\sigma = (\sigma_u, u \in \mathfrak{t})$ be such that $\sigma(\mathfrak{t}) = \mathfrak{t}'$. Assume that there exists $u \in I(\mathfrak{t}[a])$ such that $\sigma_u \neq 1$ is not the identity. Let u be such a vertex of $I(\mathfrak{t}[a])$, with minimal height. Then $\sigma_v = 1$ for all $v \prec u$, so since $\sigma(\mathfrak{t}) = \mathfrak{t}'$, it must hold that $\mathfrak{t}_u \sim \mathfrak{t}'_u$.

First assume $u \in I_0(\mathfrak{t}[a])$, then by assumption, it holds that $|\mathfrak{t}_{u1}| > |\mathfrak{t}_{u2}|$ and $|\mathfrak{t}'_{u1}| > |\mathfrak{t}'_{u2}|$. Since $\mathfrak{t}_u \sim \mathfrak{t}'_u$ and $\sigma_u \neq 1$, this entails that $\mathfrak{t}_{u1} \sim \mathfrak{t}'_{u2}$ (and $\mathfrak{t}_{u2} \sim \mathfrak{t}'_{u1}$), a contradiction with the order on the sizes of these trees.

A second possibility is that $u \in I_1(\mathfrak{t})$, so that $u1$ or $u2$ is in $S(\mathfrak{t}[a])$. Assuming e.g. that the first case holds, this says that $u2$ is in $I(\mathfrak{t}[a])$, but since $\sigma_u \neq 1$ this says that $u1 \in I(\mathfrak{t}'[a]) = I(\mathfrak{t}[a])$ which is disjoint from $S(\mathfrak{t}[a])$, a contradiction.

The last possibility is that $u \in I_2(\mathfrak{t}[a])$, and we let $v = \pi_{\mathfrak{t}[a]}(u) \in I(\mathfrak{s})$, where $(\mathfrak{s}, \mathbf{x}) = \Pi(\mathfrak{t}[a])$. But the fact that $\sigma_u \neq 1$ has the effect of switching the values of x_{v1}, x_{v2} in the (good) labeled plane tree $(\mathfrak{s}, \mathbf{x})$. Since these values are distinct, this contradicts $\mathfrak{t}[a] = \mathfrak{t}'[a]$.

This shows that $\sigma_u = 1$ for every $u \in I(\mathfrak{t}[a]) = I(\mathfrak{t}'[a])$. The conclusion follows easily by noticing that $\mathfrak{t}'_u = \sigma^{(u)}(\mathfrak{t}_u)$ for every $u \in I(\mathfrak{t}[a])$, where by definition, $\sigma^{(u)} = (\sigma_{uv} : v \in I(\mathfrak{t}_u))$.

□

We now choose, once and for all, a section Π_0^{-1} of Π , i.e. a mapping from the set of a -skeletons to \mathbb{T} such that $\Pi \circ \Pi_0^{-1}$ is the identity.

3 Local limit theorems for a -skeletons of trees

Starting from an a -good tree \mathbf{t} with a -skeleton $(\mathbf{s}, \mathbf{x}, \mathbf{y})$, we can always choose a plane representative \mathfrak{t} of \mathbf{t} such that $\mathfrak{t}[a] = \Pi_0^{-1}(\mathbf{s}, \mathbf{x})$. There might be many such choices of \mathfrak{t} , but by Lemma 5, we know there is a well-defined family of trees $\mathbf{t}_{(u)}$, $u \in L(\mathfrak{t}_0)$, such that $|\mathbf{t}_{(u1)}| > |\mathbf{t}_{(u2)}|$ whenever $u \in I_0(\mathfrak{t}_0)$, and such that \mathbf{t} is the \sim -equivalence class of the plane tree (recall that $s : \mathbf{T} \rightarrow \mathbb{T}$ is a section)

$$\mathfrak{t}[a] \cup \bigcup_{u \in L(\mathfrak{t}_0)} us(\mathbf{t}_{(u)}).$$

More precisely, letting

$$\mathbf{T}_{\leq a} = \{\mathbf{t} \in \mathbf{T} : |\mathbf{t}| \leq a\}$$

and

$$\mathbf{F}_a = \{(\mathbf{t}, \mathbf{t}') : a \geq |\mathbf{t}| > |\mathbf{t}'|, |\mathbf{f}| := |\mathbf{t}| + |\mathbf{t}'| > a\},$$

the trees $\mathbf{t}_{(u)}$, $u \in S(\mathfrak{t}[a])$ lie in $\mathbf{T}_{\leq a}$, while the pairs $(\mathbf{t}_{(u1)}, \mathbf{t}_{(u2)})$ for $u \in I_0(\mathfrak{t}[a])$ lie in \mathbf{F}_a and satisfy $|\mathbf{t}_{(u1)}| + |\mathbf{t}_{(u2)}| = y_{\pi_{\mathfrak{t}[a]}(u)}$. By convention, for $\mathbf{f} = (\mathbf{t}, \mathbf{t}') \in \mathbf{F}_a$ we let $s(\mathbf{f}) = (s(\mathbf{t}), s(\mathbf{t}')) \in \mathbb{F}_a$. Summing up our study, we get:

Lemma 6 *Any a -good tree \mathbf{t} can be obtained from the following data in a one-to-one way:*

1. a good a -skeleton $(\mathbf{s}, \mathbf{x}, \mathbf{y})$ such that $x_{u1} > x_{u2}$ for every $u \in I(\mathbf{s})$,
2. two sequences $(\mathbf{t}(1), \dots, \mathbf{t}(|\mathbf{x}|)) \in \mathbf{T}_{\leq a}^{|\mathbf{x}|}$ and $(\mathbf{f}(1), \dots, \mathbf{f}(|\mathbf{s}|)) \in \mathbf{F}_a^{|\mathbf{s}|}$ such that

$$\mathbf{f}(j) = (\mathbf{t}(j), \mathbf{t}'(j)) \quad \text{and} \quad |\mathbf{f}(j)| = y_{v_j}, \quad 1 \leq j \leq |\mathbf{s}|,$$

where $v_1, \dots, v_{|\mathbf{s}|}$ is the ordered list of elements of $L(\mathbf{s})$,

by letting \mathbf{t} be the \sim -equivalence class of the planar tree

$$\mathfrak{t} = \mathfrak{t}_0 \cup \bigcup_{i=1}^{|\mathbf{x}|} u_i s(\mathbf{t}(i)) \cup \bigcup_{j=1}^{|\mathbf{s}|} l_j s(\mathbf{f}(j)).$$

where $\mathfrak{t}_0 = \Pi_0^{-1}(\mathbf{s}, \mathbf{x})$, and $u_1, \dots, u_{|\mathbf{x}|}$ resp. $l_1, \dots, l_{|\mathbf{s}|}$ are the ordered lists of elements of $S(\mathfrak{t}_0)$ and $I_0(\mathfrak{t}_0)$.

3 Local limit theorems for a -skeletons of trees

Let us draw the probabilistic consequences of our analysis of trimmed trees, starting with plane trees.

3.1 Plane trees

Let $\mu = (\mu_k)_{k \geq 1}$ be the law of the size of a Galton-Watson tree with offspring distribution $(\delta_0 + \delta_2)/2$; one has immediately

$$\mu_n = \frac{\#\mathbb{T}_n}{2^{2n-1}}, \quad n \geq 1, \tag{10}$$

3 Local limit theorems for a -skeletons of trees

and by Stirling's formula, the following equivalent holds:

$$\mu_n \underset{n \rightarrow \infty}{\sim} \frac{1}{2\sqrt{\pi n^{3/2}}}. \quad (11)$$

We recall that conditioned on having n vertices, a Galton-Watson tree with offspring distribution $(\delta_0 + \delta_2)/2$ is uniform on \mathbb{T}_n . Consider a probability space (Ω, \mathcal{F}, P) on which is defined an i.i.d. sequence $X_1, X_2 \dots$ of μ -distributed random variables.

Lemma 7 *Let $(\mathfrak{s}, \mathbf{x}, \mathbf{y})$ be an a -skeleton. Then, seeing $\text{Sk}_a : \mathfrak{t} \mapsto \text{Sk}_a(\mathfrak{t})$ as a random variable,*

$$\begin{aligned} & \mathbb{P}_n(\text{Sk}_a = (\mathfrak{s}, \mathbf{x}, \mathbf{y})) \\ &= \frac{1}{2^{2|\mathfrak{s}|-1} \mu_n} P \left(\sum_{i=1}^{|\mathbf{x}|} X_i = n - |\mathbf{y}|, \max_{1 \leq i \leq |\mathbf{x}|} X_i \leq a \right) \prod_{u \in L(\mathfrak{s})} P(X_1 + X_2 = y_u, X_1 \vee X_2 \leq a). \end{aligned} \quad (12)$$

Proof. By Lemma 3, the number of plane trees admitting $(\mathfrak{s}, \mathbf{x}, \mathbf{y})$ for a -skeleton equals

$$2^{|\mathbf{x}|} \# \left\{ (\mathfrak{t}(1), \dots, \mathfrak{t}(|\mathbf{x}|)) \in \mathbb{T}_{\leq a}^{|\mathbf{x}|} : \sum_{i=1}^{|\mathbf{x}|} |\mathfrak{t}(i)| = n - |\mathbf{y}| \right\} \prod_{u \in L(\mathfrak{s})} \#\{\mathfrak{f} \in \mathbb{F}_a : |\mathfrak{f}| = y_u\},$$

which equals by (10)

$$\begin{aligned} & 2^{|\mathbf{x}|+2n-2|\mathbf{y}|-|\mathbf{x}|} \prod_{j=1}^{|\mathfrak{s}|} 2^{2y_j-2} \\ & \times P \left(\sum_{i=1}^{|\mathbf{x}|} X_i = n - |\mathbf{y}|, \max_{1 \leq i \leq |\mathbf{x}|} X_i \leq a \right) \prod_{u \in L(\mathfrak{s})} P(X_1 + X_2 = y_u, X_1 \vee X_2 \leq a), \end{aligned}$$

giving the result after dividing by $\#\mathbb{T}_n$. \square

From this, we deduce an important limiting result (Proposition 2), which can be thought of a kind of ‘‘local limit theorem’’ for the $\lfloor \epsilon n \rfloor$ skeleton of a \mathbb{P}_n -distributed element.

From now on, we fix a number $\epsilon \in (0, 1)$ and will usually omit its mention to allow lighter notations. The number ϵ will be allowed to move in the further Section 5.2 (also at the end of Section 4.6), but we will take care that there is no ambiguity at this moment.

For $\mathfrak{s} \in \mathbb{T}$ and $M > 1/\epsilon$, set

$$I_M(\mathfrak{s}) := \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{\mathfrak{s}} \times \mathbb{R}_+^{L(\mathfrak{s})} : \begin{array}{ll} x_u \in (M^{-1}, M) & u \in \mathfrak{s} \\ y_u \in (\epsilon + M^{-1}, 2\epsilon) & u \in L(\mathfrak{s}) \end{array} \right\}. \quad (13)$$

We also define the family of functions a_k (for $k \geq 0$) on $(0, \infty)$ by letting $a_0 \equiv 1$, $a_1(x) = (2\sqrt{\pi x^{3/2}})^{-1} \mathbb{1}_{x \geq \epsilon}$ and for $k \geq 2$,

$$a_k(x) = \frac{1}{(4\pi)^{k/2}} \int_{u_1, \dots, u_{k-1} \geq \epsilon} \frac{\mathbb{1}_{\{u_1 + \dots + u_{k-1} \leq x - \epsilon\}} du_1 \dots du_{k-1}}{(u_1 \dots u_{k-1} (x - u_1 - \dots - u_{k-1}))^{3/2}}, \quad x \geq 0, \quad (14)$$

3 Local limit theorems for a -skeletons of trees

the latter being $= 0$ on $[0, k\epsilon]$. Let also

$$b(y) = \frac{1}{4\pi} \int_{y-\epsilon}^{\epsilon} \frac{du}{(u(y-u))^{3/2}}, \quad y \in (\epsilon, 2\epsilon].$$

Note that the functions a_k, b depend tacitly on the parameter ϵ . Finally, we let

$$g(x) = \frac{\exp(-1/(4x))}{2\sqrt{\pi}x^{3/2}}, \quad x > 0 \tag{15}$$

be the density of the hitting time of $1/\sqrt{2}$ by a standard Brownian motion. This is the density of a stable law with parameter $1/2$, which has Laplace transform

$$\int_{\mathbb{R}_+} \exp(-\lambda x)g(x)dx = \exp(-\sqrt{\lambda}), \quad \lambda \geq 0.$$

It can also be expressed as the density function of the law of the total sum of the atoms of a Poisson random measure on \mathbb{R}_+ , with intensity

$$\frac{dx}{2\sqrt{\pi}x^{3/2}} \mathbb{1}_{\{x>0\}}.$$

The following statement is a ‘‘local limit theorem’’ for a -skeletons of plane trees.

Proposition 2 *Let $\mathfrak{s} \in \mathbb{T}$. Then for every $\epsilon \in (0, 1)$ and $M > 1/\epsilon$, the quantity*

$$\sup_{\substack{(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_+^{\mathfrak{s}} \times \mathbb{Z}_+^{L(\mathfrak{s})} \\ (\mathbf{x}/\sqrt{n}, \mathbf{y}/n) \in I_M(\mathfrak{s})}} \left| n^{2|\mathfrak{s}|-1/2} \mathbb{P}_n(\text{Sk}_{[\epsilon n]} = (\mathfrak{s}, \mathbf{x}, \mathbf{y})) - \psi^{\mathfrak{s}}(\mathbf{x}/\sqrt{n}, \mathbf{y}/n) \right| \tag{16}$$

converges to 0 as $n \rightarrow \infty$, where, setting $z = 1 - |\mathbf{y}|$,

$$\psi^{\mathfrak{s}}(\mathbf{x}, \mathbf{y}) = \frac{\sqrt{\pi}}{2^{2|\mathfrak{s}|-2}} \left(\sum_{k=0}^{\lfloor z/\epsilon \rfloor} \frac{(-|\mathbf{x}|)^{k-2}}{k!} \int_{k\epsilon}^z g\left(\frac{z-u}{|\mathbf{x}|^2}\right) a_k(u) du \right) \prod_{v \in L(\mathfrak{s})} b(y_v).$$

Although it is not apparent at first sight, the functions $\psi^{\mathfrak{s}}$ are equal to 0 whenever $|\mathfrak{s}| > 1/\epsilon$. Indeed, in this case \mathfrak{s} cannot be the first component of the $[\epsilon n]$ -skeleton of a plane tree, since otherwise \mathbf{y}_n would have strictly more than $1/\epsilon$ components of size at least ϵn , which is impossible.

By comparing this proposition with a statement like Proposition 1, it is tempting to interpret the functions $\psi^{\mathfrak{s}}, \mathfrak{s} \in \mathbb{T}$ as density functions associated with a trimmed version of the CRT. We are going to make this more precise in Section 4.6.

In order to prove the proposition, write using Lemma 7,

$$\begin{aligned} n^{2|\mathfrak{s}|-1/2} \mathbb{P}_n(\text{Sk}_{[\epsilon n]} = (\mathfrak{s}, \mathbf{x}, \mathbf{y})) &= \frac{n^{-3/2}}{2^{2|\mathfrak{s}|-1} \mu_n} \\ &\times n P \left(\sum_{i=1}^{|\mathbf{x}|} X_i = n - |\mathbf{y}|, \max_{1 \leq i \leq |\mathbf{x}|} X_i \leq [\epsilon n] \right) \\ &\times \prod_{u \in L(\mathfrak{s})} n^2 P(X_1 + X_2 = y_u, X_1 \vee X_2 \leq [\epsilon n]). \end{aligned}$$

3 Local limit theorems for a -skeletons of trees

The asymptotics of the first term is $\sqrt{\pi}/2^{2|s|-2}$ by (11). The two following Lemmas give an uniform approximation of the two other terms.

Lemma 8 *For any $M > 1/\epsilon$, we have the convergence*

$$\lim_{n \rightarrow \infty} \sup_{l/n \in [\epsilon + M^{-1}, 2\epsilon]} |n^2 P(X_1 + X_2 = l, X_1 \vee X_2 \leq \lfloor \epsilon n \rfloor) - b(l/n)| = 0,$$

Proof. By definition, for $l/n \in [\epsilon + M^{-1}, 2\epsilon]$

$$\begin{aligned} P(X_1 + X_2 = l, X_1 \vee X_2 \leq \lfloor \epsilon n \rfloor) &= \sum_{j=l-\lfloor \epsilon n \rfloor}^{\lfloor \epsilon n \rfloor} \mu_j \mu_{l-j} \\ &= \frac{1}{4\pi} \sum_{j=l-\lfloor \epsilon n \rfloor}^{\lfloor \epsilon n \rfloor} \frac{1}{j^{3/2}(l-j)^{3/2}} + o\left(\sum_{j=\lfloor n/M \rfloor}^{\lfloor \epsilon n \rfloor} \frac{1}{j^{3/2}(l-j)^{3/2}}\right), \end{aligned}$$

as $n \rightarrow \infty$, where we have used the equivalent (11), and the fact that $l - \lfloor \epsilon n \rfloor \geq \lfloor n/M \rfloor$. In turn,

$$\begin{aligned} \frac{1}{4\pi} \sum_{j=l-\lfloor \epsilon n \rfloor}^{\lfloor \epsilon n \rfloor} \frac{1}{j^{3/2}(l-j)^{3/2}} &= \frac{1}{4\pi} \int_{l-\lfloor \epsilon n \rfloor}^{\lfloor \epsilon n \rfloor} \frac{dx}{[x]^{3/2}(l-[x])^{3/2}} \\ &= \frac{1}{4\pi n^2} \int_{(l-\lfloor \epsilon n \rfloor)/n}^{\lfloor \epsilon n \rfloor/n} \frac{dx}{(\lfloor nx \rfloor/n)^{3/2}(l/n - \lfloor nx \rfloor/n)^{3/2}}. \end{aligned}$$

This yields the result as the function $(x, y) \mapsto x^{-3/2}(y-x)^{-3/2}$ is uniformly continuous on the compact set $\{(x, y) : (x, y-x) \in [(2M)^{-1}, 2\epsilon]^2\}$, in which the terms $(\lfloor nx \rfloor/n, l/n)$ of the above integral are constrained to lie (at least for n large enough). \square

Lemma 9 *For $M > 1$,*

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{\substack{l/\sqrt{n} \in [M^{-1}, M] \\ m/n \in [0, 1]}} \left| nP\left(\sum_{i=1}^l X_i = m, \max_{1 \leq i \leq l} X_i \leq \lfloor \epsilon n \rfloor\right) \right. \\ \left. - \sum_{k \geq 0} \frac{(-l/\sqrt{n})^{k-2}}{k!} \int_0^{m/n} du a_k(u) g\left(\frac{m/n - u}{(l/\sqrt{n})^2}\right) \right| = 0. \end{aligned}$$

Proof. For events $A, B_j, j \in J$ with finite J , one has

$$E \left[\mathbb{1}_A \prod_{j \in J} (1 - \mathbb{1}_{B_j}) \right] = \sum_{C \subseteq J} (-1)^{\#C} E \left[\mathbb{1}_A \prod_{j \in C} \mathbb{1}_{B_j} \right],$$

which is usually called the *inclusion-exclusion principle*. Using this and an elementary exchangeability argument, one has

$$nP\left(\sum_{i=1}^l X_i = m, \max_{1 \leq i \leq l} X_i \leq \lfloor \epsilon n \rfloor\right) = n \sum_{k=0}^l (-1)^k \binom{l}{k} P\left(\sum_{i=1}^l X_i = m, \min_{1 \leq i \leq k} X_i > \lfloor \epsilon n \rfloor\right);$$

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in order to let the reader follows more easily the step of the computations, we rewrite the RHS as

$$\sum_{k=0}^l (-1)^k \frac{\binom{l}{k}}{n^{k/2}} \times n^{k/2+1} P \left(\sum_{i=1}^l X_i = m, \min_{1 \leq i \leq k} X_i > \lfloor \epsilon n \rfloor \right). \quad (17)$$

Note that the sum can be reduced to its $m/\lfloor \epsilon n \rfloor$ first terms, because the constraint on the minimum of the first k variables X_i forces $k\lfloor \epsilon n \rfloor < m$. Thus, there are at most $1/\epsilon$ non-zero terms in the sum, which is independent on n . It remains to show the (uniform) convergence of these individual terms.

It is obvious that $\binom{l}{k}/n^{k/2} \sim (l/(\sqrt{n}))^k/k!$ uniformly for $l/\sqrt{n} \in [M^{-1}, M]$. To handle the probability term in (17), we introduce

$$\begin{aligned} A_k(n, r) &= P \left(X_1 + \dots + X_k = r, \min_{1 \leq i \leq k} X_i > \lfloor \epsilon n \rfloor \right) \\ G(l, r) &= P(X_1 + \dots + X_l = r), \end{aligned}$$

and write the quantity after the \times sign, in (17),

$$\begin{aligned} &\sum_{r=0}^m n^{k/2+1} A_k(n, r) G(l-k, m-r) \\ &= \int_0^{m/n} \left(n^{k/2+1} A_k(n, \lfloor rn \rfloor) \right) \left(\frac{\sqrt{n}}{l} \right)^2 (l^2 G(l-k, m - \lfloor rn \rfloor)) dr. \end{aligned} \quad (18)$$

In order to handle the terms involving $G(l, m)$, we use the Gnedenko-Kolmogorov local limit theorem for sums of i.i.d. random variables in the domain of attraction of a stable (1/2) distribution [7, Theorem 8.4.1], giving that

$$\lim_{l \rightarrow \infty} \sup_{m \geq 0} |l^2 G(l, m) - g(m/l^2)| = 0. \quad (19)$$

In particular, in the expression (18), we can omit the $-k$ inside the G terms. On the other hand, we have

$$\lim_{n \rightarrow \infty} \sup_{0 \leq m \leq n} \left| n^{1+k/2} A_k(n, m) - a_k(m/n) \right| = 0. \quad (20)$$

Indeed, using (11),

$$\begin{aligned} A_k(n, m) &\sim \sum_{\substack{j_1, \dots, j_k > \lfloor \epsilon n \rfloor \\ j_1 + \dots + j_k = m}} \prod_{l=1}^k \frac{1}{2\sqrt{\pi} j_l^{3/2}} \\ &= (4\pi)^{-k/2} \int \frac{dv_1 \dots dv_{k-1} \mathbb{1}_{D(k, m)}(\mathbf{v})}{\left(m - \sum_{i=1}^{k-1} \lfloor v_i \rfloor \right)^{3/2} \prod_{i=1}^{k-1} \lfloor v_i \rfloor^{3/2}}, \end{aligned}$$

where by definition $D(k, m) = \{(v_1, \dots, v_{k-1}) : \min_{1 \leq i \leq k-1} \lfloor v_i \rfloor \wedge (m - \sum_{i=1}^{k-1} \lfloor v_i \rfloor) > \lfloor \epsilon n \rfloor\}$. After the linear change of variables $u_i = v_i/n$, (20) boils down to a situation similar to Lemma 8, using the fact that the function

$$(u_1, \dots, u_{k-1}, x) \mapsto \left(x - u_1 - \dots - u_{k-1} \prod_{i=1}^{k-1} u_i \right)^{-3/2}$$

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is (uniformly) continuous on the compact

$$\left\{ (u_1, \dots, u_{k-1}, x) : 0 \leq x \leq 1, \min_{1 \leq i \leq k-1} u_i \geq \epsilon/2, x - u_1 \dots - u_{k-1} \geq \epsilon/2 \right\}.$$

It is now easy to conclude, by (18), (19) and (20). \square

3.2 The case of trees

We now want to make a study parallel to the previous one in case of trees. We let

$$\nu_n = \varrho^n \# \mathbf{T}_n, n \geq 1, \quad (21)$$

which defines a probability distribution on \mathbb{N} thanks to (3), satisfying

$$\nu_n \underset{n \rightarrow \infty}{\sim} \frac{\mathbf{c}}{2\sqrt{\pi n^{3/2}}}. \quad (22)$$

We now let (Ω, \mathcal{F}, P) be a probability space under which an i.i.d. sequence X'_1, X'_2, \dots of ν -distributed random variables is defined. A (non-plane) tree analog of Lemma 7 states as follows. We let G_a denote the set of a -good trees, and $\mathbf{P}_n^a := \mathbf{P}(\cdot \cap G_a)$.

Lemma 10 *Let $(\mathfrak{s}, \mathbf{x}, \mathbf{y})$ be an a -good skeleton. Then, seeing $\text{Sk}_a : \mathfrak{t} \mapsto \text{Sk}_a(\mathfrak{t})$ as a random variable,*

$$\begin{aligned} \mathbf{P}_n^a(\text{Sk}_a = (\mathfrak{s}, \mathbf{x}, \mathbf{y})) &= \frac{1}{2^{|\mathfrak{s}|} \nu_n} P \left(\sum_{i=1}^{|\mathfrak{x}|} X'_i = n - |\mathbf{y}|, \max_{1 \leq i \leq |\mathfrak{x}|} X'_i \leq a \right) \\ &\times \prod_{u \in L(\mathfrak{s})} P(X'_1 + X'_2 = y_u, X'_1 \vee X'_2 \leq a, X'_1 \neq X'_2). \end{aligned}$$

Proof. The proof goes like that of Lemma 7, applying Lemma 6 (and using ν instead of μ); the only important difference being that in the first displayed formula of the proof, we must replace plane trees with trees, the factor $2^{|\mathfrak{x}|}$ does not appear anymore, and we should consider pairs of trees $(\mathfrak{t}, \mathfrak{t}')$ such that $|\mathfrak{t}| + |\mathfrak{t}'| = y_i$ and $|\mathfrak{t}| < |\mathfrak{t}'|$, the last constraint being absent in the plane case. This can be replaced by the constraint $|\mathfrak{t}| \neq |\mathfrak{t}'|$ by introducing factors of 2, one for each $1 \leq j \leq |\mathfrak{s}|$. \square

Besides the fact that the random variables X'_i replace X_i , the main difference with Lemma 7 is the presence of the event G_a , and of the additional condition $X'_i \neq X'_j$ in the last event. We can also make a comment on the fact that a prefactor of the form $2^{-|\mathfrak{s}|}$ replaces a factor $2^{-2^{|\mathfrak{s}|+1}}$ in Lemma 7: if we were concerned with the law of the non-plane version $p(\mathcal{T}_n)$ of a \mathbb{P}_n -distributed random variable \mathcal{T}_n , then for each (non-plane) possible skeleton $(\mathfrak{s}, \mathbf{x}, \mathbf{y})$ without symmetries, there would be exactly $2^{|\mathfrak{s}|-1}$ plane skeletons contributing to its weight.

From this, one deduces a counterpart of Proposition 2 for (non-plane) trees. For $\mathfrak{s} \in \mathbb{T}$, we let $\tilde{I}_M(\mathfrak{s})$ be defined in a similar way as $I_M(\mathfrak{s})$ in (13), except that we further ask that the components of \mathbf{x} are pairwise distinct, and are such that $x_{u1} > x_{u2}$ whenever $u \in I(\mathfrak{s})$.

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Proposition 3 *Let $\mathfrak{s} \in \mathbb{T}$. Then for every $\epsilon \in (0, 1)$ and $M > 1/\epsilon$, the quantity*

$$\sup_{\substack{\mathbf{x}, \mathbf{y} \in \mathbb{Z}_+^{\mathfrak{s}} \times \mathbb{Z}_+^{L(\mathfrak{s})}: \\ (\mathbf{x}/\sqrt{n}, \mathbf{y}/n) \in \tilde{I}_M(\mathfrak{s})}} \left| n^{2|\mathfrak{s}|-1/2} \mathbf{P}_n^{[\epsilon n]}(\text{Sk}_{[\epsilon n]} = (\mathfrak{s}, \mathbf{x}, \mathbf{y})) - 2^{|\mathfrak{s}|-1} \mathbf{c}^{2|\mathfrak{s}|-1} \psi^{\mathfrak{s}}(\mathbf{c}\mathbf{x}/\sqrt{n}, \mathbf{y}/n) \right|$$

converges to 0 as $n \rightarrow \infty$.

The proof follows exactly the same route as that of Proposition 2, using the two following intermediate lemmas.

Lemma 11 *For any $M > 1/\epsilon$, we have the convergence*

$$\lim_{n \rightarrow \infty} \sup_{l/n \in [\epsilon + M^{-1}, 2\epsilon]} \left| n^2 P(X'_1 + X'_2 = l, X'_1 \vee X'_2 \leq [\epsilon n], X'_1 \neq X'_2) - \mathbf{c}^2 b(l/n) \right| = 0,$$

Lemma 12 *For $M > 1$,*

$$\lim_{n \rightarrow \infty} \sup_{\substack{l/\sqrt{n} \in [M^{-1}, M] \\ m/n \in [0, 1]}} \left| n P \left(\sum_{i=1}^l X'_i = m, \max_{1 \leq i \leq l} X'_i \leq [\epsilon n] \right) - \sum_{k \geq 0} \frac{(-\mathbf{c}l/\sqrt{n})^{k-2}}{k!} \int_0^{m/n} du a_k(u) g \left(\frac{m/n - u}{(\mathbf{c}l/\sqrt{n})^2} \right) \right| = 0.$$

In turn, the proofs of these lemmas are exactly the same as that of Lemmas 8 and 9. In the proof of Lemma 11, one just has to be careful of the extra \mathbf{c}^2 term which comes in front of b , by using the equivalent (22) instead of (11), and take into account the (asymptotically unimportant) fact that X'_1, X'_2 are constrained to take distinct values. In the proof of Lemma 12, a similar remark holds for the function a_k , which has to be replaced by $\mathbf{c}^k a_k$, while the function g must be replaced by $x \mapsto \mathbf{c}^{-2} g(\mathbf{c}^{-2}x)$, which is the density probability function of the total sum of masses of a Poisson process with intensity

$$\frac{\mathbf{c} dx}{2\sqrt{\pi}x^{3/2}} \mathbb{1}_{\{x>0\}}.$$

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We now want to have an interpretation of the functions $\psi^{\mathfrak{s}}$ in terms of the CRT. To this end, we first give another interpretation of the trimming operations discussed in Section 2.1.

4.1 Excursion functions and trimming

Recall that \mathcal{E} is the set of continuous functions f defined on a compact interval $[0, \sigma_f]$, that are non-negative, and satisfy $f(0) = f(\sigma_f) = 0$. With every $f \in \mathcal{E}$, one associates an \mathbb{R} -tree $\mathcal{T}_f = [0, \sigma_f]/\equiv_f$ endowed with the distance d_f , as in Section 1.4. This tree is naturally *rooted* (i.e. comes with a distinguished point) at the point ρ , which is the \equiv_f -equivalence class of 0.

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For I a real interval, we let $|I| = \sup I - \inf I$ denote its length. For every (nonempty, open or closed) interval I , $a = \inf I$ and $b = \sup I$, such that $\inf_I f = f(a) = f(b)$, we can define a new excursion function $f_I \in \mathcal{E}$ by

$$f_I(t) = f(t + a) - f(a), 0 \leq t \leq \sigma_{f_I} = |I|.$$

The extremities $a, b \in I$ encode the same point $x = x_I$ in \mathcal{T}_f , and f_I encodes in turn an \mathbb{R} -tree $\mathcal{T}_f(x) := \mathcal{T}_{f_I}$, which is naturally interpreted as the subtree of \mathcal{T}_f rooted at x , i.e. the set of points $y \in \mathcal{T}_f$ such that

$$d_f(\rho, y) = d_f(\rho, x) + d_f(x, y).$$

Yet otherwise said, this is the set of all descendants of x . We define the *mass* of $\mathcal{T}_f(x)$ as $|\mathcal{T}_f(x)| = |I|$. Finally, for $a \in (0, \sigma_f)$ a positive real number, we let

$$\mathcal{T}_f^{[a]} = \{x \in \mathcal{T}_f : |\mathcal{T}_f(x)| \geq a\},$$

which is also the closure of $\{x \in \mathcal{T}_f : |\mathcal{T}_f(x)| > a\}$, as is easily checked. The set $\mathcal{T}_f^{[a]}$ is an \mathbb{R} -tree which contains ρ , and is naturally rooted at this point.

As we will soon see, the operation $\mathcal{T}_f \mapsto \mathcal{T}_f^{[a]}$ is a continuous analog of the trimming operation $\mathfrak{t} \mapsto \mathfrak{t}[a]$ defined in Section 2.

First we present a slightly different way to look at $\mathcal{T}_f^{[a]}$. For any $f \in \mathcal{E}$ and $t \geq 0$, let $\mathcal{O}_f(t)$ be the set of (non-empty) connected components of the open set $\{f > t\}$. It is convenient to imagine $\mathcal{O}_f(t), t \geq 0$ as a *fragmentation process*, as the set $\{f > t\}$ decreases from $[0, \sigma_f]$ to \emptyset when t moves from 0 to ∞ . Note that, if $I \in \mathcal{O}_f(t)$, then f_I is well-defined and is positive on $(0, \sigma_{f_I})$, and as above, I encodes a point x_I in \mathcal{T}_f . This point will be in $\mathcal{T}_f^{[a]}$ if and only if $|I| \geq a$. This identifies $\mathcal{T}_f^{[a]}$ with the set of intervals $I \in \mathcal{O}_f(t)$ with $t \geq 0$ and $|I| \geq a$. Note however that this identification is not one-to-one, as several intervals can encode the same point of \mathcal{T}_f .

From this, we can state the following Lemma allowing to control the Gromov-Hausdorff distance between \mathcal{T}_f and $\mathcal{T}_f^{[a]}$.

Lemma 13 *For every $f \in \mathcal{E}$ such that $\sigma_f > a$, it holds that*

$$d_{\text{GH}}(\mathcal{T}_f, \mathcal{T}_f^{[a]}) \leq \omega(f, a),$$

where $\omega(f, a) = \sup_{|x-y| \leq a} |f(x) - f(y)|$ is the modulus of continuity of f , evaluated at a . In particular, $\mathcal{T}_f^{[a]} \rightarrow \mathcal{T}_f$ for the Gromov-Hausdorff metric as $a \rightarrow 0$.

Proof. Since $\mathcal{T}_f^{[a]}$ is defined as a subset of \mathcal{T}_f , it suffices to show that any point of \mathcal{T}_f not in $\mathcal{T}_f^{[a]}$ is at distance at most $\omega(f, a)$ from a point in $\mathcal{T}_f^{[a]}$. So let x be such a point. For every $0 < t < d_f(\rho, x)$, there is a unique interval $I_x(t) \in \mathcal{O}_f(t)$ that contains some (hence all) representative of x in $[0, \sigma_f]$ for the equivalence relation \equiv_f . Since x is not in $\mathcal{T}_f^{[a]}$ it holds that $|I_x(t)| < a$ for t close enough to $d_f(\rho, x)$, so let t_0 be the infimum of all t 's with this property. The intervals $I_x(t_0 - \epsilon)$ decrease as $\epsilon \rightarrow 0$ to some interval $I_x(t_0 -)$ with length $\geq a$, which encodes a vertex $x_0 \in \mathcal{T}_f^{[a]}$. For every $\epsilon \in (0, d_f(\rho, x) - t_0)$, the interval $I_x(t_0 + \epsilon)$ has length $< a$, so easily $d_f(x, x_{I_x(t_0 + \epsilon)}) = f(s) - f(s') \leq \omega(f, a)$ where s is a representative of x and s' is the left end of $I_x(t_0 + \epsilon)$. Hence $d_f(x, x_0) \leq \epsilon + \omega(f, a)$ for every $\epsilon > 0$, giving the result. \square

4.2 The a -real skeleton description of $\mathcal{T}_f^{[a]}$

Let us now derive an alternative representation of $\mathcal{T}_f^{[a]}$ in terms of marked trees. To avoid trivialities we let

$$\mathcal{E}_a = \{f \in \mathcal{E} : \sigma_f > a\},$$

and will assume from now on that $a > 0$ and $f \in \mathcal{E}_a$ so that in particular, $\mathcal{T}_f^{[a]}$ is non-empty and not reduced to a single point.

Assuming for a second that $\mathcal{O}_f(0) = (0, \sigma_f)$, by continuity of f , $\mathcal{O}_f(t)$ contains a single interval of length $> a$ for every small enough $t > 0$. When t increases, this property will break down either because a second interval of length $> a$ is created, or if no interval of length $> a$ remains (see Figure 7). For every $f \in \mathcal{E}_a$, we let

$$t_a(f) = \inf \left\{ t \geq 0 : \sup_{I \in \mathcal{O}_f(t)} |I| \leq a \right\} \in [0, \sup f],$$

and

$$s_a(f) = \inf \{t \geq 0 : \exists I, I' \in \mathcal{O}_f(t) : I \neq I', |I| > a \text{ and } |I'| > a\} \in [0, \infty].$$

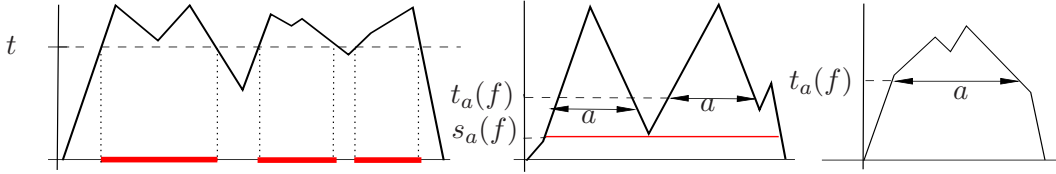


Figure 7: The set of intervals $\mathcal{O}_f(t)$ (left), the case $s_a(f) < +\infty$ (middle), the case $s_a(f) = +\infty$ (right).

By definition, for every $t < s_a(f) \wedge t_a(f)$, $\mathcal{O}_f(t)$ contains a single interval $I_f(t)$ with $|I_f(t)| > a$, and these decrease to a closed interval $I_f(s_a(f) \wedge t_a(f)-)$ as $t \uparrow s_a(f) \wedge t_a(f)$. By convention, we let $I_f(0-) = (0, \sigma_f)$ in the case $s_a(f) \wedge t_a(f) = 0$. It is immediate to see that if $s_a(f) < \infty$ then $s_a(f) < t_a(f)$. In the case $s_a(f) = \infty$, we let

$$Y_a(f) := |I_f(t_a(f)-)| \geq a.$$

In all cases, thanks to the description of $\mathcal{T}_f^{[a]}$ in terms of the intervals of $\mathcal{O}_f(t)$, one checks easily that the mapping $t \in [0, s_a(f) \wedge t_a(f)) \mapsto x_{I_f(t)} \in \mathcal{T}_f$ is an isometry, and that its image is exactly the set

$$\left\{ x \in \mathcal{T}_f^{[a]} : d_f(x, \rho) < s_a(f) \wedge t_a(f) \right\}.$$

This isometry can be extended continuously to the whole segment $[0, s_a(f) \wedge t_a(f)]$ by mapping its right end to $x_{I_f(s_a(f) \wedge t_a(f)-)}$. Otherwise said, for $f \in \mathcal{E}_a$, the “bottom” of the tree $\mathcal{T}_f^{[a]}$ is made of a line segment (possibly reduced to the single point $\{\rho\}$). Moreover, if $s_a(f) = \infty$, then the above isometry admits all of $\mathcal{T}_f^{[a]}$ as its image, so the latter is isometric to the segment $[0, t_a(f)]$. So let us study what happens above level $s_a(f)$ in the case $s_a(f) < \infty$.

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We will be interested in a particular type of functions of \mathcal{E}_a . We say that $x \in (0, \sigma_f)$ is a local minimum of f if $f(x) \leq f(y)$ for every y in an open neighborhood of x . Note that $0, \sigma_f$ are purposely excluded with this definition. We let m_f be the set of local minima of f .

For $a > 0$, we let $\overline{\mathcal{E}}_a$ be the set of functions $f \in \mathcal{E}$ such that for all $x, y \in m_f, x \leq y$ such that $f(x) = f(y)$, then $x = y$ or $\min\{f(t) : t \in [x, y]\} < f(x)$. This condition implies that \mathcal{T}_f has only binary branching points. We also let $\overline{\mathcal{E}}_a^*$ be the subset of $\overline{\mathcal{E}}_a$ made of those functions f such that

$$a \notin \text{closure}(\{|I| : \exists t \geq 0, I \in \mathcal{O}_f(t)\}).$$

The sets $\overline{\mathcal{E}}_a, \overline{\mathcal{E}}_a^*$ are not nice from a topological perspective (in particular they are neither open nor closed), but it turns out that the mapping $f \mapsto \mathcal{O}_f$ behaves well when restricted to these sets, as the following lemmas will show.

Lemma 14 *Let $a > 0$ and $f \in \overline{\mathcal{E}}_a$.*

1. *If $s_a(f) = \infty$ and $f \in \overline{\mathcal{E}}_a^*$, then $Y_a(f) > a$.*
2. *If $s_a(f) < \infty$, then $\mathcal{O}_f(s_a(f))$ has exactly two components $I_f^{(l)}, I_f^{(r)}$ with $|I_f^{(l)}| \wedge |I_f^{(r)}| > a$, and these can be ordered so that $\sup I_f^{(l)} = \inf I_f^{(r)}$. We then set $f^{(l)} = f_{I_f^{(l)}}, f^{(r)} = f_{I_f^{(r)}}$.*

Proof. 1. Assume $s_a(f) = \infty$. The possibility that $Y_a(f) = a$ is excluded by definition of $Y_a(f)$ and the fact that a is not a limit point of the set of lengths of intervals in $\cup_{t \geq 0} \mathcal{O}_f(t)$. Hence $Y_a(f) > a$.

2. Assume $s_a(f) < \infty$. Let $I_0 = I_f(s_a(f)-)$. Necessarily, it must hold that f_{I_0} attains the value 0 at some point x of $(0, \sigma_{f_{I_0}})$, which is unique as it must correspond to a local minimum of $f \in \overline{\mathcal{E}}_a$. Hence, $\mathcal{O}_{f_{I_0}}(0)$ is made of two intervals I_1, I_2 such that $\sup I_1 = \inf I_2 = x$, and $\inf I_1 = 0, \sup I_2 = |I_0|$. These two intervals correspond to two intervals

$$I_f^{(l)} = (\inf I_0, x + \inf I_0), \quad I_f^{(r)} = (x + \inf I_0, \sup I_0),$$

belonging to $\mathcal{O}_f(s_a(f))$. By definition, the latter set contains two intervals with lengths $> a$, but $I_f^{(l)}, I_f^{(r)}$ are the only possible ones, since an interval $I \in \mathcal{O}_f(s_a(f))$ is a subinterval of some interval in $\mathcal{O}_f(t)$ for $t < s_a(f)$, so in order that $|I| > a$ we must have $I \subset I_f(t)$ for every $t < s_a(f)$, giving $I \subset I_0$. \square

Point 2. in Lemma 14 entails that in the case $s_a(f) < \infty$, the subtree of \mathcal{T}_f above the point $x_{I_f(s_a(f)-)}$ splits into two branches with masses $> a$. More precisely, if $h > 0$ and $\mathcal{T}, \mathcal{T}'$ are two rooted \mathbb{R} -trees with roots ρ, ρ' , we let $\Theta(h, \mathcal{T}, \mathcal{T}')$ be the \mathbb{R} -tree obtained by identifying the roots ρ, ρ' and grafting to this point a line segment of length h . The latter can be easily formalized as the quotient of the disjoint union

$$[0, h] \sqcup \mathcal{T} \sqcup \mathcal{T}'$$

in which the point $h \in [0, h]$ is identified with ρ and ρ' , and endowed with the proper distance function that extends the metrics on each of these components. The tree $\Theta(h, \mathcal{T}, \mathcal{T}')$ is naturally rooted at the point $0 \in [0, h]$. Then if $f \in \overline{\mathcal{E}}_a$ and $s_a(f)$, with the notations of Lemma 14,

$$\mathcal{T}_f^{[a]} \stackrel{\text{isom}}{=} \Theta(s_a(f), \mathcal{T}_{f^{(l)}}^{[a]}, \mathcal{T}_{f^{(r)}}^{[a]}), \tag{23}$$

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while $\mathcal{T}_f^{[a]}$ is isometric to $[0, t_a(f)]$ if $s_a(f) = \infty$. Put together, these two facts allow a recursive construction of $\mathcal{T}_f^{[a]}$, which ends in a finite number of steps, since by definition, we have $\sigma_{f^{(l)}} \vee \sigma_{f^{(r)}} \leq \sigma_f - a$ whenever $s_a(f) < \infty$.

This description can be further simplified as follows. Recall the definition of the concatenation operation Ξ for labeled plane trees, defined in Section 2.1. We can associate, in a recursive manner, a marked plane tree $(\mathfrak{s}, \mathbf{g}) = \zeta(f, a)$ with $\mathbf{g} = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{\mathfrak{s}} \times \mathbb{R}^{\mathfrak{s}}$ with every $f \in \overline{\mathcal{E}}_a$ in the following way.

- If $s_a(f) = \infty$ then $\mathfrak{s} = \{\emptyset\}, x_{\emptyset} = t_f(a), y_{\emptyset} = Y_a(f)$.
- If $s_a(f) < \infty$ then $\zeta(f, a) = \Xi((s_a(f), 0), \zeta(f^{(l)}, a), \zeta(f^{(r)}, a))$.

Note that every \mathbf{y} -mark of an internal vertex is 0, so we simply forget these marks, identifying $\zeta(f, a)$ with the tree \mathfrak{s} with marks $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{\mathfrak{s}} \times \mathbb{R}^{L(\mathfrak{s})}$.

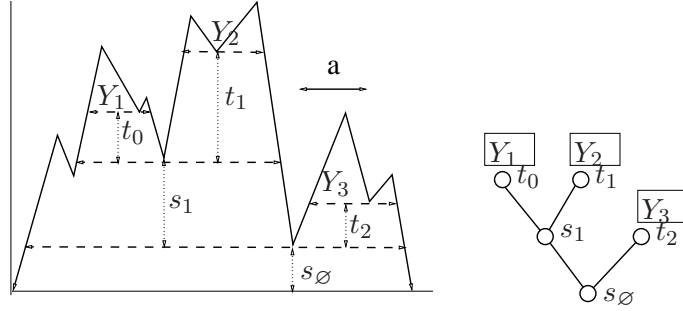


Figure 8: Representation of $\zeta(f, a)$

The marked tree $\zeta(f, a)$ is called the a -real skeleton of f (or of \mathcal{T}_f). Building the tree $\mathcal{T}_f^{[a]}$ from $\zeta(f, a)$ is now an easy matter by comparing the definition of $\zeta(f, a)$ with (23). With every pair of the form $(\mathfrak{s}, \mathbf{x})$, where $\mathfrak{s} \in \mathbb{T}$ and $\mathbf{x} = (x_u, u \in \mathfrak{s}) \in (0, \infty)^{\mathfrak{s}}$, we associate an \mathbb{R} -tree $\theta(\mathfrak{s}, \mathbf{x})$ in a natural way, by grafting segments with lengths x_u according to the genealogy of \mathfrak{s} . Formally, we let $\theta((\emptyset, x_{\emptyset}))$ be the isometry class of $[0, x_{\emptyset}]$, and recursively, if $\mathfrak{s} \neq \{\emptyset\}$,

$$\theta(\mathfrak{s}, \mathbf{x}) = \Theta(x_{\emptyset}, \theta((\mathfrak{s}, \mathbf{x})_1), \theta((\mathfrak{s}, \mathbf{x})_2)),$$

where by definition $(\mathfrak{s}, \mathbf{x})_u$ is the plane subtree \mathfrak{s}_u with labels $\mathbf{x}_u = (x_{uv}, v \in \mathfrak{s}_u)$.

Lemma 15 *Let $f \in \overline{\mathcal{E}}_a$. Then $\mathcal{T}_f^{[a]}$ and $\theta(\zeta(f, a))$ are isometric, where in the latter notation we do not take the \mathbf{y} -marks into account.*

To prove this, observe that $\theta(\zeta(f, a))$ is isometric to $[0, t_a(f)]$ if $s_a = \infty$, and that

$$\theta(\zeta(f, a)) = \Theta(s_a(f), \theta(\zeta(f^{(l)}, a)), \theta(\zeta(f^{(r)}, a))),$$

if $s_a(f) < \infty$, and compare with (23).

4.3 Continuity properties of the a -real skeleton

Lemma 16 *Let $a > 0$ and $(a_n, n \geq 1)$ be a positive sequence converging to a . Let $(f_n, n \geq 1)$ be a sequence such that $f_n \in \mathcal{E}_{a_n}$ for every n , uniformly converging to $f \in \overline{\mathcal{E}}_a^*$. Then*

1. *If $s_a(f) = \infty$, then $t_{a_n}(f_n) \rightarrow t_a(f)$, and $Y_{a_n}(f_n) \rightarrow Y_a(f)$ as $n \rightarrow \infty$.*
2. *If $s_a(f) < \infty$, then $s_{a_n}(f_n) \rightarrow s_a(f)$ and $(f_n^{(l)}, f_n^{(r)}) \rightarrow (f^{(l)}, f^{(r)})$ as $n \rightarrow \infty$.*

Proof. 1. This comes immediately from the uniform convergence of $f_n \rightarrow f$, and does not use the fact that $f \in \overline{\mathcal{E}}_a^*$.

2. Let $x = \inf I_f^{(l)}$, $y = \sup I_f^{(l)} = \inf I_f^{(r)}$ and $z = \sup I_f^{(r)}$. Let also $\epsilon \in (0, ((y-x) \wedge (z-y))/2)$. From the uniform convergence of f_n to f , the minimum of f_n over $(x + \epsilon, z - \epsilon)$ is attained at some y_n , which converges to y as $n \rightarrow \infty$ since $f(v) > f(y)$ for all $v \in [x + \epsilon, y) \cup (y, z - \epsilon]$. Let $x_n = \sup\{v < y_n : f_n(v) = f_n(y_n)\}$ and $z_n = \inf\{v > y_n : f_n(v) = f_n(y_n)\}$. Then we have $x_n \rightarrow x, z_n \rightarrow z$ as $n \rightarrow \infty$. Consequently, the intervals $(x_n, y_n), (y_n, z_n)$ have both lengths $> a_n$ for n large enough, since $y - x, z - y$ are $> a$ and $a_n \rightarrow a$. It remains to note that for large n , there can be no two intervals of length $> a_n$ in $\mathcal{O}_{f_n}(t)$ for $t < f_n(y_n)$, in which case it will follow that $s_{a_n}(f_n) = f_n(t_n) \rightarrow f(t) = s_a(f)$, and the remaining properties are easy. But assuming that two such intervals exist, one of them must contain the interval (x_n, z_n) . By extracting subsequences, we may assume that the other interval is of the form (x'_n, z'_n) (with $z'_n - x'_n > a_n$) and that its extremities converge to (x', z') . By construction $f(x') = f(z') \leq s_a(f)$ and by definition, it holds that $a = \lim a_n \leq z' - x' \leq a$. Hence, (x', z') is an interval of length a in $\mathcal{O}_f(f(x'))$ contradicting the fact that $f \in \overline{\mathcal{E}}_a^*$. \square

Corollary 1 *Let $a > 0$ and $(a_n, n \geq 1)$ be a positive sequence converging to a . Let $(f_n, n \geq 1)$ be a sequence such that $f_n \in \overline{\mathcal{E}}_{a_n}$ for every n , converging to $f \in \overline{\mathcal{E}}_a^*$. Then $\zeta(f_n, a_n) \rightarrow \zeta(f, a)$. Consequently, it holds that $d_{\text{GH}}(\mathcal{T}_{f_n}^{[a_n]}, \mathcal{T}_f^{[a]}) \rightarrow 0$.*

The first part of this statement is obtained directly by combining Lemma 16 with the recursive definition of $\zeta(f, a)$. The last sentence follows from Lemma 15 and the easy fact that $\mathbf{x} \mapsto \theta(\mathbf{s}, \mathbf{x})$ is continuous from $(0, \infty)^{\mathbb{S}}$ to $(\mathcal{T}, d_{\text{GH}})$, for every $\mathbb{S} \in \mathbb{T}$.

4.4 The case of discrete trees

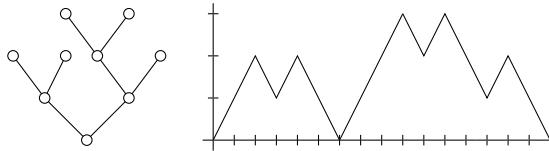


Figure 9: A plane tree and its contour process.

We now reconnect the previous discussion to our study of (discrete) trees. Let $\mathfrak{t} \in \mathbb{T}_n$. The *contour exploration* of \mathfrak{t} is the sequence $(u_{\mathfrak{t}}(0), \dots, u_{\mathfrak{t}}(4n-4))$ where $u_{\mathfrak{t}}(0) = \emptyset$, and recursively, $u_{\mathfrak{t}}(i+1)$ is the first child (in lexicographical order) of $u_{\mathfrak{t}}(i)$ not in $\{u_{\mathfrak{t}}(0), \dots, u_{\mathfrak{t}}(i)\}$, if such a child exists, or the parent of $u_{\mathfrak{t}}(i)$ otherwise. Each vertex is visited as many times as its degree in this sequence, which explains that the procedure stops at the $(4n-4)$ -th step (indeed, each

4 Trimming continuous trees

edge of the tree is visited twice, and there are $2n - 2$ edges in a binary tree with n leaves). A picture of this is to imagine a particle wrapping around the tree and moving at speed 1, starting from \emptyset (see Figure 9).

Definition 5 *The contour process of $\mathfrak{t} \in \mathbb{T}_n$ is the continuous function $C_{\mathfrak{t}}(t), 0 \leq t \leq 4n - 4$ whose graph is obtained by a linear interpolation between the points $(i, |u_{\mathfrak{t}}(i)|), 0 \leq i \leq 4n - 4$.*

The contour process of \mathfrak{t} is also called the *Dyck path encoding* of \mathfrak{t} , as the latter can be recovered from its contour process. We will only be interested in recovering the metric space $(\mathfrak{t}, d_{\mathfrak{t}})$ from $C_{\mathfrak{t}}$. If $v, w \in \mathfrak{t}$, let $0 \leq i, j \leq 4n - 4$ be such that $u_{\mathfrak{t}}(i) = v, u_{\mathfrak{t}}(j) = w$. Then it is easy to see that $|v \wedge w| = \min_{i \wedge j \leq k \leq i \vee j} C_{\mathfrak{t}}(k)$, independently of the choice of i, j . Therefore,

$$d_{\mathfrak{t}}(v, w) = C_{\mathfrak{t}}(i) + C_{\mathfrak{t}}(j) - 2 \min_{i \wedge j \leq k \leq i \vee j} C_{\mathfrak{t}}(k) = d_{C_{\mathfrak{t}}}(i, j).$$

Of course, for integers i and j , $d_{C_{\mathfrak{t}}}(i, j) = 0$ if and only if $u_{\mathfrak{t}}(i) = u_{\mathfrak{t}}(j)$, which is equivalent to $i \equiv_{C_{\mathfrak{t}}} j$, and so, the quotient space $\{0, 1, \dots, 4n - 4\} / \equiv_{C_{\mathfrak{t}}}$ endowed with the distance $d_{C_{\mathfrak{t}}}$, is isometric to $(\mathfrak{t}, d_{\mathfrak{t}})$. Therefore \mathfrak{t} is isometric to a subset of $\mathcal{T}_{C_{\mathfrak{t}}}$, and it is immediate to check that

$$d_{\text{GH}}(\mathfrak{t}, \mathcal{T}_{C_{\mathfrak{t}}}) \leq 1.$$

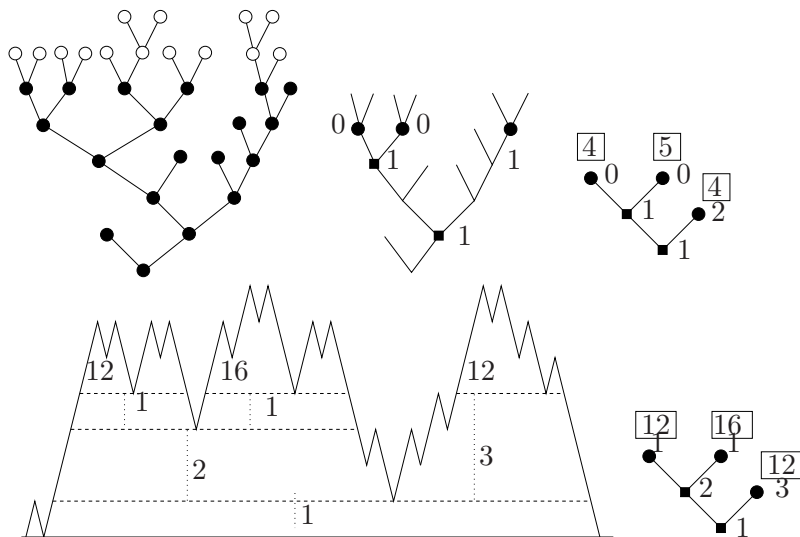


Figure 10: Comparison between the encoding via the skeleton and via the function ζ . On the first line the trimmed tree $\mathfrak{t}[3]$ and the skeleton representation. On the second line, representation of $\zeta(C_{\mathfrak{t}}, 12)$.

The trimming operation discussed in Section 2.1 has a direct interpretation in terms of trees of the form $\zeta(f, a)$, when f is the contour function of a plane tree.

Lemma 17 *Let $a > 0$ be an integer, and $\mathfrak{t} \in \mathbb{T}_n$. If $\text{Sk}_a(\mathfrak{t}) = (\mathfrak{s}, \mathbf{x}, \mathbf{y})$ then*

$$\zeta(C_{\mathfrak{t}}, 4(a - 1)) = (\mathfrak{s}, \mathbf{x}', \mathbf{y}'),$$

where $x'_u = x_u + \mathbb{1}_{\{u \neq \emptyset\}}$, for every $u \in \mathfrak{s}$, and $y'_u = 4y_u - 4$ for any $u \in L(\mathfrak{s})$.

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Figure 10 illustrates precisely the Lemma and the differences between the two points of view: the labels \mathbf{x} of the vertices are the same up to 1 due to discrete-continuous artifacts, and a subtree of size k becomes a subpath of length $4(k-1)$; the proof below formalises these observations.

Proof. We first note that since the plane trees that we consider are binary, it is immediate that the contour function $C_{\mathfrak{t}}$ is an element of $\overline{\mathcal{E}}_a$ for $a < 4|\mathfrak{t}| - 4$, so that $\zeta(C_{\mathfrak{t}}, a)$ is well-defined.

We will use the following alternative description of $\text{Sk}_a(\mathfrak{t}) = (\mathfrak{s}, \mathbf{x}, \mathbf{y})$. Recall the definition of $\pi_{\mathfrak{t}[a]} : \mathfrak{s} \rightarrow I_2(\mathfrak{t}[a])$. Then for $u \in \mathfrak{s}$ and $i \in \{1, 2\}$ such that $ui \in \mathfrak{s}$, we have

$$x_{ui} = |\pi_{\mathfrak{t}[a]}(ui)| - |\pi_{\mathfrak{t}[a]}(u)| - 1,$$

counting the number of elements of $I_1(\mathfrak{t}[a])$ comprised between $\pi_{\mathfrak{t}[a]}(u)$ and $\pi_{\mathfrak{t}[a]}(ui)$, while $x_{\emptyset} = |\pi_{\mathfrak{t}[a]}(\emptyset)|$.

Let us compare this with the construction of $\zeta(C_{\mathfrak{t}}, 4(a-1))$. For $t \geq 0$ an integer, an interval I of $\mathcal{O}_{C_{\mathfrak{t}}}(t)$ encodes a vertex u_I of \mathfrak{t} , seen as a subset of $\mathcal{T}_{C_{\mathfrak{t}}}$ as in the construction above. If the length of the interval is $\geq 4a - 4$, then the subtree \mathfrak{t}_{u_I} has at least a leaves, and consequently, this vertex belongs to $\mathfrak{t}[a]$. On the other hand, if the length is $\leq 4a - 4$, then none of the strict descendants of u_I can be in $\mathfrak{t}[a]$. From these observations, it follows that in the case $s_{4(a-1)}(C_{\mathfrak{t}}) = \infty$, it holds that

$$\text{Sk}_a(\mathfrak{t}) = (\{\emptyset\}, s_{4(a-1)}(C_{\mathfrak{t}}), Y_{4(a-1)}(C_{\mathfrak{t}})/(4(a-1))).$$

In the case $s_{4(a-1)}(C_{\mathfrak{t}}) < \infty$, the vertex of \mathfrak{t} encoded by the interval $I_{C_{\mathfrak{t}}}(s_{4(a-1)}(C_{\mathfrak{t}})-)$ is the vertex $\pi_{\mathfrak{t}[a]}(\emptyset)$ of Section 2.1.2, and it holds that $\text{Sk}_a(\mathfrak{t}) = (\mathfrak{s}, \mathbf{x}, \mathbf{y})$ satisfies $x_{\emptyset} = s_a(C_{\mathfrak{t}})$. Now, it must be noted that the functions $C_{\mathfrak{t}}^{(l)}, C_{\mathfrak{t}}^{(r)}$ defined as in Lemma 14 are not the contour functions of a binary plane tree, because they are strictly positive on their respective intervals of definitions. We let $\mathfrak{t}^{(l)}, \mathfrak{t}^{(r)}$ be the binary plane trees that admit as contour functions the paths $C_{\mathfrak{t}}^{(l)}, C_{\mathfrak{t}}^{(r)}$, to which the first and last steps have been removed. Then

$$\text{Sk}_a(\mathfrak{t}) = \Xi(s_{4(a-1)}(C_{\mathfrak{t}}), \text{Sk}_a(\mathfrak{t}^{(l)}), \text{Sk}_a(\mathfrak{t}^{(r)})).$$

Using this and an inductive argument, this indeed shows that

$$x_u = x'_u - 1, \quad u \neq \emptyset,$$

with the notations of the statement, this -1 being due to the fact that the quantities s_a, t_a associated with the functions $C_{\mathfrak{t}}^{(l)}, C_{\mathfrak{t}}^{(r)}$, must be subtracted 1 in order to yield the number of skeleton leaves comprised between two elements of $I_2(\mathfrak{t}[a])$. \square

In view of deriving scaling limits, define the rescaled contour function of \mathfrak{t} by

$$c_{\mathfrak{t}}(s) = \frac{C_{\mathfrak{t}}(4(|\mathfrak{t}| - 1)s)}{\sqrt{2|\mathfrak{t}|}}, \quad 0 \leq s \leq 1,$$

and the n -rescaled skeleton as

$$\text{SK}_a^{(n)}(\mathfrak{t}) = (\mathfrak{s}, n^{-1/2}\mathbf{x}, n^{-1}\mathbf{y}),$$

where $(\mathfrak{s}, \mathbf{x}, \mathbf{y}) = \text{Sk}_a(\mathfrak{t})$. Then by Lemma 17, it holds that for $\mathfrak{t} \in \mathbb{T}_n$,

$$\zeta\left(c_{\mathfrak{t}}, \frac{4(a-1)}{4(n-1)}\right) = (\mathfrak{s}, \mathbf{x}', n^{-1}\mathbf{y}),$$

where for any $u \in \mathfrak{s}$, $|n^{-1/2}\mathbf{x}_u - \mathbf{x}'_u| \leq n^{-1/2}$.

4.5 Scaling limits

We now discuss the scaling limits of the trimmed tree. The core result is the counterpart of Proposition 1 (in fact, this is a stronger result) in terms of contour processes; it is due to Aldous [4] (see also [21])

Proposition 4 *Let $\mathcal{T}_n, n \geq 1$ be random variables with respective distributions \mathbb{P}_n . Then $c_{\mathcal{T}_n}$ converges to $2\mathfrak{e}$, \mathfrak{e} the normalized Brownian excursion, in distribution for the topology of uniform convergence on $\mathcal{C}([0, 1])$.*

Corollary 2 *For every $\epsilon \in (0, 1)$, it holds that $\text{Sk}_{[\epsilon n]}^{(n)}(\mathfrak{t})$ under \mathbb{P}_n converges in distribution to $\zeta(2^{3/2}\mathfrak{e}, \epsilon)$.*

Note the normalization in front of the Brownian excursion in the last statement, which comes from the fact that the \mathbf{x} -components of the rescaled skeleton are obtained by dividing by \sqrt{n} the components of the skeleton (instead of $\sqrt{2n}$). From Corollary 1 and the discussion after Lemma 17, this is an immediate consequence of the following

Lemma 18 *For every $\epsilon \in (0, 1)$, it holds that with probability 1, \mathfrak{e} is an element of $\overline{\mathcal{E}}_\epsilon^*$.*

Proof. The fact that the local minima are realized once (a.s.) is a well-known property of the Brownian excursion, which is inherited from the analogous property for Brownian motion, and the property of positivity of \mathfrak{e} over $(0, 1)$ comes from the definition. Hence a.s. \mathfrak{e} belongs to $\overline{\mathcal{E}}_\epsilon$. It remains to show that ϵ is not a limit point of the lengths of intervals of $\cup_{t \geq 0} \mathcal{O}_\mathfrak{e}(t)$. To show this, we use the fact from [6] that if U is a uniform random variable in $[0, 1]$, independent of \mathfrak{e} , then for $0 \leq t < \mathfrak{e}_U$, the unique element $I_U(t)$ of $\mathcal{O}_\mathfrak{e}(t)$ containing U is such that $(|I_U(t)|, t \geq 0)$ has same distribution as $(\exp(-\xi_\tau(t)), t \geq 0)$, where

- ξ is a subordinator [5], with Lévy measure $e^x dx / (e^x - 1)^{3/2}$, and
- τ is the inverse of the increasing, continuous process $t \mapsto \int_0^t \exp(-\xi_s/2) ds$.

In particular, the closure C of the range of $(|I_U(t)|, t \geq 0)$ is the image of the closure of the range of ξ by $x \mapsto e^{-x}$, to which has been adjoined the point 0. By well-known properties of subordinators [5], points are polar for ξ , so that ϵ is not in C , almost-surely. Since U is uniform and independent of \mathfrak{e} , the wanted property easily follows — for instance, one can note that a.s. $\cup_{t \geq 0} \mathcal{O}_\mathfrak{e}(t)$ is the set of intervals $I_{U_i}(t), i \geq 1, 0 \leq t < \mathfrak{e}_{U_i}$ where $U_i, i \geq 1$ is i.i.d. uniform on $[0, 1]$ and independent of \mathfrak{e} , and with the obvious notation for $I_{U_i}(t)$. \square

4.6 Interpretation in terms of $\psi^{\mathfrak{s}}$

Now we reconnect the previous discussion to the functions $\psi^{\mathfrak{s}}$ defined in Section 3.

Lemma 19 *For every $\epsilon \in (0, 1)$, the functions $\psi^{\mathfrak{s}}, \mathfrak{s} \in \mathbb{T}$ form a “probability density”, in the sense that*

$$\sum_{\mathfrak{s} \in \mathbb{T}} \int_{\mathbb{R}_+^{\mathfrak{s}}} dx \int_{\mathbb{R}_+^{L(\mathfrak{s})}} dy \psi^{\mathfrak{s}}(\mathbf{x}, \mathbf{y}) = 1.$$

5 Proof of Theorem 1

For every $\epsilon \in (0, 1)$, we can define a probability measure $\Psi(d(\mathfrak{s}, \mathbf{x}, \mathbf{y}))$ on $\{(\mathfrak{s}, \mathbf{x}, \mathbf{y}) : \mathfrak{s} \in \mathbb{T}, \mathbf{x} \in \mathbb{R}^{\mathfrak{s}}, \mathbf{y} \in \mathbb{R}^{L(\mathfrak{s})}\}$ by the formula

$$\langle \Psi, F \rangle = \sum_{\mathfrak{s} \in \mathbb{T}} \int_{\mathbb{R}_+^{\mathfrak{s}}} d\mathbf{x} \int_{\mathbb{R}_+^{L(\mathfrak{s})}} d\mathbf{y} \psi^{\mathfrak{s}}(\mathbf{x}, \mathbf{y}) F(\mathfrak{s}, \mathbf{x}, \mathbf{y}),$$

for every non-negative measurable function F . Then Ψ is the law of $\zeta(2^{3/2}\mathfrak{e}, \epsilon)$.

Proof. Fix $\mathfrak{s} \in \mathbb{T}$. Recall the definition (13) of the open set $I_M(\mathfrak{s}) \subset \mathbb{R}_+^{\mathfrak{s}} \times \mathbb{R}_+^{L(\mathfrak{s})}$. Then, defining $\lfloor \mathbf{x} \rfloor$ by taking integer parts componentwise, we have

$$\begin{aligned} \mathbb{P}_n \left(\text{Sk}_{\lfloor \epsilon n \rfloor}^{(n)} \in \bigcup_{\mathfrak{s} \in \mathbb{T}} \{(\mathfrak{s}, \mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in I_M(\mathfrak{s})\} \right) \\ = \sum_{\mathfrak{s} \in \mathbb{T}} \int_{I_M(\mathfrak{s})} d\mathbf{x} d\mathbf{y} n^{2|\mathfrak{s}|-1/2} \mathbb{P}_n(\text{Sk}_{\lfloor \epsilon n \rfloor} = (\mathfrak{s}, \lfloor \mathbf{x} \sqrt{n} \rfloor, \lfloor n\mathbf{y} \rfloor)). \end{aligned}$$

The lim inf of the left-hand side as $n \rightarrow \infty$ is at least $P(\zeta(2^{3/2}\mathfrak{e}, \epsilon) \in \bigcup_{\mathfrak{s} \in \mathbb{T}} \{(\mathfrak{s}, \mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in I_M(\mathfrak{s})\})$, because of Corollary 2 and well-known properties of weak convergence of probability measures. And the right-hand side converges to

$$\sum_{\mathfrak{s} \in \mathbb{T}} \int_{I_M(\mathfrak{s})} d\mathbf{x} d\mathbf{y} \psi^{\mathfrak{s}}(\mathbf{x}, \mathbf{y}),$$

because of the uniform convergence stated in Proposition 2. This yields

$$P \left(\zeta(2^{3/2}\mathfrak{e}, \epsilon) \in \bigcup_{\mathfrak{s} \in \mathbb{T}} \{(\mathfrak{s}, \mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in I_M(\mathfrak{s})\} \right) \leq \sum_{\mathfrak{s} \in \mathbb{T}} \int_{I_M(\mathfrak{s})} d\mathbf{x} d\mathbf{y} \psi^{\mathfrak{s}}(\mathbf{x}, \mathbf{y}).$$

Letting $M \rightarrow \infty$, the sets $I_M(\mathfrak{s})$ respectively increase to $\{(\mathbf{x}, \mathbf{y}) \in (0, \infty)^{\mathfrak{s}} \times (\epsilon, 2\epsilon)^{L(\mathfrak{s})}\}$, and we know from Lemma 18 that the law of $\zeta(2^{3/2}\mathfrak{e}, \epsilon)$ is supported on the union of such sets as \mathfrak{s} ranges in \mathbb{T} . Therefore, the probability on the left-hand side converges to 1 as $M \rightarrow \infty$, which yields the first statement. The fact that Ψ is the law of $\zeta(2^{3/2}\mathfrak{e}, \epsilon)$ is a simple adaptation of the previous argument, and is omitted. \square

5 Proof of Theorem 1

We now finally embark in the proof of our main results. This will be done in two steps: first, we prove that the $\lfloor \epsilon n \rfloor$ -trimmed tree obtained from a \mathbf{P}_n -distributed tree converges to the ϵ -trimming of the (appropriately scaled) CRT. Then we show that the trimmed tree is “not too far” from the original tree.

5.1 Convergence of the a -skeleton under \mathbf{P}_n

The main result of this section is

5 Proof of Theorem 1

Proposition 5 For every $\epsilon \in (0, 1)$, the random variables

$$\mathbf{t} \mapsto \frac{\mathbf{c}}{\sqrt{2n}} \mathbf{t}[\lfloor \epsilon n \rfloor], \quad \text{under } \mathbf{P}_n$$

converge to $\theta(\zeta(2\mathfrak{e}, \epsilon))$ as $n \rightarrow \infty$, in distribution for the topology of $(\mathcal{M}, d_{\text{GH}})$, where \mathfrak{e} is the standard Brownian excursion.

To prove this, we study the scaling limit of $\mathbf{t} \mapsto \text{Sk}_{\lfloor \epsilon n \rfloor}(\mathbf{t})$ under \mathbf{P}_n . First, we define a probability measure, by

$$\langle \Psi^\circ, F \rangle = \sum_{\mathfrak{s} \in \mathbb{T}} 2^{|\mathfrak{s}|-1} \int_{\Delta_{\mathfrak{s}}} d\mathbf{x} \int_{\mathbb{R}_+^{L(\mathfrak{s})}} d\mathbf{y} \mathbf{c}^{2|\mathfrak{s}|-1} \psi^{\mathfrak{s}}(\mathbf{c}\mathbf{x}, \mathbf{y}) F(\mathfrak{s}, \mathbf{x}, \mathbf{y}),$$

where

$$\Delta_{\mathfrak{s}} = \{(x_u, u \in \mathfrak{s}) : x_{u_1} > x_{u_2} \text{ for all } u \in I(\mathfrak{s})\}.$$

The fact that $\langle \Psi^\circ, 1 \rangle = 1$ is a consequence of the fact that Ψ° charges only good labeled trees $(\mathfrak{s}, \mathbf{x}, \mathbf{y})$, and Lemma 4, because of the presence of the factor $2^{|\mathfrak{s}|-1}$. The following is the analog of Corollary 2 (and the end of Lemma 19) in the context of (non-plane) trees.

Lemma 20 For every $\epsilon \in (0, 1)$, the law of $\mathbf{t} \mapsto \text{Sk}_{\lfloor \epsilon n \rfloor}^{(n)}(\mathbf{t})$ under \mathbf{P}_n converges weakly to the measure Ψ° .

Proof. For $\mathfrak{s} \in \mathbb{T}$ and $\mathbf{x} \in (0, \infty)^{\mathfrak{s}}, \mathbf{y} \in (0, \infty)^{L(\mathfrak{s})}$, we set

$$g_n(\mathfrak{s}, \mathbf{x}, \mathbf{y}) = n^{2|\mathfrak{s}|-1/2} \mathbf{P}_n^{[\epsilon n]}(\text{Sk}_{\lfloor \epsilon n \rfloor} = (\mathfrak{s}, \lfloor \mathbf{x}\sqrt{n} \rfloor, \lfloor n\mathbf{y} \rfloor)).$$

Using Proposition 3, we have that $g_n(\mathfrak{s}, \mathbf{x}, \mathbf{y}) \rightarrow 2^{|\mathfrak{s}|-1} \mathbf{c}^{2|\mathfrak{s}|-1} \psi^{\mathfrak{s}}(\mathbf{c}\mathbf{x}, \mathbf{y})$ pointwise, and uniformly over $\bigcup_{\mathfrak{s}} \tilde{I}_M(\mathfrak{s})$ for every $M > 0$. Consequently, for every $M > 0$, it holds that

$$\begin{aligned} \mathbf{P}_n(G_{\lfloor \epsilon n \rfloor}) &= \sum_{\mathfrak{s} \in \mathbb{T}} \int_{\mathbb{R}_+^{\mathfrak{s}} \times \mathbb{R}_+^{L(\mathfrak{s})}} d\mathbf{x} d\mathbf{y} g_n(\mathfrak{s}, \mathbf{x}, \mathbf{y}) \\ &\geq \sum_{\mathfrak{s} \in \mathbb{T}} \int_{\tilde{I}_M(\mathfrak{s})} d\mathbf{x} d\mathbf{y} g_n(\mathfrak{s}, \mathbf{x}, \mathbf{y}) \\ &\xrightarrow{n \rightarrow \infty} \sum_{\mathfrak{s} \in \mathbb{T}} 2^{|\mathfrak{s}|-1} \int_{\tilde{I}_M(\mathfrak{s})} d\mathbf{x} d\mathbf{y} \mathbf{c}^{2|\mathfrak{s}|-1} \psi^{\mathfrak{s}}(\mathbf{c}\mathbf{x}, \mathbf{y}), \end{aligned}$$

which as $M \rightarrow \infty$ converges to

$$\sum_{\mathfrak{s} \in \mathbb{T}} 2^{|\mathfrak{s}|-1} \int_{\Delta_{\mathfrak{s}}} d\mathbf{x} \int_{(\epsilon, 2\epsilon)^{L(\mathfrak{s})}} d\mathbf{y} \mathbf{c}^{2|\mathfrak{s}|-1} \psi^{\mathfrak{s}}(\mathbf{c}\mathbf{x}, \mathbf{y}) = \langle \Psi^\circ, 1 \rangle = 1.$$

Therefore, $\lim \mathbf{P}_n(G_{\lfloor \epsilon n \rfloor}) = 1$, and by the so-called Scheffé Lemma, it holds that g_n converges in L^1 . Letting F be a uniformly continuous function bounded by K , it holds that

$$|\mathbf{P}_n(F(\text{Sk}_{\lfloor \epsilon n \rfloor}^{(n)})) - \mathbf{P}_n^{[\epsilon n]}(F(\text{Sk}_{\lfloor \epsilon n \rfloor}^{(n)}))| \leq K(1 - \mathbf{P}_n(G_{\lfloor \epsilon n \rfloor})) \rightarrow 0,$$

while

$$\mathbf{P}_n^{[\epsilon n]}(F(\text{Sk}_{\lfloor \epsilon n \rfloor}^{(n)})) = \sum_{\mathfrak{s} \in \mathbb{T}} \int_{\mathbb{R}_+^{\mathfrak{s}} \times \mathbb{R}_+^{L(\mathfrak{s})}} d\mathbf{x} d\mathbf{y} F(\mathfrak{s}, \lfloor \mathbf{x}\sqrt{n} \rfloor / \sqrt{n}, \lfloor n\mathbf{y} \rfloor / n) g_n(\mathfrak{s}, \mathbf{x}, \mathbf{y})$$

converges to $\langle \Psi^\circ, F \rangle$, using the uniform continuity of F and the L^1 -convergence of g_n . \square

5 Proof of Theorem 1

Lemma 21 *If $(\mathfrak{s}, \mathbf{x}, \mathbf{y})$ and $(\mathfrak{s}', \mathbf{x}', \mathbf{y}')$ are respectively Ψ and Ψ° distributed, then $\theta(\mathfrak{s}, \mathbf{x})/\mathbf{c}$ and $\theta(\mathfrak{s}', \mathbf{x}')$ have the same distribution.*

Proof. Note that $\theta(\mathfrak{s}, \mathbf{x}) = \theta(\mathfrak{s}', \mathbf{x}')$ whenever $(\mathfrak{s}, \mathbf{x}) \approx (\mathfrak{s}', \mathbf{x}')$, i.e. whenever these two plane labeled trees represent the same labeled tree. Therefore, for a given a -good tree \mathfrak{t} , $\theta(\text{Sk}_a(\mathfrak{t})) = \theta(\mathfrak{s}', \mathbf{x}')$ for any of the $2^{|\mathfrak{s}|-1}$ choices of $(\mathfrak{s}', \mathbf{x}') \approx \text{Sk}_a(\mathfrak{t})$, and there are exactly $2^{|\mathfrak{s}|-1}$ of them. The conclusion follows easily from the definition of Ψ° and the fact that θ is a continuous function. \square

Proof of Proposition 5. As a consequence of Lemma 20, and the easily checked fact that

$$d_{\text{GH}}(\theta(\text{Sk}_{\lfloor \epsilon n \rfloor}(\mathfrak{t})), \mathfrak{t}[\lfloor \epsilon n \rfloor]) \leq 1$$

for every $\mathfrak{t} \in \mathbf{T}_n$, we obtain that $n^{-1/2}\mathfrak{t}[\lfloor \epsilon n \rfloor]$ converges in distribution to the image by θ of a random variable with distribution Ψ° , which by Lemma 20 and Lemma 5 has same law as $\theta(\zeta(2^{3/2}\mathbf{e}/\mathbf{c}, \epsilon))$. \square

5.2 Tightness

By Proposition 5, we know that the $\lfloor \epsilon n \rfloor$ -trimming of a \mathbf{P}_n -distributed tree is close in distribution, when n is large, to $\theta(\zeta(2\mathbf{e}, \epsilon))$. In turn, the latter is close to the CRT $\mathcal{T} = \mathcal{T}_{2\mathbf{e}}$ when ϵ is small, by Lemma 13. On an intuitive basis, the proof will be complete if we are able to show that for ϵ small enough, under \mathbf{P}_n , trees \mathfrak{t} are typically close, in the Gromov-Hausdorff sense, to $\mathfrak{t}[\lfloor \epsilon n \rfloor]$ in a uniform way as $n \rightarrow \infty$. This is what the next lemma is taking care of.

Lemma 22 *For every $\eta > 0$, one has*

$$\lim_{\epsilon \rightarrow 0} \sup_{n \rightarrow \infty} \mathbf{P}_n(d_{\text{GH}}(\mathfrak{t}, \mathfrak{t}[\lfloor \epsilon n \rfloor]) > \eta\sqrt{n}) = 0.$$

Proof. Conditionally on its skeleton $\text{Sk}_{\lfloor \epsilon n \rfloor} = (\mathfrak{s}, \mathbf{x}, \mathbf{y})$, and conditionally on the set $G_{\lfloor \epsilon n \rfloor}$ of probability going to 1, the whole tree is reconstructed by grafting trees all with sizes $\leq \lfloor \epsilon n \rfloor$ to the tree $\Pi_0^{-1}(\mathfrak{s}, \mathbf{x})$, as in Lemma 6. Let r_1, r_2, \dots be these sizes, in decreasing order, and $\mathfrak{t}_1, \mathfrak{t}_2, \dots$ be the associated trees (with some arbitrary convention for ties). Of course, $\sum_i r_i = n$. Conditionally on the sizes, the trees are random elements, respectively with distribution \mathbf{P}_{r_i} , by Lemma 6.

At this point, we rely on the following particular case of [8, Theorem 5], stating that, if $H : \mathfrak{t} \mapsto H(\mathfrak{t})$ denotes the height (i.e. maximal height of elements of any $\mathfrak{t} \in \mathfrak{t}$) of the canonical random variable, then

$$\mathbf{E}_n[H^4] \leq Cn^2, \quad n \geq 1,$$

where \mathbf{E}_n denotes expectation with respect to \mathbf{P}_n , and C is some constant in $(0, \infty)$. Consequently, by the Markov inequality,

$$\mathbf{P}_n(H \geq x\sqrt{n}) \leq Cx^{-4}, \quad x > 0.$$

6 Final comments

Now, using this fact after conditioning on the sizes $r_i, i \geq 1$,

$$\begin{aligned}
\mathbf{P}_n(d_{\text{GH}}(\mathbf{t}, \mathbf{t}[\lfloor \epsilon n \rfloor]) > \eta\sqrt{n}) &\leq \mathbf{P}_n(\exists i \geq 0 : H(\mathbf{t}_i) \geq \eta\sqrt{n}) \\
&\leq \mathbf{E}_n \left[\sum_i P_{r_i}(H \geq \eta\sqrt{n}) \right] \\
&\leq C\eta^{-2} \mathbf{E}_n \left[\sum_i (r_i/n)^2 \right] \\
&\leq C\eta^{-2} \epsilon.
\end{aligned}$$

In the last step, we wrote $(r_i/n)^2 \leq \epsilon(r_i/n)$ and summed over i . Hence the result. \square

To conclude the proof of Theorem 1, simply apply the following and last lemma, with $X_n = \mathbf{c}(2n)^{-1/2}\mathbf{t}$ under \mathbf{P}_n , $X_n^{(\epsilon)} = \mathbf{c}(2n)^{-1/2}\mathbf{t}[\lfloor \epsilon n \rfloor]$ under \mathbf{P}_n , $X^{(\epsilon)} = \theta(\zeta(2e, \epsilon))$ and $X = \mathcal{T}_{2e}$. The hypotheses are enforced, by Lemma 22, and since $X_n^{(\epsilon)} \rightarrow X^{(\epsilon)}$ in distribution (Proposition 5) and $X^{(\epsilon)} \rightarrow X$ a.s. as a consequence of Lemma 13.

Lemma 23 *On some probability space (Ω, \mathcal{F}, P) , let $X_n, X, X_n^{(\epsilon)}, X^{(\epsilon)}, n \geq 1, \epsilon \in (0, 1)$ be random variables with values in some metric space (Z, d) such that $X_n^{(\epsilon)} \rightarrow X^{(\epsilon)}$ in distribution as $n \rightarrow \infty$, $X^{(\epsilon)} \rightarrow X$ in distribution as $\epsilon \rightarrow 0$, and*

$$\limsup_{\epsilon \rightarrow 0} \sup_{n \geq 1} P(d(X_n, X_n^{(\epsilon)}) > \eta) = 0.$$

Then $X_n \rightarrow X$ in distribution as $n \rightarrow \infty$.

Proof. Let F be closed in Z , then

$$P(X_n \in F) \leq P(X_n \in F, d(X_n, X_n^{(\epsilon)}) \leq \eta) + \delta_\eta^{(\epsilon)},$$

where $\delta_\eta^{(\epsilon)} = \sup_n P(d(X_n, X_n^{(\epsilon)}) > \eta)$. Thus $P(X_n \in F) \leq P(X_n^{(\epsilon)} \in F_\eta) + \delta_\eta^{(\epsilon)}$, where $F_\eta = \{z \in Z : d(z, F) \leq \eta\}$ is closed. Since $X_n^{(\epsilon)} \rightarrow X^{(\epsilon)}$ in distribution as $n \rightarrow \infty$, it holds that

$$\limsup_n P(X_n \in F) \leq P(X^{(\epsilon)} \in F_\eta) + \delta_\eta^{(\epsilon)},$$

and taking the limsup as $\epsilon \rightarrow 0$, appealing to the convergence in distribution $X^{(\epsilon)} \rightarrow X$, and finally, letting $\eta \rightarrow 0$, this gives $\limsup_n P(X_n \in F) \leq P(X \in F)$ as wanted. \square

6 Final comments

To our knowledge, the present work is the first to deal with the full scaling limit picture of a random non-labeled, non-plane tree. As was predictable from the known enumeration results, and conjectured by Aldous, the CRT is the object that arises as the limit of uniform random binary rooted trees.

In this paper, we have purposely restricted our attention to binary trees in order to make the combinatorial arguments as simple as possible when considering trimming of trees. However, it is to be expected that the kind of methods used to prove statements like Proposition 5 will still

References

work for more general families of unordered trees, for instance trees which are ‘at most m -ary’ for some $2 \leq m \leq \infty$, as considered in Otter’s work [22] (the case $m = \infty$ is sometimes referred to as *Pólya trees*). What is to be expected is that after a trimming by a mass $\lfloor \epsilon n \rfloor$ a uniform tree with n vertices, the tree that one obtains will be binary with a probability close to 1, because the CRT is itself binary. In view of the probabilistic approach taken in Section 3.2, it seems also possible to adapt this method to more general families of trees, i.e. of trees satisfying an enumeration scheme similar to (4).

We have not gone through the details of this approach in full generality, though, for several reasons. One of these is that it needs a result similar to that of Broutin and Flajolet [8] for more general trees. This might appear as a minor issue, as combinatorial methods are probably robust enough to be generalizable to more general situations. For instance, while completing the present work, we became aware of the paper [11] by Drmota and Gittenberger, in which moments estimates of the kind of [8] are obtained for the case of Pólya trees. However, one should note that the results of [8] give much more than what is actually needed to prove Lemma 22, i.e. precise moment estimates, or the derivation of the exact scaling limit of the height. From the probabilistic viewpoint, it would be very interesting to be able to decide in a slicker way whether this ‘tightness’ property of Lemma 22 holds in a general setting. Note in particular that a derivation of the scaling limit for the total height of the tree follows from Theorem 1:

$$\mathbf{P}_n(\mathbf{c}H \geq x\sqrt{2n}) \xrightarrow{n \rightarrow \infty} P(2 \max e \geq x) = 2 \sum_{k \geq 1} (k^2 x^2 - 1) e^{-k^2 x^2 / 2}, \quad \text{for } x > 0$$

(for the last inequality, see e.g. [14]), which is [8, Theorem 1].

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