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# MORSE 2-JET SPACE AND $h$ -PRINCIPLE

ALAIN CHENCINER AND FRANÇOIS LAUDENBACH

ABSTRACT. A section in the 2-jet space of Morse functions is not always homotopic to a holonomic section. We give a necessary condition for being the case and we discuss the sufficiency.

## 1. INTRODUCTION

Given a submanifold  $\Sigma$  in an  $r$ -jet space (of smooth sections of a bundle over a manifold  $M$ ), it is natural to look at the associated *differential relation*  $\mathcal{R}(\Sigma)$  formed by the  $(r+1)$ -jets *transverse* to  $\Sigma$ . For  $j^r f$  being transverse to  $\Sigma$  at  $x \in M$  is detected by  $j_x^{r+1} f$ . This is an open differential relation in the corresponding  $(r+1)$ -jet space. One can ask whether the Gromov *h-principle* holds: is any section with value in  $\mathcal{R}(\Sigma)$  homotopic to a *holonomic* section of  $\mathcal{R}(\Sigma)$ ? (We recall that a holonomic section of a  $(r+1)$ -jet space is a section of the form  $j^{r+1} f$ .)

According to M. Gromov, the answer is yes when  $M$  is an open manifold and  $\Sigma$  is *natural*, that is, invariant by a lift of  $\text{Diff}(M)$  to the considered jet space (see [3] p. 79, [1] ch. 7).

The answer is also yes when the codimension of  $\Sigma$  is higher than the dimension  $n$  of  $M$ ; this case follows easily from Thom's transversality theorem in jet spaces (see [6]). In the case of jet space of functions and when  $\Sigma$  is natural and  $\text{codim} \Sigma \geq n+1$ , it also can be seen as a baby case of a theorem of Vassiliev [9].

In this note we are interested in a codimension  $n$  case when  $M$  is a compact  $n$ -dimensional manifold. Let  $J^r(M)$  denote the space of  $r$ -jets of real functions; when the boundary of  $M$  is not empty, it is meant that we speak of jets of functions which are locally constant on the boundary. We take  $\Sigma \subset J^1(M)$  the set of critical 1-jets. Then  $\mathcal{R}(\Sigma) \subset J^2(M)$  is the open set of 2-jets of Morse functions. We shall analyze the obstructions preventing the *h-principle* to hold with this differential relation.

## 2. INDEX COCYCLES

It is more convenient to work with the *reduced* jet spaces  $\tilde{J}^r(M)$ , quotient of  $J^r(M)$  by  $\mathbb{R}$  which acts by translating the value of the jet. It is a vector bundle whose linear structure is induced by that of  $C^\infty(M)$ . For instance,  $\tilde{J}^1(M)$  is isomorphic to the cotangent space  $T^*M$ . Let  $\mathcal{M}$  denote the reduced 2-jets of Morse functions, that is the 2-jets which are transverse to the zero section  $0_M$  of  $T^*M$  (in the sequel, *jet* will mean *reduced jet*). Let  $\pi : \tilde{J}^2(M) \rightarrow \tilde{J}^1(M)$  be the projection and  $\pi_0 : \tilde{J}_0^2(M) \rightarrow 0_M$  be its restriction over the zero section of the cotangent

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space. Since it is formed of critical 2-jets, it is a vector bundle whose fiber is the space of quadratic forms,  $S^2(T_x^*M)$ ,  $x \in M$ . Let  $\mathcal{M}_0 := \mathcal{M} \cap \tilde{\mathcal{J}}_0^2(M)$ ; it is a bundle over  $0_M$  whose fiber consists of non-degenerate quadratic forms. Its complement in  $\tilde{\mathcal{J}}_0^2(M)$  is denoted  $\mathcal{D}$  (like discriminant); it is formed of 2-jets which are not transverse to  $0_M$ . When  $M$  is connected,  $\mathcal{M}_0$  has a connected component  $\mathcal{M}_0^i$  for each index  $i \in \{1, \dots, n\}$  of quadratic forms.

**2.1. Transverse orientation.** Each  $\mathcal{M}_0^i$  is a proper submanifold of codimension  $n$  in  $\mathcal{M}$ . Moreover the differential  $d\pi$  gives rise to an isomorphism of normal fiber bundles

$$\nu(\mathcal{M}_0^i, \mathcal{M}) \cong \pi^*(\nu(0_M, T^*M))|_{\mathcal{M}_0^i}.$$

Of course,  $\nu(0_M, T^*M)$  is canonically isomorphic to the cotangent bundle  $\tau^*M$ , whose total space is  $T^*M$ . When  $M$  is oriented, so is the bundle  $\tau^*M$ . When  $M$  is not orientable, one has a local system of orientations of  $\tau^*M$ . Pulling it back by  $\pi$  yields a local system of orientations of  $\nu(\mathcal{M}_0^i, \mathcal{M})$  (that is, co-orientations of  $\mathcal{M}_0^i$ ). Let us denote  $\mathcal{M}_0^{\text{even}}$  (resp.  $\mathcal{M}_0^{\text{odd}}$ ) the union of the  $\mathcal{M}_0^i$ 's for  $i$  even (resp. odd). We endow  $\mathcal{M}_0^{\text{even}}$  with the above local system of co-orientations. For reasons which clearly appear below, it is more natural to equip  $\mathcal{M}_0^{\text{odd}}$  with the opposite system of co-orientations.

**Lemma 2.2.** *Let  $s = j^2f$  be a holonomic section of  $\mathcal{M}$  meeting  $\mathcal{M}_0$  transversally. Then each intersection point of  $s(M)$  with  $\mathcal{M}_0$  is positive. The same statement holds when  $s$  is a local holonomic section only.*

**Proof.** Let  $a$  be such an intersection point in  $s(M) \cap \mathcal{M}_0^i$ ; so  $i$  is the index of the corresponding critical 2-jet. We can calculate in local coordinates  $(x, y', y'')$ , where  $x = (x_1, \dots, x_n)$  are local coordinates of  $M$ ,  $y' = (y'_1, \dots, y'_n)$  (resp.  $y'' = (y''_{jk})_{1 \leq j \leq k \leq n}$ ) are the associated coordinates of  $T_x^*M$  (resp.  $S^2T_x^*M$ ). Since  $f$  is holonomic, we have  $y''_{jk}(a) = \frac{\partial y'_j}{\partial x_k}(a)$ . Finally, the sign of  $\det y''(a)$  (positive if  $i$  is even and negative if not) gives the sign of the Jacobian determinant at  $a$  of the map  $x \mapsto y'(x)$ , that is the sign of the intersection point when  $\mathcal{M}_0^i$  is co-oriented by the canonical orientation of the  $y'$ -space. As we have reversed this co-orientation when  $i$  is odd, the intersection point is positive whatever the index is.  $\square$

**Proposition 2.3.** 1) *Each  $\mathcal{M}_0^i$  defines a degree  $n$  cocycle of  $\mathcal{M}$  with coefficients in the local system  $\mathbb{Z}^{\text{or}}$  of integers twisted by the orientation of  $M$ . Let  $\mu_i$  be its cohomology class in  $H^n(\mathcal{M}, \mathbb{Z}^{\text{or}})$ ; in particular, if  $s : M \rightarrow \mathcal{M}$  is a section,  $\langle \mu_i, [s] \rangle$  is an integer.*

2) *When  $s$  is homotopic to a holonomic section  $j^2f$ , then  $\langle \mu_i, [s] \rangle$  is positive and equals the number  $c_i(f)$  of critical points of the Morse function  $f$ . In particular the total number  $|Z|$  of zeroes of the section  $\pi \circ s$  (which, by construction, is transverse to the 0-section) satisfies:*

$$|Z| \geq \sum_{i=0}^n c_i(f).$$

**Proof.** 1) Let  $\sigma$  be a singular  $n$ -cycle with twisted coefficients of  $\mathcal{M}$ . It can be  $C^0$ -approximated by  $\sigma'$ , an  $n$ -cycle which is transverse to  $\mathcal{M}_0^i$ . As  $\mathcal{M}_0^i$  is a proper submanifold, there are finitely many intersections points in  $\sigma' \cap \mathcal{M}_0^i$ , each one having a sign with respect to the local system

of coefficients. The algebraic sum of these signs defines an integer  $c(\sigma')$ . One easily checks that  $c(\sigma') = 0$  if  $\sigma'$  is a boundary. As a consequence, if  $\sigma'_0$  and  $\sigma'_1$  are two approximations of  $\sigma$ , as  $\sigma'_1 - \sigma'_0$  is a boundary, we have  $c(\sigma'_1) - c(\sigma'_0) = 0$  which allows us to uniquely define  $c(\sigma)$  as the value of an  $n$ -cocycle on  $\sigma$ . Typically, the image of a section carries an  $n$ -cycle with twisted coefficients and this algebraic counting applies.

2) Since  $c$  defined in 1) is a cocycle, it takes the same value on  $s$  and on  $j^2 f$ . According to lemma 2.2, it counts  $+1$  for each intersection point in  $j^2 f \cap \mathcal{M}_0^i$ , that is, for each index  $i$  critical point of  $f$ .  $\square$

**Corollary 2.4.** *If  $s$  is a section of  $\mathcal{M}$  which is homotopic to a holonomic section, the integers  $m_i := \langle \mu_i, [s] \rangle$  fulfill the Morse inequalities*

$$\begin{aligned} m_0 &\geq \beta_0(F) \\ m_1 - m_0 &\geq \beta_1(F) - \beta_0(F) \\ &\dots \\ m_0 - m_1 + \dots + (-1)^n m_n &= \beta_0(F) - \beta_1(F) \dots + (-1)^n \beta_n(F) =: \chi(M) \end{aligned}$$

where  $F$  is a field of coefficients,  $\beta_i(F) = \dim_F H_i(M, F^{or})$  is the  $i$ -th Betti number with coefficients in  $F^{or}$  ( $F$  twisted by the orientation) and  $\chi(M)$  is the Euler characteristic (independent of the field  $F$ ).

**Corollary 2.5.** *The  $h$ -principle does not hold true for the sections of  $\mathcal{M}$ .*

**Proof.** It is sufficient to construct a section  $s$  of  $\mathcal{M}$  which violates the Morse inequalities, for example a section which does not intersect  $\mathcal{M}_0^0$ . Leaving the case of the circle as an exercise, we may assume  $n > 1$ . One starts with a section  $s_1$  of  $T^*M$  tranverse to  $O_M$ . Each zero of  $s_1$  has a sign (if the local orientation of  $M$  is changed, so are both local orientations of  $s_1$  and  $O_M$  the sign of the zero is unchanged). For each zero  $a$ , one can construct a homotopy fixing  $a$ , with arbitrary small support, which makes  $s_1$  linear in a small neighborhood of  $a$ . As  $GL(n, \mathbb{R})$  has exactly two connected components, one can even suppose that after the homotopy,  $s_1$  is near  $a$  the derivative of a non degenerate quadratic function whose index can be chosen arbitrarily provided it is even (resp. odd) if  $a$  is a positive (resp. a negative) zero. Finally, one can achieve by homotopy that near each zero  $a$ , one has  $s_1 = df$  with  $a$  a non-degenerate critical point of  $f$  of index 2 (resp. 1) if  $a$  is a positive (resp. negative) zero.

Near the zeroes  $s_1$  has a canonical lift to  $\mathcal{M}$  by  $s_2 = j^2 f$ . Away from the zeroes, the lift  $s_2$  extends as a lift of  $s_1$  since the fibers of  $\pi$  are contractible over  $T^*M \setminus O_M$ . By construction, we have  $\langle \mu_0, [s_2] \rangle = 0$ , violating the first Morse inequality.  $\square$

**Remark 2.6.** *Denote  $\mu_{even} = \mu_0 + \mu_2 + \dots$  and  $\mu_{odd} = \mu_1 + \dots$ . The following statement holds true:  $\mu_{even} = \mu_{odd}$  if and only if the Euler characteristic vanishes.*

**Proof.** Assume first  $\mu_{even} = \mu_{odd}$ . Proposition 2.3 yields for any holonomic section in  $\mathcal{M}$ :  $m_{even} = m_{odd}$ , that is  $\chi(M) = 0$ . Conversely, if  $\chi(M) = 0$ , there exists a non-vanishing 1-form on  $M$  and hence, by lifting it to  $\tilde{\mathcal{J}}^2(M)$ , a section  $v_0$  in  $\mathcal{M}$  avoiding  $\mathcal{M}_0$ . We form

$$W = \{z \in \tilde{\mathcal{J}}^2(M) \mid z = z_0 + tv_0, z_0 \in \mathcal{M}_0, t \geq 0 \text{ or } z_0 \in \mathcal{D}, t > 0\}.$$

It is a proper submanifold in  $\mathcal{M}$  whose boundary (with orientation twisted coefficients) is  $\mathcal{M}_0^{\text{even}} - \mathcal{M}_0^{\text{odd}}$ . Therefore, every cycle  $c$  satisfies  $\langle \mu_{\text{even}}, c \rangle = \langle \mu_{\text{odd}}, c \rangle$ , which implies the wanted equality.  $\square$

### 3. ARE MORSE INEQUALITIES SUFFICIENT?

This question is closely related to the problem of minimizing the number of critical points of a Morse function. This problem was solved by S. Smale in dimension higher than 5 for simply connected manifolds, as a consequence of the methods he developed for proving his famous  $h$ -cobordism theorem (see [8] or chapter 2 in [2]). Under the same topological assumptions we can answer our question positively. But there are other cases, discussed later, where the answer is negative.

**Proposition 3.1.** *Two sections  $s, t$  of  $\mathcal{M} \subset \tilde{J}^2(M)$  are homotopic as sections of  $\mathcal{M}$  if and only if their algebraic intersection numbers  $m_i$  with  $\mathcal{M}_0^i$  are the same.*

**Proof.** According to proposition 2.3 1), the condition is necessary. Let us prove that it is sufficient. Leaving the 1-dimensional case to the reader, we assume  $\dim M \geq 2$ . Denote  $s^1 = \pi \circ s$ . Each zero of  $s^1$  is given an index due to its lifting by  $s$  to a point of some  $\mathcal{M}_0^i$ . For each index  $i$  choose  $|m_i|$  zeroes of  $s^1$ ,  $a_i^1, \dots, a_i^{|m_i|}$ , among its zeroes of index  $i$ ; when  $m_i > 0$  (resp.  $m_i < 0$ ), we choose the  $a_i^j$  so that the corresponding intersection points of  $s(M)$  with  $\mathcal{M}_0^i$  are positive (resp. negative). When  $m_i = 0$ , no points are selected. In the same way,  $|m_i|$  zeros  $b_i^1, \dots, b_i^{|m_i|}$  of  $t^1$  are chosen.

The intersection signs being the same, one can find a homotopy of  $t$  in  $\mathcal{M}$ , which brings the  $b_i^j$  to coincide with the  $a_i^j$  and makes the two sections  $s$  and  $t$  coincide in the neighborhood of these points.

The other zeroes of  $s^1$  of index  $i$  can be matched into pairs of points  $\{a_i^{j+}, a_i^{j-}\}$  of opposite sign. A Whitney type lemma allows us to cancel all these pairs by a suitable homotopy of  $s$  in  $\mathcal{M}$ , reducing to the case when  $s^1$  has no other zeroes than the  $a_i^j$ 's,  $j = 1, \dots, |m_i|$ . A similar reduction may be assumed for  $t$ . Let us finish the proof in this case before stating and proving this lemma.

Both sections  $s^1$  and  $t^1$  of  $T^*M$  are homotopic (among sections) by a homotopy which is stationary on a neighborhood  $N(a_i^j)$ . Making this homotopy  $h : M \times [0, 1] \rightarrow T^*M$  transverse to the zero section, the preimage of  $0_M$  consists of arcs  $\{a_i^j\} \times [0, 1]$  and finitely many closed curves  $\gamma_k$ . Each of these closed curves can be arbitrarily decorated with an index  $i$ . This choice allows us to lift  $h$  to  $\tilde{J}^2(M)$  as a homotopy  $\tilde{h}$  from  $s$  to  $t$ ; this  $\tilde{h}$  is the desired homotopy. More precisely, we proceed as follows for getting  $\tilde{h}$ . First  $h|_{\gamma_k}$  is lifted to  $\mathcal{M}_0^i$  by using that the fiber of  $\pi : \mathcal{M}_0^i \rightarrow 0_M$  is connected. The transversality of  $h$  to  $0_M$  allows us to extend this lifting to a neighborhood of  $\gamma_k$ , making  $\tilde{h}$  transverse to  $\mathcal{M}_0^i$ . Now it is easy to extend  $\tilde{h}$  to  $M \times [0, 1]$ , since the fiber of  $\pi$  over any point outside  $0_M$  is contractible.

**A WHITNEY TYPE LEMMA.** *Let  $(b^+, b^-)$  be a pair of transverse intersection points of  $s$  with  $\mathcal{M}_0^i$  having opposite sign when they are thought of as zeroes of  $s_1$  in  $M$ . Let  $\alpha$  be a simple path in  $M$  joining them avoiding the other zeroes of  $s^1$  and let  $N$  be a neighborhood of  $\alpha$ . Then there exists a homotopy  $S = (s_u)_{u \in [0, 1]}$  of  $s_0 = s$  into  $\mathcal{M}$ , supported in  $N$  and cancelling the*

pair  $(b^+, b^-)$ , that is,  $\pi \circ s_1$  has no zeroes in  $N$ .

**PROOF.** We choose an embedded 2-disk (with corners)  $\Delta$  in  $N \times [0, 1[$  meeting  $N \times \{0\}$  transversally along  $\alpha$ . We first construct the homotopy  $S^1 := \pi \circ S$  of  $s^1$  among the sections of  $T^*M$ , following the cancellation process of Whitney which we are going to recall. We require  $S^1$  to be transverse to  $0_M$  with  $(S^1)^{-1}(0_M) = \beta$ , where  $\alpha \cup \beta = \partial\Delta$ . Using a trivialization of  $T^*M|_N$ ,  $S^1|_{N \times [0, 1]}$  reads  $S^1(x, u) = (x, g(x, u))$ . The requirement is that  $g$  vanishes transversally along the arc  $\beta$ ; it is possible exactly because  $\dim M \geq 2$  and the end points have opposite signs. Let  $T$  be a small tubular neighborhood of  $\beta$ ; its boundary traces an arc  $\beta'$  on  $\Delta$ , “parallel” to  $\beta$ . Let  $\alpha'$  be the subarc of  $\alpha$  whose end points are those of  $\beta'$ . The restriction  $g|_T$  is required to be a trivialization of  $T$ , but this latter may be chosen freely. We choose it so that the loop  $(g|\beta') \cup (s^1|\alpha')$  be homotopic to 0 in  $(\mathbb{R}^n)^* \setminus \{0\}$ ; of course, when  $n > 2$  this condition is automatically fulfilled. Now  $g$  can be extended to the rest of  $\Delta$  as a non-vanishing map. As  $N \times [0, 1]$  collapses onto  $N \times \{0\} \cup \Delta \cup T$ , the extension of  $g$  can be completed without adding zeroes outside  $\beta$ , yielding the desired homotopy  $S^1$ .

It remains to lift  $S^1$  to  $\mathcal{M}$ . The lifting is first performed along  $\beta$  with value in  $\mathcal{M}_0^i$ . Then it is globally extended in the same way as in the above lifting process.  $\square$

**Corollary 3.2.** *Let  $s$  be a section of  $\mathcal{M} \subset \tilde{\mathcal{J}}^2(M)$  and  $m_i$  be its algebraic intersection number with  $\mathcal{M}_0^i$ . Let  $f : M \rightarrow \mathbb{R}$  be a Morse function whose number  $c_i(f)$  of critical points of index  $i$  satisfies*

$$c_i(f) = m_i$$

*for all  $i \in \{0, \dots, n\}$ . Then  $s$  and  $j^2(f)$  are homotopic as sections of  $\mathcal{M}$ .*

**Corollary 3.3.** *We assume  $\dim M \geq 6$  and  $\pi_1(M) = 0$ . Let  $s$  be a section of  $\mathcal{M} \subset \tilde{\mathcal{J}}^2(M)$  whose algebraic intersection numbers  $m_i$  fulfills the Morse inequalities for every field of coefficients. In particular, they are non-negative. Then  $s$  is homotopic through sections in  $\mathcal{M}$  to a holonomic section.*

**Proof.** Under these topological assumptions the following result holds true: *For any set of non-negative integers  $\{c_0, c_1, \dots, c_n\}$  satisfying the Morse inequalities for any field of coefficients, there exists a Morse function on  $M$  with  $c_i$  critical points of index  $i$  (see theorem 2.3 in [2]).* So we have a Morse function  $f : M \rightarrow \mathbb{R}$  with  $m_i$  critical points of index  $i$ . According to corollary 3.2,  $s$  is homotopic in  $\mathcal{M}$  to  $j^2f$ .  $\square$

**3.4.** We end this section by recalling that the Morse inequalities are not sharp for estimating the number of critical points of a Morse function on a non-simply connected closed manifold. Typically when  $\pi_1(M)$  equals its subgroup of commutators (perfect group), some critical points of index 1 are required for generating the fundamental group, but the Morse inequalities allow  $c_1 = 0$  (see [7] for more details). On the other hand, the only constraint for a section of  $\mathcal{M}$  with intersection numbers  $m_i$  is the Euler-Poincaré identity:

$$m_0 - m_1 + \dots = \chi(M).$$



So it is possible to find a section  $s$  whose intersection number  $m_i$  is the minimal rank in degree  $i$  of a free complex whose homology is  $H_*(M, \mathbb{Z})$ , that is,

$$m_i = \beta_i + \tau_i + \tau_{i-1},$$

where  $\beta_i$  stands for the rank of the free quotient of  $H_i(M, \mathbb{Z})$  and  $\tau_i$  denotes the minimal number of generators of its torsion subgroup ([2] p. 15). Such a set of integers satisfies the Morse inequalities but is far from being realizable by a Morse function. Finally this section  $s$  is not homotopic in  $\mathcal{M}$  to a holonomic section.

#### 4. FAILURE OF THE 1-PARAMETRIC VERSION OF THE $h$ -PRINCIPLE

We thank Yasha Eliashberg who pointed out to us the failure of the  $h$ -principle in the 1-parametric version of the problem under consideration.

Here  $M$  is assumed to be a product  $M = N \times [0, 1]$ . Let  $f_0 : M \rightarrow [0, 1]$  be the projection. When  $M$  is not 1-connected and  $\dim M \geq 6$ , according to Allen Hatcher the so-called *pseudo-isotopy problem* has always a negative answer: there exists  $f$  without critical points which is not joinable to  $f_0$  among the Morse functions (see [4]). But  $j^2 f$  can be joined to  $j^2 f_0$  by a path  $\gamma$  in  $\mathcal{M}$ . Indeed, take a generic homotopy  $\gamma^1$  joining  $df$  to  $df_0$ ; then arguing as in the proof of proposition 3.1 it is possible to lift it to  $\mathcal{M}$ . When  $M$  is the  $n$ -torus  $\mathbb{T}^n$ , A. Douady showed very simply the stronger fact that the path  $\gamma^1$  can be taken among the non-singular 1-forms (see appendix to [5]). This  $\gamma$  is not homotopic in  $\mathcal{M}$  with end points fixed to a path of holonomic sections.

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