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# Maximal inequality for high-dimensional cubes: quantitative estimates

Guillaume AUBRUN

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## Abstract

We present lower estimates for the best constant appearing in the weak  $(1, 1)$  maximal inequality in the space  $(\mathbf{R}^n, \|\cdot\|_\infty)$ . We show that it grows to infinity faster than  $(\log n)^\kappa$  for any  $\kappa < 1$ . We follow the approach used by J.M. Aldaz in a recent paper. The new part of the argument relies on Donsker's theorem identifying the Brownian bridge as the limit ( $n \rightarrow \infty$ ) of the empirical distribution function associated to coordinates of a point randomly chosen in the unit cube  $[0, 1]^n$ .

## Introduction

Let  $\text{vol}(\cdot)$  denote the Lebesgue measure on  $\mathbf{R}^n$ . For  $x \in \mathbf{R}^n$  and  $r > 0$ , let  $Q(x, r)$  denote the  $n$ -dimensional cube with center  $x$  and edge length  $2r$ . For a positive Borel measure  $\mu$  on  $\mathbf{R}^n$ , let  $M\mu$  be the “cubic” centered maximal function

$$M\mu(x) := \sup_{r>0} \frac{\mu(Q(x, r))}{\text{vol}(Q(x, r))}$$

We are interested in the following weak  $(1, 1)$  inequality : for any positive Borel measure  $\mu$  on  $\mathbf{R}^n$  and any  $L > 0$ ,

$$L \text{vol}\{M\mu > L\} \leq C\mu(\mathbf{R}^n). \quad (1)$$

We denote by  $\Theta_n$  the best possible  $C$  appearing in (1). Using a mollifying argument, one checks that restricting to absolutely continuous measures (or functions in  $L^1$ ) does not alter the value of  $\Theta_n$ .

We are interested in the asymptotic behaviour of  $\Theta_n$ . The easiest proof of (1) goes through Vitali's covering lemma, and gives  $\Theta_n \leq 3^n$ . This was greatly improved by Stein and Strömberg [6], who obtained  $\Theta_n \leq Cn \log n$  for some absolute constant  $C$ . This is the best known upper bound (we refer to [1] for additional background and bibliography). Conversely, Aldaz [1] proved recently that the sequence  $(\Theta_n)$  is unbounded. However he presented no concrete lower bounds. We prove the following

**Theorem 1.** *For any  $\kappa \in (0, 1)$ , there is a constant  $\lambda(\kappa) > 0$  so that*

$$\Theta_n \geq \lambda(\kappa)(\log n)^\kappa.$$

Note also that it is unknown whether the constants defined similarly, but using Euclidean balls instead of cubes, are uniformly bounded.

Let us briefly sketch the proof, which follows closely Aldaz's approach. We consider the counting measure on a very large box inside the lattice  $\mathbf{Z}^n$ . We show that the value of the corresponding maximal function at a point  $x \in \mathbf{R}^n$  is closely related to the statistical distribution of the coordinates of  $x \bmod 1$ . The asymptotic statistical behaviour is governed by a stochastic process

$(\beta_t)_{0 \leq t \leq 1}$  called the Brownian bridge. More precisely, the typical value of the maximal function is related to the following quantity

$$\sup_{\varepsilon \leq t \leq 1-\varepsilon} \frac{\beta_t}{\sqrt{t(1-t)}}.$$

The order of magnitude of the latter is given by the law of the iterated logarithm.

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## Proof of the theorem

Consider the measure  $\mu$  obtained by putting a Dirac mass on each lattice point  $x \in \mathbf{Z}^n$ . That is, for any Borel set  $A$ ,  $\mu(A) := \text{card}(A \cap \mathbf{Z}^n)$ . Although this measure has infinite mass, it can be used to estimate the constant  $\Theta_n$ :

**Lemma 1.** *For any  $L > 1$ ,*

$$L \text{ vol}(\{x \in [0, 1]^n \text{ s.t. } M\mu(x) > L\}) \leq \Theta_n.$$

*Proof of lemma 1.* The point is that large cubes can be ignored when computing  $M\mu$ . Indeed, since a cube of edge length  $2r$  contains at most  $(2r + 1)^n$  lattice points, we get for any  $L > 1$ ,

$$\sup_{r > n/2 \log L} \frac{\mu(Q(x, r))}{\text{vol}(Q(x, r))} \leq \sup_{r > n/2 \log L} \left(1 + \frac{1}{2r}\right)^n \leq \left(1 + \frac{\log L}{n}\right)^n \leq L \quad (2)$$

Let  $\mu_R$  be the restriction of  $\mu$  to the cube  $Q(0, R)$ . It follows from (2) that for any  $x \in Q(0, R - n/2 \log L)$ ,  $M\mu(x) > L$  if and only if  $M\mu_R(x) > L$ . Using the obvious fact that  $M\mu$  is  $\mathbf{Z}^n$ -periodic, this gives

$$\Theta_n \geq \frac{L \text{ vol}(\{x \in \mathbf{R}^n \text{ s.t. } M\mu_R > L\})}{\mu_R(\mathbf{R}^n)} \geq \left(\frac{\lfloor R - n/2 \log L \rfloor}{R}\right)^n L \text{ vol}(\{x \in [0, 1]^n \text{ s.t. } M\mu(x) > L\}).$$

It remains to take  $R \rightarrow \infty$ . □

For  $x \in [0, 1]^n$ , we will relate the value of  $M\mu(x)$  to the statistical distribution of the coordinates of  $x$  in the interval  $[0, 1]$ . For  $t \in (0, 1)$ , a number  $x \in [0, 1]$  is called  $t$ -centered if it belongs to  $[\frac{1-t}{2}, \frac{1+t}{2}]$ . Define the following set

$$E_{t,K}^n := \left\{x \in [0, 1]^n \text{ with at least } nt + K\sqrt{nt(1-t)} \text{ } t\text{-centered coordinates}\right\}.$$

This definition may look strange but the reason is simple. Think of  $x$  as a random variable uniformly distributed on  $[0, 1]^n$ . The number of  $t$ -centered coordinates is a random variable with expectation  $nt$ , and standard deviation  $\sqrt{nt(1-t)}$ . In particular, it follows from the central limit theorem that

$$\lim_{n \rightarrow \infty} \text{vol}(E_{t,K}^n) = \mathbf{P}(G > K)$$

where  $G$  is a standard Gaussian random variable. The next lemma shows that the maximal function is large on the set  $E_{t,K}^n$ . This is essentially a variant on ([1], Claim 1)

**Lemma 2.** *For any  $\eta > 0$ , there exists a constant  $D(\eta)$  so that for any  $K > 0$  and  $t \in (0, 1)$ , if  $n \geq \frac{D(\eta)K^2}{t(1-t)}$ , then*

$$E_{t,K}^n \subset \left\{M\mu > e^{K^2/(2+\eta)}\right\}.$$

Since the proof of lemma 2 is essentially independent from the rest of the argument, we postpone it to the next section. We now think of  $([0, 1]^n, \text{vol})$  as a probability space, and of the coordinates  $x_1, \dots, x_n$  as independent random variables. Let  $X_i := 2|x_i - \frac{1}{2}|$ . It is easily checked that  $(X_1, \dots, X_n)$  are also independent random variables uniformly distributed on  $[0, 1]$ . We introduce new random variables, for  $0 \leq t \leq 1$

$$\alpha_t^{(n)} := \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \mathbf{1}_{\{\frac{1-t}{2} \leq x_i \leq \frac{1+t}{2}\}} - t \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \mathbf{1}_{\{X_i \leq t\}} - t \right).$$

The following identity is at the heart of our probabilistic reasoning (it is easily checked that the set involved is measurable)

$$\text{vol} \left( \bigcup_{\varepsilon \leq t \leq 1-\varepsilon} E_{t,K}^n \right) = \mathbf{P} \left( \sup_{\varepsilon \leq t \leq 1-\varepsilon} \frac{\alpha_t^{(n)}}{\sqrt{t(1-t)}} \geq K \right). \quad (3)$$

A theorem by Donsker [3] asserts that  $(\alpha_t^{(n)})_{0 \leq t \leq 1}$  converges ( $n \rightarrow \infty$ ) in the sup norm towards a Brownian bridge  $(\beta_t)_{0 \leq t \leq 1}$ . Recall that a Brownian bridge  $(\beta_t)_{0 \leq t \leq 1}$  is defined as a Gaussian process which is almost surely continuous and given by the covariance

$$\mathbf{E} \beta_t \beta_u = t(1-u) \quad \text{for } 0 \leq t \leq u \leq 1.$$

In particular,  $\beta_0 = \beta_1 = 1$  a.s. There are several ways to relate the Brownian bridge to the Brownian motion. Let  $(B_t)_{t>0}$  be a standard Brownian motion, given by the covariance  $\mathbf{E} B_t B_u = \min(t, u)$ . It is easily checked, just by computing the covariance, that the process  $(B_t - tB_1)_{0 \leq t \leq 1}$  is a Brownian bridge. Similarly, the process

$$\left( (1-t)B_{\frac{t}{1-t}} \right)_{0 \leq t \leq 1} \quad (4)$$

is also a Brownian bridge. We refer to [4] (Chapter 7.8) for more information on the Brownian bridge.

As the formula (3) hints, we need to control the supremum of  $\beta_t / \sqrt{t(1-t)}$ . Using the transformation given by (4), this amounts to controlling the supremum of  $B_t / \sqrt{t}$ , where  $(B_t)$  is a Brownian motion. It is well-known that this supremum, when taken over  $(0, \infty)$ , is almost surely  $+\infty$ . This last fact can be used to show that  $\Theta_n$  is unbounded, giving a simpler proof of Aldaz's result<sup>1</sup>. To obtain concrete lower bounds, we need a more quantitative result, given by the law of the iterated logarithm ; this is the statement of the next lemma. For completeness, we include a proof at the end of the paper.

**Lemma 3.** *Let  $(\beta_t)_{0 \leq t \leq 1}$  be a Brownian bridge. For any  $\eta \in (0, 2)$ , there exists a constant  $c(\eta) > 0$  so that for any  $0 < \varepsilon \leq 1/e$ ,*

$$\mathbf{P} \left( \sup_{\varepsilon \leq t \leq 1-\varepsilon} \frac{\beta_t}{\sqrt{t(1-t)}} \geq \sqrt{(2-\eta) \log \log(1/\varepsilon)} \right) \geq c(\eta). \quad (5)$$

Since we are interested in non-asymptotic bounds, we also need more quantitative results about the speed of convergence in Donsker's theorem. This is provided by the following theorem, due to Komlós–Major–Tusnády [5]. We use the version appearing in a paper by Bretagnolle and Massart [2]

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<sup>1</sup>This was done in a prior version of this note, which can be found on the author's webpage. This qualitative approach does not use advanced probabilistic tools beyond the multivariate central limit theorem.

**Theorem** (Komlós–Major–Tusnády). *For any  $n \geq 1$ , there exists a probability space on which are defined*

- A  $n$ -tuple of i.i.d. random variables uniformly distributed on  $[0, 1]$  :  $X_1, \dots, X_n$ ,
- a Brownian bridge :  $(\beta_t^{(n)})_{0 \leq t \leq 1}$ ,

so that, denoting

$$\alpha_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{j=1}^n (\mathbf{1}_{\{X_j \leq t\}} - t),$$

the following inequality is valid for any  $x > 0$ :

$$\mathbf{P} \left( \sup_{0 \leq t \leq 1} |\sqrt{n}(\alpha_t^{(n)} - \beta_t^{(n)})| > 12 \log n + x \right) < 2 \exp(-x/6).$$

We are now ready to prove the main theorem.

*Proof of the theorem.* Fix  $\eta > 0$  and let  $c(\eta)$  the constant given by lemma 3, which we apply with the choice  $\varepsilon := \log^2 n/n$ . Choose now  $x > 0$  so that  $2 \exp(-x/6) < c(\eta)/2$ . Applying Komlós–Major–Tusnády theorem, we obtain a coupling of  $\alpha_t^{(n)}$  and  $\beta_t^{(n)}$  so that the following events hold simultaneously with probability larger than  $c(\eta)/2$

$$\left\{ \begin{array}{l} \sup_{\varepsilon \leq t \leq 1-\varepsilon} \frac{\beta_t^{(n)}}{\sqrt{t(1-t)}} \geq \sqrt{(2-\eta) \log \log(1/\varepsilon)} \\ \forall t \in [0, 1], \quad \alpha_t^{(n)} > \beta_t^{(n)} - \frac{12 \log n + x}{\sqrt{n}} \end{array} \right.$$

This shows that

$$\mathbf{P} \left( \sup_{\varepsilon \leq t \leq 1-\varepsilon} \frac{\alpha_t^{(n)}}{\sqrt{t(1-t)}} \geq \sqrt{(2-\eta) \log \log(1/\varepsilon)} - \frac{12 \log n + x}{\sqrt{n\varepsilon(1-\varepsilon)}} \right) \geq \frac{c(\eta)}{2}.$$

Set  $K := \sqrt{(2-\eta) \log \log(1/\varepsilon)} - \frac{12 \log n + x}{\sqrt{n\varepsilon(1-\varepsilon)}}$ . Using formula (3), we obtain

$$\text{vol} \left( \bigcup_{\varepsilon \leq t \leq 1-\varepsilon} E_{t,K}^n \right) \geq \frac{c(\eta)}{2}.$$

One checks that for  $n$  large enough,  $K \geq \sqrt{(2-2\eta) \log \log n}$ . Also, for  $n$  large enough,  $n \geq \frac{D(\eta)K^2}{\varepsilon(1-\varepsilon)}$ , so that we can use lemma 2 and conclude that

$$\text{vol} \left( \left\{ x \in [0, 1]^n \text{ s.t. } M\mu(x) > e^{K^2/(2+\eta)} \right\} \right) \geq \frac{c(\eta)}{2}.$$

Using lemma 1, this implies that for  $n$  large enough,

$$\Theta_n \geq \frac{c(\eta)}{2} e^{K^2/(2+\eta)} \geq \frac{c(\eta)}{2} (\log n)^{\frac{2-2\eta}{2+\eta}}.$$

This inequality can be extended to all  $n$  by adjusting  $c(\eta)$  is necessary. Since  $\frac{2-2\eta}{2+\eta}$  is arbitrarily close to 1, this proves the theorem.  $\square$

## Proof of lemma 2

A variant of this lemma appears in [1]. Let  $s \geq 1$  be an integer. Let  $x_i \in [0, 1]$ . Note that

$$\text{card} \left( \left[ x_i - \left( s - \frac{1-t}{2} \right), x_i + \left( s - \frac{1-t}{2} \right) \right] \cap \mathbf{Z} \right) = \begin{cases} 2s & \text{if } x_i \text{ is } t\text{-centered,} \\ 2s-1 & \text{otherwise.} \end{cases}$$

Consequently, if  $x \in [0, 1]^n$  has at least  $m$   $t$ -centered coordinates, then

$$M\mu(x) \geq \sup_{s \in \mathbf{N}^*} \frac{\mu(Q(x, s - (1-t)/2))}{\text{vol}(Q(x, s - (1-t)/2))} = \sup_{s \in \mathbf{N}^*} \frac{(2s)^m (2s-1)^{n-m}}{(2s - (1-t))^n}$$

and it remains to optimize on  $s$ . Let  $F(s) := (2s)^m (2s-1)^{n-m} (2s-1+t)^{-n}$ . We first compute the supremum of  $F$  over  $s \in \mathbf{R}^+$ . We write  $m = nt + C\sqrt{n}$  and compute

$$(\log F)'(s) = \frac{-2\sqrt{n}(2Cs - C(1-t) - \sqrt{nt}(1-t))}{(2s)(2s-1+t)(2s-1)}.$$

Thus,  $F$  is maximal at the point  $s_0$  defined as

$$s_0 := \frac{\sqrt{nt}(1-t)}{2C} + \frac{1-t}{2}.$$

The maximal value of  $F$  is

$$F(s_0) = \left( 1 + \frac{C}{\sqrt{nt}} \right)^{nt+C\sqrt{n}} \left( 1 - \frac{C}{\sqrt{n}(1-t)} \right)^{n(1-t)-C\sqrt{n}}.$$

Let  $\Phi(x) = (1+x) \log(1+x)$ . One checks by consecutive differentiation that  $\Phi(x) \geq x + \frac{x^2}{2} - \frac{x^3}{6}$  for  $x > -1$ . We get

$$\log F(s_0) = nt\Phi\left(\frac{C}{\sqrt{nt}}\right) + n(1-t)\Phi\left(\frac{-C}{\sqrt{n}(1-t)}\right) \geq \frac{C^2}{2t(1-t)} + \frac{C^3}{6\sqrt{n}t^2(1-t)^2}.$$

However, since we are only allowed to consider integer-valued  $s$ , we need to evaluate  $F(\lfloor s_0 \rfloor)$ , and show that it is not very different from  $F(s_0)$ . For  $x \in [s_0 - 1, s_0]$ , we have

$$2x \geq 2x - 1 + t \geq 2x - 1 \geq 2s_0 - 3 \geq \frac{\sqrt{nt}(1-t)}{C} - 3.$$

Provided  $\sqrt{nt}(1-t) > 3C$ , this implies

$$|(\log F)'(x)| \leq \frac{4\sqrt{n}C^4}{(\sqrt{nt}(1-t) - 3C)^3}.$$

Applying the mean value theorem to  $\log F$  between  $\lfloor s_0 \rfloor$  and  $s_0$  gives

$$\log M\mu(x) \geq \log F(\lfloor s_0 \rfloor) \geq \frac{C^2}{2t(1-t)} + \frac{C^3}{6\sqrt{n}t^2(1-t)^2} - \frac{4\sqrt{n}C^4}{(\sqrt{nt}(1-t) - 3C)^3}. \quad (6)$$

If  $x \in E_{t,K}^n$ , then (6) holds with  $C = K\sqrt{t(1-t)}$ . Set  $n = \frac{DK^2}{t(1-t)}$ . Using the inequality  $2t-1 > -1$ , we get

$$\log M\mu(x) \geq \frac{K^2}{2} - \frac{K^2}{6\sqrt{D}} - \frac{4\sqrt{D}K^2}{(\sqrt{D}-3)^3}.$$

One checks that the last expression is larger than  $K^2/(2+\eta)$  when  $D$  is large enough.

### Proof of lemma 3

Fix  $\eta \in (0, 2)$  and let  $(B_t)_{t \geq 0}$  be a standard Brownian motion. Using the transformation given by (4), we check that, denoting  $A = 1/\varepsilon$ ,

$$\mathbf{P} \left( \sup_{\varepsilon \leq t \leq 1-\varepsilon} \frac{\beta_t}{\sqrt{t(1-t)}} \geq \sqrt{(2-\eta) \log \log(1/\varepsilon)} \right) = \mathbf{P} \left( \sup_{\frac{1}{A-1} \leq t \leq A-1} \frac{B_t}{\sqrt{t}} \geq \sqrt{(2-\eta) \log \log A} \right).$$

We will actually estimate the supremum of  $B_t/\sqrt{t}$  over  $[1, A-1]$ . Since  $(B_t)$  and  $(-B_t)$  have the same distribution, we have

$$\mathbf{P} \left( \sup_{1 \leq t \leq A-1} \frac{B_t}{\sqrt{t}} \geq \sqrt{(2-\eta) \log \log A} \right) \geq \frac{1}{2} \mathbf{P} \left( \sup_{1 \leq t \leq A-1} \frac{|B_t|}{\sqrt{t}} \geq \sqrt{(2-\eta) \log \log A} \right)$$

Choose  $\alpha > 1$  large enough so that  $\rho < 2$ , where  $\rho$  denotes the number

$$\rho := \frac{(2-\eta)(1+\sqrt{\alpha})^2}{\alpha-1}.$$

Let  $N := \left\lfloor \frac{\log(A-1)}{\log \alpha} \right\rfloor - 1$  and for  $1 \leq j \leq N$ , consider the event :

$$E_j := \left\langle \frac{B_{\alpha^{j+1}} - B_{\alpha^j}}{\sqrt{\alpha^{j+1} - \alpha^j}} \geq \sqrt{\rho \log \log A} \right\rangle$$

One checks that if  $E_j$  holds, then at least one of the following inequalities is true:

$$\begin{cases} \text{either} & B_{\alpha^{j+1}} \geq \sqrt{(2-\eta) \log \log A} \cdot \sqrt{\alpha^{j+1}} \\ \text{or} & B_{\alpha^j} \leq -\sqrt{(2-\eta) \log \log A} \cdot \sqrt{\alpha^j}. \end{cases}$$

Consequently,

$$\mathbf{P} \left( \sup_{1 \leq t \leq A} \frac{|B_t|}{\sqrt{t}} \geq \sqrt{(2-\eta) \log \log A} \right) \geq \mathbf{P} \left( \bigcup_{1 \leq j \leq N} E_j \right)$$

Now because the Brownian motion has independent increments, the events  $E_j$  are independent, and they all have the same probability

$$\mathbf{P}(E_j) = \mathbf{P} \left( G > \sqrt{\rho \log \log A} \right),$$

where  $G$  is a standard Gaussian random variable. We use the following estimate (see [4], Theorem 1.4) : for  $x > 1$

$$\mathbf{P}(G > x) \geq \frac{x^{-1} - x^{-3}}{\sqrt{2\pi}} e^{-x^2/2} \geq C(\rho) \exp(-x^2/\rho).$$

for some constant  $C(\rho) > 0$ . We obtain  $\mathbf{P}(E_j) \geq C(\rho)(\log A)^{-1}$ . Finally, since the events  $E_j$  are independent,

$$\mathbf{P} \left( \bigcup_{1 \leq j \leq N} E_j \right) \geq 1 - \left( 1 - \frac{C(\rho)}{\log A} \right)^N \geq 1 - \exp \left( -\frac{C(\rho)N}{\log A} \right)$$

Given our choice of  $N$ , one checks that the last expression is bounded below by some positive constant depending only on  $\eta$  when  $A$  tends to  $+\infty$ . This proves the lemma, at least for  $A$  large enough. Small values of  $A$  can be taken into account by adjusting a posteriori the constant  $c(\eta)$  if necessary.

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<sup>2</sup>This work was partly achieved while the author was on strike to protest against an unfair reform of French universities.