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MORSE 2-JET SPACE AND h -PRINCIPLE

ALAIN CHENCINER AND FRANÇOIS LAUDENBACH

ABSTRACT. A section in the 2-jet space of Morse functions is not always homotopic to a holonomic section. We give a necessary condition for being the case and we discuss the sufficiency.

1. INTRODUCTION

Given a submanifold Σ in an r -jet space (of smooth sections of a bundle over a manifold M), it is natural to look at the associated *differential relation* $\mathcal{R}(\Sigma)$ formed by the $(r+1)$ -jets *transverse* to Σ . For $j^r f$ being transverse to Σ at $x \in M$ is detected by $j_x^{r+1} f$. This is an open differential relation in the corresponding $(r+1)$ -jet space. One can ask whether the Gromov *h-principle* holds: is any section with value in $\mathcal{R}(\Sigma)$ homotopic to a *holonomic* section of $\mathcal{R}(\Sigma)$? (We recall that a holonomic section of a $(r+1)$ -jet space is a section of the form $j^{r+1} f$.)

According to M. Gromov, the answer is yes when M is an open manifold and Σ is *natural*, that is, invariant by a lift of $\text{Diff}(M)$ to the considered jet space (see [3] p. 79, [1] ch. 7).

The answer is also yes when the codimension of Σ is higher than the dimension n of M ; this case follows easily from Thom's transversality theorem in jet spaces (see [6]). In the case of jet space of functions and when Σ is natural and $\text{codim} \Sigma \geq n+1$, it also can be seen as a baby case of a theorem of Vassiliev [9].

In this note we are interested in a codimension n case when M is a compact n -dimensional manifold. Let $J^r(M)$ denote the space of r -jets of real functions; when the boundary of M is not empty, it is meant that we speak of jets of functions which are locally constant on the boundary. We take $\Sigma \subset J^1(M)$ the set of critical 1-jets. Then $\mathcal{R}(\Sigma) \subset J^2(M)$ is the open set of 2-jets of Morse functions. We shall analyze the obstructions preventing the *h-principle* to hold with this differential relation.

2. INDEX COCYCLES

It is more convenient to work with the *reduced* jet spaces $\tilde{J}^r(M)$, quotient of $J^r(M)$ by \mathbb{R} which acts by translating the value of the jet. It is a vector bundle whose linear structure is induced by that of $C^\infty(M)$. For instance, $\tilde{J}^1(M)$ is isomorphic to the cotangent space T^*M . Let \mathcal{M} denote the reduced 2-jets of Morse functions, that is the 2-jets which are transverse to the zero section 0_M of T^*M (in the sequel, *jet* will mean *reduced jet*). Let $\pi : \tilde{J}^2(M) \rightarrow \tilde{J}^1(M)$ be the projection and $\pi_0 : \tilde{J}_0^2(M) \rightarrow 0_M$ be its restriction over the zero section of the cotangent

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space. Since it is formed of critical 2-jets, it is a vector bundle whose fiber is the space of quadratic forms, $S^2(T_x^*M)$, $x \in M$. Let $\mathcal{M}_0 := \mathcal{M} \cap \tilde{\mathcal{J}}_0^2(M)$; it is a bundle over 0_M whose fiber consists of non-degenerate quadratic forms. Its complement in $\tilde{\mathcal{J}}_0^2(M)$ is denoted \mathcal{D} (like discriminant); it is formed of 2-jets which are not transverse to 0_M . When M is connected, \mathcal{M}_0 has a connected component \mathcal{M}_0^i for each index $i \in \{1, \dots, n\}$ of quadratic forms.

2.1. Transverse orientation. Each \mathcal{M}_0^i is a proper submanifold of codimension n in \mathcal{M} . Moreover the differential $d\pi$ gives rise to an isomorphism of normal fiber bundles

$$\nu(\mathcal{M}_0^i, \mathcal{M}) \cong \pi^*(\nu(0_M, T^*M))|_{\mathcal{M}_0^i}.$$

Of course, $\nu(0_M, T^*M)$ is canonically isomorphic to the cotangent bundle τ^*M , whose total space is T^*M . When M is oriented, so is the bundle τ^*M . When M is not orientable, one has a local system of orientations of τ^*M . Pulling it back by π yields a local system of orientations of $\nu(\mathcal{M}_0^i, \mathcal{M})$ (that is, co-orientations of \mathcal{M}_0^i). Let us denote \mathcal{M}_0^{even} (resp. \mathcal{M}_0^{odd}) the union of the \mathcal{M}_0^i 's for i even (resp. odd). We endow \mathcal{M}_0^{even} with the above local system of co-orientations. For reasons which clearly appear below, it is more natural to equip \mathcal{M}_0^{odd} with the opposite system of co-orientations.

Lemma 2.2. *Let $s = j^2f$ be a holonomic section of \mathcal{M} meeting \mathcal{M}_0 transversally. Then each intersection point of $s(M)$ with \mathcal{M}_0 is positive. The same statement holds when s is a local holonomic section only.*

Proof. Let a be such an intersection point in $s(M) \cap \mathcal{M}_0^i$; so i is the index of the corresponding critical 2-jet. We can calculate in local coordinates (x, y', y'') , where $x = (x_1, \dots, x_n)$ are local coordinates of M , $y' = (y'_1, \dots, y'_n)$ (resp. $y'' = (y''_{jk})_{1 \leq j \leq k \leq n}$) are the associated coordinates of T_x^*M (resp. $S^2T_x^*M$). Since f is holonomic, we have $y''_{jk}(a) = \frac{\partial y'_j}{\partial x_k}(a)$. Finally, the sign of $\det y''(a)$ (positive if i is even and negative if not) gives the sign of the Jacobian determinant at a of the map $x \mapsto y'(x)$, that is the sign of the intersection point when \mathcal{M}_0^i is co-oriented by the canonical orientation of the y' -space. As we have reversed this co-orientation when i is odd, the intersection point is positive whatever the index is. \square

Proposition 2.3. 1) *Each \mathcal{M}_0^i defines a degree n cocycle of \mathcal{M} with coefficients in the local system \mathbb{Z}^{or} of integers twisted by the orientation of M . Let μ_i be its cohomology class in $H^n(\mathcal{M}, \mathbb{Z}^{or})$; in particular, if $s : M \rightarrow \mathcal{M}$ is a section, $\langle \mu_i, [s] \rangle$ is an integer.*

2) *When s is homotopic to a holonomic section j^2f , then $\langle \mu_i, [s] \rangle$ is positive and equals the number $c_i(f)$ of critical points of the Morse function f . In particular the total number $|Z|$ of zeroes of the section $\pi \circ s$ (which, by construction, is transverse to the 0 -section) satisfies:*

$$|Z| \geq \sum_{i=0}^n c_i(f).$$

Proof. 1) Let σ be a singular n -cycle with twisted coefficients of \mathcal{M} . It can be C^0 -approximated by σ' , an n -cycle which is transverse to \mathcal{M}_0^i . As \mathcal{M}_0^i is a proper submanifold, there are finitely many intersections points in $\sigma' \cap \mathcal{M}_0^i$, each one having a sign with respect to the local system

of coefficients. The algebraic sum of these signs defines an integer $c(\sigma')$. One easily checks that $c(\sigma') = 0$ if σ' is a boundary. As a consequence, if σ'_0 and σ'_1 are two approximations of σ , as $\sigma'_1 - \sigma'_0$ is a boundary, we have $c(\sigma'_1) - c(\sigma'_0) = 0$ which allows us to uniquely define $c(\sigma)$ as the value of an n -cocycle on σ . Typically, the image of a section carries an n -cycle with twisted coefficients and this algebraic counting applies.

2) Since c defined in 1) is a cocycle, it takes the same value on s and on $j^2 f$. According to lemma 2.2, it counts +1 for each intersection point in $j^2 f \cap \mathcal{M}_0^i$, that is, for each index i critical point of f . \square

Corollary 2.4. *If s is a section of \mathcal{M} which is homotopic to a holonomic section, the integers $m_i := \langle \mu_i, [s] \rangle$ fulfill the Morse inequalities*

$$\begin{aligned} m_0 &\geq \beta_0(F) \\ m_1 - m_0 &\geq \beta_1(F) - \beta_0(F) \\ &\dots \\ m_0 - m_1 + \dots + (-1)^n m_n &= \beta_0(F) - \beta_1(F) \dots + (-1)^n \beta_n(F) =: \chi(M) \end{aligned}$$

where F is a field of coefficients, $\beta_i(F) = \dim_F H_i(M, F^{or})$ is the i -th Betti number with coefficients in F^{or} (F twisted by the orientation) and $\chi(M)$ is the Euler characteristic (independent of the field F).

Corollary 2.5. *The h -principle does not hold true for the sections of \mathcal{M} .*

Proof. It is sufficient to construct a section s of \mathcal{M} which violates the Morse inequalities, for example a section which does not intersect \mathcal{M}_0^0 . Leaving the case of the circle as an exercise, we may assume $n > 1$. One starts with a section s_1 of T^*M tranverse to O_M . Each zero of s_1 has a sign (if the local orientation of M is changed, so are both local orientations of s_1 and O_M the sign of the zero in unchanged). For each zero a , one can construct a homotopy fixing a , with arbitrary small support, which makes s_1 linear in a small neighborhood of a . As $GL(n, \mathbb{R})$ has exactly two connected components, one can even suppose that after the homotopy, s_1 is near a the derivative of a non degenerate quadratic function whose index can be chosen arbitrarily provided it is even (resp. odd) if a is a positive (resp. a negative) zero. Finally, one can achieve by homotopy that near each zero a , one has $s_1 = df$ with a a non-degenerate critical point of f of index 2 (resp. 1) if a is a positive (resp. negative) zero.

Near the zeroes s_1 has a canonical lift to \mathcal{M} by $s_2 = j^2 f$. Away from the zeroes, the lift s_2 extends as a lift of s_1 since the fibers of π are contractible over $T^*M \setminus O_M$. By construction, we have $\langle \mu_0, [s_2] \rangle = 0$, violating the first Morse inequality. \square

Remark 2.6. *Denote $\mu_{even} = \mu_0 + \mu_2 + \dots$ and $\mu_{odd} = \mu_1 + \dots$. The following statement holds true: $\mu_{even} = \mu_{odd}$ if and only if the Euler characteristic vanishes.*

Proof. Assume first $\mu_{even} = \mu_{odd}$. Proposition 2.3 yields for any holonomic section in \mathcal{M} : $m_{even} = m_{odd}$, that is $\chi(M) = 0$. Conversely, if $\chi(M) = 0$, there exists a non-vanishing 1-form on M and hence, by lifting it to $\tilde{\mathcal{J}}^2(M)$, a section v_0 in \mathcal{M} avoiding \mathcal{M}_0 . We form

$$W = \{z \in \tilde{\mathcal{J}}^2(M) \mid z = z_0 + tv_0, z_0 \in \mathcal{M}_0, t \geq 0 \text{ or } z_0 \in \mathcal{D}, t > 0\}.$$

It is a proper submanifold in \mathcal{M} whose boundary (with orientation twisted coefficients) is $\mathcal{M}_0^{\text{even}} - \mathcal{M}_0^{\text{odd}}$. Therefore, every cycle c satisfies $\langle \mu_{\text{even}}, c \rangle = \langle \mu_{\text{odd}}, c \rangle$, which implies the wanted equality. \square

3. ARE MORSE INEQUALITIES SUFFICIENT?

This question is closely related to the problem of minimizing the number of critical points of a Morse function. This problem was solved by S. Smale in dimension higher than 5 for simply connected manifolds, as a consequence of the methods he developed for proving his famous h -cobordism theorem (see [8] or chapter 2 in [2]). Under the same topological assumptions we can answer our question positively. But there are other cases, discussed later, where the answer is negative.

Proposition 3.1. *Two sections s, t of $\mathcal{M} \subset \tilde{\mathcal{J}}^2(M)$ are homotopic as sections of \mathcal{M} if and only if their algebraic intersection numbers m_i with \mathcal{M}_0^i are the same.*

Proof. According to proposition 2.3 1), the condition is necessary. Let us prove that it is sufficient. Leaving the 1-dimensional case to the reader, we assume $\dim M \geq 2$. Denote $s^1 = \pi \circ s$. Each zero of s^1 is given an index due to its lifting by s to a point of some \mathcal{M}_0^i . For each index i choose $|m_i|$ zeroes of s^1 , $a_i^1, \dots, a_i^{|m_i|}$, among its zeroes of index i ; when $m_i > 0$ (resp. $m_i < 0$), we choose the a_i^j so that the corresponding intersection points of $s(M)$ with \mathcal{M}_0^i are positive (resp. negative). When $m_i = 0$, no points are selected. In the same way, $|m_i|$ zeros $b_i^1, \dots, b_i^{|m_i|}$ of t^1 are chosen.

The intersection signs being the same, one can find a homotopy of t in \mathcal{M} , which brings the b_i^j to coincide with the a_i^j and makes the two sections s and t coincide in the neighborhood of these points.

The other zeroes of s^1 of index i can be matched into pairs of points $\{a_i^{j+}, a_i^{j-}\}$ of opposite sign. A Whitney type lemma allows us to cancel all these pairs by a suitable homotopy of s in \mathcal{M} , reducing to the case when s^1 has no other zeroes than the a_i^j 's, $j = 1, \dots, |m_i|$. A similar reduction may be assumed for t . Let us finish the proof in this case before stating and proving this lemma.

Both sections s^1 and t^1 of T^*M are homotopic (among sections) by a homotopy which is stationary on a neighborhood $N(a_i^j)$. Making this homotopy $h : M \times [0, 1] \rightarrow T^*M$ transverse to the zero section, the preimage of 0_M consists of arcs $\{a_i^j\} \times [0, 1]$ and finitely many closed curves γ_k . Each of these closed curves can be arbitrarily decorated with an index i . This choice allows us to lift h to $\tilde{\mathcal{J}}^2(M)$ as a homotopy \tilde{h} from s to t ; this \tilde{h} is the desired homotopy. More precisely, we proceed as follows for getting \tilde{h} . First $h|_{\gamma_k}$ is lifted to \mathcal{M}_0^i by using that the fiber of $\pi : \mathcal{M}_0^i \rightarrow 0_M$ is connected. The transversality of h to 0_M allows us to extend this lifting to a neighborhood of γ_k , making \tilde{h} transverse to \mathcal{M}_0^i . Now it is easy to extend \tilde{h} to $M \times [0, 1]$, since the fiber of π over any point outside 0_M is contractible.

A WHITNEY TYPE LEMMA. *Let (b^+, b^-) be a pair of transverse intersection points of s with \mathcal{M}_0^i having opposite sign when they are thought of as zeroes of s_1 in M . Let α be a simple path in M joining them avoiding the other zeroes of s^1 and let N be a neighborhood of α . Then there exists a homotopy $S = (s_u)_{u \in [0, 1]}$ of $s_0 = s$ into \mathcal{M} , supported in N and cancelling the*

pair (b^+, b^-) , that is, $\pi \circ s_1$ has no zeroes in N .

PROOF. We choose an embedded 2-disk (with corners) Δ in $N \times [0, 1[$ meeting $N \times \{0\}$ transversally along α . We first construct the homotopy $S^1 := \pi \circ S$ of s^1 among the sections of T^*M , following the cancellation process of Whitney which we are going to recall. We require S^1 to be transverse to 0_M with $(S^1)^{-1}(0_M) = \beta$, where $\alpha \cup \beta = \partial\Delta$. Using a trivialization of $T^*M|N$, $S^1|N \times [0, 1]$ reads $S^1(x, u) = (x, g(x, u))$. The requirement is that g vanishes transversally along the arc β ; it is possible exactly because $\dim M \geq 2$ and the end points have opposite signs. Let T be a small tubular neighborhood of β ; its boundary traces an arc β' on Δ , “parallel” to β . Let α' be the subarc of α whose end points are those of β' . The restriction $g|T$ is required to be a trivialization of T , but this latter may be chosen freely. We choose it so that the loop $(g|\beta') \cup (s^1|\alpha')$ be homotopic to 0 in $(\mathbb{R}^n)^* \setminus \{0\}$; of course, when $n > 2$ this condition is automatically fulfilled. Now g can be extended to the rest of Δ as a non-vanishing map. As $N \times [0, 1]$ collapses onto $N \times \{0\} \cup \Delta \cup T$, the extension of g can be completed without adding zeroes outside β , yielding the desired homotopy S^1 .

It remains to lift S^1 to \mathcal{M} . The lifting is first performed along β with value in \mathcal{M}_0^i . Then it is globally extended in the same way as in the above lifting process. \square

Corollary 3.2. *Let s be a section of $\mathcal{M} \subset \tilde{\mathcal{J}}^2(M)$ and m_i be its algebraic intersection number with \mathcal{M}_0^i . Let $f : M \rightarrow \mathbb{R}$ be a Morse function whose number $c_i(f)$ of critical points of index i satisfies*

$$c_i(f) = m_i$$

for all $i \in \{0, \dots, n\}$. Then s and $j^2(f)$ are homotopic as sections of \mathcal{M} .

Corollary 3.3. *We assume $\dim M \geq 6$ and $\pi_1(M) = 0$. Let s be a section of $\mathcal{M} \subset \tilde{\mathcal{J}}^2(M)$ whose algebraic intersection numbers m_i fulfills the Morse inequalities for every field of coefficients. In particular, they are non-negative. Then s is homotopic through sections in \mathcal{M} to a holonomic section.*

Proof. Under these topological assumptions the following result holds true: *For any set of non-negative integers $\{c_0, c_1, \dots, c_n\}$ satisfying the Morse inequalities for any field of coefficients, there exists a Morse function on M with c_i critical points of index i (see theorem 2.3 in [2]).* So we have a Morse function $f : M \rightarrow \mathbb{R}$ with m_i critical points of index i . According to corollary 3.2, s is homotopic in \mathcal{M} to j^2f . \square

3.4. We end this section by recalling that the Morse inequalities are not sharp for estimating the number of critical points of a Morse function on a non-simply connected closed manifold. Typically when $\pi_1(M)$ equals its subgroup of commutators (perfect group), some critical points of index 1 are required for generating the fundamental group, but the Morse inequalities allow $c_1 = 0$ (see [7] for more details). On the other hand, the only constraint for a section of \mathcal{M} with intersection numbers m_i is the Euler-Poincaré identity:

$$m_0 - m_1 + \dots = \chi(M).$$

So it is possible to find a section s whose intersection number m_i is the minimal rank in degree i of a free complex whose homology is $H_*(M, \mathbb{Z})$, that is,

$$m_i = \beta_i + \tau_i + \tau_{i-1},$$

where β_i stands for the rank of the free quotient of $H_i(M, \mathbb{Z})$ and τ_i denotes the minimal number of generators of its torsion subgroup ([2] p. 15). Such a set of integers satisfies the Morse inequalities but is far from being realizable by a Morse function. Finally this section s is not homotopic in \mathcal{M} to a holonomic section.

4. FAILURE OF THE 1-PARAMETRIC VERSION OF THE h -PRINCIPLE

We thank Yasha Eliashberg who pointed out to us the failure of the h -principle in the 1-parametric version of the problem under consideration.

Here M is assumed to be a product $M = N \times [0, 1]$. Let $f_0 : M \rightarrow [0, 1]$ be the projection. When M is not 1-connected and $\dim M \geq 6$, according to Allen Hatcher the so-called *pseudo-isotopy problem* has always a negative answer: there exists f without critical points which is not joinable to f_0 among the Morse functions (see [4]). But $j^2 f$ can be joined to $j^2 f_0$ by a path γ in \mathcal{M} . Indeed, take a generic homotopy γ^1 joining df to df_0 ; then arguing as in the proof of proposition 3.1 it is possible to lift it to \mathcal{M} . When M is the n -torus \mathbb{T}^n , A. Douady showed very simply the stronger fact that the path γ^1 can be taken among the non-singular 1-forms (see appendix to [5]). This γ is not homotopic in \mathcal{M} with end points fixed to a path of holonomic sections.

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